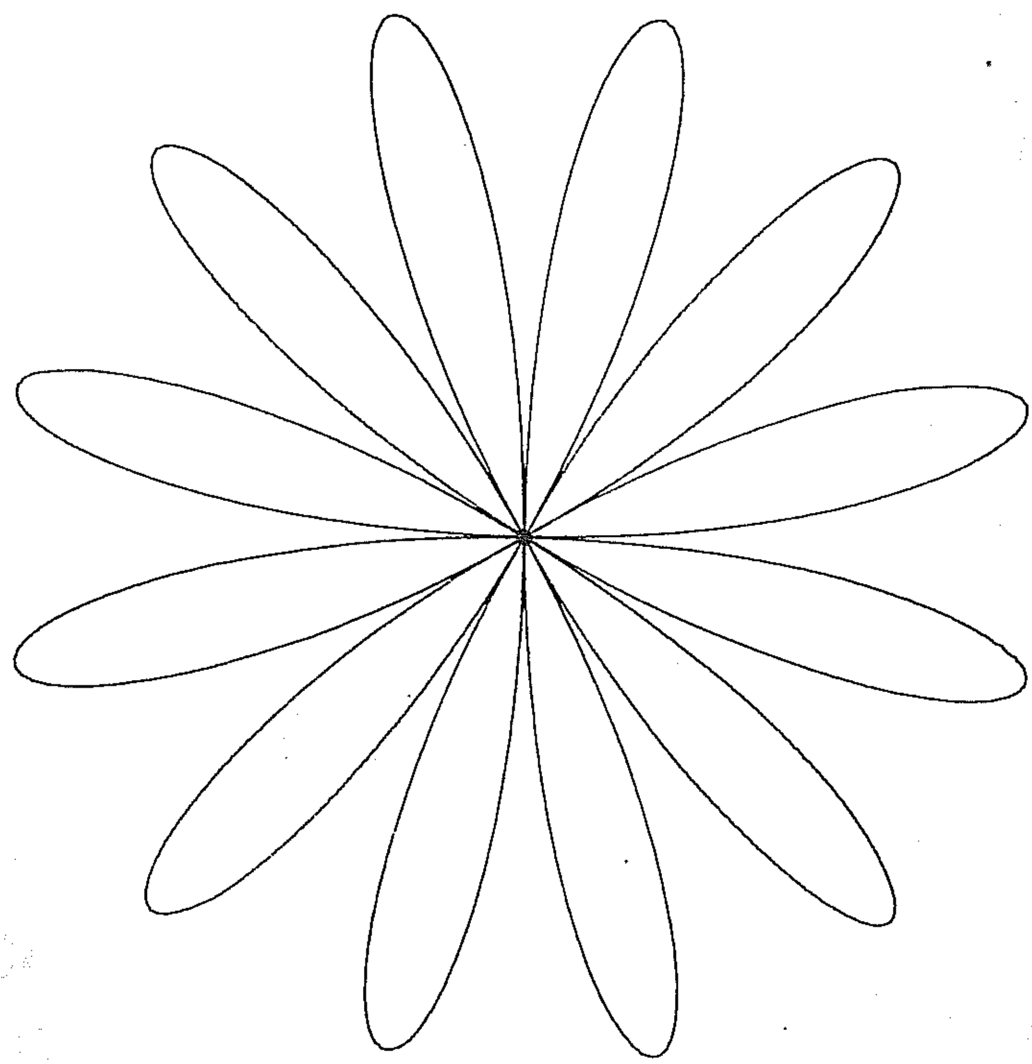
UNE

Department of Mathematics,
Statistics and Computing Science

Pure Mathematics 111-2

Introductory Calculus, Complex Numbers, Algebra and Analytic Geometry

Second Semester



THE UNIVERSITY OF NEW ENGLAND

Pure Mathematics 111-2

Course information, Second semester 1988

Dear Pure Mathematics 111-2 Student,

Enclosed are notes covering the course for second semester. The bulk of the notes are those written by Dr. B Sims for Pure Mathematics 111-2 in previous years. Some notations in the notes with which you may not be familiar are:

$\simeq$ for "approximately equals", as in $\pi \simeq \frac{22}{7}$;

$\doteq$ for "equals to the number of decimal places quoted", as in $\pi \doteq 3.14159$;

$:=$ for "equals by definition";

$f : A \to B : x \mapsto f(x)$ to specify the function f mapping the set A (the domain of f) into the set B (its codomain), which assigns to the element x of A the element $f(x)$ of B. Often $f(x)$ is given by a formula; for example

$$f : [-1, 1] \to \mathbb{R} : x \mapsto \sqrt{1 - x^2} \ .$$

This year the following sections of the main notes will be omitted from the course: §§9.3–9.4, p.186–191 and exercises 7–8, p.192–193; §§10.4–10.5, p.209–217 and exercises 9–11, p.221; §11.4, p.243–245. They will be replaced by the material on functions of two variables in the supplementary notes. These notes also contain miscellaneous explanatory material and a simplified treatment of §8.3, p.162–169, which will be covered more lightly than in previous years.

For internal students there will be a *compulsory* test on integration, to be held during the lecture period on Friday, 23rd September. A mark of well over 50% will have to be scored in order to pass, and students who do not achieve this mark will be required to repeat the test at a date and time to be determined.

A number of assignments are set throughout the semester; these should be submitted by internal students to their tutor, or posted by external students, by the dates shown on p.ii–iii. Please note that submission of assignments (except for the last which is optional) is a requirement for completing this course.

External students should send their assignments (in an assignment folder if possible) to the Department of External Studies; any written queries should be directed to me personally at the Department of Mathematics. You are also welcome to inquire by telephone ((067) 73 2350) about any problems you may have. Internal students may see me in my office (Booth Block 163) any time I am free.

Hoping you enjoy your studies.

David Angell

David Angell
1st July 1988

CONTENTS

LECTURE SCHEDULE, SECOND SEMESTER 1988

Date		Lecture	Topic	Pages in Notes	Pages in Supplement
August	1	38	Integration	116–120	
	3	39	Integration	120–124	
	5	40	Integration	125–128	S1
	8	41	Integration	128–130	
	10	42	Integration	131–134	
	12	43	Integration	135–136	S1
			Assignment 13 is due on August 12th		
August	15	44	Integration	137–139	
	17	45	Numerical integration	141–144	S1
	19	46	The exponential function	146–149	
			Assignment 14 is due on August 19th		
August	22	47	The exponential function	149–151	S1
	24	48	The natural logarithm	156–158	S1–S2
	26	49	The natural logarithm	158–159	
			Assignment 15 is due on August 26th		
August	29	50	Partial fractions	[162–166]	S2–S4
	31	51	Partial fractions	[166–169]	S4–S5
September	2	52	Applications of integration	171–174	
			Assignment 16 is due on September 2nd		
September	5	53	Applications of integration	174–176	
	7	54	Applications of integration	177–182	
	9	55	Applications of integration	182–186	
			Assignment 17 is due on September 9th		
September	12	56	Polynomial approximation	194–197	
	14	57	Polynomial approximation	197–200	
	16	58	Polynomial approximation	201–204	
			Assignment 18 is due on September 16th		
September	19	59	Polynomial approximation	204–206	S6
	21	60	Polynomial approximation	206–209	S6–S8
	23		Integration test		
			Assignment 19 is due on September 23rd		
			VACATION		

Date		Lecture	Topic	Pages in Notes	Pages in Supplement
October	10	61	Complex numbers	223–226	S8
	12	62	Complex numbers	226–234	S8
	14	63	Complex numbers	235–238	
			Assignment 20 is due on October 14th		
October	17	64	Complex numbers	238–241	S8
	19	65	Functions of two variables		S9–S12
	21	66	Functions of two variables		S13–S15
			Assignment 21 is due on October 21st		
October	24	67	Functions of two variables		S16–S18
	26	68	Functions of two variables		S19–S21
	28	69	Functions of two variables		S22–S24
			Assignment 22 is due on October 28th		
October	31	70	Differential equations	247–249	
November	2	71	Differential equations	250–252	
	4	72	Differential equations	255–257	
			Assignment 23 is due on November 4th		
November	7	73	Differential equations	258–259	
	9	74	Differential equations	260–261	S27–S29
	11	75	Exact differential equations		S30–S33
			Assignment 24 is due on November 11th		

Assignment 25 is optional; however, if you would like it marked and returned before the examination you should submit it by the end of semester, or as soon as possible afterwards.

Assignment 13

Question 1

(a) Using properties (1) – (6) of the definite integral prove that if f and g are integrable on $[a,b]$ and $f(x) \geq g(x)$ for all x in $[a,b]$, then

$$\int_a^b f \geq \int_a^b g\,.$$

(b) Prove that

$$\int_0^b x^3\,dx = \frac{b^4}{4}$$

from 'first principles'. (That is, use 7.1.2. Include a proof by induction that $\displaystyle\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$.)

Question 2

(a) Show that

$$\int \sec^2 2x\,dx = \frac{\tan 2x}{2} + C\,.$$

(b) Show that

$$\int x^3\sqrt{1+x^4}\,dx = \tfrac{1}{6}(1+x^4)^{3/2} + C\,.$$

(c) Derive formulæ for $\sin^2 x$ and $\cos^2 x$ in terms of $\cos 2x$. Hence find

$$\int_a^b \sin^2 x\,dx \quad\text{and}\quad \int_a^b \cos^2 x\,dx\,.$$

Question 3

If f is increasing and integrable then

$$\int_a^b f^{-1} = bf^{-1}(b) - af^{-1}(a) - \int_{f^{-1}(a)}^{f^{-1}(b)} f\,.$$

(Do not prove this result.)

(a) Give an interpretation of this result in terms of areas and graphs.

(b) Use the result to find $\displaystyle\int_a^b \sqrt[n]{x}\,dx$ for $0 \leq a \leq b$. (Here n is a natural number.)

Assignment 14

Question 1

(a) Find $\displaystyle\int \frac{x}{\sqrt{1-x^4}}\,dx$ by substitution.

(b) Find $\displaystyle\int \sqrt{1-x^2}\,dx$. (Hint: you might like to use the substitution $x = \sin u$.)

(c) Find $\displaystyle\int \frac{dx}{\sqrt{x}(1+\sqrt{x})^5}$.

Question 2

Find

$$\text{(a)} \int \frac{dx}{\sqrt{2x-x^2}}\,; \qquad \text{(b)} \int \frac{x\,dx}{x^4+3}\,; \qquad \text{(c)} \int \frac{dx}{1+\cos x}\,.$$

(Hint for (c): use the 't-substitution' $t = \tan(\frac{1}{2}x)$ in conjunction with the formulæ of §6.5.)

Assignment 15

Question 1

Evaluate the following integrals by algebraic manipulations and/or substitutions:

$$\text{(a)} \int \frac{dx}{x^2+6x+10}\,; \qquad \text{(b)} \int \frac{dx}{2+\sin x}\,; \qquad \text{(c)} \int \frac{x^2}{\sqrt{1-x^2}}\,dx\,.$$

Question 2

Evaluate the following integrals:

$$\text{(a)} \int x^2 \cos x\,dx\,; \qquad \text{(b)} \int x\sqrt{1-x^2}\,dx\,; \qquad \text{(c)} \int x\cos(2x+1)\,dx\,.$$

Question 3

(a) Derive a reduction formula for $\int \cos^n x\,dx$. Hence or otherwise evaluate the integral $\int \cos^4 x\,dx$.

(b) Use integration by parts to show that

$$\int_a^b \frac{\sin x}{x}\,dx = \frac{\cos a}{a} - \frac{\cos b}{b} - \int_a^b \frac{\cos x}{x^2}\,dx\,.$$

Assignment 16

Question 1

(a) Use Simpson's formula with four intervals to find the approximate value of $\ln 2$ which is by definition

$$\int_1^2 \frac{1}{x}\,dx\,.$$

(b) Show that Simpson's rule for $\int_{-h}^{h}$ is exact for any cubic polynomial.

(c) Find an approximate numerical value for

$$\int_0^{\pi} \frac{1}{2+\cos x}\,dx$$

using Simpson's formula with four intervals. Compare your answer with the exact value.

Question 2

(a) Evaluate $\int e^x \cos 2x\,dx$.

(b) Graph the function

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

Is f continuous at $x = 0$? (Hint: consider $\lim_{x\to 0} e^{-1/x^2}$.) Evaluate $f'(0)$ from first principles.

(d) Use L'Hôpital's rule (twice) to show that

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \infty\,.$$

Question 3

(See exercise 7, p.153.)

(a) Graph the functions $y = \sinh x$, $y = \cosh x$.

(b) Prove that $\cosh^2 x - \sinh^2 x = 1$.

(c) Show that $\sinh' = \cosh$ and $\cosh' = \sinh$.

Question 4

Exercises 7(iv) on p.154 and (4) on p.160.

Assignment 17

Question 1

Differentiate the following functions:

(i) $f(x) = \ln(\sin^2 x)$; (ii) $f(x) = 2^x$; (iii) $f(x) = (\ln x)^{\ln x}$.

Question 2

Find $\displaystyle\int \frac{\ln x}{x}\,dx$ in two ways: (i) by a substitution, and (ii) by integration by parts.

Question 3

Find the following integrals:

(i) $\displaystyle\int \frac{dx}{x(1+x)}$; (ii) $\displaystyle\int \frac{\sin\theta}{\cos^2\theta + \cos\theta - 2}\,d\theta$; (iii) $\displaystyle\int \frac{x+4}{x^2+1}\,dx$;

(iv) (*harder*) $\displaystyle\int \frac{2x}{(x^2+x+1)^2}\,dx$.

Assignment 20

Question 1

Determine the n-th degree MacLaurin polynomial for $f(x) = (1+x)^p$, where p is not an integer.

Question 2

(a) $\sqrt{e}$ is to be computed from the series

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_n(x) .$$

How large should one choose n in order to guarantee that $|R_n(x)| < 0.0005$?

(b) For what range of values of x can one replace $\sin x$ by $x - \frac{1}{6}x^3$ with an error not greater than 5×10^{-4}?

Question 3

Find the Taylor series expansion of $f(x) = \cos x$ about the point $x = -\frac{\pi}{4}$.

Assignment 21

Question 1

(a) Express $\dfrac{(1+2i)(3-4i)}{-1-i}$ in the form $a + bi$.

(b) Find the modulus and argument of

$$\text{(i)}\quad \frac{2+3i}{5-i}\ ; \qquad \text{(ii)}\quad 1 + i\sqrt{3}\ .$$

(c) Find the modulus and argument of

$$z = (1 + i\sqrt{3})^{30} .$$

Hence find $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$.

(d) Prove that the real part of z is $\frac{1}{2}(z + \bar{z})$, while the imaginary part is $\frac{1}{2i}(z - \bar{z})$.

Question 2

(a) Let $z = \frac{1}{2}(-1 + i\sqrt{3})$. Find, and plot in the complex plane, z, z^2, z^3, z^4 and z^5.

(b) Prove that if z_1 and z_2 are complex numbers then

$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\mathrm{Re}(z_1 \bar{z}_2)$$

and

$$|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\mathrm{Re}(z_1 \bar{z}_2) .$$

(Hint: use Question 2(d) at some stage.) Deduce that

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2) .$$

Give a geometric interpretation of this last equality.

Question 3

For distinct complex numbers a and b describe the curve specified by

$$|z - a| = |z - b| .$$

Assignment 22

Question 1

(a) Find all fourth roots of i. Without using a calculator, express the root having smallest positive argument in a form that does not involve any trigonometric functions.

(b) Find four distinct values for $(1-\sqrt{3}i)^{3/4}$.

Question 2

(a) Express $\sin(5\psi)$ in terms of $\cos\psi$ and $\sin\psi$.

(b) Express $\sin^5\psi$ in the form $a_0 + a_1\sin\psi + a_2\sin 2\psi + \cdots$ for some constants $a_0, a_1, a_2, \ldots$

Question 3

Find the (maximum possible) domain of the function

$$f : D \to \mathbb{R}\,,\; f(x,y) = \cos^{-1}(x+2y-3)\,.$$

Sketch D as a region in the x-y plane.

Assignment 23

Question 1

For the following functions, find f_x, f_y, f_{xx}, f_{xy}, f_{yx} and f_{yy}. Does $f_{xy} = f_{yx}$?

(a) $f(x,y) = (x+y)(x^2+2y^2)$; (b) $f(x,y) = e^x(\cos 3y - \sin y)$;

(c) $f(x,y) = \sqrt{\dfrac{1+x^2}{1+y^2}}$.

Question 2

Find $\nabla f(x,y)$, and find the derivative of f at (a,b) in the direction of $\mathbf{u}$, where

(a) $f(x,y) = xy^2e^{x-1}$, $(a,b) = (1,2)$, $\mathbf{u} = \mathbf{i}+\mathbf{j}$;

(b) $f(x,y) = \cos(5x-3y)$, $(a,b) = (\frac{\pi}{4}, \frac{\pi}{4})$, $\mathbf{u} = -3\mathbf{i} - 4\mathbf{j}$.

Question 3

Find the rate of steepest increase of $f(x,y)$ at the point (a,b), and the direction in which this rate is achieved, for

(a) $f(x,y) = (1/x) + (1/y)$, $(a,b) = (1,1)$;

(b) $f(x,y) = \cos x \sin y$, $(a,b) = (0, \frac{\pi}{2})$.

What can you say about the behaviour of f in the second part?

Question 4

(a) Find $\frac{df}{dt}$ if $f(x,y) = x^2y$, $x = e^t + t$, $y = -e^{2t}$.

(b) Find $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial \theta}$ if $f(x,y) = x^4 - y^4$, $x = r\cos\theta$, $y = r\sin\theta$. Simplify your answers as much as possible.

Assignment 24

Question 1

For each of the following differential equations find the solution satisfying the given initial condition(s).

(a) $\dfrac{dy}{dx} + y = e^x , \quad y(0) = 2 ;$

(b) $\dfrac{dy}{dx} = x\sqrt{y} , \quad y(0) = 1 ;$

(c) $\dfrac{dy}{dx} + (\tan x)y = \sin x \cos x , \quad y(0) = 0 .$

Question 2

Exercise 3 on p.253.

Question 3

For each of the following differential equations find the solution satisfying the given initial conditions.

(a) $y'' + 2y' - 3y = 0 , \quad y(0) = 1 , \ y'(0) = 0 ;$

(b) $y'' - 2y' - y = 0 , \quad y(0) = 0 , \ y'(0) = 4 ;$

(c) $y'' - 6y' + 8y = 0 , \quad y(0) = 3 , \ y'(0) = 8 .$

Assignment 25

This assignment is not compulsory. If you wish to have it marked before the examination you should submit it before the end of semester, or as soon as possible after.

Question 1

Solve

(a) $y'' - 8y' + 16y = 0 , \quad y(0) = \frac{2}{3} , \ y'(0) = -2 ;$

(b) $y'' + 2y' + 2y = 0 , \quad y(0) = 2 , \ y'(0) = 1 ;$

(c) $y'' + 4y = 8\cos 2x , \quad y(0) = 0 , \ y'(0) = -4 ;$

(d) $y'' - y' - 2y = 8x^2 , \quad y(0) = y'(0) = 0 .$

Question 2

Solve the following exact differential equations.

(a) $y(1 + \tan^2 x)\, dx + (\tan x - \tan^2 y - 1)\, dy = 0 ;$

(b) $\left(\dfrac{3x^2}{y} - y\right) - \left(x + \dfrac{x^3}{y^2}\right)\dfrac{dy}{dx} = 2 .$

TUTORIAL AND PRACTICE PROBLEMS

These tutorial problems were included with last year's notes. Those not relevant to this year's course are marked with the symbol #.

The Definite Integral

1. If k is a constant, use the axioms (1) - (6) of section 7.1 of the notes to evaluate

$$\int_a^b k.$$

2. Using the axioms (1) - (6), prove the inequality

$$m(b - a) \leq \int_a^b f$$

of the Integral Mean Value Theorem (p 120 of the notes).

3. Use the result

$$\int_a^b f = \lim_{n\to\infty} \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + k\left[\frac{b-a}{n}\right]\right)$$

to evaluate $\int_0^1 (1 - x^2)\,dx$, and illustrate (by means of a graph) what you are doing. (You'll need the formula $\sum_{k=1}^{n} k^2 = \frac{n}{6}(n+1)(2n+1)$.)

4. Using only the axioms (1) - (6), and the result $\int_a^b x^2 dx = \frac{b^3}{3} - \frac{a^3}{3}$, evaluate

(i) $\int_{[illegible]}^{[illegible]} (4 - (x + 3)^{[illegible]})\,dx$ (ii) $\int_1^{[illegible]} (2x^{[illegible]} - 6x + 7)\,dx$

5. By regarding $\int_a^b f$ as an area (with regions below the x-axis being assigned negative areas), evaluate

(i) $\int_0^2 \sqrt{4 - x^2}\,dx$ (ii) $\int_0^{2\pi} \sin^3 x\,dx$ (iii) $\int_1^3 \sqrt{(x-1)(5-x)}\,dx$

The Fundamental Theorem of Calculus

1. Find the primitives of:

   (i) $x^4 - 3x^3 + 5x^2$ (ii) $\sqrt[3]{x}$ (iii) $\dfrac{1}{x\sqrt{x}}$

   (iv) $(1 - 2x)(1 + 3x)$ (v) $a \cos bx$ (vi) $\dfrac{1}{1 + 9x^2}$.

2. Find (where possible) the following improper integrals:

   (i) $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$ (ii) $\displaystyle\int_0^2 \frac{1}{x^2}\,dx$ (iii) $\displaystyle\int_2^\infty \frac{1}{x^3}\,dx$

   (iv) $\displaystyle\int_5^\infty \frac{1}{\sqrt{x}}\,dx$ (v) $\displaystyle\int_{\sqrt{3}}^\infty \frac{1}{1 + x^2}\,dx$

3. Evaluate $\displaystyle\int_0^{2\pi} \sin mx \sin nx\,dx$ for all integers m and n.

   (Hint: Use the identity $2 \sin A \sin B = \cos(A-B) - \cos(A+B)$)

4. Find the derivative of $x \cos x$, and hence evaluate $\displaystyle\int_0^{\pi/2} x \sin x\,dx$.

5. To evaluate $\displaystyle\int_{-1}^1 \frac{dx}{x^4}$, we can say

   $$\int_{-1}^1 \frac{dx}{x^4} = \left[-\frac{1}{3x^3}\right]_{-1}^1 = -\frac{1}{3} - \frac{1}{3} = -\frac{2}{3}. \quad \text{i.e.} \int_{-1}^1 \frac{dx}{x^4} < 0.$$

   However, $\dfrac{1}{x^4}$ is a positive function, and so $\displaystyle\int_{-1}^1 \frac{dx}{x^4} \geq 0$ by Axiom(3) of §7.1 in the notes. Explain this apparent contradiction.

6. Find the area enclosed by the two curves $y = 3 + 2x - x^2$ and $y = x^2 - 4x + 3$

Techniques of Integration (Substitution, parts, etc.):

1. Find the integrals of the following functions using the method of substitution:

   (i) $\dfrac{\sin x}{\sqrt{\cos^3 x}}$ (ii) $\dfrac{\sin\sqrt{x}}{\sqrt{x}}$ (iii) $\dfrac{x + 1}{(x^2 + 2x + 2)^3}$

   (iv) $\dfrac{\sin x + \cos x}{(\sin x - \cos x)^{1/3}}$ (v) $\dfrac{1}{(5 + 2x)^2 + 9}$ (vi) $\sin^3 x \cos^6 x$

2. Find the integrals of the following functions using integration by parts:

   (i) $x \cos 2x$ (ii) $x^2 \sin 2x$ (iii) $x^3 \cos 2x$

3. Use integration by parts to show that

   $$\int_0^{\pi/2} \sin^2 x\,dx = \frac{1}{2}\int_0^{\pi/2} 1.dx = \frac{\pi}{4}, \quad \text{and}$$

   $$\int_0^{\pi/2} \sin^4 x\,dx = \frac{3}{4}\int_0^{\pi/2} \sin^2 x\,dx = \frac{3\pi}{16}.$$

   Similarly, evaluate $\displaystyle\int_0^{\pi/2} \sin^6 x\,dx$.

4. Evaluate the following integrals by using the given substitutions:

   (i) $\displaystyle\int \frac{x}{(2x + 1)^3}\,dx$, $z = 2x + 1$ (ii) $\displaystyle\int_1^2 \frac{dx}{x^2\sqrt{5x^2 - 4}}$, $x^2 = \dfrac{1}{z}$.

5. Determine (i) $\displaystyle\int \frac{dx}{\sqrt{7 - 6x - x^2}}$ (ii) $\displaystyle\int \frac{dx}{\sqrt{x(4 - x)}}$

   (Hint: Firstly, complete the square.)

6. In the integration by parts formula,

$$\int uv'\,dx = uv - \int u'v\,dx,$$

put $u = \frac{1}{x}$, $v = x$ (so $u' = -\frac{1}{x^2}$, $v' = 1$).

Then $\int \frac{1}{x}.1.dx = \frac{1}{x}.x - \int \left(\frac{-1}{x^2}\right).x\,dx$

i.e. $\int \frac{dx}{x} = 1 + \int \frac{dx}{x}$.

Subtracting $\int \frac{dx}{x}$ from both sides gives $0 = 1$.

Question : Where's the error?

Simpson's Rule

1. Use Simpson's Rule with four strips to obtain values for

(i) $\int_{-2}^{2} (12x^2 + 6)\,dx$ (ii) $\int_{-2}^{2} (x^4 + 5x^2 + 12 - 6\cos\frac{\pi x}{2})\,dx$.

Then evaluate the integrals exactly.

The two results for $\int_{-2}^{2} (12x^2 + 6)dx$ should be the same - why?

The values for $\int_{-2}^{2} (12x^2 + 6)dx$ and $\int_{-2}^{2} (x^4 + 5x^2 + 12 - 6\cos\frac{\pi x}{2})dx$ that were obtained from Simpson's Rule are also the same (although the functions are different) - why isn't this result too surprising?

2. Use Simpson's Rule with two strips, then with four strips, to find approximate values for $\int_0^1 \frac{dx}{1+x^2}$. Hence give approx. values for π .

Exponential and Logarithmic Functions

1. Differentiate with respect to x:

   (i) e^{x^3} (ii) $e^{2x}\sin x$ (iii) $\dfrac{e^x-1}{e^x+1}$ (iv) $x\,e^x$

   (v) $x^{1/x}$ (vi) $(\ln x)^x$ (vii) 3^{x^2}

2. Find the integrals:

   (i) $\int e^{x/2}\,dx$ (ii) $\int e^{\cos x}\sin x\,dx$ (iii) $\int x^2 e^x\,dx$

   (iv) $\int e^x \sin(e^x)\,dx$ (v) $\int x^{10}\ln x\,dx$

3. The hyperbolic sine of x (written sinh x) is defined by

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

   Similarly, the hyperbolic cosine of x (written cosh x) is defined by

$$\cosh x = \frac{e^x + e^{-x}}{2}.$$

   Use these definitions to prove that the derivative of sinh x is cosh x, and of cosh x is sinh x.

4. Find a simple expression for $\cosh^2 x - \sinh^2 x$ in two ways:

   (i) by using the definitions in Q.3

   (ii) by differentiating $\cosh^2 x - \sinh^2 x$.

5. Sketch the graphs of $y = \cosh x$ and $y = \sinh x$, indicating any turning points.

6. Show that the function sinh x is invertible, with inverse

$$\sinh^{-1} x = \ln[x + \sqrt{1 + x^2}\,].$$

   Hence find the derivative of $\sinh^{-1} x$.

7. Prove that $\cosh 2x = 2\cosh^2 x - 1$, and hence solve $\cosh 2x - 4\cosh x + 3 = 0$.

Partial Fractions

1. Integrate with respect to x:

   (i) $\dfrac{1}{x^2 - 5x + 6}$ (ii) $\dfrac{1}{x^2 + 2x + 3}$ (iii) $\dfrac{1}{x^2 - 8x + 16}$

2. Find the integrals of:

   (i) $\dfrac{x}{2x^2 - x - 3}$ (ii) $\dfrac{2 - 3x}{3x^2 - 4x + 1}$ (iii) $\dfrac{4x^2 - x + 12}{x(x^2 + 4)}$

   (iv) $\dfrac{x^2 + 1}{(x + 1)^2}$

Polar Coordinates and Area

1. Sketch the curves (i) $r = 2(1 - \cos\theta)$, (ii) $r = 1 + 2\cos\theta$
2. Sketch the curve $r^3 = a^3 \sin 3\theta$ $(a > 0)$, showing it consists of 3 loops.
3. Find the area of one loop of the curve $r = \cos^2\theta$
4. Sketch the curve $r = a(1 + \sin\theta)$, and find the area enclosed by it.
5. Express the equation $(x^2 + y^2)^2 = 2xy$ in polar coordinates, sketch the graph of the function, and find the area enclosed by one loop of the curve.

Arc Length, Volume, and Surface Area

#1. For the curve $y = \cosh x$, find the length of the curve from the point where $x = 0$, to the point where $x = 1$. (Hint: Use $\cosh^2 x - \sinh^2 x = 1$)

#2. Show that $\int \sec x\,dx = \ln(\sec x + \tan x) + C$, and hence find the length of the curve $y = \ln(\cos x)$ from $x = 0$ to $x = \frac{\pi}{4}$.

#3. For the curve $x^4 - 6xy + 3 = 0$, show that $1 + (\frac{dy}{dx})^2 = \frac{(x^4 + 1)^2}{4x^4}$. Hence find the length of the curve from the point where $x = 1$ to the point where $x = 2$.

