The Douglas Rachford Reflection Method and Generalizations

Laureate Prof. Jonathan Borwein with Matthew Tam

http://carma.newcastle.edu.au/DRmethods/paseky.html

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Feasibility Problem

Given closed sets $C_1, C_2, \ldots, C_N \subseteq \mathcal{H}$ the feasibility problem asks

$$\text{find } x \in \bigcap_{j=1}^{N} C_j.$$

Many problems can be cast is this form. Three examples:

1. Linear systems "$Ax = b$": $C_j = \{x : \langle a_j, x\rangle = b_j\}$.
2. Phase retrieval: $C_1 = \{f : |\hat{f}| = m \text{ a.e.}\}$ and $C_2 = \{f : f = 0 \text{ on } D\}$.
3. **Matrix completion problems:** more on this later!

Projection algorithms are a popular approach to solving feasibility problems. They work on the following principle:

1. While the intersection might be difficult to deal with directly, the individual constraint sets are sufficiently "simple".
2. "Simple" means we can efficiently compute nearest points.
3. Use an iterative scheme which employs nearest points to individual constraint sets at each stage, and obtain a solution in the limit.

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Douglas, Rachford & Peaceman

Jim Douglas Jnr (1927 –)

Henry Rachford

Donald Peaceman

Let $S \subseteq \mathcal{H}$ be non-empty. The (nearest point) projection onto S is the (set-valued) mapping,

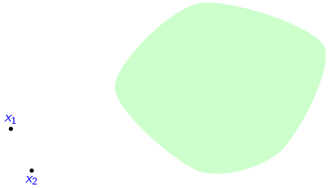
$$P_S x := \left\{ s \in S : \|x - s\| \leq \inf_{s \in S} \|x - s\| \right\}.$$

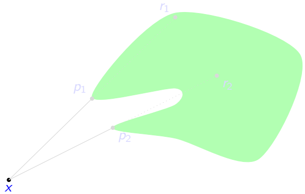
If S is closed and convex then projections exists uniquely with

$$P_S(x) = p \iff \langle x - p, s - p \rangle \leq 0 \text{ for all } s \in S.$$

The reflection w.r.t. S is the (set-valued) mapping,

$$R_S := 2P_S - I.$$





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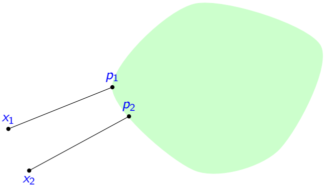

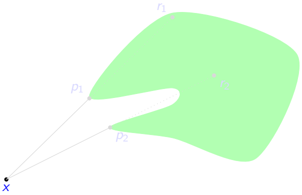

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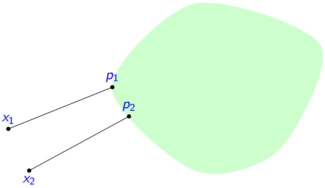

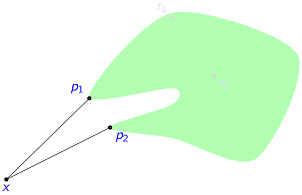

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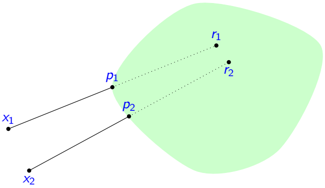

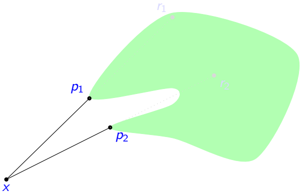

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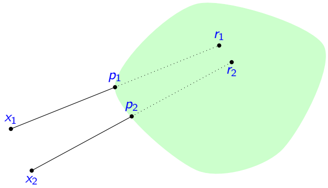
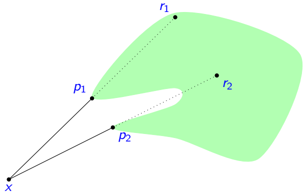


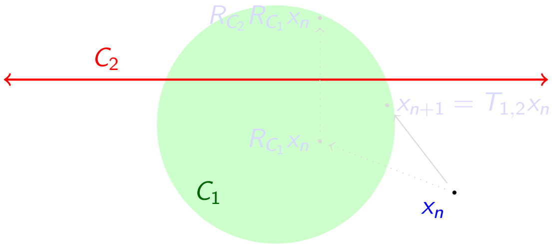


The Douglas–Rachford Algorithm

Given an initial point $x_0 \in \mathcal{H}$, the Douglas–Rachford method is the fixed-point iteration given by

$$x_{n+1} \in T_{C_1,C_2} x_n \quad \text{where} \quad T_{C_1,C_2} := \frac{Id + R_{C_2} R_{C_1}}{2}.$$

We hope that (x_n) converges to a fixed point of of the operator T_{C_1,C_2}.



