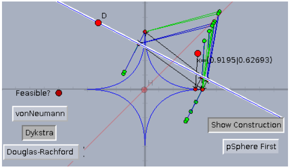
The Method of Alternating Projections

Matthew Tam

Variational Analysis Session
56th Annual AustMS Meeting
24th – 27th September 2012

My Year So Far...



Honours student supervised by Jon Borwein.
Thesis topic: alternating projections.

Over the past year I've learnt about:

- Classical alternating projection results.
- Difficulties of nonconvex alternating projections (AMSI Vacation) including development of an interactive *Cinderella* interface. `http://carma.newcastle.edu.au/summer/matt/`
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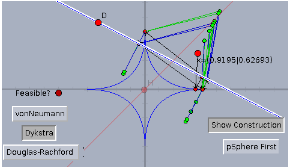

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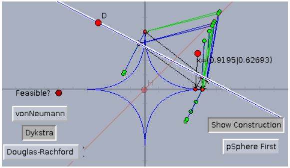

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- We are designing large scale experiments to understand this better.

Introduction

Let $\mathcal{H}$ be a Hilbert space. The (metric) projection of $x \in \mathcal{H}$ onto the set M is a point $p \in M$ such that

$$\|p - x\| \leq \|m - x\| \quad \text{for all } m \in M$$

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Given sets M, N such that $M \cap N \neq \emptyset$ can we:

- Compute $P_{M \cap N}(x)$ given $x \in \mathcal{H}$? (Best approximation)
- Find a point $x^* \in M \cap N$? (Feasibility)

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We address the question:

$$\boxed{\text{Can these problems be solved knowing only } P_M \text{ and } P_N\text{?}}$$

Two Closed Subspaces

Let M, N be closed subspaces. Then:

Fact

P_M, P_N commute if and only if their composition is equal to $P_{M \cap N}$.

If the projections commute then their composition gives $P_{M \cap N}$.

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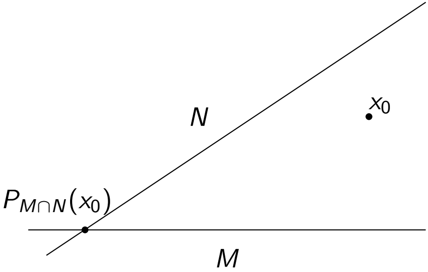
Otherwise, try projecting alternatively:

$$x_0 \overset{P_M}{\mapsto} x_1 \overset{P_N}{\mapsto} x_2 \overset{P_M}{\mapsto} x_3 \overset{P_N}{\mapsto} x_4 \overset{P_M}{\mapsto} x_5 \overset{P_N}{\mapsto} \dots$$

What happens in the limit?

Why It Works?



If H is the hyperplane given by $\langle a, x\rangle = b$ then

$$P_H(x) = x - \frac{\langle a, x\rangle}{\|a\|^2}a$$

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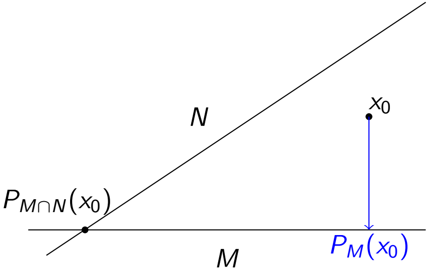

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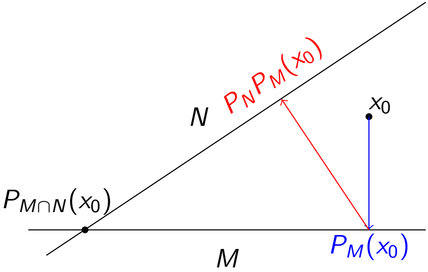

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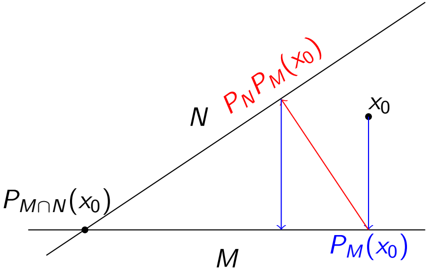