4. Find the volumes of the solids generated by revolution through 2π radians about the x-axis of the area bounded by:
   (i) $y = x^4$, $x = 2$, and $y = 0$ (ii) $y^4 = 16x$, $x = 1$, $x = 4$ and $y = 0$.

5. As for Q.4 but rotating about the y-axis:
   (i) $y^2 = x^3$, $y = 8$, and $x = 0$ (ii) $y^2 = \frac{1 - x^2}{x^2}$, $y = 0$, $y = 1$, and $x = 0$.

6. Sketch the area in the first quadrant bounded by the curve $y = 4x - x^2$ and the x-axis. Find the volume of the solid formed when this area is revolved through 360° about (i) the x-axis, and (ii) the y-axis.

7. Sketch $y = \frac{1}{\sqrt{1 + x^2}}$ for $x \geq 0$. Find the area of the region under this curve between $x = 0$ and $x = B$, and say what happens as $B \to \infty$. Now find the volume of the solid generated when the region under the curve between $x = 0$ and $x = B$ is rotated one revolution about the x-axis. What happens to the volume as $B \to \infty$? Do you find anything strange about the results for area and volume as $B \to \infty$?

#8. The portion of the parabola $y^2 = 4ax$ between $x = 0$ and $x = 3a$ is rotated through 180° about the x-axis. Find the surface area generated.

MacLaurin Series and the Remainder.

1. The MacLaurin series for $\cos x$ is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots\ldots$$

Let S_n denote the sum of n terms of this series. Taking $x = 2$, use your calculator to evaluate $\cos x$ exactly (remember to work in radians!), and then draw up a table showing values of S_n and $|S_n - \cos x|$ for $n = 1, 2, 3, \ldots$ (work to 6 or 7 decimal places).

i.e., $\cos 2 = -0.4161468$

n	S_n	$\|S_n - \cos x\|$
1	1	1.416147
2	-1	0.583853
3	-0.3333333	0.082814
.	.	.
.	.	.
.	.	.

Continue the table until the error term is 0 to 6 decimal places. Then draw up similar tables for $x = 0.2$ and $x = 0.02$. What do these results suggest about the number of terms of the MacLaurin series necessary to obtain accurate results for $\cos x$?

2. The error estimate for the MacLaurin series states that

$$\left|f(x) - \left\{f(0) + f'(0).x + f''(0)\frac{x^2}{2!} + \ldots + f^{(n)}(0)\frac{x^n}{n!}\right\}\right|$$

$$\leq \frac{|x|^{n+1}}{(n+1)!} \cdot \max_{0\leq t\leq x} |f^{(n+1)}(t)|, \quad (x \geq 0) \;\ldots.\; (*)$$

(if $x < 0$ the term on the RHS becomes $\frac{|x|^{n+1}}{(n+1)!} \cdot \max_{x\leq t\leq 0} |f^{(n+1)}(t)|$

(a) Use (*) to show that

$$\left|\cos x - \left\{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!}\right\}\right| \leq \frac{|x|^{2n+1}}{(2n+1)!}$$

(b) How many terms of its MacLaurin expansion will be sufficient to obtain a value for $\cos(0.5)$ that is accurate to 5 decimal places?

(Hint: You need to find the lowest value of n for which $\frac{(0.5)^{2n+1}}{(2n+1)!}$ is less than 0.000005. So evaluate the expression for $n = 1, 2, \ldots$ using a calculator). (Ans: 4 terms are sufficient).

(c) Prove that for each real x the series $\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$ converges to $\cos x$. (You can assume $\lim_{n\to\infty} \frac{r^n}{n!} = 0$ for any real r).

3. Use the error estimate (*) of Q.2 to determine a number of terms of the MacLaurin expansion of $f(x) = \ln(1 + x)$ that will be sufficient to obtain a value for $\ln(5/4)$ that is accurate to 3 decimal places. (Ans: 4 terms are sufficient).

4. Determine the MacLaurin series for $f(x) = \ln(1 - x)$. Then use (*) of Q.2 to show that the MacLaurin series for $\ln(1 - x)$ converges to $\ln(1 - x)$ for $-1\leq x\leq \frac{1}{2}$. (It's easiest to consider separately the cases $-1\leq x\leq 0$ and $0\leq x\leq \frac{1}{2}$).

5. (a) Use the error estimate:

$$\left|f(x) - \left\{f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \ldots + f^{(n)}(0)\frac{x^n}{n!}\right\}\right|$$

$$\leq \frac{|x|^{n+1}}{(n+1)!} \cdot \max_{-|x|\leq t\leq |x|} |f^{(n+1)}(t)|,$$

to show that, for $-1\leq x\leq 1$:

$$\frac{-|x|^{n+1}}{(n+1)!}e \leq e^x - \left\{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}\right\} \leq \frac{|x|^{n+1}}{(n+1)!}e$$

(b) By replacing x by $-t^2$ in the result of (a), and then integrating from 0 to x with respect to t, show that, for $0 \le x \le 1$:

$$\frac{-x^{2n+3}}{(2n+3)(n+1)!}\, e \le \int_0^x e^{-t^2}\,dt - \left\{x - \frac{x^3}{3} + \frac{x^5}{5.2!} - \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)n!}\right\}$$
$$\le \frac{x^{2n+3}}{(2n+3)(n+1)!}\, e$$

(c) Hence determine a number of terms of the series

$1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \ldots + \frac{(-1)^n}{(2n+1)n!} + \ldots$ that will be sufficient to obtain a value for $\int_0^1 e^{-t^2}\,dt$ that is accurate to 4 decimal places

(Ans: 7 terms are sufficient).

Taylor Series.

1. Determine the first four non-zero terms in the Taylor series expansion of $f(x) = \cos x$ about the point $x_0 = {}^{3\pi}/4$. Hence determine an approximate value for $\cos(133°)$.

2. Find the Taylor expansion of $f(x) = \sqrt{x}$ about any point $x_0 > 0$, and find the first 3 non-zero terms in the Taylor expansion of $\sqrt[3]{x}$ about the point $x_0 = 27$.

3. Verify, up to fourth order terms, that

$$(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots)(1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots)$$
$$= (1 + (x+y) + \frac{(x+y)^2}{2!} + \frac{(x+y)^3}{3!} + \ldots).$$

Why should this result be expected?

For problems on the remainder of the course, please see Chapters 11 - 12 of the lecture notes, and various sections of the supplementary notes.

PREVIOUS EXAMINATION

PAPERS

THE UNIVERSITY OF NEW ENGLAND

UNIT NAME: PURE MATHEMATICS 111-2, 112-2, 173-2

PAPER NUMBER: -

PAPER TITLE: SECOND SEMESTER

DATE: Friday, 18th November, 1983 TIME: 9.30 a.m. to 12.30 p.m.

TIME ALLOWED: THREE HOURS (3) plus fifteen minutes reading time

NUMBER OF PAGES IN PAPER: FOUR (4)

NUMBER OF QUESTIONS ON PAPER: SIX (6)

NUMBER OF QUESTIONS TO BE ANSWERED: SIX (6)

STATIONERY PER CANDIDATE: [-] X 6 LEAF A4 BOOKS [2] X 12 LEAF A4 BOOKS

[-] X ROUGH WORK BOOKS

GRAPH: ___-___ (NUMBER OF SHEETS)

SLIDE RULES PERMITTED: YES/~~NO~~ POCKET CALCULATORS PERMITTED: YES/~~NO~~ (silent t

MATHEMATICAL TABLES PERMITTED: ~~YES~~/NO; IF YES, SUPPLIED BY STUDENT/UNIVERSITY

OTHER AIDS REQUIRED: NIL

INSTRUCTIONS FOR CANDIDATES: ATTEMPT ALL QUESTIONS.

Candidates may retain their copy of this examination question paper.
This paper may be annotated during the fifteen minutes reading time.

TEXTBOOKS OR NOTES PERMITTED: NIL

Question 1

(a) (i) Differentiate the following functions:

$f(x) = e^x \sin x$

$g(x) = 3^{x^2}$

(ii) Show that the function

$\sinh x = \dfrac{e^x - e^{-x}}{2}$

is invertible, with inverse

$\sinh^{-1} x = \ln[x + \sqrt{1 + x^2}]$.

(b) Find the following integrals:

(i) $\int x^5 \ln x \, dx$

(ii) $\int (\sin x - \sin^3 x) \, dx$

(iii) $\displaystyle\int \frac{x \, dx}{x^2 - 3x + 2}$

Question 2

(a) For the cycloid $x = (\theta - \sin \theta)$, $y = (1 - \cos \theta)$

(i) sketch the graph for $0 \le x \le 5\pi$,

(ii) find the volume of the solid obtained by revolving one arch of the curve about the x-axis,

(iii) find the arc-length of one arch.

(b) For the complex numbers $u = \sqrt{3} + i$, $v = 1 - i\sqrt{3}$ find:

(i) $\bar{u}$, the conjugate of u ; (ii) the modulus of v ;

(iii) $\dfrac{u}{v}$; (iv) $\left|\dfrac{v^3}{u^2}\right|$;

(v) the polar form of u; (vi) u^7 .

Question 3

(a) Sketch the curve $r = \cos 2\theta$.

(Question 3 is continued on page 3)

Question 3 (cont.)

(b) (i) Determine the MacLaurin expansion for $y = \ln(1 - x)$, including an expression for the general term.

(ii) Find a range of x-values for which the MacLaurin expansion of $y = \ln(1 - x)$ represents (converges to) the function.

(iii) Determine how many terms in the MacLaurin expansion of $y = \ln(1 -$ will be sufficient to compute the value of $\ln \frac{1}{2}$ accurate to five decimal places.

Question 4

(a) Find the real and imaginary parts of

$$e^{2 + i\pi/6}$$

(b) Find an expression for $\sin 6\theta$ in terms of $\sin\theta$.

(c) In about half a page, briefly outline Newton's method for obtaining approximations to a zero of the function $y = f(x)$ commenting on its convergence.

Question 5

Find the solution of each of the following differential equations which satisfies the given initial conditions.

(i) $\frac{dy}{dx} = \frac{2y}{x+1}$; $y(1) = 1$

(ii) $\frac{dy}{dx} = y(2 - y)$; $y(0) = \frac{1}{2}$

(iii) $\frac{dy}{dx} = e^x - y$; $y(0) = 1$

(iv) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$; $y(0) = 0$, $\frac{dy}{dx}(0) = 1$

(v) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = 0$; $y(0) = 1$, $\frac{dy}{dx}(0) = 0$.

(Question 6 is on page 4)

Question 6

(a) A population of fish is subject to "fishing" at the constant rate of c catches per year. If $N(t)$ denotes the size at time t then

$$\frac{dN}{dt} = 0.2N - c .$$

If the initial size of the population is $N(0) = 1000$:

(i) find an expression for $N(t)$ in terms of c;

(ii) find a value for the rate of fishing, c which will maintain the population at a constant size. Show that for higher levels of fishing the population will eventually become extinct.

(b) For the function $f(x,y) = 4x^2 + 9y^2 - 12xy$ find

$$\frac{\partial f}{\partial x} \quad \text{and} \quad \frac{\partial^2 f}{\partial y^2}$$

and verify that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} .$$

If $x = \frac{u + v}{2}$ and $y = \frac{u - v}{3}$ use the chain rule to find $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$.

THE UNIVERSITY OF NEW ENGLAND

UNIT NAME: PURE MATHEMATICS 111-2, 112-2, 173-2

PAPER NUMBER: -

PAPER TITLE: SECOND SEMESTER

DATE: Wednesday 14th November, 1984 TIME: 9.30 a.m. to 12.30 p.m.

TIME ALLOWED: THREE HOURS (3) plus fifteen minutes reading time.

NUMBER OF PAGES IN PAPER: FOUR (4)

NUMBER OF QUESTIONS ON PAPER: SIX (6)

NUMBER OF QUESTIONS TO BE ANSWERED: SIX (6)

STATIONERY PER CANDIDATE: [-] X 6 LEAF A4 BOOKS [2] X 12 LEAF A4 BOOKS

[-] X ROUGH WORK BOOKS

GRAPH: - (NUMBER OF SHEETS)

SLIDE RULES PERMITTED: ~~YES~~/NO POCKET CALCULATORS PERMITTED: ~~YES~~/NO (silent

MATHEMATICAL TABLES PERMITTED: ~~YES~~/NO; IF YES, SUPPLIED BY STUDENT/UNIVERSITY

OTHER AIDS REQUIRED: Nil

INSTRUCTIONS FOR CANDIDATES:
Candidates may retain their copy of this examination question paper.
This paper may be annotated during the fifteen minutes reading time.

TEXTBOOKS OR NOTES PERMITTED: Nil

Question 1

(i) Find

(a) $$\int \frac{dx}{x^2 + 2x + 2}$$

(b) $$\int \cos^5 x \, dx$$

(c) $$\int \frac{dx}{\sqrt{\sqrt{x} + 1}}$$

(ii) Using the substitution $u = \sin x$ show that

$$\int \sin x \cos x \, dx = \frac{1}{2} \sin^2 x + C.$$

Also show that the substitution $v = \cos x$ leads to

$$\int \sin x \cos x \, dx = -\frac{1}{2} \cos^2 x + C .$$

Reconcile these answers.

(iii) Find $$\int \frac{dx}{(1 + x^2)^2}$$

Question 2

(i) Assuming only that g is a function satisfying $g' = g$

$$g' = g \; ; \; g(0) = 1$$

deduce that

$$g(-x) = g(x)^{-1}$$

(ii) Differentiate

$$f(x) = 2^{x^2 \sin x}$$

(iii) Show that the function

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

is invertible for $x \geq 0$ and find an expression for the inverse in terms of ln.

(iv) Find $$\int \frac{x \, dx}{x^2 - 2x + 1}$$

Question 3

(i) Use Simpson's Formula with four strips to evaluate approximately

$$\int_0^1 \frac{dx}{1 + x}$$

(ii) Sketch the curve $r = 1 + \sin\theta$, and find the area enclosed by it.

(iii) For the cycloid $x = (\theta - \sin\theta)$, $y = (1 - \cos\theta)$

(a) sketch the graph for $0 \le x \le 5\pi$,

(b) find the volume of the solid obtained by revolving one arch or the curve about the x-axis.

Question 4

(i) Determine the MacLaurin expansion for $y = (1 + x)^{\frac{1}{2}}$, including an expression for the general term.

(ii) Find a range of positive x-values for which the MacLaurin expansion of $y = (1 + x)^{\frac{1}{2}}$ represents (converges to) the function.

(iii) Determine how many terms in the MacLaurin expansion of $y = (1 + x)^{\frac{1}{2}}$ will be sufficient to compute the value of $\sqrt{1.5}$ accurate to five decimal places.

[You may assume the expression

$$\frac{1}{n!}\int_0^x t^n f^{(n+1)}(x - t)\,dt$$

for the *remainder*.]

Question 5

(i) For the complex numbers $u = 1 + \sqrt{3}\,i$, $v = 1 - \sqrt{3}\,i$ find:

(a) $\bar{u}$, the conjugate of u; (b) the modulus of v;

(c) $\frac{u}{v}$ in the form $x + iy$ with x,y real;

(d) $\left|\frac{v^3}{u^2}\right|$; (e) the polar form of u;

(f) $\text{Im}\,(u^{19})$.

(ii) Find the 5 fifth roots of i and plot them on an argand diagram.

(iii) Find an expression for $\cos 8\theta$ in terms of $\cos\theta$.

Question 6

(i) Find the solution of each of the following differential eqatuions which satisfies the given initial conditions:

(a) $\frac{dy}{dx} = -e^x y$; $y(1) = 1$

(b) $\frac{dy}{dx} - \frac{2y}{x} = x^2 \sin 3x$; $y(\frac{\pi}{2}) = 0$

(c) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - y = 0$; $y(0) = 1$, $\frac{dy}{dx}(0) = 0$.

(ii) A population of fish is subject to "fishing" at the constant rate of f catches per year. If $y(t)$ denotes the size at time t then

$$\frac{dy}{dt} = 0.2y - f.$$

If the initial size of the population $y(0) = 1000$:

(a) find an expression for $y(t)$ in terms of f;

(b) find a value for the rate of fishing, f which will maintain the population at a constant size. For higher levels of fishing find an expression for the time at which the population will become extinct.

THE UNIVERSITY OF NEW ENGLAND

UNIT NAME: PURE MATHEMATICS 111-2, 173-2

PAPER NUMBER:

PAPER TITLE: SECOND SEMESTER

DATE: Thursday, 14th November, 1985 TIME: 9.30 a.m. to 12.30 p.m.

TIME ALLOWED: THREE HOURS (3) plus fifteen minutes reading time.

NUMBER OF PAGES IN PAPER: FOUR (4)

NUMBER OF QUESTIONS ON PAPER: SIX (6)

NUMBER OF QUESTIONS TO BE ANSWERED: SIX (6)

STATIONERY PER CANDIDATE: [-] X 6 LEAF A4 BOOKS [2] X 12 LEAF A4 BOOKS

[-] X ROUGH WORK BOOKS

GRAPH: - (NUMBER OF SHEETS)

SLIDE RULES PERMITTED: YES/NO POCKET CALCULATORS PERMITTED: YES/NO (silent

MATHEMATICAL TABLES PERMITTED: YES/NO; IF YES, SUPPLIED BY STUDENT/UNIVERSITY

OTHER AIDS REQUIRED: NIL

INSTRUCTIONS FOR CANDIDATES:

Candidates may retain their copy of this examination question paper.
This paper may be annotated during the fifteen minutes reading time.

TEXTBOOKS OR NOTES PERMITTED: NIL

Question 1

Find the following indefinite integrals

(a) $\displaystyle\int \frac{dx}{x^2 - 4x + 5}$,

(b) $\displaystyle\int \frac{(x - 7)}{(x - 1)(x - 3)}\, dx$,

(c) $\displaystyle\int x e^{-x}\, dx$,

(d) $\displaystyle\int \frac{2x}{1 + x^2}\, dx$.

Question 2

(a) The function cosh is defined by

$$\cosh x = \frac{e^x + e^{-x}}{2} .$$

Find its first and second derivatives and sketch its graph.

(b) Prove that $\cosh x \geq 1 + \frac{x^2}{2}$ for all x.

(c) Show that the area enclosed by the above graph, the x-axis and any two vertical lines is always equal to the length of the graph between the same verticals.

Question 3

(a) Find the volume of revolution obtained by rotating the curve $y = \frac{1}{x}$ about the x-axis between x = 1 and x = 2.

(b) Show that the area of the surface generated in part (a) is given by

$$\int_1^2 2\pi x^{-3}\sqrt{x^4 + 1}\, dx.$$

(c) Use Simpson's formula with two strips to find an approximate value for the integral in part (b). [Don't worry about giving the final numerical answer, especially if you don't have a calculator; but leave your answer in a simple form appropriate for computation.]

(d) If the same curve, $y = \frac{1}{x}$, is rotated between x = 1 and x = M, show that the volume approaches a finite value as $M \to +\infty$, but the surface area can be made as large as desired.

Question 4

(a) Sketch the cardioid, given in polar coordinates by

$$r = 1 + \cos\theta, \quad 0 \le \theta \le 2\pi .$$

Find the length of the cardioid and the area enclosed by it.

(b) Find the Maclaurin expansion for $\ln(1 - x)$ and state where the series converges.

(c) From (b) deduce the Maclaurin expansions of $\ln(1 + x)$ and $\ln \dfrac{1 + x}{1 - x}$.

Use the first two terms in the second of these to find an approximate value of $\ln 2$ and find an upper bound for the error.

Question 5

(a) Express

$$\frac{-10i}{(2 + i)(1 - 3i)}$$

in the form $x + iy$, with x, y real.

Verify that

$$\left|\frac{-10i}{(2 + i)(1 - 3i)}\right| = \frac{|-10i|}{|2 + i|\,|1 - 3i|} .$$

With minimal effort express

$$\frac{10i}{(2 - i)(1 + 3i)}$$

in the form $x + iy$, with x, y real.

(b) Using the formula $e^{i\theta} = \cos\theta + i\sin\theta$, show that $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

and hence that

$$\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta .$$

(c) Locate the fifth roots of 1 in the complex plane.

Show that the unreal fifth roots of 1 are the solutions of the quartic equation

$$u^4 + u^3 + u^2 + u + 1 = 0.$$

Using this equation, and the formula for $\cos\theta$ in part (b), prove that

$$\cos\frac{2\pi}{5} = \frac{-1 + \sqrt{5}}{4}$$

and

$$\cos\frac{4\pi}{5} = \frac{-1 - \sqrt{5}}{4} .$$

Question 6

Find the solution of each of the following differential equations which satisfies the given initial conditions

(a) $\dfrac{dy}{dx} + \dfrac{2x}{1 + x^2}\,y = 0 \;; \qquad y(0) = 2$

(b) $\dfrac{dy}{dx} + y = e^{-x}\cos x \;; \qquad y(0) = 0.$

(c) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 0 \;; \qquad y(0) = 1, \quad y'(0) = 0$

(d) $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = 0 \;; \qquad y(0) = 1, \quad y'(0) = 0.$

THE UNIVERSITY OF NEW ENGLAND

UNIT NAME: PURE MATHEMATICS 111-2

PAPER NUMBER: -

PAPER TITLE: SECOND SEMESTER

DATE: Saturday, 15th November, 1986. TIME: 9.30 a.m. to 12.30 p.m.

TIME ALLOWED: THREE HOURS (3) plus fifteen minutes reading time.

NUMBER OF PAGES IN PAPER: THREE (3)

NUMBER OF QUESTIONS ON PAPER: SIX (6)

NUMBER OF QUESTIONS TO BE ANSWERED: SIX (6)

STATIONERY PER CANDIDATE: [-] X 6 LEAF A4 BOOKS [2] X 12 LEAF A4 BOOKS [-] X ROUGH WORK BOOKS

GRAPH PAPER: - (NUMBER OF SHEETS)

POCKET CALCULATORS PERMITTED: YES/~~NO~~ (silent type)

MATHEMATICAL TABLES PERMITTED: ~~YES~~/NO; IF YES, SUPPLIED BY STUDENT/UNIVERSI

OTHER AIDS REQUIRED: NIL

INSTRUCTIONS FOR CANDIDATES: Attempt all questions. All questions are of equal value.

Candidates may make notes on this paper during the fifteen minutes reading
Candidates may retain their copy of this examination question paper.

TEXTBOOKS OR NOTES PERMITTED: NIL

Question 1

Find (i) $\int \frac{x}{\sqrt{x}-1}\,dx$,

(ii) $\int x^2 \ln x\,dx$,

(iii) $\int \frac{dx}{x^3 - x^2 + x - 1}$.

Question 2

(i) Find an expression for $\sinh^{-1}x := \frac{e^x - e^{-x}}{2}$ in terms of the function ln .

(ii) Let $I_n := \int_0^\infty x^n e^{-x}\,dx$, $n \ge 0$

(a) Show that $I_n = nI_{n-1}$

(b) By first finding I_0, show that $I_5 = 120$.

(iii) Find $\int \frac{x}{\sqrt{x}-1}\,dx$.

Question 3

Find the volume of the solid of revolution generated when one arch of the cycloid

$x = \theta - \sin\theta$

$y = 1 - \cos\theta$

is rotated about the x-axis.

Question 4

(i) Find the MacLaurin expansion for the function $\cos 2x$, including an expression for the general term.

(ii) Show that the remainder between a function f and its second degree MacLaurin polynomial,

$$R_2(x) := f(x) - [f(0) + f'(0)x + \tfrac{1}{2}f''(0)x^2] ,$$

is given by

$$R_2(x) = \tfrac{1}{2}\int_0^x t^2 f'''(x-t)\,dt.$$

[Hint consider $\int_0^x f'(x-t)\,dt$.]

(iii) Illustrate Newton's method by finding 2 successive approximations to the root of $f(x) = x^3 - 26$ near $x = 3$.

Question 5

(i) Sketch the curve in the complex plane described by

$$|(z - 1)/(2z + i)| = 1$$

(ii) Find $\mathrm{Im}(1 + i)^7$

(iii) Find the four fourth roots of -1 and plot them on an argand diagram.

(iv) Express $\sin^3\theta$ in terms of $\sin\theta$ and $\sin 3\theta$.

Question 6

For each of the following differential equations find the solution satisfying the given initial conditions.

(i) $\frac{dy}{dx} = xy(1 - y)$; $y(0) = \frac{1}{2}$

(ii) $\frac{dy}{dx} - 2xy = x$; $y(0) = 1$

(iii) $\frac{dy}{dx} = \frac{x + y}{x}$; $y(1) = 1$

(iv) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$; $y(0) = 1$, $y'(0) = 0$

(v) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0$; $y(0) = 0$, $y'(0) = 1$

(vi) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = e^x$; $y(0) = 0$, $y'(0) = 0$

THE UNIVERSITY OF NEW ENGLAND

UNIT NAME: PURE MATHEMATICS 111-2

PAPER NUMBER: -

PAPER TITLE: SECOND SEMESTER

DATE: Friday, 27th November, 1987 TIME: 9.30 a.m. to 12.30 p.m.

TIME ALLOWED: THREE HOURS (3) plus fifteen minutes reading time.

NUMBER OF PAGES IN PAPER: FOUR (4)

NUMBER OF QUESTIONS ON PAPER: SIX (6)

NUMBER OF QUESTIONS TO BE ANSWERED: SIX (6)

STATIONERY PER CANDIDATE: [-] X 6 LEAF A4 BOOKS [2] X 12 LEAF A4 BOOKS

[-] X ROUGH WORK BOOKS

GRAPH PAPER: - (NUMBER OF SHEETS)

POCKET CALCULATORS PERMITTED: ~~YES~~/NO (silent type)

MATHEMATICAL TABLES PERMITTED: ~~YES~~/NO; IF YES, SUPPLIED BY STUDENT/UNIVERSITY

OTHER AIDS REQUIRED: Nil

INSTRUCTIONS FOR CANDIDATES: Approximately equal time should be devoted to each question.

Candidates may make notes on this paper during the fifteen minutes reading time.
Candidates may retain their copy of this examination question paper.

TEXTBOOKS OR NOTES PERMITTED: Nil

Question 1)

Find

(i) $\displaystyle\int \frac{dx}{1+\sqrt{x}}$

(ii) $\displaystyle\int (\ln(x))^2 dx$

(iii) $\displaystyle\int \frac{x}{\sqrt{1 - x^4}}\, dx.$

Question 2)

(a) Evaluate $\displaystyle\int_0^{\frac{1}{2}} \frac{dx}{x^3 - x^2 + x - 1}$

(b) For $a > 0$ we know that

$$\int_1^a t^{x-1}\, dt = \begin{cases} \dfrac{a^x - 1}{x} & \text{if } x \neq 0 \\ \ln(a) & \text{if } x = 0. \end{cases}$$

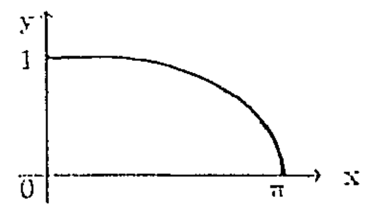
Use L'Hôpital's rule to show that

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \ln(a).$$

Question 3)

Find the volume of the figure of revolution generated by rotating about the y-axis the segment of a cycloid given parametrically by

$$x = \theta + \sin\theta \quad ; \quad 0 \leq \theta \leq \pi.$$
$$y = 1 + \cos\theta$$



Question 4)

(a) Find the MacLaurin expansion, including an expression for the general term, for the function

$$f(x) := \sqrt{2}\, \sin(x + \pi/4).$$

(b) Briefly describe Newton's method for approximating a zero of the function f, and state conditions which are sufficient to ensure convergence of the iterates to a zero.

Question 5)

(a) For the complex number $z := 2 - 2\sqrt{3}\, i$, find

(i) $|z|$

(ii) z^2

(iii) $\bar{z}/z$

(iv) a complex number ω, with $|\omega| = |z|$, such that the angle between the lines from the origin to z and the origin to ω is a right-angle

(v) the two square roots of z, and plot these together with z in the Argand plane.

(b) In the Argand plane the distance from the origin to the complex point ω equals the perpendicular distance from ω to the line $\Re e\, z = -1$. Write down a relationship between ω and $\bar{\omega}$ which expresses this fact.

(c) Show that the set of complex numbers z which satisfy the relationship

$$|z|^3 = \Re e\, z^2$$

correspond to those points on the curve with equation

$$r = \cos(2\theta)$$

in polar coordinates for which $\cos(2\theta) \geq 0$. Hence sketch the curve in the Argand plane.

Question 6)

In each of the following cases find the unique solution of the given differential equation which satisfies the given initial condition(s).

(i) $\frac{dy}{dx} = (1 + y)^{3/4}$, $y(0) = 0$

(ii) $\frac{dy}{dx} + y/x = 2e^{x^2}$, $y(1) = e$

(iii) $\frac{dy}{dx} = 1 - y^2$, $y(0) = 0$

(iv) $y'' - y' - 2y = 0$, $y(0) = 1$, $y'(0) = 1/2$

(v) $y'' + 6y' + 10y = 0$, $y(0) = 0$, $y'(0) = 6$

(vi) $y'' + 6y' + 9y = 0$, $y(0) = 0$, $y'(0) = 1$.

CONTENTS

LECTURE NOTES

CHAPTER 7

INTEGRATION

7.1 The definite integral

To each function f defined on some given closed interval [a,b] of real numbers we aim to associate a real number $\int_a^b f$ which we call the definite integral of f from a to b.