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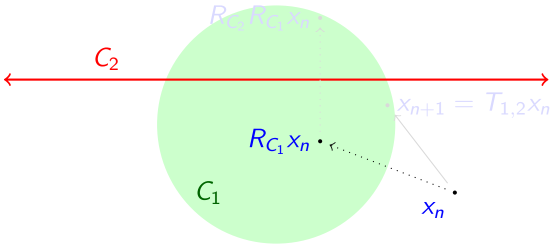

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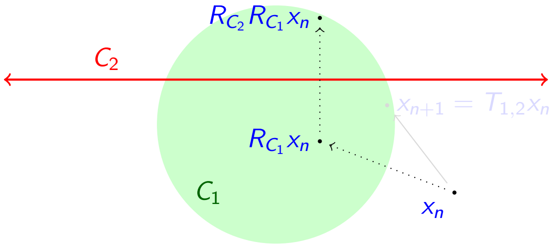

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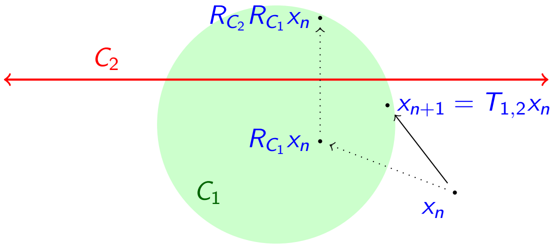

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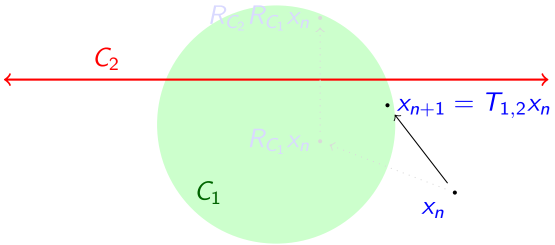

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Douglas–Rachford Fixed Points

Why $\mathrm{Fix}\, T_{C_1,C_2}$**?** Assuming single-valueness of R_{C_1} and R_{C_2} we have:

$$x \in \mathrm{Fix}\, T_{C_1,C_2} \qquad \Longleftrightarrow \qquad x = \frac{x + R_{C_2} R_{C_1} x}{2}$$

The same argument for the set-valued case yields:

- If $x \in T_{C_1,C_2} x$ then there is an element of $P_{C_1} x$ contained in $C_1 \cap C_2$.

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&\iff \quad P_{C_1} x = P_{C_2} R_{C_1} x \\
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Tools from Nonexpansive Mapping Theory

Let $T : \mathcal{H} \to \mathcal{H}$. Then T is:

- nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in \mathcal{H}.$$

- firmly nonexpansive if

$$\|Tx - Ty\|^2 + \|(I - T)x - (I - T)y\|^2 \leq \|x - y\|^2, \quad \forall x, y \in \mathcal{H}.$$

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Proposition (Nonexpansive properties)

The following are equivalent.

- T is firmly nonexpansive.
- $I - T$ is firmly nonexpansive.
- $2T - I$ is nonexpansive.
- $T = \alpha I + (1 - \alpha)R$, for $\alpha \in (0, 1/2]$ and some nonexpansive R.
- $\langle x - y, Tx - Ty\rangle \geq \|Tx - Ty\|^2$ for all $x, y \in \mathcal{H}$.
- Other characterisations.

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Nonexpansive properties of projections

Let $C_1, C_2 \subseteq \mathcal{H}$ be closed and convex. Then

- $P_{C_1} := \arg\min_{c \in C_1} \|\cdot - c\|$ is firmly nonexpansive.
- $R_{C_1} := 2P_{C_1} - I$ is nonexpansive.
- $T_{C_1,C_2} := \frac{1}{2}(I + R_{C_2} R_{C_1})$ is firmly nonexpansive.