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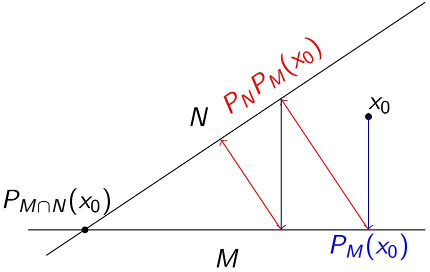

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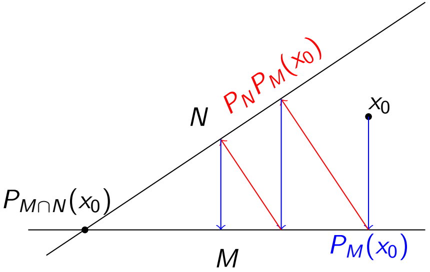

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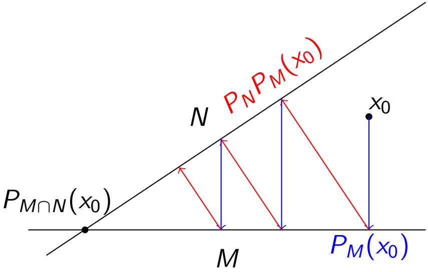

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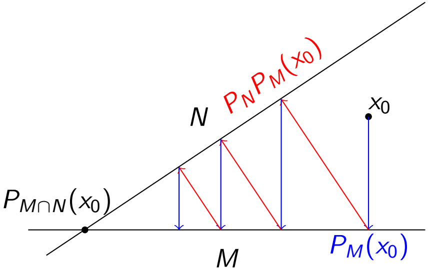

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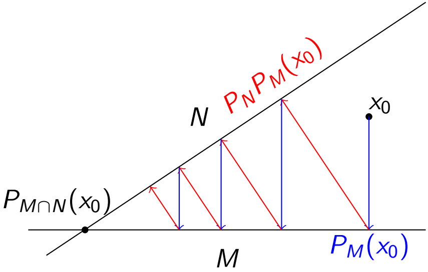

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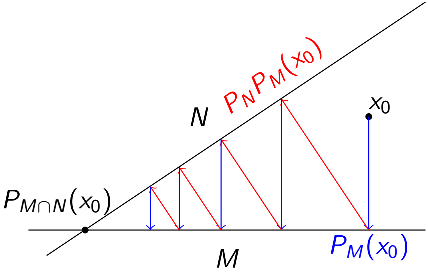

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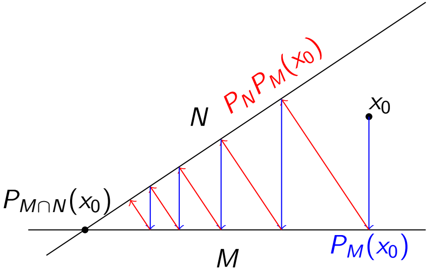

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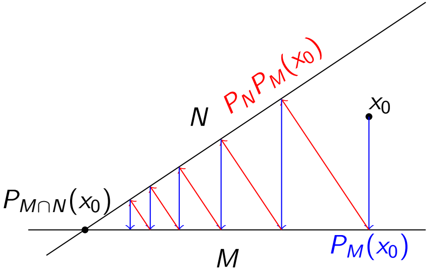

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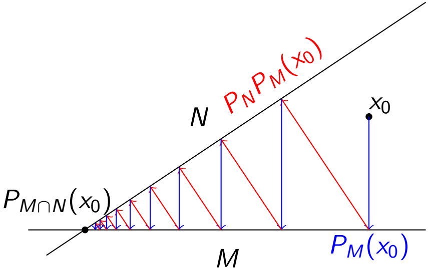

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von Neumann's Alternating Projections

Theorem (von Neumann, 1933)

Let M, $N \subseteq H$ be closed subspaces then $\forall x \in \mathcal{H}$:

$$(P_M P_N)^n(x) \to P_{M \cap N}(x)$$

Proof.