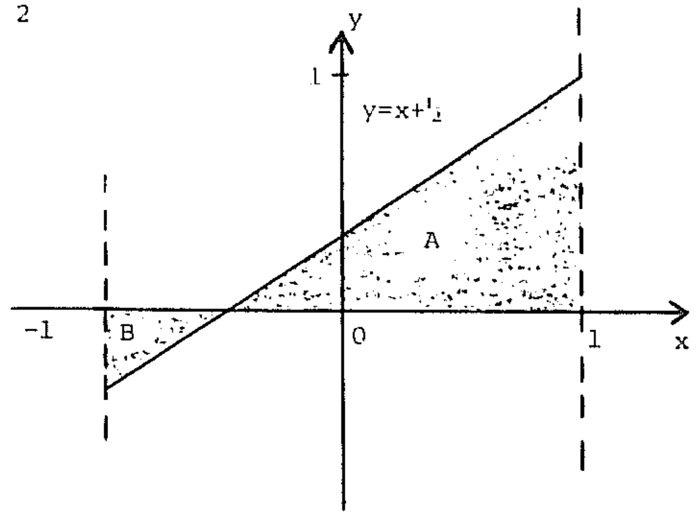
Intuitively the idea is very simple, the number $\int_a^b f$ should be the *area* of the region enclosed by the curve $y = f(x)$, the lines $x = a$, $x = b$ and the x-axis. Regions "below" the x-axis being assigned negative areas.

For example, the definite integral from -1 to 1 of the function

$$f(x) := x + \frac{1}{2}$$

is the number

$$\begin{aligned} \int_a^b f &= \text{shaded area} \\ &= \text{area A + area B} \\ &= \frac{9}{8} - \frac{1}{8}, \quad \text{as region B is below the x-axis} \\ &= 1 . \end{aligned}$$



The difficulty lies in making sense of what we mean by "area" for more complicated functions.

What would we mean by the area of the region enclosed by $x = 0$, $x = 1$, the x-axis and $y = f(x)$ when

$$f(x) := \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational ?} \end{cases}$$

Indeed it is not possible to assign an area to every function f.

We will only go part way toward answering such questions in the present course. To begin with we identify some "self-evident" properties which the definite integral must satisfy.

(1) *For any real number* c; $\int_a^b cf = c\int_a^b f$.

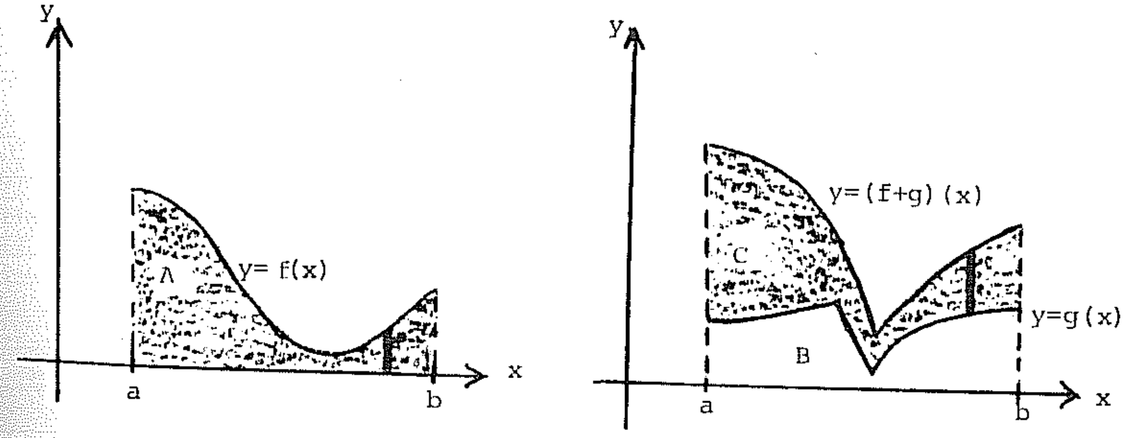
Here cf denotes the function whose value at any x is c times the value of f at x. (1) asserts that if the vertical distances between the x-axis and the graph of $y = f(x)$ are all scaled by a factor of c, then so too will the area be.

(2) *For two functions* f *and* g *defined on the interval* [a,b];

$\int_a^b f + g = \int_a^b f + \int_a^b g.$

Here f + g is the "point-wise sum" of f and g;

$(f + g)(x) := f(x) + g(x)$.



Region C is Region A "laid on top of" Region B and so has the same area as A.

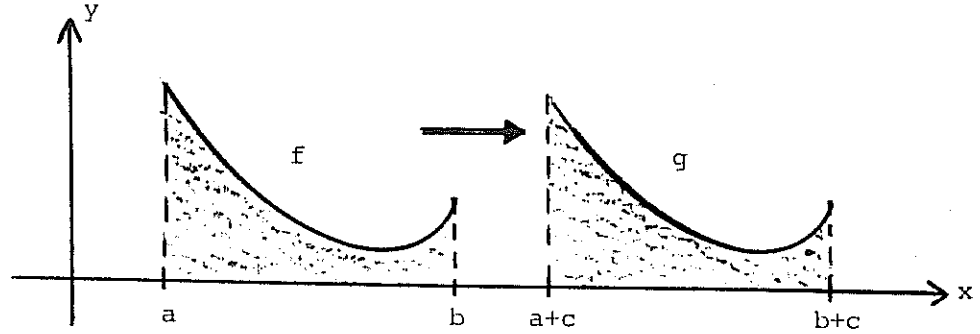
(3) *If* f *is a positive function on* [a,b]; that is, $f(x) \geq 0$ for all x in [a,b], *then*

$$\int_a^b f \geq 0 .$$

(4) *The Definite integral is translationally invariant;* that is, if $g(x) := f(x - c)$, so g is the function obtained by "sliding f to

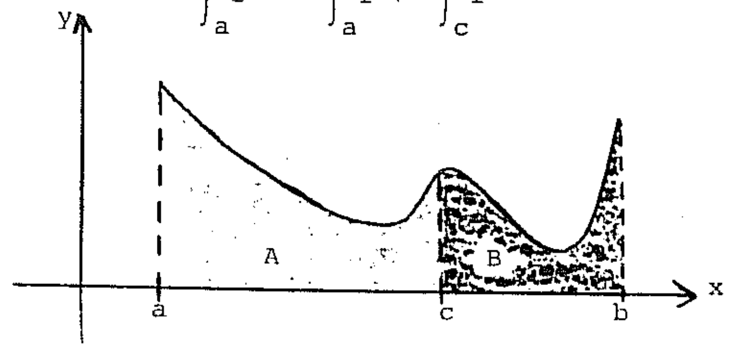
the right a distance c", then

$$\int_{a+c}^{b+c} g = \int_a^b f \quad .$$



(5) *If* c *is a real number with* $a \le c \le b$ *then*

$$\int_a^b f = \int_a^c f + \int_c^b f$$



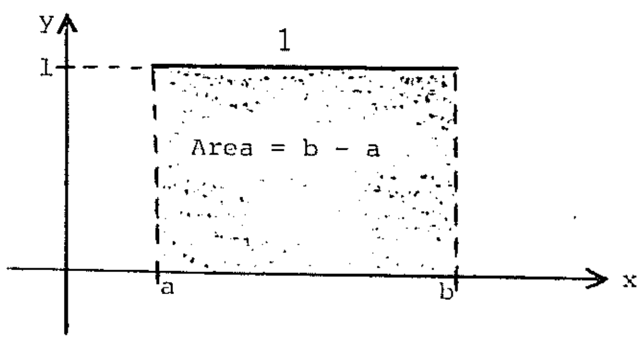
$$\int_a^b f = \text{total shaded area}$$

$$= \text{Area A + Area B}$$

$$= \int_a^c f + \int_c^b f \quad .$$

(6) $\int_a^b 1 = b - a,$

here 1 denotes the constant function which is identically equal to 1 on [a,b]



It is convenient to have (5) hold for all real numbers a,b and c regardless of their order. This is achieved by taking

$$\int_b^a f := -\int_a^b f \quad , \quad \text{whenever} \quad a < b.$$

A number of further properties of the definite integral can be deduced from the above six.

<u>For Example</u>

If f *is identically zero on the interval* [a,b] *, then* $\int_a^b f = 0.$

<u>Proof</u>. If f is identically zero, then

$$f = 0.1$$

where 0 is the real number *zero* and 1 is the constant function which is identically equal to 1 on [a,b].

Hence,

$$\int_a^b f = \int_a^b 0.1$$

$$= 0 \int_a^b 1 \quad , \text{ by (1)}$$

$$= 0\ (b - a) \quad \text{by} \quad (6)$$

$$= 0.$$

□

If $f(x) \le 0$ *for all* x *in* [a,b] *then*

$$\int_a^b f \le 0$$

<u>Proof</u>. If f is 'negative' on [a,b] then $-1f$ is positive and so by (3)

$$0 \le \int_a^b -1f$$

$$= -1\int_a^b f \quad , \quad \text{by (1)}$$

and so $\int_a^b f \le 0$ as required.

□

One of the most useful consequences is the

INTEGRAL MEAN VALUE THEOREM 7.1.1.

If m,M *are real numbers such that* $m \le f(x) \le M$ *for all* x *in* [a,b] , *then*

$$m(b - a) \le \int_a^b f \le M(b - a) .$$

Proof. Since $f(x) \le M$ for all x in [a,b] the function $g(x):=M-f(x)$ is positive on [a,b] and so by (3)

$$\begin{aligned} 0 \le \int_a^b g &= \int_a^b (M - f) \\ &= \int_a^b M.1 + (-1.f) \\ &= M\int_a^b 1 + -1\int_a^b f \quad , \text{ by (2) and (1)} \\ &= M(b - a) - \int_a^b f \quad , \text{ by (6).} \end{aligned}$$

Rearranging gives

$$\int_a^b f \le M(b - a).$$

The inequality $m(b - a) \le \int_a^b f$ is deduced similarly, and is left as an exercise. □

A few words of explanation about what we've been doing are probably desparately awaited.

It can be shown that for a large class of functions, including all the continuous ones, the properties (1) to (6) uniquely determine the definite integral, in the sense that for each function f and each pair of real numbers a,b with $a \le b$ there is only one way of assigning the real number

$$\int_a^b f$$

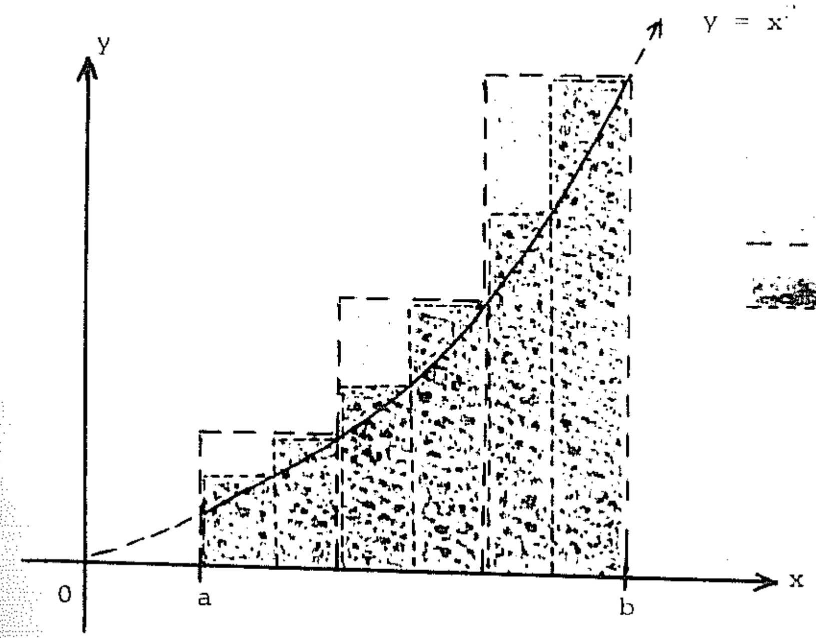
so that (1) to (6) always hold.

Thus we take (1) to (6) as the **axioms** (defining properties) for the definite integral.

Further for a function f which is continuous on the closed interval [a,b] it can be proved that

$$\int_a^b f = \lim_{n\to\infty} \frac{b - a}{n} \sum_{k=1}^{n} f\left(a + k\left[\frac{b - a}{n}\right]\right) \quad \ldots\ldots \quad (7.1.2)$$

A result which intuitively should not be surprising (see diagram below) and from which we can determine the definite integral in a number of special cases.



For EXAMPLE: If $f(x) := x^2$, to find the definite integral from a to b we first note, by (5) that

$$\int_a^b f = \int_0^b f - \int_0^a f$$

and then use the above expression to obtain

$$\begin{aligned} \int_0^b f &= \lim_{n\to\infty} \frac{b}{n} \cdot \sum_{k=1}^{n} \left(\frac{kb}{n}\right)^2 \\ &= b^3 \lim_{n\to\infty} \frac{1}{n^3} \sum_{k=1}^{n} k^2 \ . \end{aligned}$$

Recalling the formula

$$\sum_{k=1}^{n} k^2 = \frac{1}{6} n (n + 1)(2n + 1)$$

(a result you should prove by induction, if you are not already familiar with it), we therefore have

$$\int_0^b f = b^3 \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3} = \frac{b^3}{3} .$$

Similarly, replacing b by a

$$\int_0^a f = \frac{a^3}{3}$$

and so we deduce

$$\int_a^b f = \frac{b^3}{3} - \frac{a^3}{3}$$

When a function f is defined by a formula such as $f(x) := x^2$ it is convenient to write

$$\int_a^b f \quad \text{as} \quad \int_a^b x^2 \quad \text{or even} \quad \int_a^b x^2\,dx,$$

here the symbol dx has no meaning in isolation but serves to indicate that the variable (argument) of the formula is x.

Thus in the notation

$$\int_a^b [\,\textit{formula}\,]\,d\,[\,\textit{argument of formula}\,]$$

we have

$$\int_a^b x^2\,dx = \frac{b^3}{3} - \frac{a^3}{3}$$

and equivalently

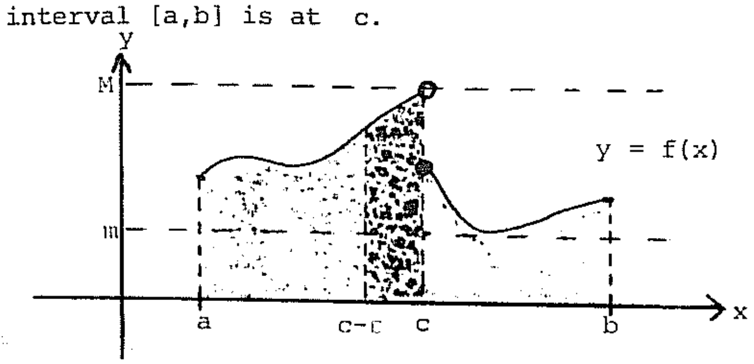
$$\int_a^b t^2\,dt = \frac{b^3}{3} - \frac{a^3}{3} .$$

Because any symbol may be substituted for x without affecting the answer it is sometimes referred to as a *dummy variable*.

Definite integrals for many of the simple functions can be calculated in this way. The procedure is however tedious and part of our aim is to find an alternative technique (section 7.2).

The general question of determining which functions can have definite integrals assigned to them in such a way that (1) to (6), and hence their consequences, continue to hold is non-elementary. It has been the object of extensive mathematical investigations which to some extent culminated in the theory of Riemann integrable functions (circa 1850) and Lebesgue's theory of integration (1904). Functions for which such an assignment is possible are termed **integrable** functions. All continuous functions are integrable, as are bounded functions with only a finite number of discontinuities.

To illustrate this, let f be a bounded function whose only discontinuity in the interval [a,b] is at c.



Then for (5) to hold we require

$$\int_a^b f = \int_a^c f + \int_c^b f$$

and so we need only assign a value to integrals of the form

$$\int_a^c f$$

where f is continuous everywhere in the interval [a,c] except at the end point c.

To do this note that for any real number $\varepsilon > 0$ we require

$$\int_a^c f = \int_a^{c-\varepsilon} f + \int_{c-\varepsilon}^c f$$

Further if the integral mean value theorem is to hold, then

$$m\varepsilon \le \int_{c-\varepsilon}^c f \le M\varepsilon$$

(where m and M are lower and upper bounds for f on [a,b]) and so

$$\int_{c-\varepsilon}^{c} f \to 0 \qquad \text{as} \qquad \varepsilon \to 0,$$

yielding

$$\int_a^c f = \lim_{\varepsilon \to 0} \int_a^{c-\varepsilon} f .$$

Extending this to a bounded function f whose only points of discontinuity in the interval $[a,b]$ are $c_1, c_2, \ldots, c_n$ we have

$$\int_a^b f = \int_a^{c_1} f + \int_c^{c_2} f + \ldots + \int_{c_n}^b f .$$

In general, for a function f which is discontinuous (or undefined) at the point b we take

$$\int_a^b f := \lim_{\varepsilon \to 0} \int_a^{b-\varepsilon} f ,$$

provided the limit exists.

Similarly, we take

$$\int_a^\infty f := \lim_{b \to \infty} \int_a^b f$$

whenever the limit exists. Such integrals are sometimes referred to as *improper integrals*. We will meet examples of this type of integral later in the course.

EXERCISES:

(1) (a) Show that all but the last of the axioms (1) to (6) would be satisfied by taking

$$\int_a^b f = 0$$

for every function f and pair of numbers a,b.

(b) Show that for every function f and pair of numbers a, b w[illegible] $a < b$, all but the fifth axiom would be satisfied by taking

$$\int_a^b f = Af(a) + Bf\left(\frac{a+b}{2}\right) + Cf(b)$$

where A,B and C are positive constants with $A+B+C = b-a$.

(2) Find, using only the results of this section

(a) $\int_1^2 8x^2 + 3x + 2\, dx$

(b) $\int_a^b x^3$ [Hint: Show by induction $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$]

(c) $\int_0^a ax + tx\, dt$ (d) $\int_1^0 x^2$

7.2 The Fundamental Theorem of Calculus

From any integrable function f and real number c we can construct a new function F, by letting the upper limit of integration vary and taking

$$F(x) := \int_c^x f .$$

For example, if $f(x) := x^2$ and $c = 0$, then

$$F(x) = \frac{x^3}{3} .$$

$F(x)$ is termed a Primitive function (or indefinite integral) for f.

If we used a different constant k for the lower limit of integration we would obtain a different primitive function

$$\mathsf{F}(x) := \int_k^x f .$$

By the extended form of (5) we have however that

$$\mathsf{F}(x) - F(x) = \int_k^x f - \int_c^x f = \int_k^x f + \int_x^c f$$
$$= \int_k^c f$$

a number depending only on c and k but independent of x. Thus, any two primitive functions for f differ only by a constant and so, if F is any primitive function for f then every other primitive function for f has the form

$$F(x) + C , \quad \text{where } C \text{ is constant.}$$

We will sometimes denote this expression by

$$\int^x f \quad \text{or} \quad \int f \ .$$

When f is given by a formula $f(x)$ we will also write

$$\int f(x)\, dx.$$

For example $\int x^2\, dx = \frac{x^3}{3} + C.$

The problem of finding the definite integral of f from a to b is easily "reduced" to that of finding a primitive function for f.

Indeed if $F(x)$ is any primitive function, so $F(x) = \int_c^x f$ for some c, then

$$\begin{aligned}\int_a^b f &= \int_a^c f + \int_c^b f \\ &= \int_c^b f - \int_c^a f \\ &= F(b) - F(a)\end{aligned}$$

It therefore seems reasonable to pursue the connection between a function f and its primitive.

As a general rule, the primitives of a function are "better" behaved than the function itself. For example

<u>Theorem 7.2.1</u> *Any primitive function of a bounded integrable function is continuous.*

Proof. Let $F(x) = \int_c^x f$ be a primitive for the bounded integrable function f and let M be a bound such that $-M \le f(x) \le M$ for all x. Then for any pair of points x and x_0

$$\begin{aligned}|F(x) - F(x_0)| &= \left|\int_c^x f - \int_c^{x_0} f\right| \\ &= \left|\int_{x_0}^x f\right| \\ &\le M|x - x_0| \ , \quad \text{by the integral mean value Theorem}\end{aligned}$$

and so as $x \to x_0$ we have $F(x) \to F(x_0)$, establishing the continuity of F at each point x_0. □

Perhaps more surprising is the central result of this section and much of our subsequent work;

<u>The Fundamental Theorem of Calculus 7.2.2</u>

If F *is any primitive function for* f *and* f *is continuous at the point* x_0, *then* F *is differentiable at* x_0 *with derivative*

$$F'(x_0) = f(x_0).$$

<u>Proof</u>

$$\left|\frac{F(x_0 + h) - F(x_0)}{h} - f(x_0)\right|$$

$$= \left|\frac{\int_c^{x_0+h} f(x) - \int_c^{x_0} f(x) - f(x_0)h}{h}\right| \qquad \text{for some } c\ .$$

$$= \left|\frac{\int_{x_0}^{x_0+h} f(x) - \int_{x_0}^{x_0+h} f(x_0)}{h}\right|$$

The second integral equals $f(x_0)h$ as $f(x_0)$ is a constant.

$$= \left|\frac{\int_{x_0}^{x_0+h} [f(x) - f(x_0)]}{h}\right|$$

$$\le \underset{\substack{\text{for } x \text{ between} \\ x_0 \text{ and } x_0 + h}}{\text{Maximum}} |f(x) - f(x_0)| \ ,$$

by the integral mean value theorem since for x between x_0 and $x_0 + h$, $-M_h \le f(x) - f(x_0) \le M_h$ where

$$M_h := \underset{\substack{\text{for } x \text{ between} \\ x_0 \text{ and } x_0 + h}}{\text{Maximum}} |f(x) - f(x_0)|.$$

Now, by the continuity of f at x_0, as $x \to x_0$ $f(x) \to f(x_0)$ and so as $h \to 0$ we see that $M_h \to 0$.

Thus

$$\lim_{h \to 0} \left| \frac{F(x_0 + h) - F(x_0)}{h} - f(x_0) \right| = 0,$$

establishing that F is differentiable at x_0 with $F'(x_0) = f(x_0)$. □

This connection between integration and differentiation, first observed by Newton and Leibniz (circa 1670), is seminal for much of mathematics and its many applications. It provides the key to determining integrals for many of the elementary functions. Indeed, since two functions with equal derivatives differ only by a constant, if F is an antiderivative for f: that is, a function which differentiates to give f, $f = F'$; then F is a primitive function for f. Thus the problem of finding a primitive (and hence integrating) can in many cases be reduced to that of finding an antiderivative. Since the rules of differentiation are *mechanical* it is possible to build up a "table" of standard derivatives and hence, reading it in reverse, also a table of integrals.

For example: Since $\frac{d}{dx}(\sin x) = \cos x$ we see that $\sin x$ is an antiderivative (and hence a primitive) for $\cos x$,

$$\int \cos x \, dx = \sin x + C$$

and so in particular

$$\int_0^{\frac{\pi}{2}} \cos x \, dx = \sin(\tfrac{\pi}{2}) - \sin(0) = 1$$

At this stage it is convenient to introduce the notation

$$[F(x)]_a^b := F(b) - F(a),$$

so we could write $\int_0^{\frac{\pi}{2}} \cos x \, dx = [\sin x]_0^{\frac{\pi}{2}}$.

Similarly, since for a real number we know $\frac{d}{dx}(x^m) = mx^{m-1}$ ($x \neq 0$ when $m<0$) we readily see that

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n \quad \text{whenever } n \neq -1 \quad (x \neq 0 \text{ when } n < 0)$$

and so

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C.$$

Whence, in particular

$$\int_a^b x^2 \, dx = \left[\frac{x^3}{3}\right]_a^b = \frac{b^3}{3} - \frac{a^3}{3} .$$

A less painful derivation than that given in §7.1 (assuming you know how to differentiate), and one indicative of the power of the last theorem.

In these cases, the problem of integration is reduced to answering the question:

What function will differentiate to give $f(x)$?

The same idea readily establishes the results tabulated below, with which you should be familiar.

Function	Primitive
$f(x)$	$F(x) = \int f(x)\,dx$
x^n $(n \neq -1)$ ($x \neq 0$ when $n < 0$)	$\frac{x^{n+1}}{n+1} + C$
$\cos x$	$\sin x + C$
$\sin x$	$-\cos x + C$
$\frac{1}{\sqrt{1-x^2}}$ $\quad |x| < 1$	$\sin^{-1}(x) + C$
$\frac{1}{1+x^2}$	$\tan^{-1}(x) + C$

You should add to this list as more of the "elementary" functions are introduced.

In the next section we will see how such a list, combined with some rules for integrals, allow us to *integrate* (that is, find primitives for) a large number of simple functions.

EXERCISES:

(1) Show that

(a) $\int \sec^2 x \, dx = \tan x + C$

(b) $\int \sin^{-1} x \, dx = x \sin^{-1} x + \sqrt{1 - x^2} + C$

(2) Find

(a) $\int_0^1 \frac{dt}{1 + t^2}$

(b) $\int_0^\infty \frac{dt}{1 + t^2}$

(c) $\int_1^\infty \frac{1}{x^2} \, dx$

(3) (i) Show that for any numbers a and b, with $b \neq 0$,

$$\int a \sin bx \, dx = -\frac{a}{b} \cos bx + C$$

(ii) Using the identity

$$\sin mx \cos nx = \frac{1}{2}[\sin(m+n)x + \sin(m-n)x]$$

deduce that

$$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0$$

for all integers m and n.

(4) From your knowledge of differentiation we readily deduce that for $|x| < 1$

$$\int \frac{-1}{\sqrt{1 - x^2}} \, dx = \cos^{-1} x + C .$$

This has been left out of our table, however

$$\int \frac{-1}{\sqrt{1 - x^2}} \, dx = \int -1 \frac{1}{\sqrt{1 - x^2}} \, dx$$

$$= -1 \int \frac{1}{\sqrt{1 - x^2}} \, dx , \quad \text{by axiom (1)}$$

$$= -\sin^{-1} x + C.$$

How do you reconcile these two apparently conflicting answers?

(5) If f is an integrable, invertible function show that

$$\int^x f^{-1} = x f^{-1}(x) - \int^{f^{-1}(x)} f$$

Give an interpretation of this result in terms of areas and graphs.

Observe that (1)(b) is a special case of this general rule.

7.3 Techniques of Integration

Two of the most frequently used rules for integrals are provided by our first pair of axioms translated in terms of primitives:

For any real number c $\quad \int cf = c\int f$.

For functions f and g $\quad \int (f + g) = \int f + \int g$.

[In the case when f and g are continuous functions not just integrable ones, so antiderivatives exist, these two rules also reflect the corresponding rules for differentiation $(cf)' = cf'$ and $(f + g)' = f' + g'$.]

Using these two rules we can for example integrate any polynomial. For example:

$$\int 12x^3 + \frac{3}{4}x^2 + 9 \, dx = 12\int x^3 dx + \frac{3}{4}\int x^2 dx + 9\int 1 \, dx$$

$$= 3x^4 + \frac{1}{4}x^3 + 9x + C.$$

Similarly we have

$$\int(8 \sin x - \frac{1}{\sqrt{1 - x^2}} + 11x)\, dx = 8\int \sin x\, dx - \int \frac{1}{\sqrt{1 - x^2}}\, dx + 11\int x\, dx$$

$$= - 8 \cos x - \sin^{-1}x + \frac{11}{2}x^2 + C.$$

Perhaps the singly most important method of finding primitives as antiderivatives is a consequence of the composite function rule for differentiation: $(G \circ f)'(x) = (G' \circ f)(x).f'(x)$, known as

Integration by Substitution.

Unfortunately this method is not easy to explain and certainly requires ingenuity if it is to be used successfully.

To determine $\int h$ the idea is to find (guess!) suitable functions f and g so that h may be rewritten as

$$h(x) = (g \circ f)(x).f'(x)$$

[that is, $h(x) = g(f(x))f'(x)$]

as then

$$\int h(x) = \int (g \circ f)(x).f'(x)$$

$$= \int (G \circ f)'(x), \quad \text{where } G' = g, \text{ or } G = \int g$$

$$= (G \circ f)(x) + C.$$

This last expression could also be written as $\int^{f(x)} g$,

so summarizing we have:

If $h(x) = g(f(x))f'(x)$ *then* $\int h(x) = \int^{f(x)} g$.

Of course this will only prove useful if we have been able to choose a function g for which it is within our ambit to find an integral.

For <u>example</u>:

To find $\int x \cos(x^2)\, dx$

we may note that

$$x \cos(x^2) = \frac{1}{2} \cos(x^2).2x$$

$$= g(f(x)).f'(x)$$

where $g := \frac{1}{2} \cos$ and $f(x) := x^2$,

so

$$\int x \cos(x^2) = \int^{f(x)} g$$

$$= \int^{x^2} \tfrac{1}{2} \cos$$

$$= \frac{1}{2}\int^{x^2} \cos$$

$$= \frac{1}{2} \sin(x^2) + C.$$

It is useful to rephrase these results in terms of the "substitution" (hence the name of the method) $u = f(x)$.