Nonexpansive maps are closed under composition, convex combinations, etc. Firmly nonexpansive maps need not be. E.g., Composition of two projections onto subspace in $\mathbb{R}^2$ (Bauschke–Borwein–Lewis, 1997).

Tools from Nonexpansive Mapping Theory (cont.)

- asymptotically regular if, for all $x \in \mathcal{H}$,

$$\|T^{n+1}x - T^n x\| \to 0.$$

Lemma (Asymptotic regularity)

Every firmly nonexpansive mapping with at least one fixed point is asymptotically regular.

Proof. Let $z \in \operatorname{Fix} T$ then, for any $x \in \mathcal{H}$, we have

$$\|T^{n+1}x - z\|^2 + \|(I-T)(T^n x)\|^2 = \|T(T^n x) - Tz\|^2 + \|(I-T)(T^n x) - (I-T)z\|^2 \leq \|T^n x - z\|^2.$$

Hence $\lim_{n\to\infty} \|T^n x - z\|$ exists, and thus $\|(I-T)(T^n x)\| \to 0$. ●

A useful Theorem for building iterative schemes:

Theorem (Opial, 1967)

Let $T : \mathcal{H} \to \mathcal{H}$ be nonexpansive and asymptotically regular with $\operatorname{Fix} T \neq \emptyset$. Set $x_{n+1} = Tx_n$. Then $x_n \xrightarrow{w} x$ such that $x \in \operatorname{Fix} T$.

→ Design a non-expansive operator with a useful fixed point set.

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Proof of Opial's Theorem

Lemma (Demiclosedness)

Let $T : \mathcal{H} \to \mathcal{H}$ be nonexpansive and denote $x_n := T^n x_0$ for some initial point $x_0 \in \mathcal{H}$. Suppose $x_n \xrightarrow{w} x$ and $x_n - Tx_n \to 0$. Then $x \in \operatorname{Fix} T$.

Proof. Since T is nonexpansive,

$$\begin{aligned} \|x - Tx\|^2 &= \|x_n - Tx\|^2 - \|x_n - x\|^2 - 2\langle x_n - x, x - Tx\rangle \\ &= \|x_n - Tx_n\|^2 + 2\langle x_n - Tx_n, Tx_n - Tx\rangle + \|Tx_n - Tx\|^2 \\ &\qquad - \|x_n - x\|^2 - 2\langle x_n - x, x - Tx\rangle \\ &\leq \|x_n - Tx_n\|^2 + 2\langle x_n - Tx_n, \underbrace{Tx_n}_{x_{n+1}} - Tx\rangle - 2\langle x_n - x, x - Tx\rangle. \end{aligned}$$

Since $x_n \xrightarrow{w} x$ and $x_n - Tx_n \to 0$, it follows that each term tends to 0. ∎

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Indeed, as $\{x_n\}$ is bounded, it contains at least one weak cluster point. Let z be any weak cluster point and denote by $\{x_{n_k}\}_{k\in\mathbb{N}}$ a subsequence weakly convergent to z. Since T is asymptotically regular,

$$\|x_{n_k} - Tx_{n_k}\| \to 0.$$

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The Douglas–Rachford Algorithm

The basic result which we have proven is the following.

Theorem (Douglas–Rachford '56, Lions–Mercier '79, Eckstein–Bertsekas '92, ...)

Suppose $C_1, C_2 \subseteq \mathcal{H}$ are closed and convex with non-empty intersection. Given $x_0 \in \mathcal{H}$ define

$$x_{n+1} := T_{C_1,C_2} x_n \quad \text{where} \quad T_{C_1,C_2} := \frac{I + R_{C_2} R_{C_1}}{2}.$$

Then (x_n) converges weakly to some $x \in \operatorname{Fix} T_{C_1,C_2}$ with $P_{C_1} x \in C_1 \cap C_2$.

Proof. Since $C_1 \cap C_2 \subseteq \operatorname{Fix} T_{C_1,C_2}$, the latter is non-empty. Thus T_{C_1,C_2} is (firmly) nonexpansive with a fixed point, hence asymptotically regular by the previous lemma. The result follows from Opial's Theorem.