To show that (x_n) is Cauchy:

$$\color{red}{P_N}\underbrace{(\ldots P_M P_N P_M P_N)}_{k \text{ terms}} = \underbrace{(\color{red}{P_N} P_N \ldots P_N P_M P_N)}_{(k+1) \text{ terms}} \text{ or } \underbrace{(\color{red}{P_N} P_M \ldots P_N P_M P_N)}_{(\text{k}+1) \text{ terms}}$$

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Halperin's Extension

Theorem (Halperin, 1962)

Let $S_1, S_2, \dots, S_r \subseteq \mathcal{H}$ be closed subspaces then $\forall x \in \mathcal{H}$:

$$(P_{S_r} \dots P_{S_2} P_{S_1})^n(x) \to (P_{\cap_{i=1}^r S_i})(x)$$

Proof.

If T linear, nonexpansive then $\mathcal{H} = \ker(I - T) \bigoplus \mathrm{cl}(\mathrm{range}(I - T))$.

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$\|T^n x - T^{n+1} x\| \to 0$ hence $T^n(I - T)x \to 0$, $\square$

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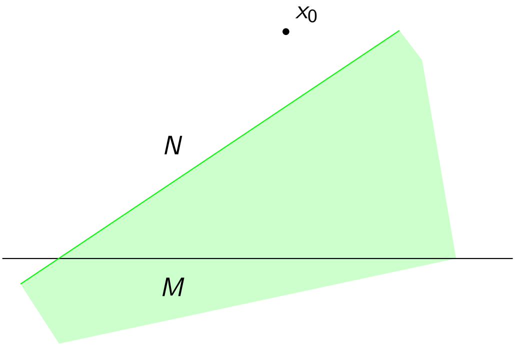

Beyond hyperplanes, half-spaces, spheres, balls,... projections are difficult to compute. Even for the ellipse in $\mathbb{R}^2$, given by $x^2/a^2 + y^2/b^2 = 1$, we have:

$$P_E(u,v) = \left(\frac{a^2 u}{a^2 - t}, \frac{b^2 v}{b^2 - t}\right) \quad \text{where} \quad \frac{a^2 u^2}{a^2 - t} + \frac{b^2 v^2}{b^2 - t} = 1$$

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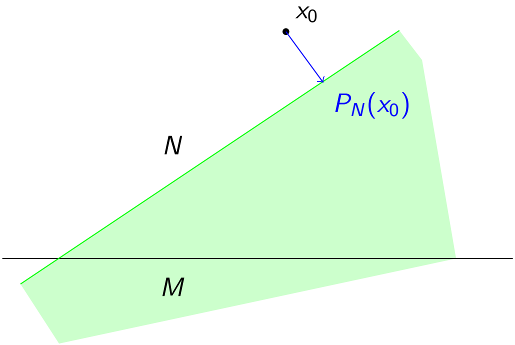

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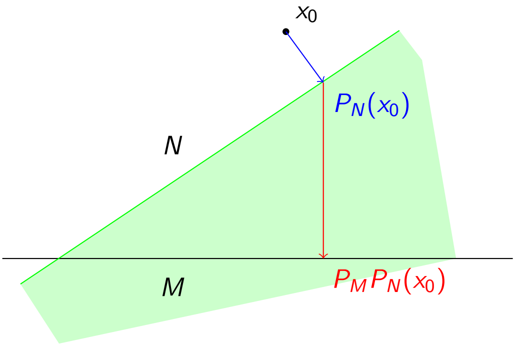