In these terms, we seek a substitution $u = f(x)$ so that

$$h(x) = g(u) \frac{du}{dx}$$

and then

$$\int h(x)\, dx = \int g(u) \frac{du}{dx}\, dx$$

$$= \int g(u)\, du.$$

In order to make sense of the last equality we need the implied resubstitution of $f(x)$ for u, which yields the answer,

$$\int^{f(x)} g .$$

<u>For example:</u>

(1) To find $\int x\sqrt{1 + x^2}\, dx$,

if we substitute $u = 1 + x^2$, noting that $\frac{du}{dx} = 2x$, we have $x\sqrt{1 + x^2} = \frac{1}{2}\sqrt{u}\ \frac{du}{dx}$ and so

$$\int x\sqrt{1+x^2}\,dx = \int \frac{1}{2}\sqrt{u}\,\frac{du}{dx}\,dx$$

$$= \frac{1}{2}\int u^{\frac{1}{2}}\,du = \frac{1}{3}u^{\frac{3}{2}} + C$$

resubstituting $1 + x^2$ for u we therefore have

$$\int x\sqrt{1+x^2}\,dx = \frac{1}{3}(1+x^2)^{\frac{3}{2}} + C$$

$$\left[= \frac{1}{3}(1+x^2)\sqrt{1+x^2} + C \text{ , if you prefer} \right].$$

(2) To find $\int \sin^5 x\,dx$ note that

$$\int \sin^5 x\,dx = \int (1-\cos^2 x)^2 \sin x\,dx$$

$$= \int \sin x\,dx - 2\int \cos^2 x \sin x\,dx + \int \cos^4 x \sin x\,dx$$

$$= -\cos x + 2\int u^2\,du - \int u^4\,du$$

where both the last two integrals have used the substitution $u = \cos x$.

Thus, $\int \sin^5 x\,dx = -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C.$

[Provided your knowledge of the Binomial theorem is up to scratch, a similar treatment allows us to find any integral of the form

$$\int \sin^m x \cos^n x\,dx$$

if at least one of the exponents m and n is odd.]

The basic problem is to find a substitution (if one exists) for which the resulting integral $\int g(u)\,du$ is tractable. There are no general rules here, one must proceed by trial-and-error guided by experience gained from practice (see exercises).

In some instances a useful start can be the observation that $h(x) = g(f(x))$ where g is a "simpler" function than h and f is fairly tame. In this case making the substitution $u = f(x)$ we have

$$\int h(x)\,dx = \int g(u)\,dx$$

where the sought for factor $\frac{du}{dx}$ is missing. This may be artificially cured by writing

$$\int g(u)\,dx = \int \left[g(u) \cdot \frac{1}{\frac{du}{dx}} \right] \frac{du}{dx}\,dx$$

$$= \int \left[g(u)\,\frac{dx}{du} \right] du,$$

by the inverse function theorem (see chapter 6), provided f is invertible.

For example: To find $\int \sqrt{1-x^2}\,dx$, making the "inverse substitution" $x = \sin u$ we have

$$\int \sqrt{1-x^2}\,dx = \int \sqrt{1-\sin^2 u}\,\frac{d(\sin u)}{du}\,du$$

$$= \int \cos^2 u\,du.$$

This last integral may be found using the identity

$$\cos^2 u = \frac{1}{2}(1 + \cos 2u)$$

so that

$$\int \cos^2 u\,du = \frac{1}{2}\int du + \frac{1}{2}\int \cos 2u\,du$$

$$= \frac{1}{2}u + \frac{1}{4}\sin 2u + C.$$

(The last integral may be found via the substitution $v = 2u$ - see Exercise 7.3.(1)) and so we have, substituting $\sin x$ for u that

$$\int \sqrt{1-x^2}\,dx = \frac{1}{2}\sin^{-1}x + \frac{1}{4}\sin(2\sin^{-1}x) + C$$

$$= \frac{1}{2}\sin^{-1}x + \frac{1}{2}\sin(\sin^{-1}x)\cos(\sin^{-1}x) + C$$

$$= \frac{1}{2}\sin^{-1}x + \frac{1}{2}x\sqrt{1-\sin^2(\sin^{-1}x)} + C$$

$$= \frac{1}{2}\sin^{-1}x + \frac{1}{2}x\sqrt{1-x^2} + C$$

In general finding a workable substitution is not easy. Its choice is to some extent determined by the form of the *integrand* (that is, the function to be integrated), however frequently more than one substitution will work. For example the integral

$$\int x\sqrt{1-x^2}\,dx$$

may be found using either of the substitutions $u = 1 - x^2$ or $x = \sin u$, see Exercise 7.3.(2)

A few suggestions are:-

Integrand	Substitution
A function of ax + b (by completing the square, functions of a quadratic in x are reduced to this form - further substitutions are usually necessary).	$u = ax + b$
A function involving $\sqrt{c^2 - x^2}$	$x = c \sin u$ } or $x = cu^{-\frac{1}{2}}$
A function involving $\sqrt{c^2 + x^2}$	$x = c \tan u$ }
A function involving $\sqrt{x^2 - c^2}$	$x = c \sec u$
A function of sin x and cos x only	The "t-substitution": $t = \tan x/2$ used in cojunction with the formulas of §6.5. (Sometimes $u = \tan x$ works better.)
A function involving fractional powers of x	$x = u^n$, where n is the common denominator of the exponents.

Another important method of integration;

Integration by Parts

derives from the product rule for differentiation: $(u.v)' = u.v' + v.u'$.

From this we see that for two continuously differentiable functions

u and v, the product u.v is an antiderivative for $u.v' + v.u'$ and so

$$u(x)v(x) = \int u(x)v'(x) + v(x)u'(x)\,dx$$
$$= \int u(x)v'(x)\,dx + \int v(x)u'(x)\,dx$$

rearranging this we obtain the basic formula

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int v(x)u'(x)\,dx$$

or in terms of definite integrals

$$\int_a^b u(x)v'(x)\,dx = \Big[u(x)v(x)\Big]_a^b - \int_a^b v(x)u'(x)\,dx$$

To apply these formulae we need to recognise the integrand as a product of two functions (u and v') in such a way that an antiderivative (v) for one factor (v') can be found and also, if the application is to be fruitful, so that the resulting integral $\int v.u'$ is in some sense simpler than the original one.

Operationally then the rule might be expressed better by writing f for u and g for v' to obtain

$$\int f.g = f.\int g - \int (f'.\int g).$$

(Here it is implied that the same primitive for g is used at both occurences.)

<u>For example</u>:

$$\int \underset{u}{x}\, \underset{v'}{\cos x}\,dx = \underset{u}{x}\, \underset{v}{\sin x} - \int \underset{u'}{1}\, \underset{v}{\sin x}\,dx$$

(so $v = \sin x$)

$$= x \sin x + \cos x + C.$$

Notice that we might have selected cos x for u and x for v', however this would have led to

$$\int x\cos x\,dx = \tfrac{1}{2}x^2\cos x + \tfrac{1}{2}\int x^2 \sin x\,dx$$

and resulted in a less tractable integral than the original one. Clearly some care (and intelligent trial-and-error) is necessary.

Integration by parts may be used to derive various reduction formulae, for example:

If $I_n = \int \sin^n x \, dx$, then

$$I_n = \int \underset{u}{\sin^{n-1} x} \, . \underset{v'}{\sin x} \, dx$$

$$= \underset{u}{\sin^{n-1} x} \, . \underset{v}{(-\cos x)} - \int \underset{u'}{(n-1)\sin^{n-2} x \cos x} \, . \underset{v}{(-\cos x)} \, dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x \, dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x (1-\sin^2 x) \, dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \, dx - (n-1)\int \sin^n x \, dx$$

$$= -\sin^{n-1} \cos x + (n-1) I_{n-2} - (n-1) I_n$$

rearranging and solving for I_n we have

$$I_n = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} I_{n-2}$$

or $$\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx \, .$$

Repeated application of this reduction formulae allows us to find $\int \sin^n x$ for any n (even or odd).

Thus

$$\int \sin^4 x = -\frac{1}{4}\sin^3 x \cos x + \frac{3}{4}\int \sin^2 x$$

$$= -\frac{1}{4}\sin^3 x \cos x + \frac{3}{4}\left[-\frac{1}{2}\sin x \cos x + \frac{1}{2}\int \sin^0 x\right]$$

$$= -\frac{1}{4}\sin^3 x \cos - \frac{3}{8}\sin x \cos x + \frac{3}{8}\int 1 \, dx.$$

$$= -\left[\frac{1}{4}\sin^3 x + \frac{3}{8}\sin x\right]\cos x + \frac{3}{8}x + C \, .$$

The trick used to find I_n in the last example often works. That is, we use integration by parts to find $\int h$ in terms of $\int h$ again, and then solve the resulting expression for $\int h$.

EXERCISES

(1) For a continuous function f and constant a show that

$$\int^x f(ax) \, dx = \frac{1}{a}\int^{ax} f(x) \, dx$$

(2) Find

$$\int x\sqrt{1 - x^2} \, dx$$

by using (a) the substitution $u = 1 - x^2$

and (b) the substitution $x = \sin u$.

Observe that (b) leads to $\int \sin u \cos^2 u \, du$. Making the further substitution $v = \cos u$ gives

$$\int x\sqrt{1 - x^2} \, dx = \int \sin u \cos^2 u \, du = -\int v^2 dv = -\frac{1}{3} v^3 + C$$

$$= -\frac{1}{3}\cos^3 u + C = -\frac{1}{3}\cos^3 (\sin^{-1} x) + C.$$

On the other hand, using $x = \cos u$ gives

$$\int x\sqrt{1 - x^2} \, dx = -\int \sin^2 u \cos u \, du = -\frac{1}{3}\sin^3 u + C = -\frac{1}{3}\sin^3(\cos^{-1} x) + C.$$

Reconcile the three apparently different answers for $\int x\sqrt{1 - x^2} \, dx$ that were obtained in (a) and (b).

(3)(a) Using algebraic manipulations and/or substitutions find the following integrals.

(i) $\displaystyle\int \frac{dx}{\sqrt{4 - 9x^2}}$

(ii) $\displaystyle\int \sin x \sin 2x \, dx$

(iii) $\displaystyle\int \frac{\sqrt{1 - x}}{1 - \sqrt{x}} \, dx$

(iv) $\displaystyle\int \frac{x \, dx}{\sqrt{1 - x^4}}$

(v) $\displaystyle\int \frac{dx}{x^2 + 2x + 2}$

(vi) $\displaystyle\int \frac{dx}{1 + \cos x}$

(vii) $\displaystyle\int \frac{dx}{2 + \sin x}$

(b) Use integration by parts to find:

(i) $\int x^2 \sin x \, dx$

(ii) $\int x \tan^{-1} x \, dx$

(iii) $\int \sin^{-1} x \, dx$,

compare your answer with 7.2.(1)(b).

[Hint: rewrite the integral as $\int 1.\sin^{-1} x \, dx$.
Considering the factor v' to be the constant 1 which can always be done is a trick which often works with integration by parts.]

(c) Find, using any method

(i) $\int \frac{x + 2}{\sqrt{5 - x^2 + 2x}} dx$

(ii) $\int \sec^{-1}\sqrt{x} \, dx$

(iii) $\int \frac{dx}{\sqrt{\sqrt{x} + 1}}$

(4) (a) Derive the answer obtained in the notes for $\int \sin^5 x \, dx$ by using the reduction formula for $\int \sin^n x \, dx$.

(b) Derive a reduction formula for

$$\int \cos^n x \, dx.$$

(c) (i) Show that

$$\int_0^{\frac{\pi}{2}} \sin^{2n+1} x \, dx = \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n + 1}$$

and

$$\int_0^{\frac{\pi}{2}} \sin^{2n} x \, dx = \frac{\pi}{2} \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n - 1}{2n}$$

(ii) Starting with the "obvious" inequalities

$$0 < \sin^{2n+1} x < \sin^{2n} x < \sin^{2n-1} x$$

for $0 < x < \frac{\pi}{2}$, show that

$$1 \le \frac{\int_0^{\frac{\pi}{2}} \sin^{2n} x \, dx}{\int_0^{\frac{\pi}{2}} \sin^{2n+1} x \, dx} \le 1 + \frac{1}{2n}$$

Hence conclude that

$$\frac{\pi}{2} = \lim_{n \to \infty} \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots \frac{2n}{2n - 1} \cdot \frac{2n}{2n + 1}$$

A result known as Wallis' (1616-1703) product.

Using this we obtain the following approximations to π.

n	$2 \cdot \frac{2}{1} \cdot \frac{2}{3} \cdots \frac{2n}{2n - 1} \cdot \frac{2n}{2n + 1}$
10	3.06
100	3.134
1000	3.1408

While this result is aesthetically pleasing, the product converges slowly and is not very useful computationally.

7.4 Numerical Integration

Integrals for a great many functions cannot be found in terms of the elementary functions. Some intrinsically cannot be expressed in such a way, for example

$$\int \frac{dx}{\sqrt{\cos x - \cos \theta}}$$

(the corresponding definite integral from 0 to θ determines the period of a simple pendulum swinging to an angle of θ with the vertical.)

Other integrals are simply too complicated for us to be able to find.

In still other cases an explicit functional form for the integrand may not be known only a table of values, collected from an experiment say, may be available. In these situations it is useful to calculate an approximate value for the definite integral

$$\int_a^b f$$

from the numerical values of f at several points between a and b.

Since the definite integral for a continuous function is given by

$$\int_a^b f = \lim_{n \to \infty} \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + k\left[\frac{b-a}{n}\right]\right),$$

one way is to simply evaluate the expression inside the limit for a sufficiently large value of n.

If we divide $[a,b]$ into n equal intervals, each of length $h = (b-a)/$ and write

$$x_0 = a, \quad x_1 = a + h, \; \ldots, \; x_n = a + nh = b$$

then we have

$$\int_a^b f \approx h[f(x_1) + f(x_2) + \ldots + f(x_n)] \; .$$

For example, using $n = 4$ we have

$$\int_0^1 x^5 \, dx = \frac{1}{4}[(\frac{1}{4})^5 + (\frac{1}{2})^5 + (\frac{3}{4})^5 + 1]$$

$$\doteq 0.3 \quad \text{(c.f. the exact answer of } 0.1\dot{6})$$

This idea may be developed into the Mid-ordinate and the Trapezoidal rules for numerical integration. Rather than pursue these

further we will develop an alternative approach. We begin by finding an approximation to

$$\int_{-h}^{h} f$$

in terms of the values $f(-h)$, $f(0)$ and $f(h)$. In view of Exercise 7.1.(1)(b) it is reasonable to seek an approximation of the form

$$Af(-h) + Bf(0) + Cf(h).$$

The coefficients A,B and C will be determined so that the approximation is exact for the three functions $1, x$ and x^2 (and hence by Axioms 7.1(1)&(2) for any quadratic). That is we require that

$$A + B + C = \int_{-h}^{h} 1 = 2h$$

$$-hA + hC = \int_{-h}^{h} x = 0$$

$$h^2A + h^2C = \int_{-h}^{h} x^2 = \frac{2h^3}{3}$$

These are easily solved to obtain $A = C = {}^h/_3$, $B = \frac{4}{3}h$, and so we have

$$\int_{-h}^{h} f \approx \frac{h}{3}[f(-h) + 4f(0) + f(h)].$$

An approximation known as Simpson's (1710-1761) Rule.

Using the notation of the previous page: $h = (b-a)/n$, $x_k = a+kh$; axiom 7.1.(4) then gives

$$\int_{x_{k-1}}^{x_{k+1}} f(x)\,dx = \int_{-h}^{h} f(x + a + kh)\,dx$$

$$\approx \frac{h}{3}[f(x_{k-1}) + 4f(x_k) + f(x_{k+1})]$$

and so provided n is even we have

$$\int_a^b f = \int_{x_0}^{x_2} f + \int_{x_2}^{x_4} f + \ldots + \int_{x_{n-2}}^{x_n} f$$

$$\approx \frac{h}{3}[f(x_0)+4f(x_1)+f(x_2)] + \frac{h}{3}[f(x_2)+4f(x_3)+f(x_4)] + \ldots$$

$$+ \frac{h}{3}[f(x_{n-2})+4f(x_{n-1})+f(x_n)],$$

yielding Simpson's formula for n intervals (n even):

$$\int_a^b f \approx \frac{h}{3}[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+\ldots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)]$$

For example: Using $n = 4$ we have

$$\int_0^1 x^5 \approx \frac{1}{12}[0^5 + 4(\tfrac{1}{4})^5+2(\tfrac{1}{2})^5+4(\tfrac{3}{4})^5+1]$$

$$\doteq 0.168 \quad \text{(c.f. our earlier approximation of 0.3 and the exact answer } 0.1\dot{6}\text{)}$$

In general Simpson's Formula leads to better approximations, for the same number of intervals, than the other methods mentioned. Indeed we will later see that the error in Simpson's Formula is proportional to $\frac{1}{n^4}$ while the other methods have errors proportional to $\frac{1}{n}$ or $\frac{1}{n^2}$.

EXERCISES (a calculator may be handy):

(1) (a) Use Simpson's Formula with 2, 4 and 6 intervals to obtain approximations for $\int_0^1 x^5$. Calculate the percentage error in each case; does it decrease in the way you would expect?

(b) Use Simpson's Formula with 4 intervals to estimate

$$\int_0^{\frac{\pi}{4}} \frac{dx}{\sqrt{\cos x}}$$

(2) Tabulate approximate values for the primitive function

$$F(x) = \int_1^x \frac{1}{t}\,dt$$

at $x = 1, 1.5, 2, 2.5, 3, 3.5$ and 4 using Simpson's Forumula with $h = 0.25$.

Use this information to prepare a graph of $F(x)$ for $1 \le x \le 4$ and so estimate the value for x at which $F(x) = 1$.

(3) (a) Show that Simpson's Rule for $\int_{-h}^{h} f$ is exact for any polynomial of the form

$$f(x) := a_0 + a_1x + a_2x^2 + \underbrace{a_3x^3 + a_5x^5 + \ldots + a_{2n+1}x^{2n+1}}_{\text{odd degrees only}}$$

(b) By noting that

$$\int_{x_{k-1}}^{x_{k+1}} ax^2 + bx + c\,dx = a\int_{-h}^{h} x^2dx + (2ax_k + b)\int_{-h}^{h} x\,dx + (ax_k^2+bx_k+c)\int_{-h}^{h} 1\,dx$$

deduce that Simpson's formula is exact for any quadratic.

(c) Extend the argument in (b) to show that Simpson's formula is exact for any cubic.

(4) In deriving Simpson's Rule the coefficients A, B and C were chosen to make the approximation exact for 1, x and x^2. It is possible to use alternative functions than these three.

It might be sensible to choose functions whose form and graphs resemble that of the integrand.

Derive an approximate integration Rule of the form $Af(1) + Bf(1.25) + Cf(1.5)$ which is exact for $\int_1^{1.5} f$ when $f = 1$, $x^{-\frac{1}{2}}$ and x^{-2}.

Use this to estimate

$$\int_1^{1.5} \frac{1}{x}\,dx .$$

(Compare your answer with the approximation obtained in (2) and the correct value to 7 decimal places of 0.405465.)

CHAPTER 8

THE FUNCTIONS EXP AND Ln

8.1 The Exponential Function

The solution of many simply occurring problems requires a function f satisfying;

$$f' = kf \quad \text{and} \quad f(0) = f_0 \qquad \ldots\ldots \quad (*)$$

(Here k and f_0 are known constants.)

For example:-

(1) *Population Growth* (Biology). Let $P(t)$ denote the size of a population at time t; if we make the assumption that the number of off-spring per individual per unit of time is a constant r, then we have;

$$\text{the rate of change of population size, } \frac{dP}{dt} = rP.$$

Also $P(0) = P_0$, the known initial size of the population.

(2) *Radio-active (or organic) decay* (Physics/Medicine).

Starting with an initial amount of matter M and assuming that a fixed fraction k of the matter $M(t)$ present by time t decays per unit of time, we have

$$\frac{dM}{dt} = -kM, \quad m(0) = M_0 .$$

(3) *Accumulation of Wealth* (Economics), or how money breeds money. If an initial investment of I_0 compounds interest continuously at a rate r, then if $I(t)$ denotes the value of our investment by time t we have

$$\frac{dI}{dt} = (1+r)I \quad , \quad I(0) = I_0$$

Let $g(x) = f(\frac{x}{k})/f_0$, then $g(0) = 1$,

and $\quad g'(x) = \frac{1}{k} f'(\frac{x}{k})/f_0$

$$= f(\tfrac{x}{k})/f_0 \qquad \text{(by (*))}$$

$$= g(x)$$

also, $\quad f(x) = f_0\, g(kx)$.

Therefore to solve (*) for any value of k and f_0 it is only necessary to know a function g satisfying the "differential equation"

$$g' = g \quad \text{and} \quad g(0) = 1.$$

For the moment we will assume that such a function is possible (and so the models given above are sensible at least in so far as they admit solutions). We will refer to this function as the exponential function and denote it by $\exp(x)$.

Thus

$$\exp'(x) = \exp(x) \text{ and } \exp(0) = 1 \ldots\ldots\ldots \quad (**)$$

We will soon see that there can be only one such function, so it makes sense to give it a particular name. A proof that such a function does indeed exist is deferred to § 8.2.

Properties of $\exp(x)$.

Lemma 1: *For each* x, $\exp(x).\exp(-x) = 1$.

Proof. By the product rule for differentiation $\exp(x)\exp(-x)$ is differentiable and

$$[\exp(x).\exp(-x)]' = \exp'(x)\exp(-x) + \exp(x)[\exp(-x)]'$$

$$= \exp'(x)\exp(-x) - \exp(x)\exp'(-x)$$

$$= \exp(x)\exp(-x) - \exp(x)\exp(-x) \qquad \text{(by **)}$$

$$= 0.$$

Thus $\exp(x).\exp(-x)$ is a constant independent of x. To find the value of this constant we need only evaluate $\exp(x).\exp(-x)$ for a particular x, using $x = 0$ we therefore get

$$\exp(x).\exp(-x) = 1.$$

□

The result of this lemma is often used in the form

$$\exp(-x) = \exp(x)^{-1} = \frac{1}{\exp(x)} .$$

Corollary 2: $\exp(x) \gneq 0$ *for all* x.

Proof. First note that lemma 1 implies $\exp(x) \neq 0$ for all x. Now suppose there were some point x_0 with $\exp(x_0) < 0$. Then since $\exp(0) = 1 > 0$ and exp is by assumption differentiable and therefore continuous, the intermediate value theorem (5.4.1) givesa point between 1 and x_0 where exp is zero, contradicting our first observation, so no such point x_0 can exist and we have $\exp(x) > 0$ for all x.

□

Corollary 3: exp *is a strictly increasing, and hence invertible function.*

Proof. This follows immediately since $\exp'(x) = \exp(x) > 0$ by Corollary 2.

□

We next show that exp is the *unique* solution of (**).

Lemma 4: *There is at most one function which satisfies* (**).

Proof. If there were more than the one solution exp(x) to (**) then we could certainly find a second function g with $g' = g$ and $g(0) = 1$. We show this is impossible by proving that g = exp. By Corollary 2 the quotient $^g/\exp$ exists, indeed by lemma 1

$$\frac{g(x)}{\exp(x)} = g(x)\exp(-x)$$

and so by the product rule for differentiation

$$\left[\frac{g(x)}{\exp(x)}\right]' = g'(x)\exp(-x) - g(x)\exp'(-x)$$

$$= 0 \text{ as } \exp' = \exp.$$

Thus $\frac{g(x)}{\exp(x)}$ equals a constant. To find the value of this constant we put $x = 0$ and conclude that

$$\frac{g(x)}{\exp(x)} = \frac{g(0)}{\exp(0)} = 1 , \text{ or } g(x) = \exp(x).$$

□

Lemma 5: *For any* x *and* a, $\exp(a+x) = \exp(a)\exp(x)$.

Proof. Let $g(x) = \frac{\exp(a+x)}{\exp(a)}$ (possible as $\exp(a) \neq 0$) then

$$g'(x) = \frac{1}{\exp(a)}[\exp(a+x)]'$$

$$= \frac{1}{\exp(a)} . \exp'(a+x).1$$

$$= \frac{\exp(a+x)}{\exp(a)} , \text{ as } \exp' = \exp$$

$$= g(x).$$

That is $g' = g$, further $g(0) = \frac{\exp(a+0)}{\exp(a)} = 1$ and so we conclude that g is the unique solution exp to (**).

Thus $\frac{\exp(a+x)}{\exp(a)} = \exp(x)$ or $\exp(a+x) = \exp(a)\exp(x)$.

□

The previous five results provide us with an almost complete picture of the exponential function.

If we denote by e the value of exp(x) at 1 then we have:

(i) $e = \exp(1) > \exp(0) = 1$, by Corollary 3.

[Indeed to 7 decimal places $e \doteq 2.7182818$. We will look at ways whereby e may be calculated on several subsequent occasions.]

(ii) Using $\exp(1) = e$ and lemma 5 we have

$$\exp(2) = \exp(1+1) = \exp(1)^2 = e^2$$

$$\exp(3) = \exp(2+1) = \exp(2)\,\exp(1) = e^3$$

and in general, by an easy induction and lemma 1,

$$\exp(n) = e^n \quad \text{for any whole number } n.$$

Indeed, for any rational value $\frac{p}{q}$ (p,q integers)

$$\exp(\tfrac{p}{q}) = e^{p/q}.$$

To see this observe that for $p,q > 0$

$$e = \exp(1) = \exp(\underbrace{\tfrac{1}{q} + \tfrac{1}{q} + \ldots + \tfrac{1}{q}}_{q \text{ terms}}) = [\exp(\tfrac{1}{q})]^q$$

So $\exp(\frac{1}{q}) = e^{1/q}$ (the q'th root of e)

and then

$$\exp(\tfrac{p}{q}) = \exp(\underbrace{\tfrac{1}{q} + \tfrac{1}{q} + \ldots + \tfrac{1}{q}}_{p \text{ terms}}) = [\exp(\tfrac{1}{q})]^p$$

$$= e^{p/q}.$$

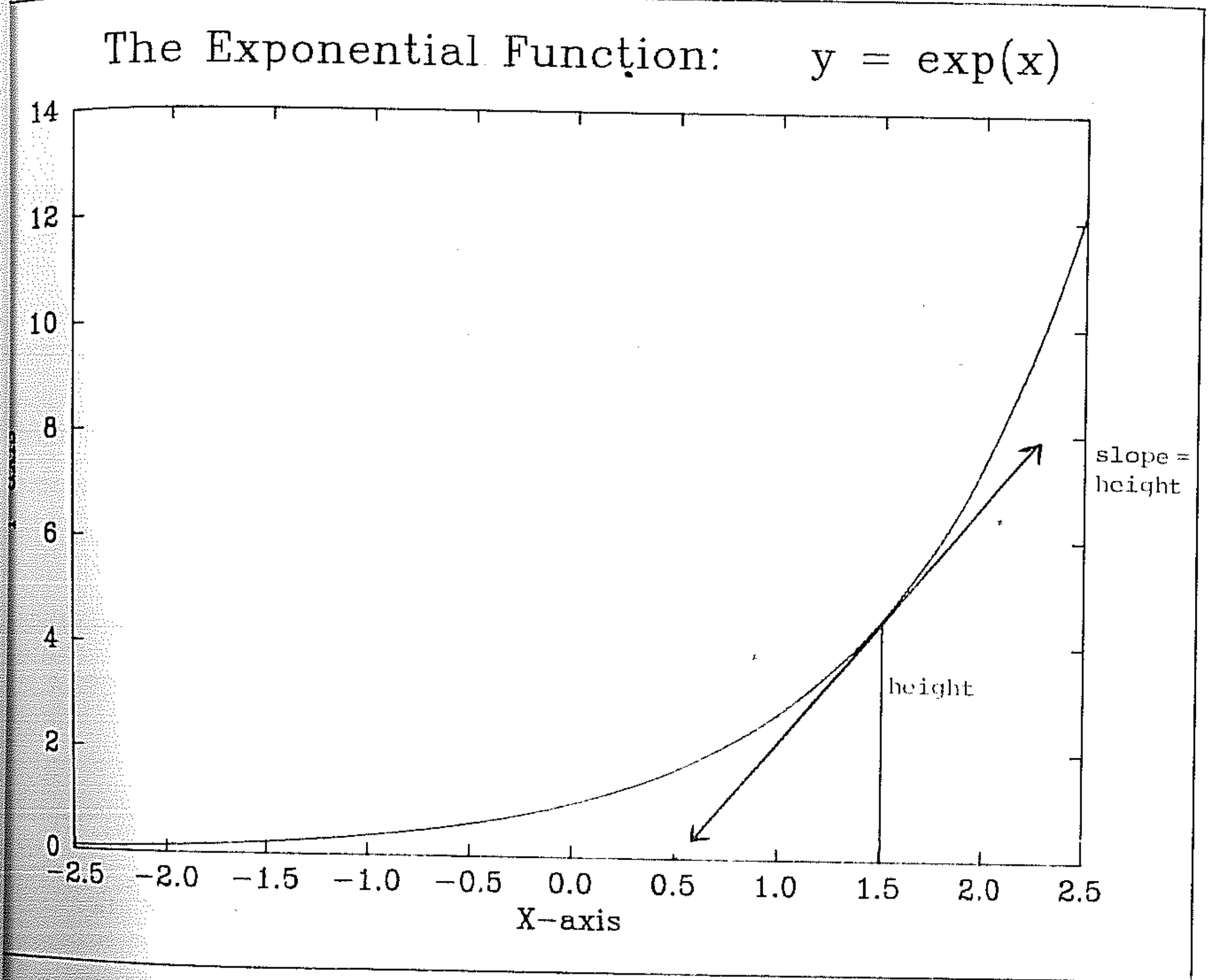
The general result, including negatives, now follows from lemma 1.

Combining (ii) with the requirement of continuity leads us to define

$$e^r := \exp(r) \quad \text{for all real numbers } r.$$

For this reason the exponential function is often denoted by e^x instead of $\exp(x)$.

Putting the above information together we see that the graph of $y = e^x$ has the form illustrated below.



EXERCISES

(1) Differentiate each of the following functions.