- If the intersection is empty the iterates diverge: $\|x_n\| \to \infty$.
- Bauschke–Combettes–Luke (2004): Thorough analysis of convex case.
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The following generalization include potentially empty intersections. Let

$$V := \overline{C_1 - C_2}, \qquad v := P_V(0), \qquad F := C_1 \cap (C_2 + v).$$

Theorem (Bauschke–Combettes–Luke 2004)

Suppose $C_1, C_2 \subseteq \mathcal{H}$ are closed and convex. Given $x_0 \in \mathcal{H}$ define $x_{n+1} := T_{C_2,C_1} x_n$. Then the following hold.

(a) $x_n - x_{n+1} = P_{C_1} x_n - P_{C_2} R_{C_1} \to v$ and $P_{C_1} x_n - P_{C_2} P_{C_1} \to v$.

(b) If $C_1 \cap C_2 \neq \emptyset$ then (x_n) converges weakly to a point in

$$\mathrm{Fix}\, T_{C_1,C_2} = C_1 \cap C_2 + N_V(0);$$

otherwise, $\|x_n\| \to +\infty$.

(c) Exactly one of the following alternatives holds:

(i) $F = \emptyset$, $\|P_{C_1} x_n\| \to +\infty$ and $\|P_{C_2} P_{C_1} x_n\| \to +\infty$.

(ii) $F \neq \emptyset$, the sequence $(P_{C_1} x_n)$ and $(P_{C_2} P_{C_1} x_n)$ are bounded and their weak cluster points are best approximation pairs relative to (C_1, C_2).

The Douglas–Rachford Algorithm: Moment Problem

Recall the moment problem from Lecture I for linear map $A : X \to \mathbb{R}^M$ and a point $y \in \mathbb{R}^M$ has constraints:

$$C_1 := \mathcal{H}^+, \qquad C_2 := \{x \in \mathcal{H} : A(x) = y\}.$$

The following theorem gives conditions for norm convergence.

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Let $\mathcal{H}$ be a Hilbert lattice, $C_1 := \mathcal{H}^+$, C_2 be a closed affine subspace with finite codimensions, and $C_1 \cap C_2 \neq \emptyset$. For $x_0 \in \mathcal{H}$ define $x_{n+1} = T_{C_1,C_2} x_n$. Let Q denote the projection onto the subspace parallel to C_2. Then (x_n) converges in norm whenever:

(a) $C_1 \cap \text{range}(Q) = \{0\}$,

(b) $Q(C_2 - C_1) \subseteq C_1 \cup (-C_1)$ and $Q(C_1) \subseteq C_1$.

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Pierra's Product Space Reformulation

For our constraint sets $C_1, C_2, \ldots, C_N \subseteq \mathcal{H}$ we define

$$\mathbf{D} := \{(x, x, \ldots, x) \in \mathcal{H}^N : x \in \mathcal{H}\}, \quad \mathbf{C} := \prod_{j=1}^{N} C_j.$$

We now have an equivalent two set feasibility problem since

$$x \in \bigcap_{j=1}^{N} C_j \subseteq \mathcal{H} \iff (x, x, \ldots, x) \in \mathbf{D} \cap \mathbf{C} \subseteq \mathcal{H}^N.$$

Moreover the projections onto the new sets can be computed whenever $P_{C_1}, P_{C_2}, \ldots, P_{C_N}$. Denote $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ they are given by

$$P_{\mathbf{D}}\mathbf{x} = \left(\frac{1}{N}\sum_{j=1}^{N} x_i\right)^N \quad \text{and} \quad P_{\mathbf{C}}\mathbf{x} = \prod_{j=1}^{N} P_{C_j} x_j.$$

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Is there a Douglas–Rachford variant which can be used to solve the problem in the original space? *i.e.*, Without recourse to a product space formulation?

An obvious candidate is the following: Given $x_0 \in \mathcal{H}$ define

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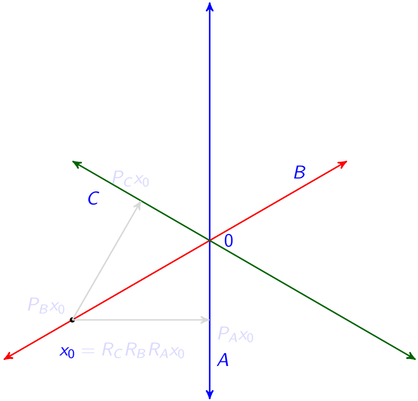