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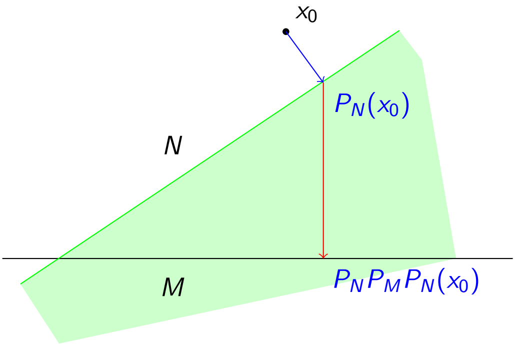

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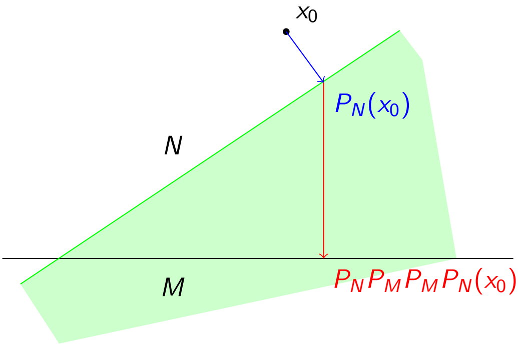

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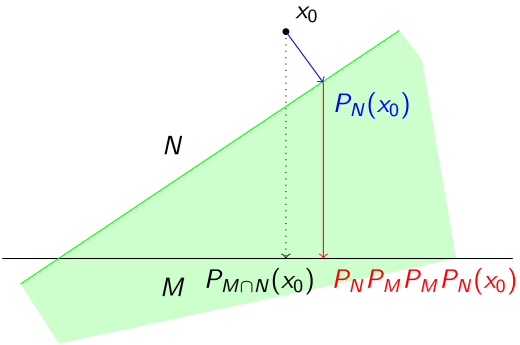

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Bregman's Generalisation

Theorem (Bregman, 1965)

Let $C_1, C_2, \dots, C_r \subseteq \mathcal{H}$ be closed convex sets then $\forall x \in \mathcal{H}$:

$$(P_{C_r} \dots P_{C_2} P_{C_1})^n(x) \xrightarrow{w} x^* \in \bigcap_{i=1}^{r} C_i$$

Proof.

Use weak compactness to extract a weakly convergence subsequence. □

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$\boxed{\text{Can MAP fail to converge in norm?}}$

The Hundal Example (Revisited)

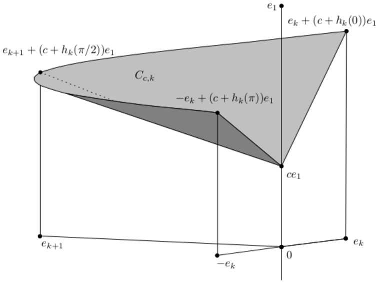
Failure of Norm Convergence (Hundal, 2004)

Let $\mathcal{H} = \ell_2$ and $\{e_i\}$ an orthonormal basis. Take $x_0 = e_3$ and

$$C_1 = \ker(e_1) \text{ and } C_2 = \operatorname{cl}\operatorname{conv}\bigcup_{k=2}^{\infty} \operatorname{epi} C_{0,k}$$

then MAP fails to converge is norm.

Note: C_1 is a hyperplane and C_2 a closed convex cone.



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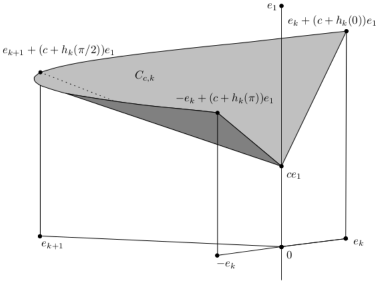

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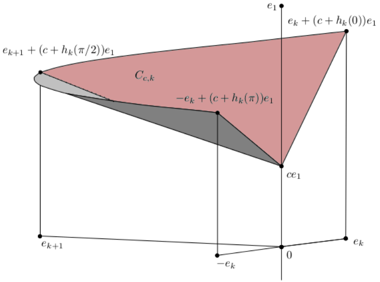