(i) e^{x^2}

(ii) $xe^{\sin x}$

(iii) e^{e^x}

(iv) e^{-1/x^2} $(x \neq 0)$

(v) $e^{3x} \cdot e^{7x}$

(vi) $(1 + e^{-x})^{-1}$

(2) Graph each of the functions:

(i) e^{-x}

(ii) e^{-x^2}

(iii) e^{-1/x^2} $(x \neq 0)$

(iv) $e^{\sin x}$

(3) Find the following integrals:

(i) $\int \cos x \, e^{\sin x} \, dx$ (iii) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx$

(ii) $\int \frac{e^x \, dx}{e^{2x}+2e^x+1}$ (iv) $\int e^x \sin x \, dx$

(v) $\int x^3 e^{x^2} \, dx.$

(4) (i) Use l'Hôpital's Rule to show:

(a) $\lim_{x \to \infty} \frac{e^x}{x} = \infty$

and in general

(b) $\lim_{x \to \infty} \frac{e^x}{x^n} = \infty$ for any integer $n > 0$.

(ii) Find $\lim_{x \to 0} x \, e^{1/x}$

(5) Let $f(x) := \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$

(i) Show that f is continuous.

(ii) Find from first principles $f'(0)$ and conclude that f is differentiable.

(iii) Show $f''(0) = 0$.

(6) *(Half-Lives)*

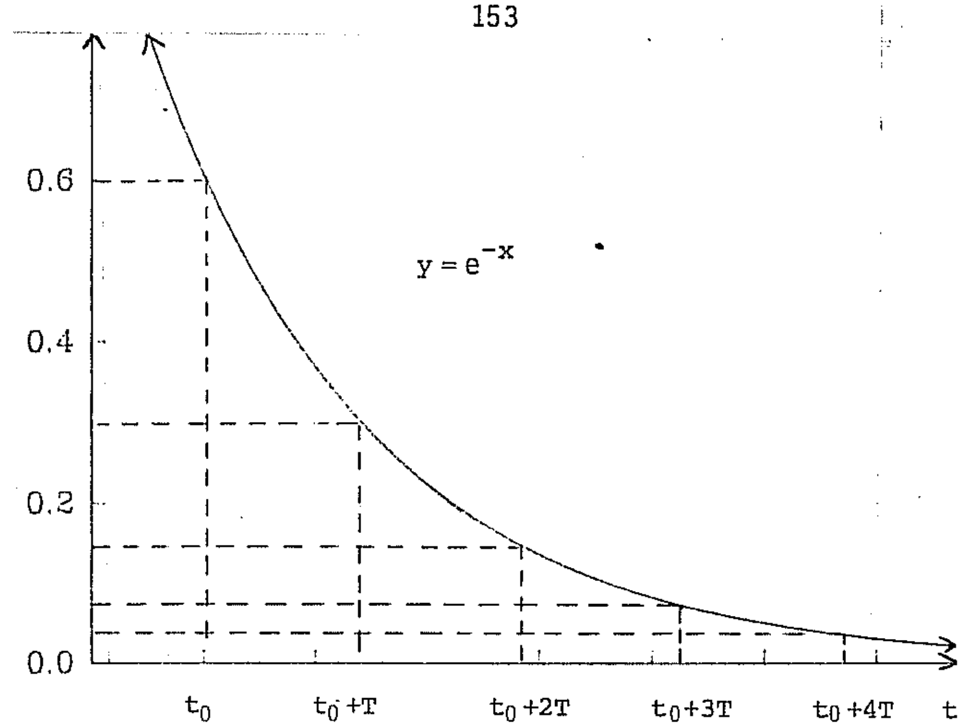
For the decay phenomenon described in example (2) on p.146 show that

$$M(t) = M_0 e^{-kt} .$$

How long will it take for all the material to have decayed away?

Use the intermediate value theorem to deduce that there is a time T at which $M(T) = \frac{1}{2} M_0$.

For any time t_0, show that the amount of material left by time $t_0 + T$ is half the amount present at time t_0.



T is referred to as the $\frac{1}{2}$-*life* of the decay process. It provides a convenient measure of the rate of decay. We will later show that

$$T \doteq 0.693/k.$$

(7) *(The Hyperbolic functions)*

Let

$$\sinh x := \frac{e^x - e^{-x}}{2} \quad \text{(pronounced "schine")}$$

$$\cosh x := \frac{e^x + e^{-x}}{2} \quad \text{(pronounced as written)}$$

and

$$\tanh x := \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = 1 - \frac{2}{e^{2x}+1}$$

(pronounced "than")

(i) Graph each of the above functions.

(ii) Prove each of the identities

(a) $\cosh^2 x - \sinh^2 x = 1$

(b) $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

(c) $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$.

[The similarity with trigonometric identities should not have escaped you. It is partly as a result of this that hyperbolic functions are useful. The connection is made precise by *Osborne's Rule:*

A trigonomic identity is converted into a valid identity for hyperbolic functions by replacing each trigonometric function by the corresponding hyperbolic one and where-ever a product (or implied product, as in $\tanh^2 x$ *) of two sinh's occurs change the sign of the term.*

Thus, from $\cos 2x = 1 - 2\sin^2 x$ we have

$$\cosh 2x = 1 + 2\sinh^2 x .$$

We will later see a justification for this rule.]

(iii) Prove that

(a) $\sinh'(x) = \cosh(x)$

(b) $\cosh'(x) = \sinh(x)$

and (c) $\tanh'(x) = \dfrac{1}{\cosh^2(x)}$ $[= \operatorname{sech}^2(x)]$

(iv) The functions sinh and tanh are one-to-one and so have inverses $\sinh^{-1}$ and $\tanh^{-1}$. If cosh is restricted to $[0,\infty)$ it also has an inverse $\cosh^{-1}$ defined on $[1,\infty)$.

(a) Graph these inverse functions.

(b) Show that $\sinh(\cosh^{-1}(x)) = \sqrt{x^2 - 1}$

(c) Prove:

$$(\sinh^{-1})'(x) = \frac{1}{\sqrt{x^2 + 1}}$$

$$(\cosh^{-1})'(x) = \frac{1}{\sqrt{x^2 - 1}} \qquad (x > 1)$$

$$(\tanh^{-1})'(x) = \frac{1}{1 - x^2} \qquad (|x| < 1)$$

(v) Find

$$\int \frac{1}{\sqrt{x^2 \pm 1}}\, dx$$

and

$$\int \frac{1}{1 - x^2}\, dx$$

noting any restrictions which might apply to the range of integration in each case.

(8) (*The Gamma function*, $\Gamma(x)$)

Define Γ by

$$\Gamma(x) := \int_0^\infty e^{-t} \,.\, t^{x-1}\, dt \qquad (x > 0)$$

(a) (optional) Show that the improper integral of the right hand side exists.

[Recall the discussion on p124.]

(b) Use integration by parts to show that

$$\Gamma(x + 1) = x\Gamma(x) \quad \text{for any } x > 0.$$

(c) By first showing that $\Gamma(1) = 1$ use (b) to conclude that

$$\Gamma(n + 1) = n! \quad \text{for any natural number } n.$$

Thus $\Gamma(x)$ provides an "extension" of the *factorial function*, $n!$, to all positive real numbers.

Some values of $\Gamma(x)$ are tabulated below

x	$\Gamma(x)$
1.0	1.00000
1.2	0.91817
1.4	0.88726
1.5	0.88623
1.6	0.89352
1.8	0.93138
2.0	1.00000

(d) Use these values to graph $\Gamma(x)$ for $1 \le x \le 3$. [Don't forget the information in (b).]

(e) Use Simpson's formula on 6 intervals to approximate

$$\int_0^6 e^{-t} \sqrt{t}\, dt .$$

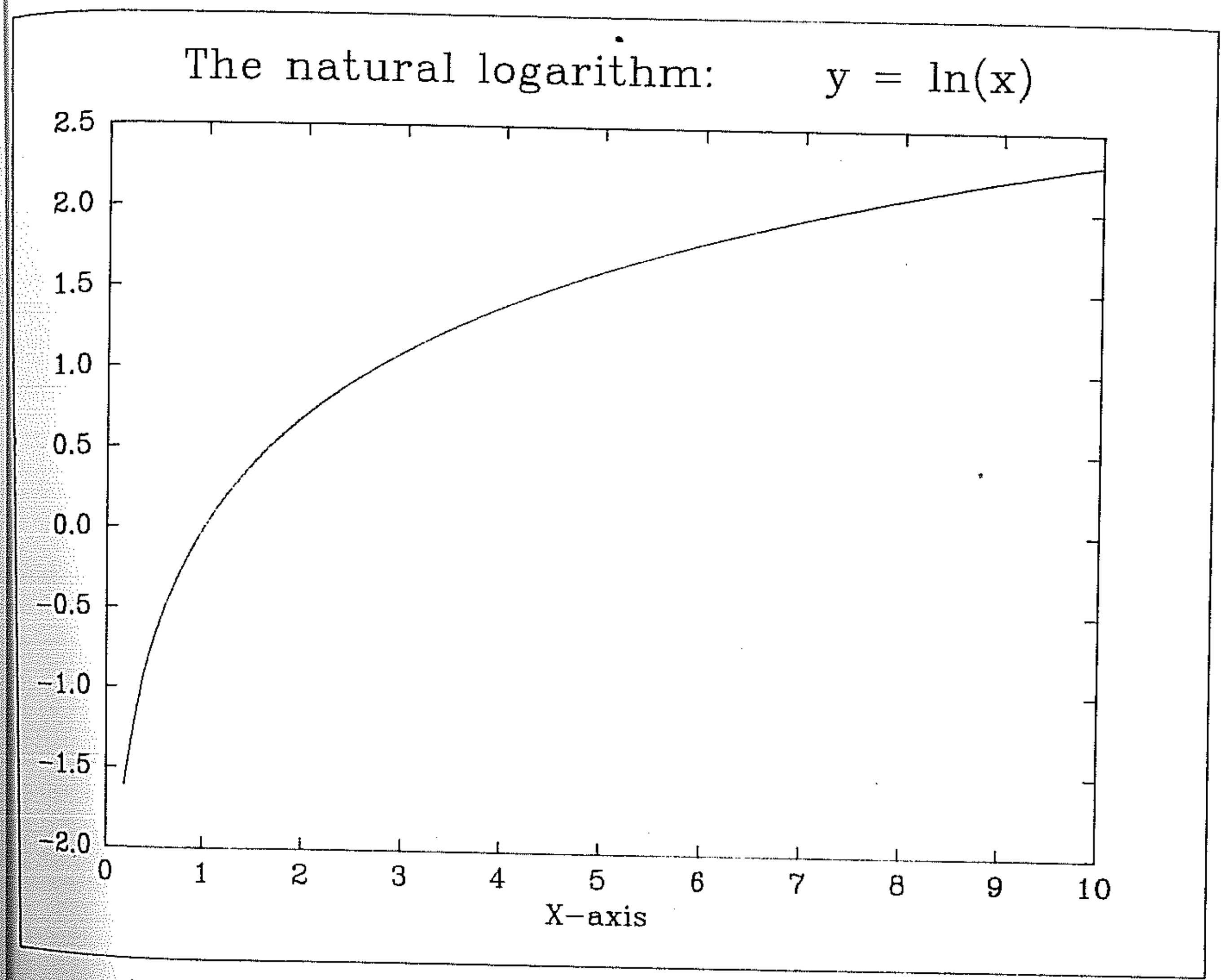
Note that this provides an approximation for $\Gamma(1.5)$, [compare it to the value tabulated above], obtained by neglecting the integral from 6 onward. Can you estimate what error this neglect has entailed?

8.2 The Natural Logarithm

By Corollary 8.1.(3) the exponential function has an inverse $\exp^{-1}$ defined on $(0,\infty)$.

Since $e^{\exp^{-1}(x)} = \exp(\exp^{-1}(x)) = x$, we have that "$\exp^{-1}(x)$ is the power to which e must be raised to obtain x". For this reason $\exp^{-1}(x)$ is referred to as the **natural logarithm** of x (or the logarithm of x to the base e) and denoted by ln x. [Some books use $\log_e x$ or just log x, however we will reserve log x for the ordinary or Naperian logarithm $\log_{10} x$ although we will hardly ever need to use it.]

The graph of $y = \ln x$ illustrated below may be obtained from that of the exponential function by reflection in the line $y = x$.



Note in particular that from the definition we have;

$$\ln(1) = 0, \ \ln(e) = 1, \ \ln(e^2) = 2, \text{ etc.}$$

Further, by the inverse function Theorem (chapter 6), ln is differentiable with

$$\ln'(x) = \frac{1}{\exp'(\ln x)} = \frac{1}{\exp(\ln x)} = \frac{1}{x}$$

Thus we see that ln x is an antiderivative of $\frac{1}{x}$ and so

$$\int_1^x \frac{1}{t}\,dt = \ln(x) - \ln(1) = \ln x.$$

At last we are in a position to demonstrate the *existence* of the exponential function.

Since the function $\frac{1}{x}$ is continuous (indeed differentiable) on $(0,\infty)$ the discussion in §7.1 shows that the primitive

$$\ln(x) = \int_1^x \frac{1}{t}\,dt$$

exists for all $x > 0$. Thus the natural logarithm and hence *ipso facto* its inverse exp(x) exist.

A basic property of the logarithm is

Lemma 1: *For* $x, a > 0$ *we have*

$$\ln ax = \ln a + \ln x.$$

Proof. Since exp and ln are inverses, putting $z = \ln a + \ln x$ we have

$$z = \ln(e^z) = \ln(e^{(\ln a + \ln x)})$$

$$= \ln(e^{\ln a} . e^{\ln x}), \quad \text{by lemma 8.1.(5)}$$

$$= \ln(a . x).$$

That is,

$$\ln a + \ln x = \ln ax$$

□

In §8.1 we defined e^r to be exp(r) for any real number r. We now extend this to powers of numbers other than e, by defining

$$a^r := e^{r \ln a}, \quad \text{for any } a > 0 \text{ and real number } r.$$

Note: This could also be expressed as $\ln(a^r) = r \ln a$.

This is what we would expect when r is a rational number $\frac{p}{q}$. Indeed we have

$$a^{p/q} = (e^{\ln a})^{p/q}$$

$$= e^{p/q \ln a}, \quad \text{by the usual rule for}$$

rational exponents: $(a^b)^c = a^{bc}$.

Further this law continues to hold (see exercise 6), as does the law

$$a^{b+c} = a^b . a^c .$$

To prove the latter for all real b and c observe that

$$a^{b+c} := e^{(b+c)\ln a} = e^{b \ln a + c \ln a}$$

$$= e^{b \ln a} . e^{c \ln a}$$

$$= a^b . a^c \quad \text{(by the above definition).}$$

Armed with this definition we can now easily differentiate the function

$$f(x) := a^x \qquad (a > 0)$$

Indeed $\quad f(x) = \exp(x \ln a)$

so $\quad f'(x) = \exp'(x \ln a) . \ln a$

$$= (\ln a)\exp(x \ln a)$$

$$= (\ln a) . a^x.$$

EXERCISES:

(1) Show that

$$\ln x = \frac{\log x}{\log e} = (\ln 10)\log x.$$

(2) Differentiate

(i) $\ln(\sin x)^2$

(ii) 3^{x^2}

(iii) x^x

(iv) $x \ln x - x$

(v) $\log x$.

(3) Use l'Hôpital's Rule to find

$$\lim_{x \to \infty} \frac{\ln x}{x}$$

hence determine

$$\lim_{x \to 0} t \ln t$$

and

$$\lim_{x \to \infty} x^{1/x}$$

(4) Find explicit formulae for $\sinh^{-1} x$, $\cosh^{-1} x$ and $\tanh^{-1}(x)$.

[Hint: In the case of $\sinh^{-1} x$, solve $y = \frac{e^x - e^{-x}}{2}$ for x by first multiplying both sides by e^x to obtain a quadratic equation in e^x.]

(5) (a) Find

(i) $\int \frac{1}{x \ln x} dx$ (iii) $\int \frac{x}{1 + x^2} dx$ (v) $\int \frac{\sqrt{x}\, dx}{1 + \sqrt[4]{x}}$

(ii) $\int x\, 3^{x^2} dx$ (iv) $\int \tan x\, dx$

(b) Use integration by parts to find

(i) $\int \ln x\, dx$ (ii) $\int \tan^{-1} x\, dx$.

[Check your answer to (i) by comparing with (2)(iv).]

(6) Prove that $(a^b)^c = a^{bc}$ for all real numbers b and c and $a > 0$.

(7) (*Evaluation of* e - First Try)

Note that the value of x for which $F(x) = 1$ in Exercise 7.4.(2) provides us with a crude estimate for e.

(a) Show that $e = \lim_{n \to \infty} (1 + \frac{1}{n})^n$.

[Hint: Observe that

$$\begin{aligned} 1 &= \ln'(1) \\ &= \lim_{h \to 0} \frac{\ln(1 + h) - \ln(1)}{h} \\ &= \lim_{h \to 0} \ln(1 + h)^{1/h} \\ &= \ln[\lim_{h \to 0} (1 + h)^{1/h}], \text{ as } \ln \text{ is continuous.} \end{aligned}$$

Take exponentials of both sides and let $h = 1/n$.]

Verify that this yields the following estimates for e

n	1	10	100	1000	10,000	1,000,000
$(1+\frac{1}{n})^n$	2	2.6	2.7	2.72	2.718	2.71828

Problems with this calculation of e include

(i) the rate of convergence is slow (large n are required)

and

(ii) we have no easy way of estimating the error in any particular approximation.

(b) From the above result deduce that for each x

$$\begin{aligned} e^x &= \lim_{n \to \infty} (1 + \frac{1}{n})^{nx} \\ &= \lim_{n \to \infty} (1 + \frac{x}{n})^n . \end{aligned}$$

(8) Show that the $\frac{1}{2}$-life of the decay process discussed in Exercise 8.1.(6) is given by

$$T = \frac{\ln 2}{k}$$

[Estimate $\ln 2$ from the information obtained in Exercise 7.4.(2); c.f. the more precise value $\ln 2 \doteq 0.6931472$ obtained from tables or a calculator.]

8.3 Further Integration

In this section we consider the integration of *rational functions*; that is, functions of the form $p(x)/q(x)$ where p and q are polynomials, and of course integrals which can be reduced to this form by a substitution.

Partial Fraction Decomposition

First note that by dividing we can express our rational function in the form

$$\frac{p(x)}{q(x)} = p_1(x) + \frac{p_2(x)}{q(x)}$$

where p_1 is a polynomial and the degree of the polynomial p_2 is less than that of the denominator q.

For example:

$$\frac{x^4 + 2x^3 - x - 1}{x^3 + x^2 - x - 1} = x + 1 + \frac{x}{x^3 + x^2 - x - 1}$$

$$\begin{array}{r|l} & x + 1 \\ x^3 + x^2 - x - 1 & x^4 + 2x^3 - x - 1 \\ & x^4 + x^3 - x^2 - x \\ \hline & x^3 + x^2 - 1 \\ & x^3 + x^2 - x - 1 \\ \hline & x \end{array}$$

Our second reduction is a consequence of the following general result.

A rational function of the form $\frac{p(x)}{q_1(x)q_2(x)}$, *where the degree of the numerator is less than that of the denominator and* q_1, q_2 *are relatively prime polynomials* (that is, q_1 and q_2 have no common factor of degree greater than or equal to 1) *may be written as*

$$\frac{p(x)}{q_1(x)\ q_2(x)} = \frac{p_1(x)}{q_1(x)} + \frac{p_2(x)}{q_2(x)}$$

where p_1 *and* p_2 *are polynomials whose degree are respectively at least one less than that of* q_1 *and* q_2.

The proof of this result relies on showing there is sufficient

freedom to choose the coefficients of p_1 and p_2 so that the identity is satisfied.

Note: By repeated application, this result extends to allow any rational function of the form

$$\frac{p(x)}{q_1(x)q_2(x)q_3(x)\ \ldots\ q_n(x)},$$

where the degree of the numerator is less than that of the denominator and the $q_1, q_2, \ldots, q_n$ are mutually relatively prime polynomials, to be expressed as

$$\frac{p_1(x)}{q_1(x)} + \frac{p_2(x)}{q_2(x)} + \ldots + \frac{p_n(x)}{q_n(x)}.$$

where in general the degree of $p_i(x)$ is less than that of $q_i(x)$ for $i = 1,2,\ldots,n$.

For example: In our last illustration the term

$$\frac{x}{x^3 + x^2 - x - 1} = \frac{x}{(x - 1)(x + 1)^2}$$

may be expressed in the form

$$\frac{x}{(x - 1)(x + 1)^2} = \frac{A}{x - 1} + \frac{Bx + C}{(x + 1)^2}.$$

What we illustrate next may be applied in general. To determine the coefficients A, B and C, putting the right hand side over a common denominator we have

$$\frac{x}{(x - 1)(x + 1)^2} = \frac{A(x + 1)^2 + (Bx+C)(x-1)}{(x - 1)(x + 1)^2}$$

For this to be an identity in x we require

$$x = A(x + 1)^2 + (Bx + C)(x - 1)$$

We may now proceed by either

(a) *Equating coefficients:* That is, collecting equal powers of x together on the right hand side we have

$$x = (A + B)x^2 + (2A - B + C)x + (A - C)$$

and require

$$A + B = 0$$

$$2A - B + C = 1$$

$$A - C = 0$$

From which we deduce $A = C = \frac{1}{4}$, $B = -\frac{1}{4}$

(b) Substituting "key" values of x to obtain equations for A, B and C. (Values which ensure that one of the terms vanish are often a good choice.)

Thus in our example

$x = -1$ gives

$$-1 = 0 + (-B + C) \times (-2) \quad \text{or} \quad C - B = \frac{1}{2}$$

$x = 1$ gives

$$1 = A \times 4 + 0 \quad \text{or} \quad A = \frac{1}{4}.$$

To obtain a third relationship we could use $x = 0$ which gives

$$0 = A + C - 1, \quad \text{so} \quad C = A = \frac{1}{4}$$

and then $B = -\frac{1}{4}$.

By either way we obtain

$$\frac{x}{(x-1)(x+1)^2} = \frac{1}{4}\left[\frac{1}{x-1} + \frac{1-x}{(x+1)^2}\right]$$

The third and final step in our reduction is to handle terms of the form

$\frac{p(x)}{q(x)^n}$, such as $\frac{1-x}{(x+1)^2}$ in the last example, where again the degree of the numerator is less than that of the denominator. Here, repeated division of $q(x)$ into $p(x)$ shows that we can write

$$\frac{p(x)}{q(x)^n} = \frac{p_1(x)}{q(x)} + \frac{p_2(x)}{q(x)^2} + \frac{p_3(x)}{q(x)^3} + \dots + \frac{p_n(x)}{q(x)^n}$$

where each of the numerators $p_1(x), p_2(x), \dots, p_n(x)$ has degree at least one less than the degree of $q(x)$.

<u>For example:</u>

We can write

$$\frac{1-x}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2}$$

where the A and B can be found as before. Indeed, equating coefficients in $1 - x = A(x + 1) + B$ gives $A = -1$, $A + B = 1$ so $B = 2$ and we have

$$\frac{1-x}{(x+1)^2} = \frac{2}{(x+1)^2} - \frac{1}{x+1}.$$

Combining all our worked examples we obtain the partial fraction decomposition:

$$\frac{x^4 + 2x^3 - x - 1}{x^3 + x^2 - x - 1} = x + 1 + \frac{1}{4}\cdot\frac{1}{x-1} + \frac{1}{2}\cdot\frac{1}{(x+1)^2} - \frac{1}{4}\cdot\frac{1}{x+1}$$

We will later demonstrate (*Fundamental Theorem of algebra*) that any polynomial can be written as a product of linear factors of the form $x + \alpha$ and quadratic factors of the form $x^2 + \beta x + \gamma$ (which have no real roots), with the possibility that some of the factors are repeated. Combining this with the above discussion we see that any rational function can be written as the sum of a polynomial and terms of the form

$$\frac{A}{(x+\alpha)^r} \qquad \text{or} \qquad \frac{Bx+C}{(x^2+\beta x+\gamma)^s} \; .$$

This was illustrated by our last example; as a further illustration consider the following.

<u>Example</u>. Find the partial fraction decompositon of

$$\frac{x^3+x}{x^6+2x^5+2x^4-2x^2-2x-1}$$

Pretending that after a little playing around we noticed that both 1 and -1 are roots of the denominator and so after some further frolics arrived at the factorization

$$x^6+2x^5+2x^4-2x^2-2x-1 = (x+1)(x-1)(x^2+x+1)^2 \, ,$$

we have from the above discussion that

$$\frac{x^3+x}{x^6+2x^5+2x^4-2x^2-2x-1} = \frac{A}{x+1}+\frac{B}{x-1}+\frac{Cx+D}{x^2+x+1}+\frac{Ex+F}{(x^2+x+1)^2}$$

and we require

$$x^3+x = A(x-1)(x^2+x+1)^2 + B(x+1)(x^2+x+1)^2 + (Cx+D)(x+1)(x-1)(x^2+x+1)$$
$$+ (Ex+F)(x+1)(x-1)$$

Multiplying out, collecting terms and equating coefficients (you should do this, I had to!) we obtain

$$\begin{aligned}
A + B + C \qquad\qquad &= 0\\
A + 3B + C + D \qquad &= 0\\
A + 5B \quad + D + E &= 1\\
-A + 5B - C \qquad + F &= 0\\
-A + 3B - C - D - E &= 1\\
-A + B \quad - D \quad - F &= 0
\end{aligned}$$

Solving these we have

$$A = 1,\ B = \frac{1}{9},\ C = -\frac{10}{9},\ D = -\frac{2}{9},\ E = -\frac{1}{3} \text{ and } F = -\frac{2}{3}.$$

[Here I actually "cheated" by noting that substituting $x = 1$ gives

$B = \frac{1}{9}$ and $x = -1$ gives $A = 1$ after which we can readily find C, D, E and F from the first four equations.]

Thus;

$$\frac{x^3+x}{x^6+2x^5+2x^4-2x^2-2x-1} = \frac{1}{x+1}+\frac{1}{9}\cdot\frac{1}{x-1}-\frac{1}{9}\cdot\frac{10x+2}{x^2+x+1}-\frac{1}{3}\cdot\frac{x+2}{(x^2+x+1)^2} \; .$$

Integration of Rational Functions

We have just seen that any rational function can be written as the sum of a polynomial and terms of the form

$$\frac{A}{(x+\alpha)^r} \qquad \text{or} \qquad \frac{Bx+C}{(x^2+\beta x+\gamma)^s} \; .$$

Thus to integrate a rational function it suffices to integrate terms of this form. Since this is always possible we conclude that <u>any rational function may be elementarily integrated.</u>

Indeed

$$\int \frac{A\,dx}{(x+\alpha)^r} = \begin{cases} A \ln(x+\alpha) + C & \text{if } r = 1 \\ \dfrac{(x+\alpha)^{1-r}}{1-r} + C & \text{if } r \neq 1 \end{cases}$$

while

$$\int \frac{Bx+C}{(x^2+\beta x+\gamma)^s} \qquad \text{may be found by}$$

first completing the square and making an appropriate substitution to obtain

$$D\int \frac{u\,du}{(u^2+1)^s} + E\int \frac{du}{(u^2+1)^s}$$

$$= E\int \frac{du}{(u^2+1)^s} + \frac{D}{2}\begin{cases} \dfrac{(u^2+1)^{1-s}}{1-s} & s \neq 1 \\ \ln(u^2+1) & s = 1 \end{cases}$$

and then using the reduction formula obtained from integration by parts;

$$\int \frac{1}{(u^2+1)^s}\,du = \frac{1}{2s-2}\,\frac{u}{(u^2+1)^{s-1}} + \frac{2s-3}{2s-2}\int \frac{1}{(u^2+1)^{s-1}}\,du \, ,$$

to find the first integral. Remember

$$\int \frac{du}{u^2 + 1} = \tan^{-1} u + C$$

For example, using the partial fraction decomposition found previously we have:

(1) $$\int \frac{x^4 + 2x^3 - x - 1}{x^3 + x^2 - x - 1}\, dx$$

$$= \int x + 1\, dx + \frac{1}{4}\int \frac{1}{x - 1}\, dx + \frac{1}{2}\int \frac{1}{(x + 1)^2}\, dx - \frac{1}{4}\int \frac{1}{x + 1}\, dx$$

$$= \frac{x^2}{2} + x + \frac{1}{4}\ln(x - 1) - \frac{1}{2}\,\frac{1}{x + 1} - \frac{1}{4}\ln(x + 1) + C.$$

$$= \frac{x^2}{2} + x - \frac{1}{2x + 2} + \frac{1}{4}\ln\left(\frac{x - 1}{x + 1}\right) + C.$$

(2) $$\int \frac{x^3 + x}{x^6 + 2x^5 + 2x^4 - 2x^2 - 2x - 1}\, dx$$

$$= \int \frac{dx}{x + 1} + \frac{1}{9}\int \frac{dx}{x - 1} - \frac{1}{9}\int \frac{10x + 2}{x^2 + x + 1}\, dx - \frac{1}{3}\int \frac{x + 2}{(x^2 + x + 1)^2}\, dx$$

$$= \ln[(x + 1)(x - 1)^{1/9}] - \frac{1}{9}\int \frac{10x + 2}{x^2 + x + 1}\, dx - \frac{1}{3}\int \frac{x + 2}{(x^2 + x + 1)^2}\, dx$$

To find these last two integrals we proceed as follows.