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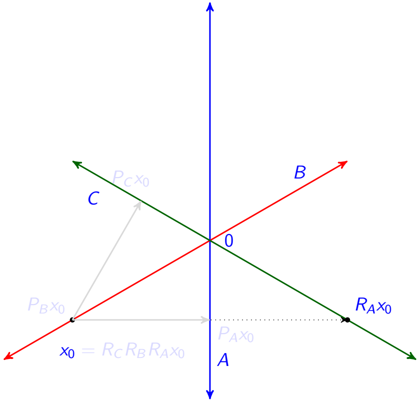

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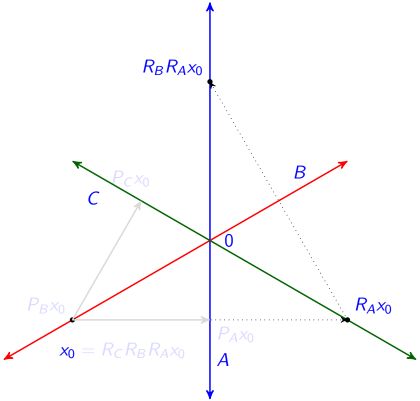

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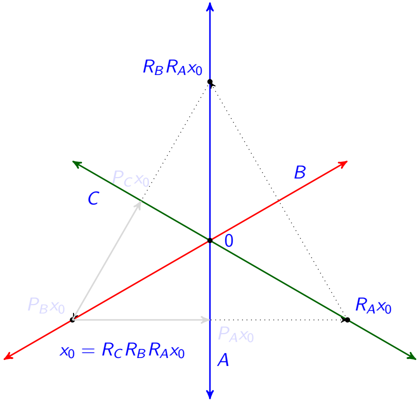

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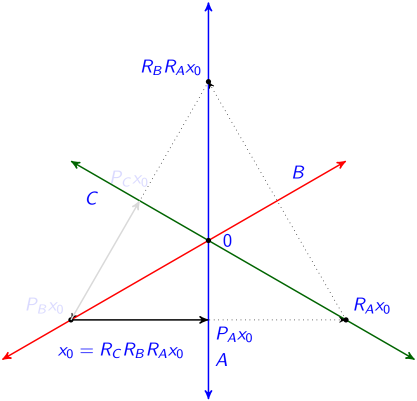

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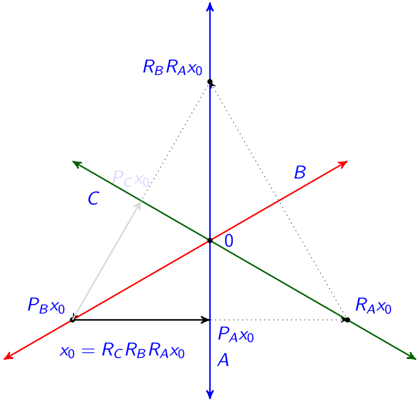

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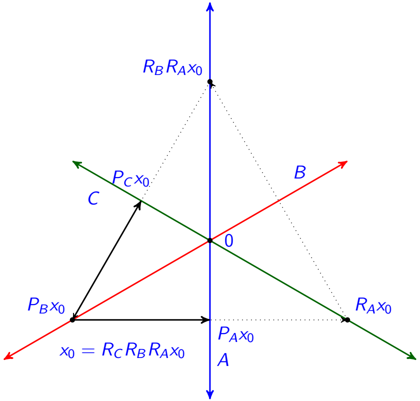

A Common Framework

Theorem (Borwein–Tam 2013)

Let $C_1, \ldots, C_N \subseteq \mathcal{H}$ be closed convex sets with nonempty intersection, let $T_j : \mathcal{H} \to \mathcal{H}$ and denote $T := T_M \ldots T_2 T_1$. Suppose the following three properties hold.

(i) T is nonexpansive and asymptotically regular,

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Then, for any $x_0 \in \mathcal{H}$, the sequence $x_n := T^n x_0$ converges weakly to some x such that $P_{C_1} x = P_{C_2} x = \cdots = P_{C_N} x$. In particular, $P_{C_1} x \in \bigcap_{i=1}^{N} C_i$.