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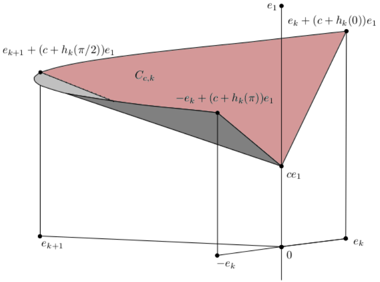

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Final step:

$$\|(P_{C_2}P_{C_1})^{N_m} e_{k_0} - e_m\| < 1/7$$
$$\implies \|(P_{C_2}P_{C_1})^{N_m} e_{k_0}\| > 6/7$$

The Hundal Example (Revisited)

Can it fail to converge in norm on a 'real' problem?

Conjecture (Borwein & Bauschke, 1993)

If C_1 is closed and affine with finite codimension, C_2 is the nonnegative cone in $\ell_2(\mathbb{N})$ then MAP is norm convergent.

- True when C_1 is a hyperplane (unlike Hundal).
- This captures most concrete applications.

Averaged Projections (The Crucial Product Space *Trick*)

Given $C_1, C_2, \dots, C_r \subseteq \mathcal{H}$ consider $\mathcal{H}^r = \mathcal{H} \times \cdots \times \mathcal{H}$. Define:

$$C = \{(x_1, x_2, \dots, x_r) : x_i \in C_i\}, \quad D = \{(x_1, x_2, \dots, x_r) : x_1 = x_i\}$$

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It is easily verified that:

$$(P_C \mathbf{x})_i = P_{C_i} x_i, \quad (P_D \mathbf{x})_i = \frac{1}{r} \sum_{j=1}^{r} x_j$$

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Each iteration, $T : \mathcal{H} \to \mathcal{H}$, can be described by:

$$\boxed{Tx = \frac{1}{r}\sum_{j=1}^{r} P_{C_j}x}$$

Douglas-Rachford and Dykstra Methods

Theorem (Lions & Mercier, 1979)

Let $C_1,\ C_2 \subseteq \mathcal{H}$ be closed convex sets, $\forall x \in \mathcal{H}$ iterate:

$$x_{n+1} := \frac{x_n + R_{C_2}R_{C_1}(x_n)}{2} \qquad \text{where } R_{C_i}(x) := 2P_{C_i}(x) - x$$

then $x_n \xrightarrow{w.} x$, a fixed point, with $P_{C_1}(x) \in C_1 \cap C_2$.

Theorem (Boyle & Dykstra, 1980)

Let $C_1, \ldots, C_r \subseteq \mathcal{H}$ be closed convex sets, $\forall x \in \mathcal{H}$ iterate:

$$x_n^i := P_{C_i}(x_n^{i-1} - I_{n-1}^i), \quad I_n^i := x_n^i - (x_n^{i-1} - I_{n-1}^i), \quad x_n^0 := x_{n-1}^r$$

with initial values $x_1^0 := x$, $I_0^i := 0$ then $x_n \to (P_{\cap_{i=1}^r C_i})(x)$.

Non-Convex Sets

Projections onto non-convex sets are no longer guaranteed to be:

- Nonexpansive/Firmly nonexpansive
- Unique (i.e. P_C is set-valued). The method becomes:

$$x_{2n+1} \in P_{C_1}(x_{2n}), \quad x_{2n} \in P_{C_2}(x_{2n-1})$$

In $\mathbb{R}^n$:

- "*Local linear convergence for alternating and averaged nonconvex projections*", Lewis, Luke & Malick (2009).
- "*Restricted normal cones and the method of alternating projections*", Bauschke, Luke, Phan & Wang (2012).
- "*The Douglas-Rachford algorithm in the absence of convexity*", Borwein & Sims (2011).