Since $x^2 + x + 1 = (x + \frac{1}{2})^2 + \frac{3}{4}$, making the substitution

$$u = \frac{2x + 1}{\sqrt{3}}$$

we have

(i) $$\int \frac{10x + 2}{x^2 + x + 1}\, dx = 2\int \frac{5u - \sqrt{3}}{u^2 + 1}\, du$$

$$= 5\ln(u^2 + 1) - 2\sqrt{3}\tan^{-1} u + C$$

$$= 5\ln\left[\frac{4}{3}(x^2 + x + 1)\right] - 2\sqrt{3}\tan^{-1}\frac{2x+1}{\sqrt{3}} + C$$

and

(ii) $$\int \frac{x + 2}{(x^2 + x + 1)^2}\, dx = \frac{4}{3}\int \frac{u + \sqrt{3}}{(u^2 + 1)^2}\, du$$

$$= -\frac{-2}{3(u^2 + 1)} + \frac{4}{\sqrt{3}}\int \frac{du}{(u^2 + 1)^2}$$

$$= -\frac{-2}{3(u^2 + 1)} + \frac{4}{\sqrt{3}}\left(\frac{1}{2}\,\frac{u}{u^2 + 1} + \frac{1}{2}\int \frac{du}{u^2 + 1}\right)$$

$$= \frac{2\sqrt{3}\,u - 1}{3(u^2 + 1)} + \frac{2}{\sqrt{3}}\tan^{-1} u + C$$

$$= \frac{x}{x^2 + x + 1} + \frac{2}{\sqrt{3}}\tan^{-1}\frac{2x + 1}{\sqrt{3}} + C.$$

Combining these results we have:

$$\int \frac{x^3 + x}{x^6 + 2x^5 + 2x^4 - 2x^2 - 2x - 1}\, dx$$

$$= \ln[(x + 1)(x - 1)^{1/9}] - \frac{5}{9}\ln\frac{4}{3}(x^2 + x + 1)$$

$$- \frac{x}{3(x^2 + x + 1)} + C$$

a thoroughly forgettable result. What you shouldn't forget are the techniques used to derive it.

We have illustrated all these techniques in just two examples. You will be expected to know and apply the same ideas in simpler cases. For complicated rational functions integration in elementary terms is almost a theoretical curiosity, certainly a *tour de force*, except that since the procedures are mechanical there now-days exist "artificial intelligence" packages which would perform the above integrations in a matter of seconds on a micro-computer.

EXERCISES:

(1) Find the following integrals.

(i) $\displaystyle\int \frac{dx}{x(1-x)}$

(ii) $\displaystyle\int \frac{dx}{x^2 + x^3}$

(iii) $\displaystyle\int \frac{x}{(x^2 + 2x + 2)^2}\, dx$

(iv) $\displaystyle\int \frac{dx}{1 + x^4}$ [Hint: $x^2 + \sqrt{2}\,x + 1$ is a factor]

(v) $\displaystyle\int \frac{1 + e^x}{1 - e^x}\, dx$

(vi) $\displaystyle\int \frac{x^3 + x + 2}{x^4 + 2x^2 + 1}\, dx$

(2) Prove the Reduction Formula

$$\int \frac{du}{(u^2+1)^s} = \frac{1}{2s-2}\,\frac{u}{(u^2+1)^{s-1}} + \frac{2s-3}{2s-2}\int \frac{1}{(u^2+1)^{s-1}}\, du.$$

CHAPTER 9

APPLICATIONS OF INTEGRATION

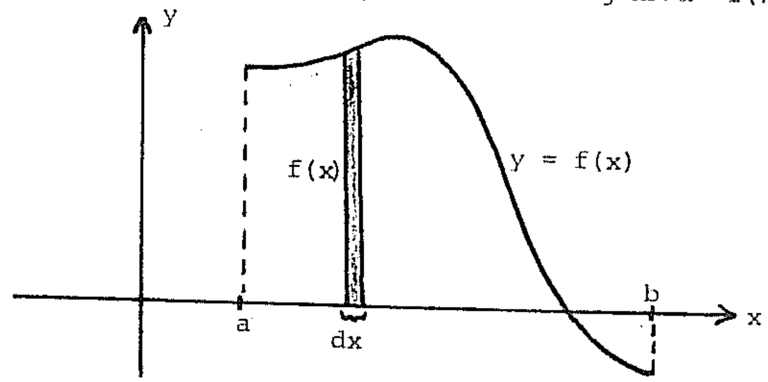
9.1 Areas

From its inception the definite integral

$\displaystyle\int_a^b f(x)\, dx$ may be interpreted as the

area of the region enclosed by the curve $y = f(x)$, the lines $x = a$, $x = b$ and the x-axis (subregions "below" the x-axis being assigned negative areas), see §7.1.

Intuitively it is useful to think of this integral representing the sum of infinitely many infinitesimal "strips" of the form illustrated below. The strip sitting on the point x having *area* $f(x)\, dx$.



Henceforth we will refer to this as the 'signed' area of the region, to distinguish it from the ('actual') area given by

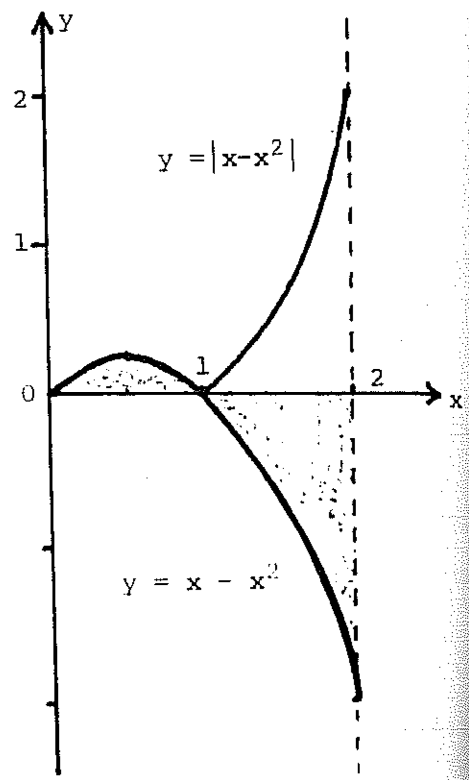
$$A = \int_a^b |f(x)|\, dx,$$

where all subregions are counted as positive.

Evaluation of this last integral involves writing it as the sum of integrals over sub-intervals throughout each of which the integrand has constant sign.

For example: If $f(x) = x - x^2$, the signed area

$$\int_0^2 f = -\frac{2}{3}$$



while

$$A = \int_0^2 |f|$$

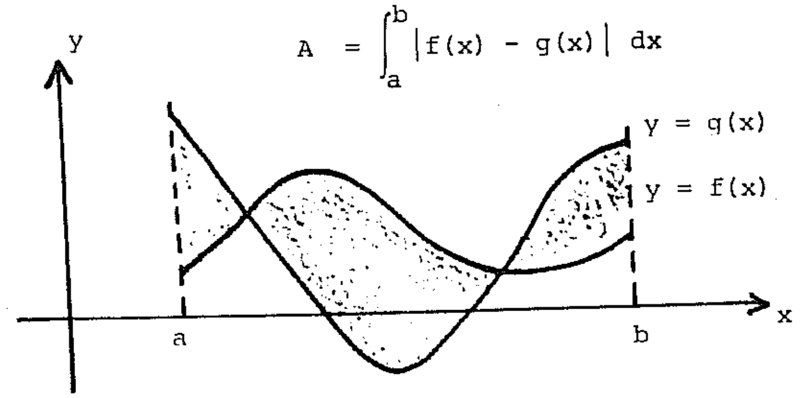
$$= \underbrace{\int_0^1 f}_{f > 0} + \underbrace{\int_1^2 -f}_{f < 0}$$

$$= \int_0^1 x - x^2 \, dx + \int_1^2 x^2 - x \, dx$$

$= 1.$

Similarly, the area of the region enclosed by the two curves $y = f(x)$, $y = g(x)$ and the lines $x = a$, $x = b$ is given by

$$A = \int_a^b |f(x) - g(x)| \, dx$$

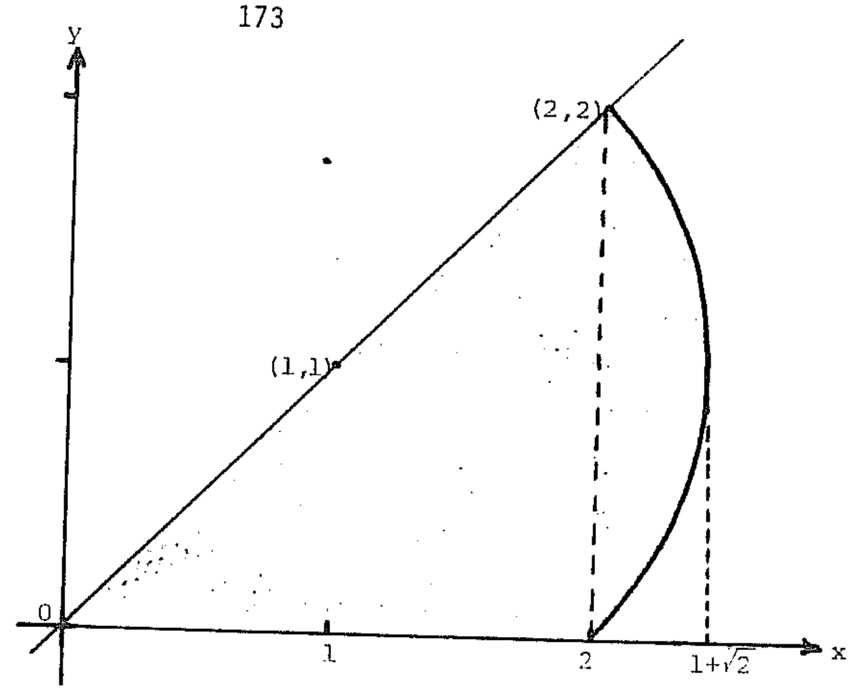


Using these results it is possible to calculate the area of quite complicated regions, usually by decomposition into "simpler" sub-regions, the area of each one being determined in the above way.

In almost every case an essential preliminary is the preparation of a good diagram.

For example: The area of the region

$$\{(x,y) : y \le x, \ x \ge 0 \ \text{and} \ (x-1)^2 + (y-1)^2 \le 2\}$$



is given by

$$A = \int_0^2 x \, dx + \int_2^{1+\sqrt{2}} \left(1 + \sqrt{2 - (x-1)^2}\right) - \left(1 - \sqrt{2 - (x-1)^2}\right) dx$$

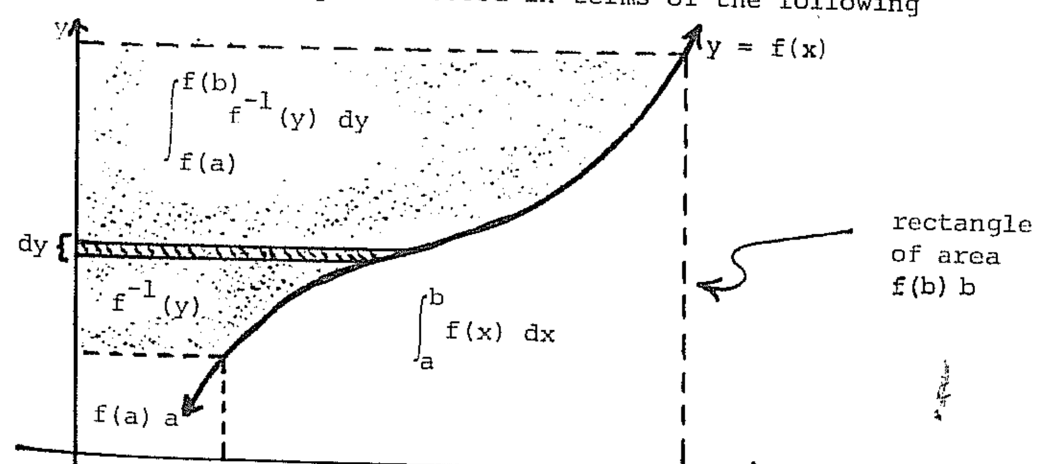
$$= 2 + 2\int_2^{1+\sqrt{2}} \sqrt{2 - (x-1)^2} \, dx.$$

$$= 1 + \frac{\pi}{2}$$

Note: In some cases it is expedient to work with the inverse function; that is consider our area built from "infinitesimal strips" parallel to the x-axis. Then using the formula of exercise 7.2(5) – a result which may also be derived using integration by parts – we have

$$\int_a^b f = f(b)b - f(a)a - \int_{f(a)}^{f(b)} f^{-1}$$

an expression which may be readily understood in terms of the following diagram.



For example: In the last illustration we would have

$$A = \int_0^2 1 + \sqrt{2 - (y - 1)^2} - y \, dy$$

$$= \int_0^2 \sqrt{2 - (y - 1)^2} \, dy$$

$$= 1 + \frac{\pi}{2} .$$

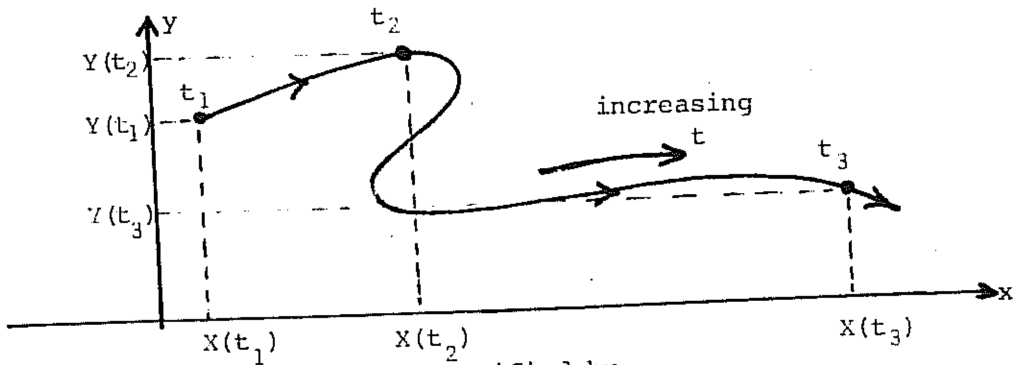
Curves Specified Parametrically

It is sometimes convenient to specify a curve by a pair of equations

$$x = X(t)$$

$$y = Y(t).$$

The curve is then the set of points $(X(t), Y(t))$, one point for each value of the *parameter* t. We may think of each point on the curve being labelled by a corresponding value of t. [If we interpret t as time and $(X(t), Y(t))$ as the position of a point at time t, then as time progresses the point is seen to trace out our curve - this corresponds to the archaic notion of a "locus".]



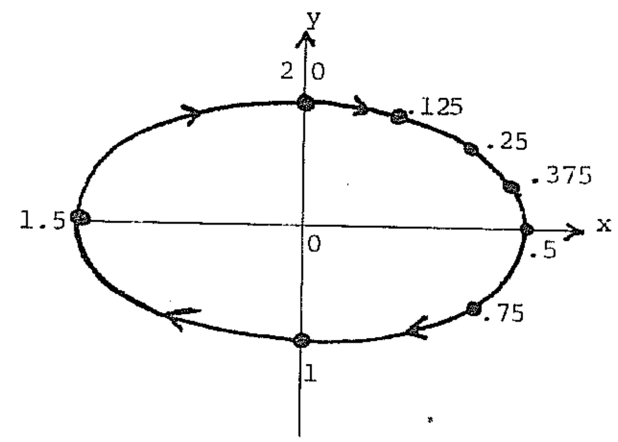
For example: For the curve specified by

$$x = 2 \sin \pi t$$

$$y = \cos \pi t$$

we have

t	corresponding point x	y
0	0	1
.125	0.77	0.92
.25	$\sqrt{2}$	$\frac{1}{\sqrt{2}}$
.375	1.85	0.38
.5	2	0
.75	$\sqrt{2}$	$-\frac{1}{\sqrt{2}}$
1	0	- 1
1.5	- 2	0
2	0	1



Because of the periodicity of sine and cosine for larger values of t the curve retraces itself.

(The points in the diagram are labelled with the corresponding values of t.)

We therefore see that these are parametric equations for an ellipse; indeed

$$\left(\frac{x}{2}\right)^2 + y^2 = \sin^2 \pi t + \cos^2 \pi t = 1.$$

As the above example illustrates, a curve expressed parametrically need not correspond to a functional relationship between x and y, at least not over the whole range of parameter values.

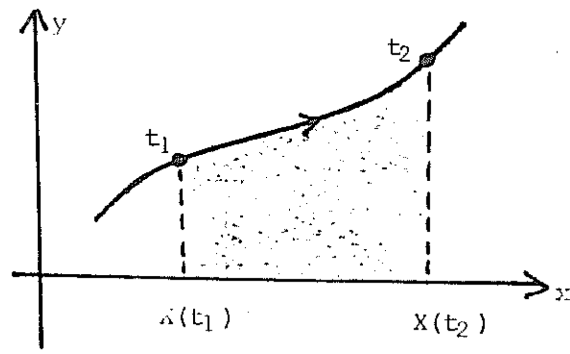
If however X is an invertible function then we have

$$y = Y(X^{-1}(x))$$

and the area

$$\int_{X(t_1)}^{X(t_2)} Y(X^{-1}(x)) \, dx$$

$$= \int_{t_1}^{t_2} Y(t) \frac{dX}{dt} \, dt ,$$



by the method of "inverse substitution" discussed on page 135.

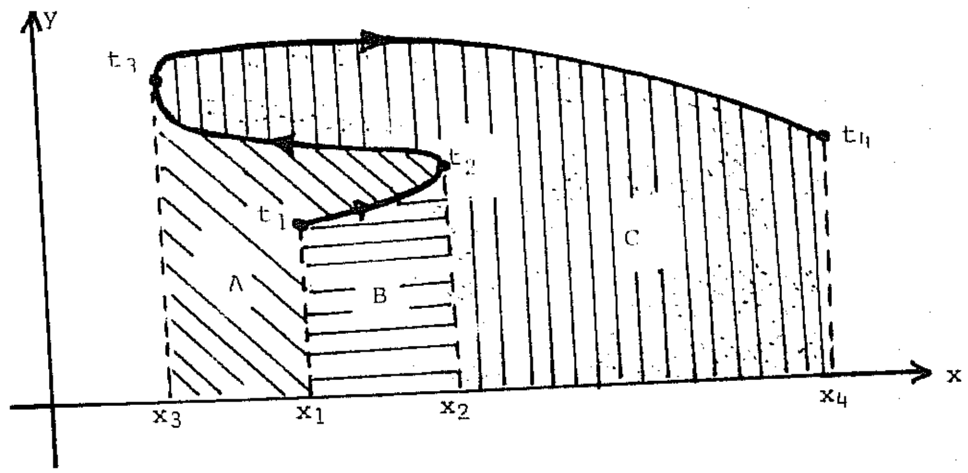
For example: In the last example $X(t) = 2 \sin \pi t$ is invertible for $0 \le t \le \frac{1}{2}$, so the area of the ellipse in the positive quadrant is given by

$$2 \int_0^{\frac{1}{2}} \cos \pi t \frac{d}{dt}(\sin \pi t)\, dt$$

$$= 2\pi \int_0^{\frac{1}{2}} \cos^2 \pi t \, dt = \frac{1}{2}\pi$$

(So, total area of ellipse is 2π)

In fact the above formula for area remains true even when X is not invertible. To see why this might be so consider the following schematic illustration.



First, note that in the parameter ranges; $t_1 \le t \le t_2$, $t_2 \le t \le t_3$, $t_3 \le t \le t_4$, $X(t)$ is invertible so the above formula applies to give

$$\int_{t_1}^{t_2} Y \frac{dX}{dt}\, dt = \int_{x_1}^{x_2} y\, dx = B$$

$$\int_{t_3}^{t_2} Y \frac{dX}{dt}\, dt = \int_{x_3}^{x_2} y\, dx = A + B$$

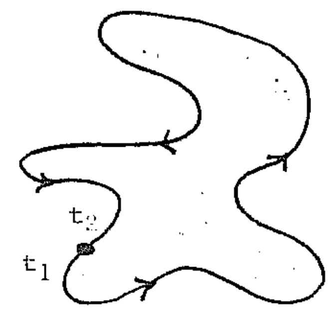
$$\int_{t_3}^{t_4} Y \frac{dX}{dt}\, dt = \int_{x_3}^{x_4} y\, dx = A + B + C.$$

But then

$$\int_{t_1}^{t_4} Y \frac{dX}{dt}\, dt = \int_{t_1}^{t_2} Y \frac{dX}{dt}\, dt + \int_{t_2}^{t_3} Y \frac{dX}{dt}\, dt + \int_{t_3}^{t_4} Y \frac{dX}{dt}\, dt$$

$$= B \qquad - (A + B) \qquad + A + B + C$$

$$= B + C.$$



In particular, if

$$x = X(t)$$

$$y = Y(t) \; ; \quad t_1 \le t \le t_2$$

parameterizes a simple closed loop [thus, the points $(X(t_1), Y(t_1))$ and $(X(t_2), Y(t_2))$ coincide and as t varies from t_1 to t_2 we make one circuit around the loop - for example the ellipse in our previous example with $0 \le t \le 2$] then

$$\left| \int_{t_1}^{t_2} Y(t) \frac{dX}{dt}\, dt \right| \quad \text{equals the area enclosed by the loop.}$$

Thus the area of the ellipse considered earlier is:

$$A = 2\pi \int_0^2 \cos^2 \pi t \, dt = 2\pi.$$

Curves Specified in other Coordinates - Polar Coordinates

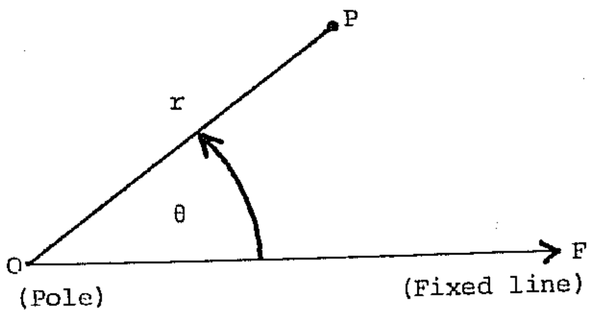
Frequently a curve is more naturally described using a coordinate system other than the Cartesian one.

To illustrate this we will consider the Polar coordinate system; in which the position of a point P is specified by the ordered pair (r, θ) where

r is the distance from P to a fixed point O (the pole)

and

θ is the angle (measured counter-clockwise) between the line OP and a fixed direction OF



Note that the polar coordinates (r, θ), $(-r, \theta + \pi)$ and $(r, \theta + 2n\pi)$, $n = \pm 1, \pm 2, \ldots$ all specify the same point. To remove this ambiguity we sometimes impose the restrictions $r > 0$, $0 \le \theta < 2\pi$. For each point P there is only one pair of coordinates (r, θ) satisfying these restrictions, we will refer to these as the polar coordinates of P.

In order that the coordinates can vary continuously with the position of the point it is in many situations useful to allow r and θ to range over all possible values.

For a Cartesian System of axes, with origin at the pole and x-axis in the direction OF, we have the following relationship between the Cartesian and the polar coordinates of a point P.

$$x = r\cos\theta \qquad y = r\sin\theta$$

$r = \sqrt{x^2 + y^2}$, and

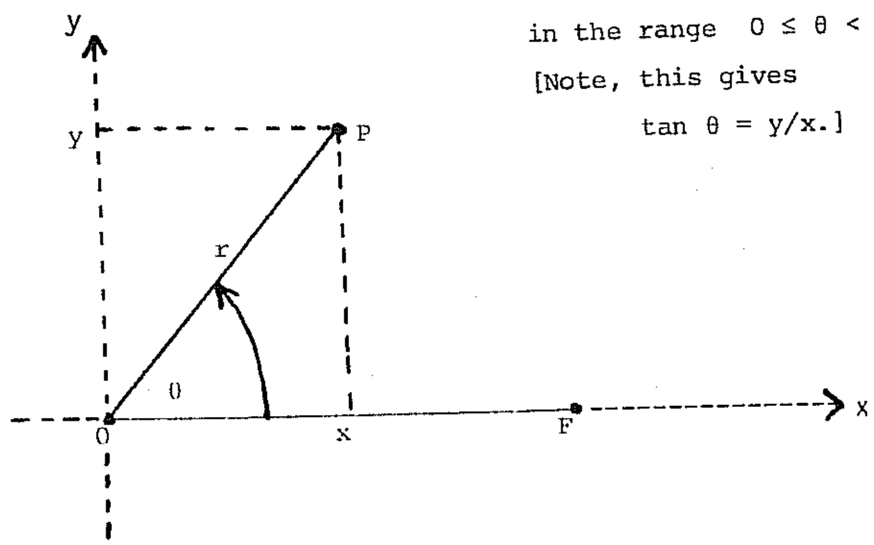
θ is the unique solution of

$$\cos\theta = x/r$$
$$\sin\theta = y/r$$

in the range $0 \le \theta < 2\pi$.

[Note, this gives

$$\tan\theta = y/x.]$$

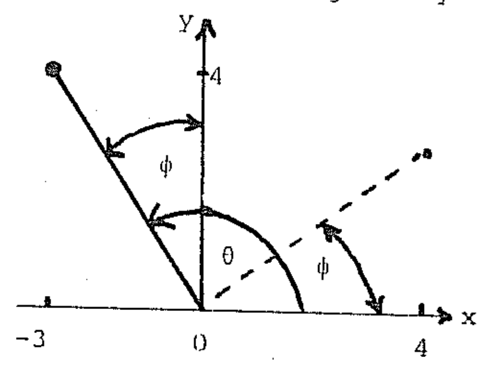


For example: The polar coordinates of the point $(-3, 4)$ are given by

$$r = \sqrt{9 + 16} = 5$$

$$\theta = \frac{\pi}{2} + \tan^{-1}\frac{3}{4}$$

$$\doteq 2.214$$



In general a curve is described in polar coordinates by specifying a relationship between r and θ.

For example, you should verify the following.

Relationship	Description of Curve
$\theta = \theta_0$ (a constant)	line through the pole inclined at an angle θ_0 to the fixed direction
$r = R_0$ (a constant)	circle; centre the pole, radius R_0
$r\cos(\theta - \theta_0) = r_0$	line through the point $P_0(r_0, \theta_0)$ and perpendicular to OP_0
$r^2 - 2r_0 r\cos(\theta - \theta_0) + r_0^2 = R_0^2$	circle; centre (r_0, θ_0), radius R_0
In particular:	
$r = 2R_0\cos\theta$	circle; centre $(R_0, 0)$, radius R_0
$r = 2R_0\sin\theta$	circle; centre $(R_0, \frac{\pi}{2})$, radius R_0

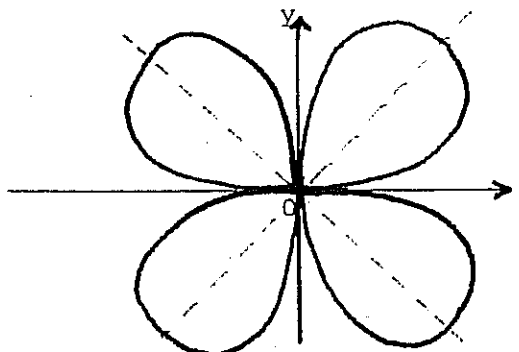
We will concern ourselves with curves given by relations of the form:

$$r = R(\theta)$$

For example:

The Limaçon; $r = b + a\cos\theta$, or the special case $(b = a)$ of the *Cardioid* $r = a(1 + \cos\theta)$.

The Rose curves; $r = a\sin n\theta$ consisting of n petals, when n is odd, and $2n$ petals, when n is even, equally spaced round the pole



The rose curve $r = \sin 2\theta$

The Archimedean Spiral; $r = a^2\,\theta$.

In sketching and analysing these curves you might make use of the following.

(a) *Symmetries.* For example, if $R(-\theta) = R(\theta)$ then the curve is symmetric about the line $\theta = 0$. Similarly if $R(-\theta) = -R(\theta)$ it is symmetric about $\theta = \pi/2$ – for instance this would be the case when r is a function of $\sin\theta$ only. Useful symmetries can also arise from the periodicity of the trigonometric functions (this is the case for the rose curves).

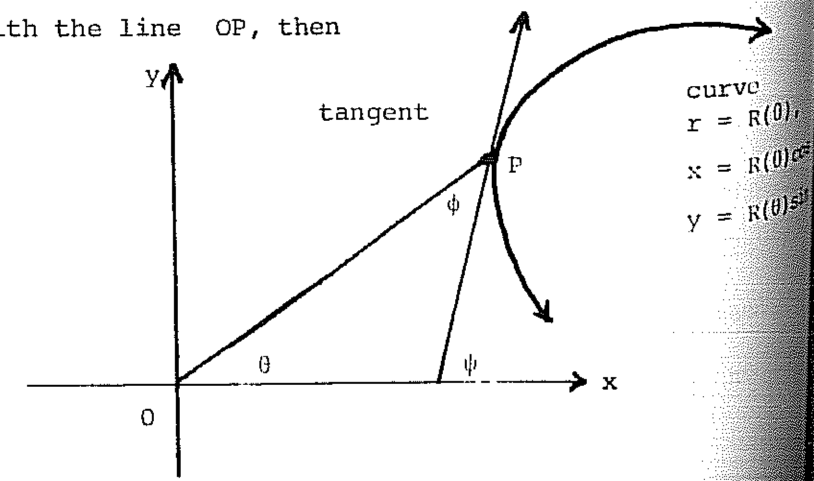
(b) *Bounds on the value of* r *and* θ.

For example, the curve $r = 2 + \sin\theta$ must lie within the annulus $1 \le r \le 3$.

(c) *The direction of the tangent(s) to the curve*

(i) At the pole: given by the roots of $R(\theta) = 0$.