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$$\langle x - P_{C_{j+1}} x, P_{C_j} x - P_{C_{j+1}} x \rangle \leq 0 \text{ for all } j$$

A Common Framework

To complete the proof observe

$$
\begin{aligned}
&\frac{1}{2}\sum_{j=1}^{N}\|P_{C_{j+1}}x - P_{C_j}x\|^2 \\
&\quad = \langle x, 0\rangle + \frac{1}{2}\sum_{j=1}^{N}\left(\|P_{C_{j+1}}x\|^2 - 2\langle P_{C_{j+1}}x, P_{C_j}x\rangle + \|P_{C_j}x\|^2\right) \\
&\quad = \left\langle x, \sum_{j=1}^{N}(P_{C_j}x - P_{C_{j+1}}x)\right\rangle - \sum_{j=1}^{N}\langle P_{C_{j+1}}x, P_{C_j}x\rangle + \sum_{j=1}^{N}\|P_{C_{j+1}}x\|^2 \\
&\quad = \sum_{j=1}^{N}\langle x, (P_{C_j}x - P_{C_{j+1}}x)\rangle - \sum_{j=1}^{N}\langle P_{C_{j+1}}x, P_{C_j}x - P_{C_{j+1}}x\rangle \\
&\quad = \sum_{j=1}^{N}\langle x - P_{C_{j+1}}x, P_{C_j}x - P_{C_{j+1}}x\rangle \le 0.
\end{aligned}
$$

Composition of DR-Operators

We require one final theorem.

Theorem (Bauschke *et al.* 2012)

Suppose that each $T_i : \mathcal{H} \to \mathcal{H}$ is firmly nonexpansive and asymptotically regular. Then $T_m T_{m-1} \dots T_1$ is also asymptotically regular.

The proof can be found in:

H.H. Bauschke, V. Martin-Marquez, S.M. Moffat, and X. Wang. **Compositions and convex combinations of asymptotically regular firmly nonexpansive mappings are also asymptotically regular**, *Fixed Point Theory and Applications* 2012, 2012:53.

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Cyclic Douglas–Rachford Method

Corollary (Borwein–Tam 2013)

Let $C_1, C_2, \ldots, C_N \subseteq \mathcal{H}$ be closed and convex with non-empty intersection. Given $x_0 \in \mathcal{H}$ define

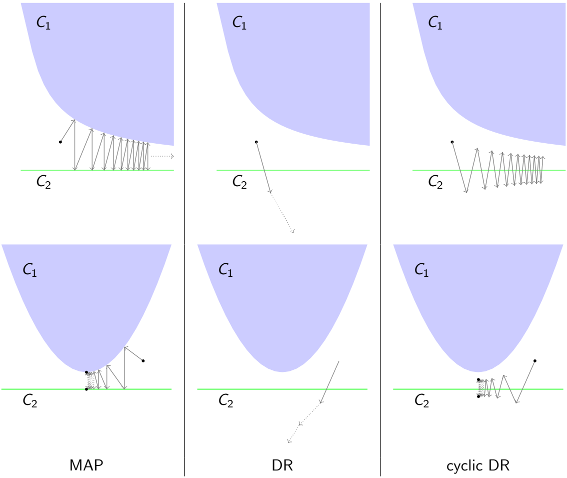
$$x_{n+1} := \underbrace{\left(T_{C_N,C_1} T_{C_{N-1},C_N} \ldots T_{C_2,C_3} T_{C_1,C_2}\right)}_{=:T_{[1\,2\,\ldots\,N]}} x_n \text{ where } T_{C_j,C_{j+1}} = \frac{I + R_{C_{j+1}} R_{C_j}}{2}.$$

Then (x_n) converges weakly to a point x such that $P_{C_1}x = \cdots = P_{C_N}x$.

- Borwein–Tam (arXiv:1310.2195): Analysed behaviour for empty intersections.
- Using Hundal (2004): There exists a hyperplane and convex cone with nonempty intersection such that convergence is not strong.
- Bauschke–Noll–Phan (2014): If $\dim \mathcal{H} < \infty$ and $\cap_{j=1}^{N} \operatorname{ri} C_j \neq \emptyset$ then convergence is linear.
- Bauschke–Noll–Phan (2014): If $\operatorname{Fix} T_{[1\,2\,\ldots\,N]}$ is bounded linearly regular and $C_j + C_{j+1}$ is closed, for each j, then convergence is linear.

Consider the following examples with $C_2 := 0 \times \mathbb{R}$, and

$$C_1 := \text{epi}(\exp(\cdot) + 1) \quad \text{or} \quad \text{epi}((\cdot)^2 + 1).$$



Averaged Douglas–Rachford Method

The following variant lends itself to parallel implementation.