Sudoku: Modelling an NP-Complete Non-Convex Problem

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

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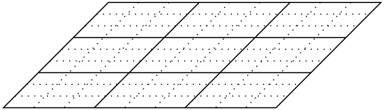

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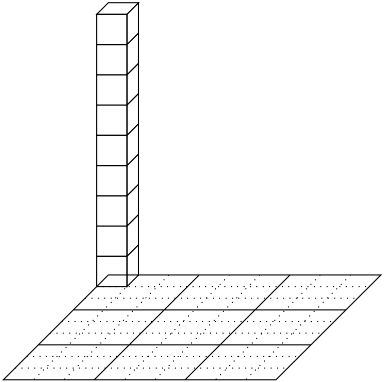

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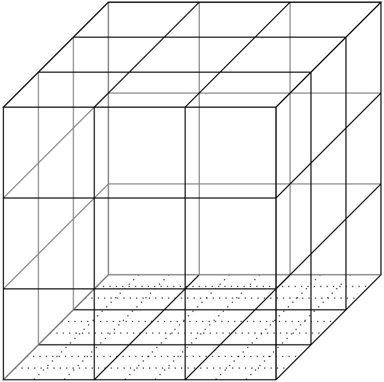

Let $A \in (\mathbb{R}^9)^3$ indexed by (i, j, k).
Constraint types are:

$$C_1 = \{A_{ij} \text{ is a standard unit vector}\}$$
$$C_2 = \{A_{ik} \text{ is a standard unit vector}\}$$
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A solution is $x^* \in C_1 \cap C_2 \cap C_3 \cap C_4$.

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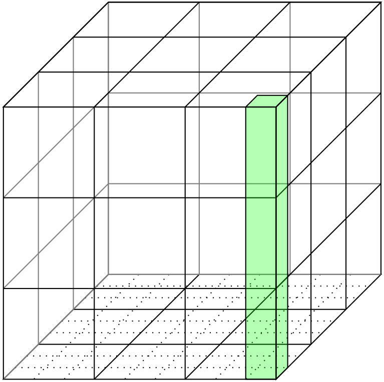

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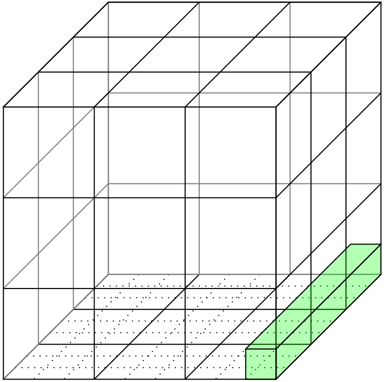

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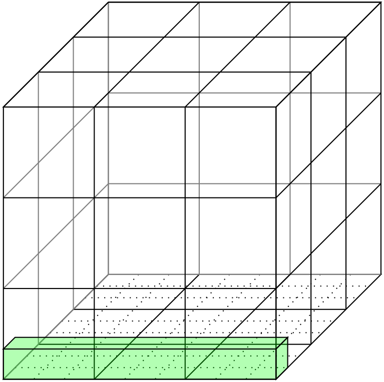

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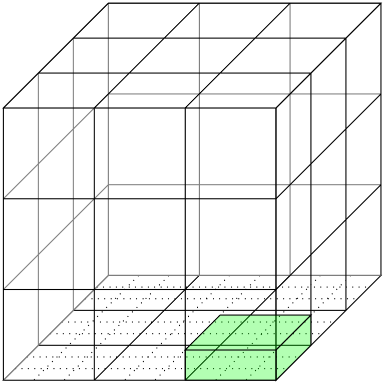

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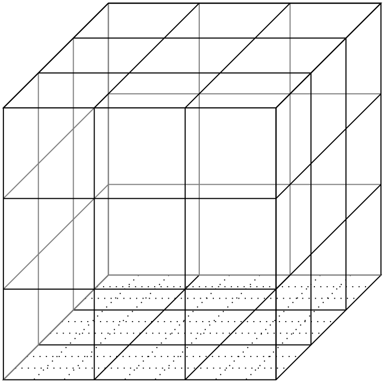

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Similar modelling can be done for:

- N-queens
- 3-SAT (NP-Complete)
- TetraVex (NP-Complete)

Solves large instances! (Sudoku $= \mathbb{R}^{2916}$)

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