(ii) At the point $P(r,\theta)$ where $r \neq 0$; if the tangent makes an angle ϕ with the line OP, then



$$\tan\phi = \tan(\psi - \theta)$$

$$= \frac{\sin\psi\cos\theta - \cos\psi\sin\theta}{\cos\psi\cos\theta + \sin\psi\sin\theta}$$

$$= \frac{\tan\psi\cos\theta - \sin\theta}{\cos\theta + \tan\psi\sin\theta}$$

But, $\tan\psi = \dfrac{dy}{dx} = \dfrac{dy}{d\theta}\Big/\dfrac{dx}{d\theta}$

$$= \frac{\frac{dR}{d\theta}\sin\theta + R\cos\theta}{\frac{dR}{d\theta}\cos\theta - R\sin\theta}$$

So,

$$\tan\phi = \frac{\dfrac{\frac{dR}{d\theta}\sin\theta\cos\theta + R\cos^2\theta}{\frac{dR}{d\theta}\cos\theta - R\sin\theta} - \sin\theta}{\cos\theta + \dfrac{\frac{dR}{d\theta}\sin^2\theta + R\sin\theta\cos\theta}{\frac{dR}{d\theta}\cos\theta - R\sin\theta}}$$

$$= \frac{R}{\frac{dR}{d\theta}}$$

or

$$\boxed{\tan\phi = R\,\frac{d\theta}{dR}\,.}$$

For example: To sketch the cardioid

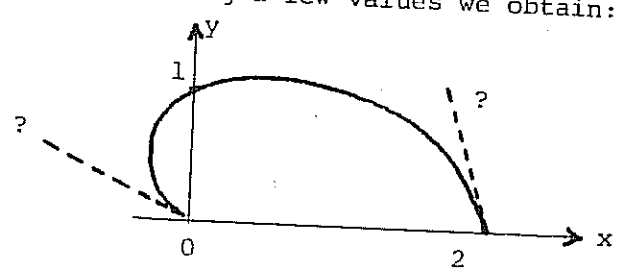
$$r = R(\theta) := 1 + \cos\theta$$

we have;

(a) $R(-\theta) = R(\theta)$, so symmetry about $\theta = 0$ (hence we need only work on the upper half of the graph).

(b) The curve is confined to the disk $|r| < 2$.

Using this and plotting a few values we obtain:



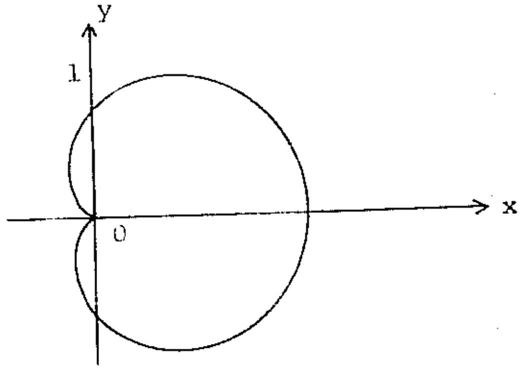
To complete the picture by reflecting in the x-axis we also need the following information.

(c) (i) The direction of the tangent(s) at $r = 0$. Namely, the roots of $1 + \cos\theta = 0$; $\theta = \pi$.

(ii) How the two halves join at (2,0). Since $R/\frac{dR}{d\theta} = \frac{1 + \cos\theta}{-\sin\theta}$

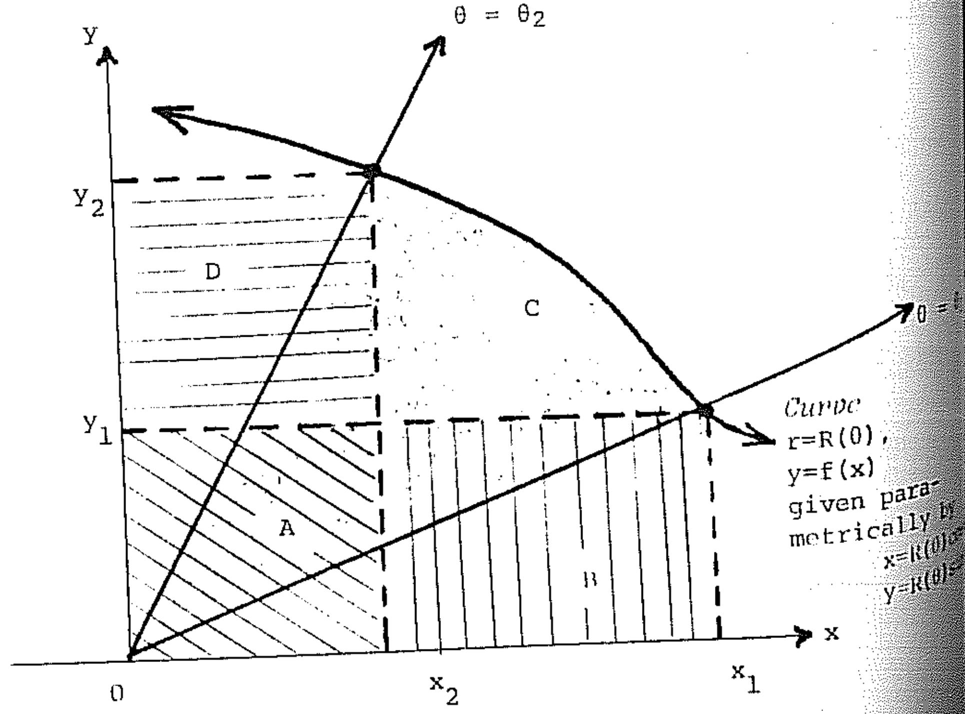
$$\to \pm\infty \quad \text{as} \quad \theta \to 0 \text{ or } 2\pi ,$$

the tangent is vertical and so we have the completed picture



Areas associated with curves given in Polar Coordinates

It is natural to seek the area enclosed by the curve $r = R(\theta)$ and the lines $\theta = \theta_1$ and $\theta = \theta_2$



From the above diagram we see that this (shaded) area equals

$$A + B + D + C - \frac{1}{2}(A + B) - \frac{1}{2}(A + D)$$

$$= \frac{1}{2}[(B + C) + (D + C)].$$

$$= \frac{1}{2}\left[\int_{x_2}^{x_1} f + \int_{y_1}^{y_2} f^{-1}\right]$$

$$= \frac{1}{2}\left[\int_{y_1}^{y_2} f^{-1} - \int_{x_1}^{x_2} f\right]$$

$$= \frac{1}{2}\left[\int_{\theta_1}^{\theta_2} x(\theta)\frac{dy}{d\theta}\,d\theta - \int_{\theta_1}^{\theta_2} y(\theta)\frac{dx}{d\theta}\,d\theta\right],$$

by the parametric expression for area.

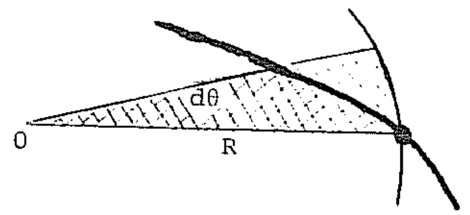
Substituting $x(\theta) = R(\theta)\cos\theta$, $y(\theta) = R(\theta)\sin\theta$ we obtain

$$\text{Area} = \frac{1}{2}\int_{\theta_1}^{\theta_2} R\cos\theta\left(\frac{dR}{d\theta}\sin\theta + R\cos\theta\right) - R\sin\theta\left(\frac{dR}{d\theta}\cos\theta - R\sin\theta\right)d\theta ,$$

which simplifies to

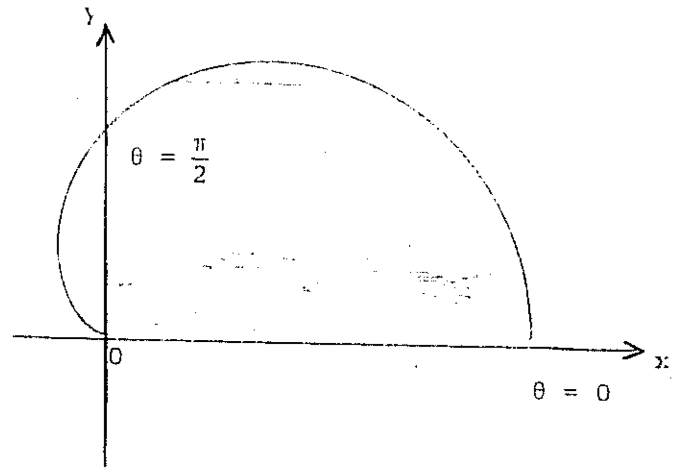
$$\boxed{\text{Area} = \frac{1}{2}\int_{\theta_1}^{\theta_2} R^2\,d\theta}$$

This can also be remembered by the following *elementary triangle*.



Area of circular segment is $\frac{1}{2}R^2 d\theta$

For example: The area of the Cardioid $r = 1 + \cos\theta$ in the positive quadrant is



$$\frac{1}{2}\int_0^{\frac{\pi}{2}} (1 + \cos\theta)^2 \, d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} 1 + 2\cos\theta + \cos^2\theta \, d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} 3/2 + 2\cos\theta + 1/2\cos 2\theta \, d\theta$$

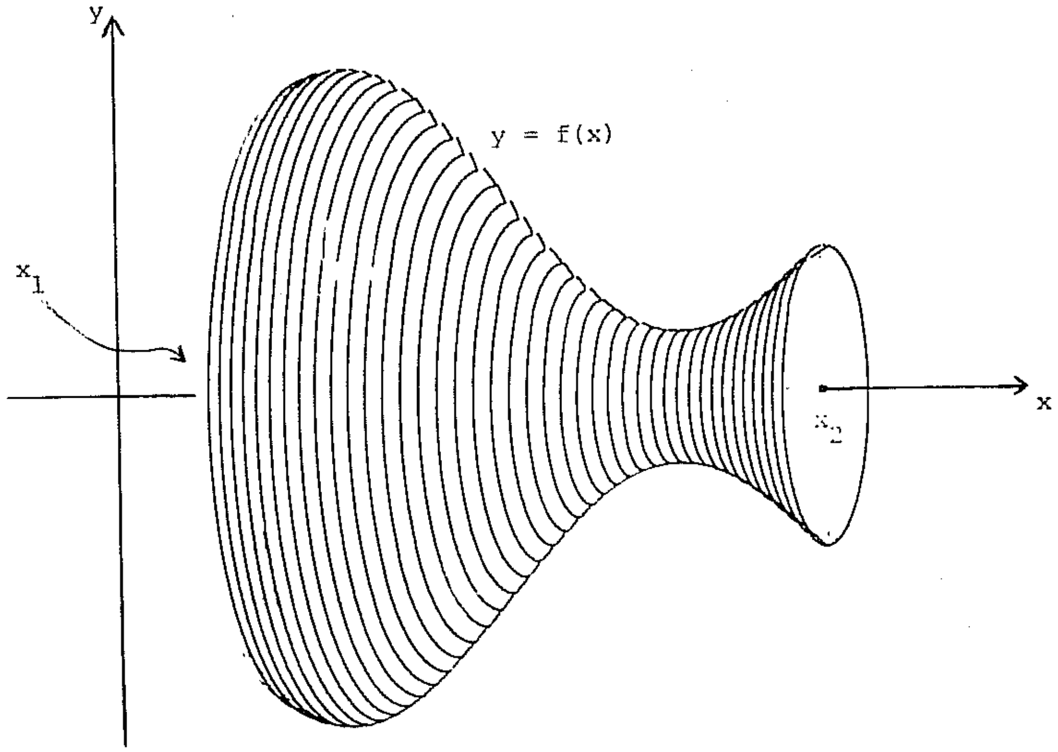
$$= \frac{3\pi}{8} + 1 .$$

Similarly the total area enclosed by the Cardioid is

$$\frac{1}{2}\int_0^{2\pi} (1 + \cos\theta)^2 \, d\theta = \frac{3\pi}{2} .$$

9.2 Volumes of Revolution

If for $x_1 \le x \le x_2$ we rotate a curve $y = f(x)$ about the x-axis we obtain a so called solid of revolution.



Building on our intuitive view of areas as the sum of infinitely many infinitessimally thin strips, we may consider our solid to be composed of an infinite number of infinitesimally thin circular lamina each perpendicular to the x-axis. The lamina centred at x having *volume* $\pi f(x)^2 dx$ - see diagram below. The volume of our solid is then the "sum"

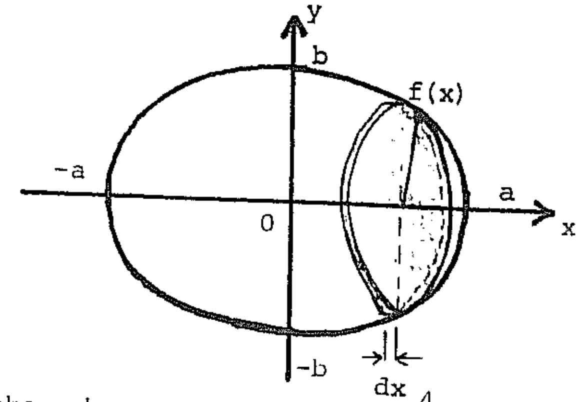
$$V = \pi \int_{x_1}^{x_2} f(x)^2 \, dx .$$

If our curve is given parametrically by $x = X(t)$, $y = Y(t)$, inverse substitution gives

$$V = \pi \int_{t_1}^{t_2} Y(t)^2 \frac{dX}{dt} \, dt$$

For example: The volume of the ellipsoid of revolution obtained by rotating the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



about the x-axis, is

$$V = \pi b^2 \int_{-a}^{a} 1 - \frac{x^2}{a^2} \, dx = \frac{4}{3}\pi b^2 a$$

[Note the particular case of the sphere, $a = b = r$; $V = \frac{4}{3}\pi r^3$.]

Alternatively, we may regard our solid of revolution to be composed of infinitessimally thin 'cylindrical shells' each of volume

$$2\pi \times \text{length of cylinder} \times y \, dy$$

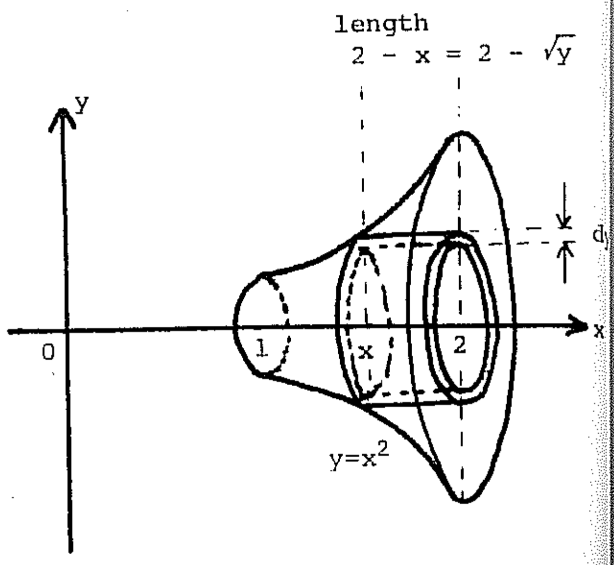
See diagram below. This sometimes leads to more tractable integrals.

For example: The volume of the solid of revolution obtained by rotating about the x-axis the curve $y = x^2$ for $1 \le x \le 2$ is

$$2\pi\int_0^1 y\,dy + 2\pi\int_1^4 (2-\sqrt{y})y\,dy$$

$$= \pi + 2\pi\left[y^2 - \frac{2}{5}y^{5/2}\right]_1^4$$

$$= \frac{31}{5}\pi .$$



REMARKS:

(1) In setting up the volume integrals, particularly when using the cylindrical shell approach, it is best to work from a good diagram.

(2) Similar formulae (obtained by interchanging the roles of x and y) apply for the volumes of solids obtained by revolution about the y-axis.

9.3 Arc length of a Curve

For a continuously differentiable function f and real numbers a, b $(a < b)$, if we let $x_k = a + k(b-a)/n$, it can be shown that

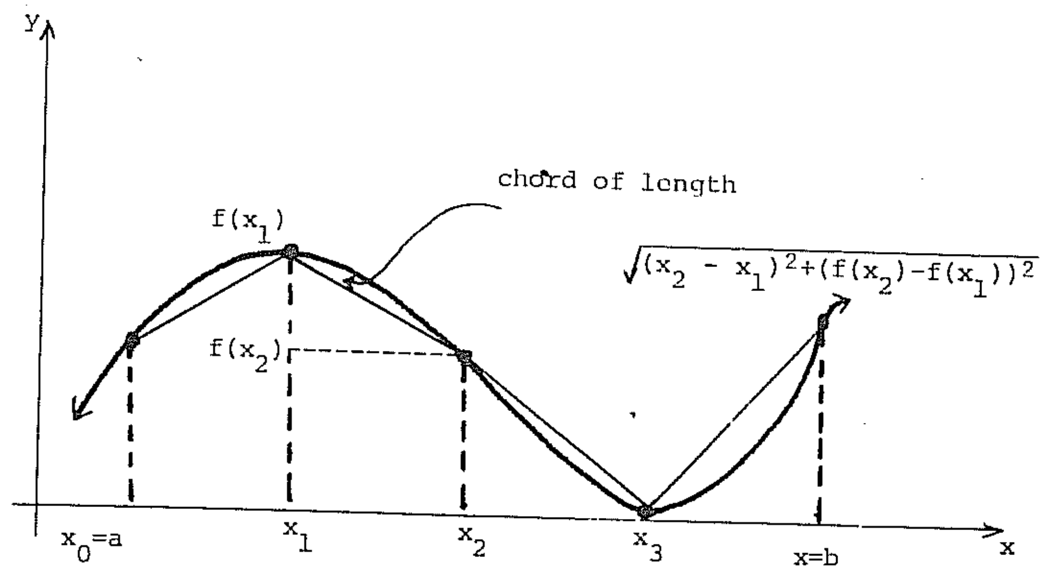
$$\lim_{n\to\infty} \sum_{k=0}^{n-1} \sqrt{(x_{k+1}-x_k)^2 + (f(x_{k+1}) - f(x_k))^2}$$

$$= \lim_{n\to\infty} \frac{b-a}{n}\sum_{k=0}^{n-1}\sqrt{1+\left[\frac{f(x_{k+1}) - f(x_k)}{x_{k+1}-x_k}\right]^2}$$

exists and equals

$$\int_a^b \sqrt{1+f'(x)^2}\,dx$$

From the diagram below (with $n = 4$) we see that it is reasonable to interpret this limit as the length of the curve between the points $(a, f(a))$ and $(b, f(b))$.



Accordingly we define the **arc length** of the curve $y = f(x)$ between the points $(a, f(a))$ and $(b, f(b))$ to be

$$S_a^b := \int_a^b \sqrt{1+(f')^2}$$

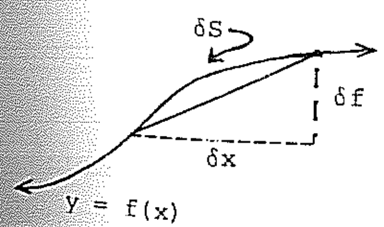
REMARK: If we define $S(x)$ to be the primitive

$$S(x) := \int_a^x \sqrt{1+(f')^2}$$

then we have

$$\frac{dS}{dx} = \sqrt{1+f'(x)^2}$$

a result which may intuitively be interpreted as follows.



$$(\delta S)^2 \approx (\delta x)^2 + (\delta f)^2$$

or

$$\frac{\delta S}{\delta x} \approx \sqrt{1+\left(\frac{\delta f}{\delta x}\right)^2}$$

For a curve given *parametrically* by

$$x = X(t), \quad y = Y(t)$$

we have for the arc length between the points $(x(t_1), y(t_1))$, $(x(t_2), y(t_2))$

$$s_{t_1}^{t_2} = \int_{x(t_1)}^{x(t_2)} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_{t_1}^{t_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, \frac{dX}{dt}\, dt$$

(making the inverse substitution $x=X(t)$)

$$= \int_{t_1}^{t_2} \sqrt{\left(\frac{dX}{dt}\right)^2 + \left(\frac{dy}{dx}\frac{dX}{dt}\right)^2}\, dt$$

So

$$s_{t_1}^{t_2} = \int_{t_1}^{t_2} \sqrt{\left(\frac{dX}{dt}\right)^2 + \left(\frac{dY}{dt}\right)^2}\, dt$$

In particular, for the *Polar* curve

$$r = R(\theta)$$

we have
$$\left(\frac{dX}{d\theta}\right)^2 + \left(\frac{dY}{d\theta}\right)^2$$

$$= (-R \sin\theta + \frac{dR}{d\theta}\cos\theta)^2 + (R\cos\theta + \frac{dR}{d\theta}\sin\theta)^2$$

$$= R^2 + \left(\frac{dR}{d\theta}\right)^2$$

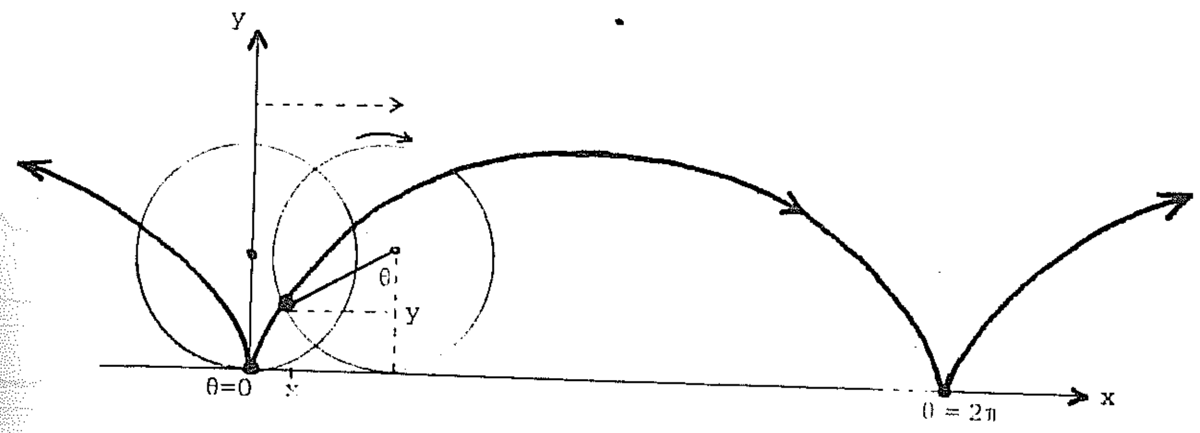
So

$$s_{\theta_1}^{\theta_2} = \int_{\theta_1}^{\theta_2} \sqrt{R(\theta)^2 + \left(\frac{dR}{d\theta}\right)^2}\, d\theta$$

Example: *The cycloid* given parametrically by

$$x = a(\theta - \sin\theta), \quad y = a(1 - \cos\theta)$$

is the curve traced out by a point on the circumference of a circle, radius r, as it rolls along the x-axis.



The length of one arch is given by

$$\int_0^{2\pi} \sqrt{\left(\frac{dX}{d\theta}\right)^2 + \left(\frac{dY}{d\theta}\right)^2}\, d\theta$$

$$= a\int_0^{2\pi} \sqrt{(1 - \cos\theta)^2 + \sin^2\theta}\, d\theta$$

$$= a\int_0^{2\pi} \sqrt{2(1 - \cos\theta)}\, d\theta$$

$$= 2a\int_0^{2\pi} \sin\frac{\theta}{2}\, d\theta$$

$$= \left[-4a\cos\frac{\theta}{2}\right]_0^{2\pi} = 8a.$$

9.4 Surfaces of Revolution

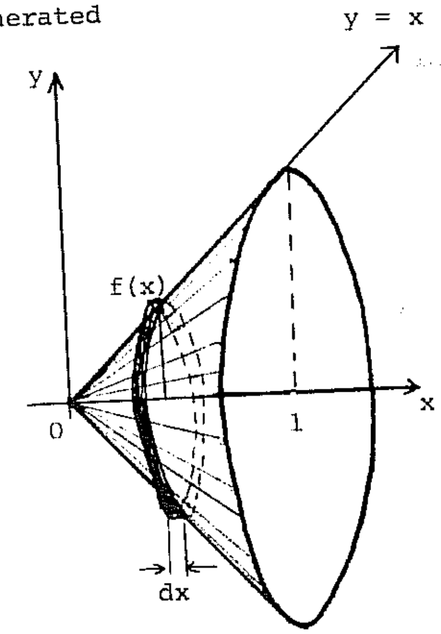
We investigate the surface area of the solid of revolution generated by rotating the curve $y = f(x)$, $a \le x \le b$, about the x-axis.

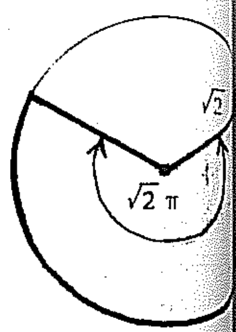
A False Start (Not to be learned)

In Section 9.2 we regarded such a solid as the union of infinitessimal lamina, each lamina has a surface area of $2\pi f(x)\, dx$, thus we might imagine using $2\pi\int_a^b f(x)\,dx$ to compute the surface area. Applying this to the curve $y = x \quad 0 \le x \le 1$, we obtain for the surface area of the cone generated

$$2\pi \int_0^1 x\,dx = \pi .$$

However, as is easily verified (do so) such a cone has a surface area of $\sqrt{2}\,\pi$!





Our error was regarding the surface to be composed of infinitessimal bands with *area* $2\pi f(x)\,dx$ obtained by rotating the "horizontal" segment of length dx about the x-axis.

<u>The Correct Expression</u>

We need to view the surface as a union of bands obtained by rotating the infinitessimal segment of the curve at $(x, f(x))$ with *length* $ds = \sqrt{1 + f'(x)^2}\,dx$ about the x-axis. The *area* of the band is then $2\pi f(x)\,ds$ and we have for the surface area

$$SA = 2\pi \int_a^b f(x)\,ds = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\,dx$$

or for the curve given parametrically by

$x = X(t)$, $y = Y(t)$ we have

$$SA = 2\pi \int_{t_1}^{t_2} Y(t)\sqrt{\left(\frac{dX}{dt}\right)^2 + \left(\frac{dY}{dt}\right)^2}\,dt$$

EXAMPLE: The area of the surface generated by rotating one arch of the cycloid

$$x = a(\theta - \sin\theta)\ , \quad y = a(1 - \cos\theta)$$

about the x-axis is

$$2\pi \int_0^{2\pi} Y(t)\sqrt{\left(\frac{dX}{dt}\right)^2 + \left(\frac{dY}{dt}\right)^2}\,dt$$

$$= 2\pi a^2\int_0^{2\pi} (1 - \cos\theta)\sqrt{(1 - \cos\theta)^2 + \sin^2\theta}\;d\theta$$

$$= 8\pi a^2 \int_0^{2\pi} \sin^3\frac{\theta}{2}\,d\theta \qquad \text{(see page 188)}$$

$$= 16\pi a^2 \int_0^{\pi} \sin^3\phi\,d\phi \qquad (\phi = \tfrac{\theta}{2})$$

$$= 16\pi a^2 \int_{-1}^{1} (1 - u^2)\,du \qquad (u = \cos\phi)$$

$$= \frac{64}{3}\pi a^2.$$

REMARK: In the formula

$$SA = 2\pi \int_a^b f(x)\,ds \qquad (a \le b)$$

negative areas are assigned to those portions of the surface generated by segments of the curve where $f(x) < 0$. To obtain the actual area we should use

$$2\pi\int_a^b |f(x)|\;ds\ .$$

EXERCISES:

(1) (a) Find the signed area

$$\int_0^2 x^2 - 3x + 2\,dx$$

(b) The average value of the function $y = f(x)$ over the interval $[a,b]$ is defined to be

$$\bar{f} = \frac{1}{b-a}\int_a^b f$$

Find the average value of $f(x) := x^2 - 3x + 2$ over $[0,2]$

Give a geometrical interpretation of $\bar{f}$.

(c) Find the area

$$A = \int_0^2 |x^2 - 3x + 2|\, dx$$

(2) By evaluating

$$\int_0^r \sqrt{r^2 - x^2}\, dx$$

deduce that the ratio of a circle's area to the square of its radius is π , where $\frac{\pi}{2}$ is the smallest positive solution to $\cos x = 0$ (see §6.5).

(3) Find

(i) The area bounded by $y = \sinh^{-1} x$, $x = 0$, $x = 1$ and the x-axis.

(ii) The area enclosed by the curves $y = 2x$, $y = 2x^3 - 3x^2$ and the lines $x = -\frac{1}{2}$, $x = 1\frac{1}{2}$.

(iii) The area enclosed by the ellipse $x = a \sin t$, $y = b \sin t$.

(iv) The area enclosed by one arch of the cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$ and the x-axis. (see p.188)

(4) Derive the polar relationship for each of the curves described in the table on p. 179. In each case; sketch the curve and find a Cartesian equation for it.

(5) Find the area of one petal of the rose curve $r = \sin 3\theta$.

(6) Find the volume of the solid obtained by rotating:

(i) The curve $y^2 = a^2x/(2a - x)$, $0 \le x \le a$

and

(ii) One arch of the cycloid

$$x = a(\theta - \sin \theta), \quad y = a(1 - \cos \theta)$$

about the x-axis.

(7) (i) Show that the perimeter of a circle of radius r is $2\pi r$.

(ii) Find the total length of the cardioid $r = (1 + \cos \theta)$.

(8) (i) Find the surface area of a sphere of radius r.

(ii) Find the area of the surface of revolution obtained by rotating about the x-axis the segment of the *caternary*

$$y = \cosh x$$

between $x = a$ and $x = b$.

(iii) Find the area of the surface generated when the portion of the curve

$$x = \cos^3 t, \quad y = \sin^3 t$$

which corresponds to the range $0 \le t \le \frac{\pi}{2}$ is rotated about the x-axis.