Corollary (Borwein-Tam 2013)

Let $C_1, C_2, \ldots, C_N \subseteq \mathcal{H}$ be closed and convex with non-empty intersection. Given $x_0 \in \mathcal{H}$ define

$$x_{n+1} := \frac{1}{N}\left(\sum_{j=1}^{N} T_{C_j,C_{j+1}}\right)x_n \quad \text{where} \quad T_{C_j,C_{j+1}} = \frac{I + R_{C_{j+1}}R_{C_j}}{2}.$$

Then (x_n) converges weakly to a point x such that $P_{C_1}x = \cdots = P_{C_N}x$.

Proof sketch. For $x_0 \in \mathcal{H}$, set $\mathbf{x}_0 = (x_0, \ldots, x_0) \in \mathcal{H}^N$. Apply the theorem to the product-space iteration

$$\mathbf{x}_{n+1} = P_D\left(\prod_{i=1}^{N} T_{C_i,C_{i+1}}\right)\mathbf{x}_n \in D \subseteq \mathcal{H}^N.$$

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Cyclically Anchored Douglas–Rachford Method

Choose the first set C_1 to be the anchor set, and think of

$$\bigcap_{j=1}^{N} C_j = C_1 \cap \left(\bigcap_{j=2}^{N} C_j \right).$$

Theorem (Bauschke–Noll–Phan 2014)

Let $C_1, C_2, \ldots, C_N \subseteq \mathcal{H}$ be closed and convex with non-empty intersection. Given $x_0 \in \mathcal{H}$ define

$$x_{n+1} := \prod_{j=2}^{N} T_{C_1,C_j} x_n \quad \text{where} \quad T_{C_1,C_j} = \frac{I + R_{C_j} R_{C_1}}{2}.$$

Then (x_n) converges weakly to a point x such that $P_{C_1} x \in \bigcap_{j=1}^{N} C_j$.

- Bauschke–Noll–Phan (2014): If $\dim \mathcal{H} < \infty$ and $\cap_{j=1}^{N} \operatorname{ri} C_j \neq \emptyset$ then convergence is linear.
- Bauschke–Noll–Phan (2014): For subspaces, if $\operatorname{Fix} T_{C_1,C_j}$ is bounded linearly regular and $C_1 + C_j$ is closed then convergence is linear.

Averaged Anchored Douglas–Rachford Method

The scheme also has a parallel counterpart:

Theorem

Let $C_1, C_2, \dots, C_N \subseteq \mathcal{H}$ be closed and convex with non-empty intersection. Given $x_0 \in \mathcal{H}$ define

$$x_{n+1} := \frac{1}{N-1}\left(\sum_{j=1}^{N} T_{C_1,C_j}\right) x_n \quad \text{where} \quad T_{C_1,C_j} = \frac{I + R_{C_j} R_{C_i}}{2}.$$

Then (x_n) converges weakly to a point x such that $P_{C_1} x \in \bigcap_{j=1}^{N} C_j$.

Proof sketch. Use the product space (as we did for the averaged DR iteration) up the iteration:

$$\mathbf{x}_{n+1} = P_D\left(\prod_{i=1}^{N} T_{C_1,C_j}\right)\mathbf{x}_n \in D \subseteq \mathcal{H}^{N-1}.$$

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Commentary and Open Questions

- The (classical) Douglas–Rachford method better than theory suggests performance on non-convex problems. Consequently many variants and extensions have recently been proposed.
- Even in the convex setting there are many subtleties and open questions.
  - Norm convergence for realistic moment problems with codimension greater than 1?
- Experimental comparison of the variants needed.

Exercises

1. Let $T_j : \mathcal{H} \to \mathcal{H}$ be firmly nonexpansive, for $j = 1, \ldots, r$, and define $T := T_r \ldots T_2 T_1$. If $\mathrm{Fix}\, T \neq \emptyset$ show that T is asymptotically regular.
2. Show that the cyclic DR method becomes MAP in certain cases. Hence find an example where convergence in cyclic DR is only weak.
3. (**Hard**) Prove or disprove: The Douglas–Rachford algorithm converges in norm for the moment problem when the affine set has codimension 2.

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Many resources available at:

http://carma.newcastle.edu.au/DRmethods