CHAPTER 10

POLYNOMIAL APPROXIMATION

10.1 Criteria for approximating a function by a polynomial

In this chapter we address the basic question of finding a polynomial p, of specified degree n, which *approximates* a specified function f in some neighbourhood of a given point x_0.

Such approximations are extremely useful in both pure and applied mathematics. For example, I am sure that you have made use of the approximations

$$\sin x \approx x$$

$$\frac{1}{1+x} \approx 1 - x$$

for values of x near $x_0 = 0$.

We will assume that the given point is $x_0=0$. By the change of variable $X = x - x_0$ we can always transform the point of interest into the origin, so this assumption leads to no loss of generality. In the statement of our problem the word "approximates" could be made precise in many different ways.

We could for example take "p approximates f for x near 0" to mean for some $\varepsilon > 0$ that

- p is chosen to minimize the largest difference in values between p and f at any point in the interval $[-\varepsilon,\varepsilon]$; that is p is the polynomial (of degree n) for which

$$\underset{-\varepsilon \le x \le \varepsilon}{\text{Maximum}} \; |f(x) - p(x)|$$

is a minimum;

OR

- p is chosen so that

$$\int_{-\varepsilon}^{\varepsilon} (f(x) - p(x))^2 \, dx$$

is a minimum.

Both of these lead to useful theories of approximation, however we will settle on the following more "heuristic" criteria for p being a good approximation to f, namely that we require:

(0) p and f should have the same value at 0; that is, the two graphs $y = p(x)$ and $y = f(x)$ should coincide at the point $x = 0$.

(1) The two graphs $y = p(x)$ and $y = f(x)$ should have the same slope at $x = 0$. Thinking of the graphs as pathways, the two paths should head in the same direction at 0.

(2) The rate of change of direction for the two graphs should be the same at $x = 0$. That is, our two pathways should "curve" in the same way at $x = 0$.

(3) The rate of change of the rate of change of direction for the two graphs should be the same at $x = 0$.

(4) ....

etc, as far as possible.

Setting $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n$,

so

$$p'(x) = a_1 + 2a_2x + 3a_3x^2 + \ldots + na_nx^{n-1}$$

$$p''(x) = 2a_2 + 3.2a_3x + \ldots + n(n-1)a_nx^{n-2}$$

$$p'''(x) = 3.2a_3 + \ldots + n(n-1)(n-2)a_nx^{n-3}$$

$$\cdots$$

$$p^{(n)}(x) = n(n-1)(n-2)(n-3)\ldots 2.1\,a_n\,,$$

and translating the above criteria into statements about derivatives we have:

(0) $p(0) = f(0)$ so $a_0 = f(0)$

(1) $p'(0) = f'(0)$ so $a_1 = f'(0)$

(2) $p''(0) = f''(0)$ so $a_2 = f''(0)/2$

(3) $p'''(0) = f'''(0)$ so $a_3 = f'''(0)/3.2$

$\cdots$

(n) $p^{(n)}(0) = f^{(n)}$ ...

In general it is not possible to maintain agreement for any higher derivatives, since after $m = n$ (the degree of p) we have $p^{(m)}(x) \equiv 0$.

Summarizing: Our criteria give, as the *n'th degree polynomial approximation to* f *at* $x = 0$

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k$$

where $k! = k(k-1)(k-2)\ldots 2.1$ is the factorial function of k.

We will refer to this as the n'th degree MacLaurin polynomial for f.

EXAMPLES:(1) For $f(x) := \sin x$ we have:

$f(x) = \sin x$	so	$f(0) = 0$
$f'(x) = \cos x$	so	$f'(0) = 1$
$f''(x) = -\sin x$	so	$f''(0) = 0$
$f'''(x) = -\cos x$	so	$f'''(0) = -1$
$f^{iv}(x) = \sin x$	so	$f^{iv}(0) = 0$

the above pattern now repeats itself indefinitely.

Using these we have as the MacLaurin polynomials for sin x

degree	polynomial
1 :	$p_1(x) = x$
2 :	$p_2(x) = x$
3 :	$p_3(x) = x - x^3/3! = x - x^3/6$
4 :	$p_4(x) = x - x^3/3!$
5 :	$p_5(x) = x - x^3/3! + x^5/5! = x - x^3/6 + x^5/120$
6 :	$p_6(x) = x - x^3/3! + x^5/5!$
7 :	$p_7(x) = x - x^3/3! + x^5/5! - x^7/7!$

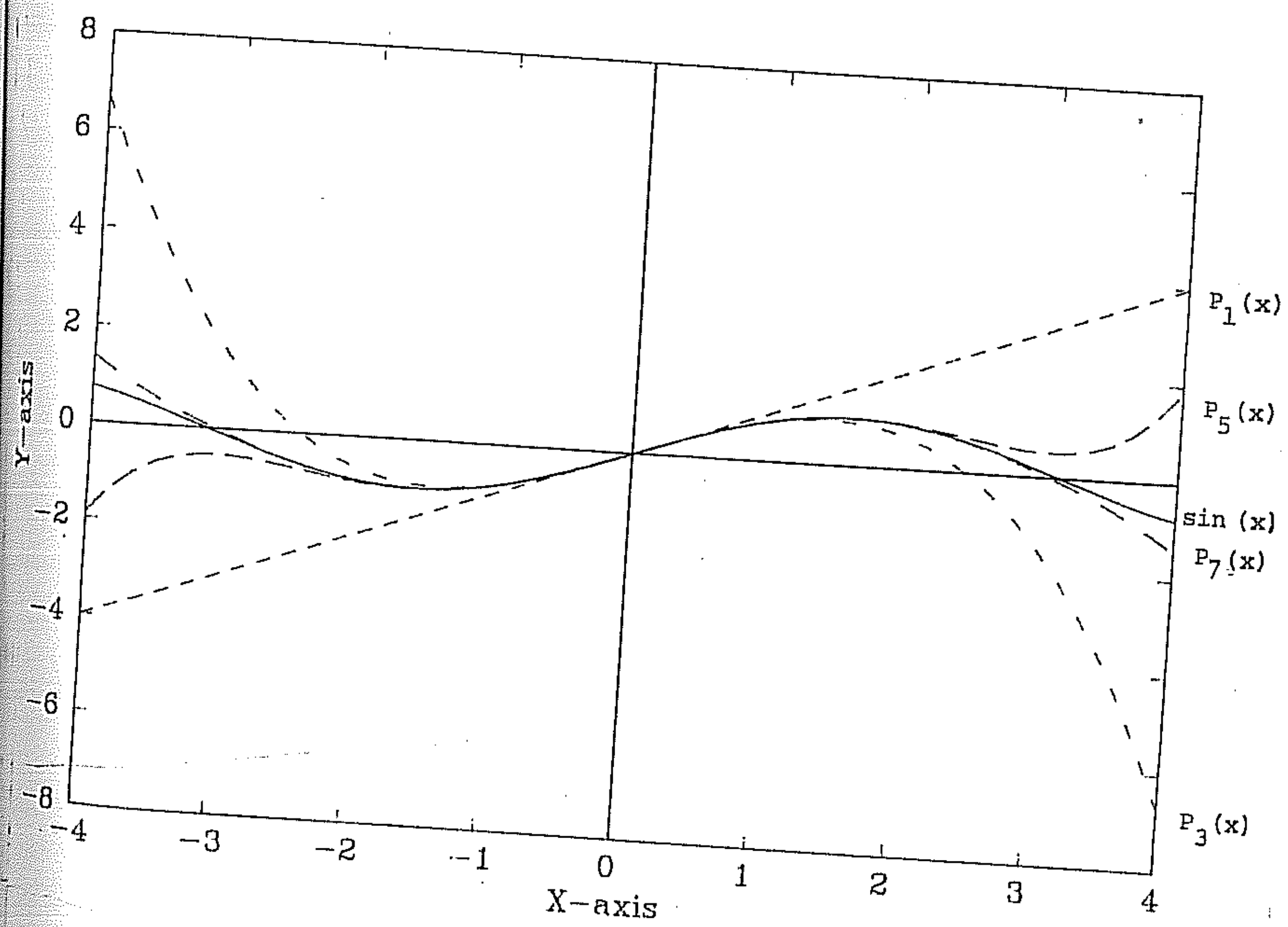
(and in general, the polynomial of odd degree)

$$2m+1 : P_{2m+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^m \frac{x^{2m+1}}{(2m+1)!}$$

$$= \sum_{k=0}^{m} (-1)^k x^{2k+1}/(2k+1)!$$

The first four distinct approximations, together with sin x are graphed below.

Successive MacLaurin Approximations To Sin(x).



(2) For $f(x) := e^x$ we have for all n

$$f^{(n)}(x) = e^x \quad \text{so} \quad f^{(n)}(0) = 1$$

and the successive MacLaurin polynomials for e^x are

(and in general)

n : $\quad p_n(x) = -(x + x^2/2 + x^3/3 + \ldots + x^n/n)$

$$= -\sum_{k=1}^{n} x^k/k$$

Evaluating these for $x = 0.5$ we obtain as successive approximations to $\ln(0.5)$;

$-0.5, -0.625, -0.667, -0.682, -0.689, -0.691, -0.692$ (c.f. $\ln(0.5) \doteq -0.693$).

(4) For $f(x) := (1 + x)^r$ we have

$f(x) = (1 + x)^r$ $\quad$ so $f(0) = 1$

$f'(x) = r(1 + x)^{r-1}$ $\quad$ so $f'(0) = r$

$f''(x) = r(r - 1)(1 + x)^{r-2}$ $\quad$ so $f''(0) = r(r - 1)$

$f'''(x) = r(r - 1)(r - 2)(1 + x)^{r-3}$ $\quad$ so $f'''(0) = r(r - 1)(r - 2)$

If r is a positive integer, $r = m$ say, then this terminates at the m'th step with

$$f^{(m)}(x) = m(m - 1)(m - 2) \ldots 2.1 = m!$$

and all subsequent derivatives are 0.

However, for values of r other than a positive integer the process continues indefinitely with

$$f^{(n)}(0) = r(r - 1)(r - 2) \ldots (r - n + 1).$$

In the case when $r = m$ a positive integer we have as the successive MacLaurin polynomials for $(1 + x)^m$

degree	polynomial
1 :	$p_1(x) = 1 + mx$
2 :	$p_2(x) = 1 + mx + \frac{m(m-1)}{2!} x^2$
3 :	$p_3(x) = 1 + mx + \frac{m(m-1)}{2!} x^2 + \frac{m(m-1)(m-2)}{3!} x^3$
. . .	
m (and all higher values) :	$p_m(x) = 1 + mx + \frac{m(m-1)}{2!} x^2 + \ldots + mx^{m-1} + x^m$

$$= 1 + mx + \binom{m}{2} x^2 + \binom{m}{3} x^3 + \ldots + x^m$$

$$= \sum \binom{m}{k} x^k$$

where $\binom{m}{k}$ is the combinatorial coefficient "m *choose* k", sometimes denoted by ${}^mC_k$,

$$\binom{m}{k} := \frac{m(m-1)(m-2)\ldots(m-k+1)}{k!}$$

$$= \frac{m!}{k!(m-k)!}$$

You should recognise the last line of the above table as the Binomial expansion for $(1 + x)^m$.

When r is not a positive integer we have, as the n'th degree MacLaurin polynomial for $(1 + x)^r$, the truncated result of the general *Binomial Theorem*, namely

$$1 + rx + \frac{r(r-1)}{2!}x^2 + \frac{r(r-1)(r-2)}{3!}x^3 + \ldots + \frac{r(r-1)\ldots(r-n+1)}{n!}x^n$$

Putting $r = \frac{1}{2}$, $n = 5$, for example, the fifth degree MacLaurin polynomial for $\sqrt{1+x}$ is seen to be

$$1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5 .$$

As these examples illustrate, one advantage of our criteria for "approximation" (which lead to the MacLaurin polynomials) over other possbile criteria mentioned at the start of the section is that, after the n'th degree polynomial approximation has been calculated, to find approximations of higher degree m all the coefficients do not have to be recomputed. The terms up to x^n remain the same, we only need to add on the higher degree terms.

It remains to investigate the error committed when a function is approximated by its n'th degree MacLaurin polynomial.

10.2 The Error in Approximation by MacLaurin Polynomials

We begin by finding an expression for the remainder

$$R_n(x) := f(x) - [f(0) + f'(0)x + f''(0)x^2/2! + \ldots + f^{(n)}(0)x^n/n!]$$

between a function f and its n'th degree MacLaurin polynomial.

The "trick" is to consider for a given (fixed) value of x the integral:

$$\int_0^x f'(x-t)\,dt.$$

Using the substitution $u = x - t$ we find;

$$\int_0^x f'(x-t)\,dt = -\int_x^0 f'(u)\,du$$

$$= \int_0^x f'(u)\,du$$

$$= f(x) - f(0) \quad \text{(by the Fundamental Theorem of Calculus).}$$

On the other hand writing

$$\int_0^x f'(x-t)\,dt = \int_0^x \underset{v'}{1} \cdot \underset{u}{f'(x-t)}\,dt$$

and integrating by parts we obtain

$$\int_0^x f'(x-t)\,dt = \Big[tf'(x-t)\Big]_0^x - \int_0^x t\Big[-f''(x-t)\Big]\,dt$$

$$= f'(0)x + \int_0^x tf''(x-t)\,dt.$$

[Note that, $\frac{d}{dt} f'(x-t) = f''(x-t) \times -1$.]

Continuing in this way; integrating

$\int_0^x \underset{v'}{t} \cdot \underset{u}{f''(x-t)}\,dt$, by parts, etc. we obtain:

$$\begin{aligned}\int_0^x f'(x-t)\,dt &= f'(0)x + \int_0^x t f''(x-t)\,dt \\ &= f'(0)x + \left[\frac{t^2}{2} f''(x-t)\right]_0^x + \int_0^x \frac{t^2}{2} f'''(x-t)\,dt \\ &= f'(0)x + f''(0)x^2/2 + \int_0^x \frac{t^2}{2} f'''(x-t)\,dt \\ &= f'(0)x + f''(0)x^2/2 + \left[\frac{t^3}{3.2} f'''(x-t)\right]_0^x + \int_0^x \frac{t^3}{3.2} f^{iv}(x-t)\,dt \\ &= f'(0)x + f''(0)x^2/2! + f'''(0)x^3/3! + \int_0^x \frac{t^3}{3!} f^{iv}(x-t)\,dt \\ &= \ldots \\ &= f'(0)x + f''(0)x^2/2! + \ldots + f^{(n)}(0)x^n/n! + \int_0^x \frac{t^n}{n!} f^{(n+1)}(x-t)\,dt\end{aligned}$$

Equating these two alternative expressions for $\int_0^x f'(x-t)\,dt$ we obtain

$$f(x) = [f(0) + f'(0)x + f''(0)x^2/2! + \ldots + f^{(n)}(0)x^n/n!] + \int_0^x \frac{t^n}{n!} f^{(n+1)}(x-t)\,dt$$

or

$$\boxed{\begin{aligned}R_n(x) &:= f(x) - [f(0) + f'(0)x + f''(0)x^2/2! + \ldots + f^{(n)}(0)x^n/n!] \\ &= \frac{1}{n!}\int_0^x t^n f^{(n+1)}(x-t)\,dt.\end{aligned}}$$

This last expression is known as the integral form of the Remainder. From it we can produce a useful *error estimate* as follows.

In case $x > 0$ note that, as t ranges from 0 to x, $x-t$ ranges from x to 0 and so

$$\max_{0\le t\le x} |f^{(n+1)}(x-t)| = \max_{0\le s\le x} |f^{(n+1)}(s)|.$$

Thus:

$$\begin{aligned}&|f(x) - [f(0) + f'(0)x + f''(0)\,x^2/2! + \ldots + f^{(n)}(0)x^n/n!]| \\ &= \left|\frac{1}{n!}\int_0^x t^n f^{(n+1)}(x-t)\,dt\right|\end{aligned}$$

$$\le \frac{1}{n!}\int_0^x |t^n f^{(n+1)}(x-t)|\,dt$$

[Thinking of the integral as a sum, we note that this is just a reflection of the fact that the absolute value of a sum is less than or equal to the sum of the absolute vales.]

$$\begin{aligned}&= \frac{1}{n!}\int_0^x |t|^n |f^{(n+1)}(x-t)|\,dt \\ &\le \frac{1}{n!}\cdot \max_{0\le s\le x} |f^{(n+1)}(s)| \cdot \int_0^x t^n\,dt \qquad (\text{as } x > 0) \\ &= \frac{1}{n!}\cdot \max_{0\le s\le x} |f^{(n+1)}(s)| \cdot \frac{x^{n+1}}{n+1} \\ &= \frac{x^{n+1}}{(n+1)!}\cdot \max_{0\le s\le x} |f^{(n+1)}(s)|\end{aligned}$$

A similar analysis in the case when $x < 0$ leads to the general result:

Error Estimate:

$$\boxed{\begin{aligned}&|f(x) - [f(0) + f'(0)x + f''(0)x^2/2! + \ldots + f^{(n)}(0)x^n/n!]| \\ &\le \frac{|x|^{n+1}}{(n+1)!}\cdot \max_{-|x|\le s\le |x|} |f^{(n+1)}(s)|\end{aligned}}$$

To illustrate how this estimate may be used consider the following.

EXAMPLE:

For $f(x) = e^x$ the n'th degree MacLaurin polynomial is:

$$1 + x + x^2/2! + x^3/3! + \ldots + x^n/n!$$

Substituting into the above error estimate we obtain the more precise information:

$$\left| e^x - [1 + x + x^2/2! + \ldots + x^n/n!] \right| \le \frac{|x|^{n+1}}{(n+1)!} \max_{-|x| \le s \le |x|} |e^s|$$

$$= \frac{|x|^{n+1}}{(n+1)!} e^{|x|}$$

We may use this to:

(1) <u>Determine e to any given accuracy</u>

Putting $x = 1$ we have

$$\left| e - [2 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}] \right| \le \frac{e}{(n+1)!} \quad \ldots\ldots (*)$$

Of course this is only useful provided we have some upper bound on the value of e. To obtain such an estimate, note that for $n = 1$ (*) becomes

$$|e - 2| \le \frac{e}{2}$$

or

$$-\frac{e}{2} \le e - 2 \le \frac{e}{2}$$

and so, in particular $\frac{e}{2} \le 2$ or $e \le 4$.

(What lower bound for e do we get from these inequalities?)

Thus, using $e \le 4$ in (*) we obtain:

$$\left| e - [2 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}] \right| \le \frac{4}{(n+1)!} \quad \ldots\ldots\ldots\ldots (**)$$

To obtain the value of e accurate to say five decimal places it is therefore only necessary to determine a value of n for which

$$\frac{4}{(n+1)!} < 0.000005$$

and then add up $2 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}$ for that value of n.

One way of determining such an n is to compute the following table (preferably using a calculator or table of factorials).

n	$\frac{4}{(n+1)!}$
1	2
2	2/3
3	1/6
4	1/30
5	1/180
6	1/1260
7	1/10080
8	1/90720
9	1/907200 (<0.000005)

Thus, accurate to five decimal places

$$e \doteq 2 + \frac{1}{2} + \frac{1}{6} + \ldots + \frac{1}{9!}$$

$$\doteq \underline{2.71828}(1525)$$

correct
digits

Given the computing power, we may in this way find e to any given accuracy.

(2) <u>Prove e is irrational</u>

It was known to the early Greeks, and I hope also to you, that $\sqrt{2}$ is an irrational number.

We now prove that e is also irrational. That is, there do not exist whole numbers a and b such that $e = a/b$.

Assume this were not the case, that is for some pair of whole numbers a and b we have $a/b = e$, then from (**) we have for any n that

$$\left| \frac{a}{b} - [2 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}] \right| < \frac{4}{(n+1)!} .$$

Choosing n to be greater than b and 4 we therefore have

$$\left| \frac{n!a}{b} - [2 \times n! + \frac{n!}{2!} + \frac{n!}{3!} + \ldots + 1] \right| < \frac{4}{n+1} < 1.$$

Now all the terms in

$$k := \frac{n!a}{b} - [2 \times n! + \frac{n!}{2!} + \frac{n!}{3!} + \ldots + 1]$$

are integers [$n > b$ means the whole number b occurs as one of the factors in $n!$ and for $m = 2,3\ldots,n$ $\frac{n!}{m!} = n(n-1)\ldots(n-m+1)$].

Thus k itself is a whole number with $|k| < 1$ and so the only possibility is $k = 0$, but this is impossible as we would have for the fixed number $\frac{a}{b}$ that

$$\frac{a}{b} = 2 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}$$

for infinitely many different values of n.

Thus no such numbers as a and b can exist and so e is irrational.

10.3 Power Series Representation of Functions

Our work with MacLaurin polynomials suggests associating with any (infinitely differentiable) function f the "formal" power series

$$f(0) + f'(0)x + f''(0)x^2/2! + f'''(0)x^3/3! + \ldots + f^{(n)}(0)x^n/n! + \ldots.$$

$$= \sum_{n=0}^{\infty} f^{(n)}(0)x^n/n!$$

whose partial sums are the successive MacLaurin polynomials for f. We will refer to this series as the MacLaurin Expansion for f.

The expansion need not converge, and even when it does the sum need not equal the function value (see exercises). In general we will therefore use the symbol $\sim$ to relate a function and its expansion, reserving = or ≃ for cases when more information is available.

We could analyse the convergence of the above power series using the results of §4.5, however it is more useful to know when the series converges to the function value. From the definition of convergence this happens when the partial sums converge to $f(x)$. That is, when the limit as $n \to \infty$ of the remainder

$$R_n(x) = |f(x) - [f(0) + f'(0)x + f''(0)x^2/2! \ldots + f^{(n)}(0)x^n/n!]|$$

is zero.

We can usually determine a range of x-values for which this is the case using the error estimate of the last section.

For example:

(1) *Representation of e^x by its MacLaurin Expansion*

Since for any given value of x, e^x is a finite number and

$$\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0 \qquad \text{(convince yourself of this!)}$$

we have that:

$$\lim_{n\to\infty} |e^x - [1 + x + x^2/2! + \ldots + x^n/n!]| = 0.$$

That is, the sequence of partial sums

$$S_n = 1 + x + x^2/2! + \ldots + x^n/n!$$

converges to e^x and so we may express e^x as the infinite series

$$e^x = 1 + x + x^2/2! + \ldots + x^n/n! + \ldots.$$

$$= \sum_{n=0}^{\infty} x^n/n!$$

(2) *The MacLaurin Expansion for $ln(1 - x)$*

We have

$$\ln(1 - x) \sim -\sum_{k=1}^{\infty} x^k/k$$

with the error estimate

$$\left|\ln(1 - x) + \sum_{k=1}^{n} x^k/k\right| \le \frac{|x|^{n+1}}{(n+1)!} \max_{-|x|\le s\le |x|} \frac{n!}{(1-|x|)^{n+1}}$$

which certainly converges to 0 as $n \to \infty$ provided $|x| \le \frac{1}{2}$ (as then, $|x|/(1 - |x|) \le 1$).

Thus we have, for $|x| \le \frac{1}{2}$

$$\ln(1 - x) = -\sum_{k=1}^{\infty} x^k/k.$$

A more careful analysis in fact establishes the convergence for $-1 \le x < 1$. (Though for $x = -1$ the convergence to $\ln 2$ is conditional.)

Given a function f, if there exists a power series $\sum_{n=0}^{\infty} a_n x^n$ for which

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{for} \quad |x| < R$$

[remember this means: For $|x| < R$, $\lim_{N\to\infty}(f(x) - \sum_{n=0}^{N} a_n x^n) = 0$]

we will refer to $\sum_{n=0}^{\infty} a_n x^n$ as a power series representation of f for $|x| < R$.

A general result which we will assume without proof states that:

If for $|x| < R$ *the function* f *has a power series representation* $\sum_{n=0}^{\infty} a_n x^n$ *then that representation is unique;* that is, if we also have $f(x) = \sum_{n=0}^{\infty} b_n x^n$, then $b_0 = a_0$, $b_1 = a_1$, ..., $b_n = a_n$, ...

(you may think of this as a sort of glorified "equating of coefficients").

The significance of this for our purpose is that, if f has a MacLaurin expansion and we can find by any means a power series representation for f then that representation is its MacLaurin expansion.

For example:

(1) From $e^x = \sum_{n=0}^{\infty} x^n/n!$ we have

$$e^{-x^2} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \dots + \frac{(-x^2)^n}{n!} + \dots.$$

$$= 1 - x^2 + \frac{x^4}{2!} - \dots + (-1)^n x^{2n}/n! + \dots.$$

as the MacLaurin expansion for e^{-x^2}.

(2) Since for $|x| < 1$ we have

$$\frac{1}{1 + x^2} = 1 - x^2 + x^4 - \dots + (-1)^n x^{2n} + \dots.$$

On integrating, we obtain the MacLaurin expansion

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + (-1)^n x^{2n+1}/(2n+1) + \dots.$$

Here, some justification for the "term by term" integration of the power series is necessary.

This is easily done using the error estimate for the Binomial expansion derived from the work in §9.2 (see exercise 5).

The general Taylor expansion of f

If we expand the function f about a point x_0 (other than the origin) then we obtain

$$f(x) \sim f(x_0) + f'(x_0)(x - x_0) + f''(x_0)(x - x_0)^2/2! \dots$$
$$+ f^{(n)}(x_0)(x - x_0)^n/n! + \dots.$$

a result easily derived by expanding the function

$g(X) := f(X + x_0)$ about the origin to obtain

$$g(X) \sim g(0) + g'(0)X + g''(0)X^2/2! + \dots + g^{(n)}(0)X^n/n! + \dots.$$

and then making the substitution $x = X + x_0$, so $X = x - x_0$.

APPENDIX 10.4 The Error in Simpson's Rule

We neglect errors due to round-off in computation.

Assume the function f has a bounded fourth derivative with $|f^{IV}(x)| \le M$ for $-h \le x \le h$. Then, from §9.2 we have for $-h \le x \le h$ that

$$|R_3(x)| \le \frac{|x^4|}{4!} M \le \frac{h^4}{4!} M$$

where $p_3(x)$ is the 3'rd degree MacLaurin polynomial for f and $R_3(x) = f(x) - p_3(x)$.

Now, if we denote by Simp(f) the approximation to $\int_{-h}^{h} f$ given by Simpson's Rule; namely,

$$\text{Simp}(f) = \frac{h}{3}[f(-h) + 4f(0) + f(h)]$$

then,

$$\left|\int_{-h}^{h} f - \text{Simp}(f)\right|$$

$$= \left|\int_{-h}^{h} (p_3 + R_3) - \text{Simp}(p_3 + R_3)\right|$$

$$= \left|\int_{-h}^{h} R_3 - \text{Simp}(R_3)\right|$$

(as $\text{Simp}(p_3) = \int_{-h}^{h} p_3$, Simpson's Rule is exact for cubics)

$$\leq \left|\int_{-h}^{h} R_3\right| + \left|\text{Simp}(R_3)\right|$$

$$\leq \frac{2h^5}{5!} M + \frac{h}{3}\left[\frac{2h^4}{4!} M\right] \qquad \text{(using } R_3(0) = 0\text{)}$$

$$= Kh^5 \qquad (K \text{ a constant})$$

(Thus, the error in Simpson's Rule is of the *order* of h^5.)

Using Simpson's formula to estimate $\int_a^b f$ with $2n$ strips, requires summing n applications of Simpson's Rule with $h = \frac{b-a}{n}$, thus the maximum error is

$$n \times K\left(\frac{b-a}{n}\right)^5$$

$$= \frac{K(b-a)^5}{n^4},$$

an error of order $\frac{1}{n^4}$.

In other words, doubling the number of strips should reduce the error by a factor of $\frac{1}{2^4}$ so the new error should be about 6% of the original one!

APPENDIX 10.5 Newton's Method

Given a function f we seek values x_∞ at which $f(x_\infty) = 0$; that is, x_∞ is a zero (or in the case when f is a polynomial, a root) of the function.

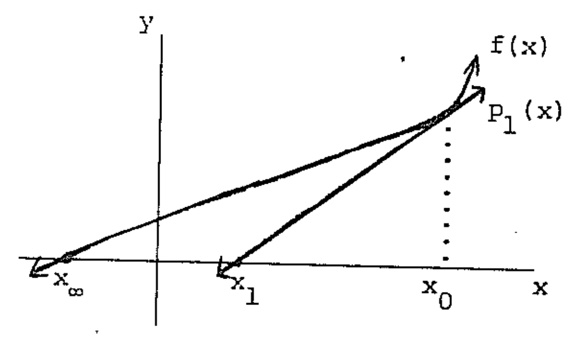
If we start with a guess x_0 for the value of x it seems reasonable to take as a "better" estimate for x_∞ the root x_1 of the 1st degree (Taylor/MacLaurin) polynomial approximation to f at x_0:

$$p_1(x) = f(x_0) + f'(x_0)(x - x_0).$$

That is

$$x_1 = x_0 - f(x_0)/f'(x_0)$$

[Note: x_1 may be interpreted as the point where the tangent to $y = f(x)$ at $(x_0, f(x_0))$ cuts the x-axis.]



Newton's method consists of iterating the procedure; using x_1 as a new initial estimate to obtain

$$x_2 = x_1 - f(x_1)/f'(x_1)$$

as a second approximation for x_∞ etc.

This leads to the general iterative scheme

$$x_{n+1} = x_n - f(x_n)/f'(x_n) \qquad (n = 0,1,2,\ldots.)$$

for successive approximations to x_∞.

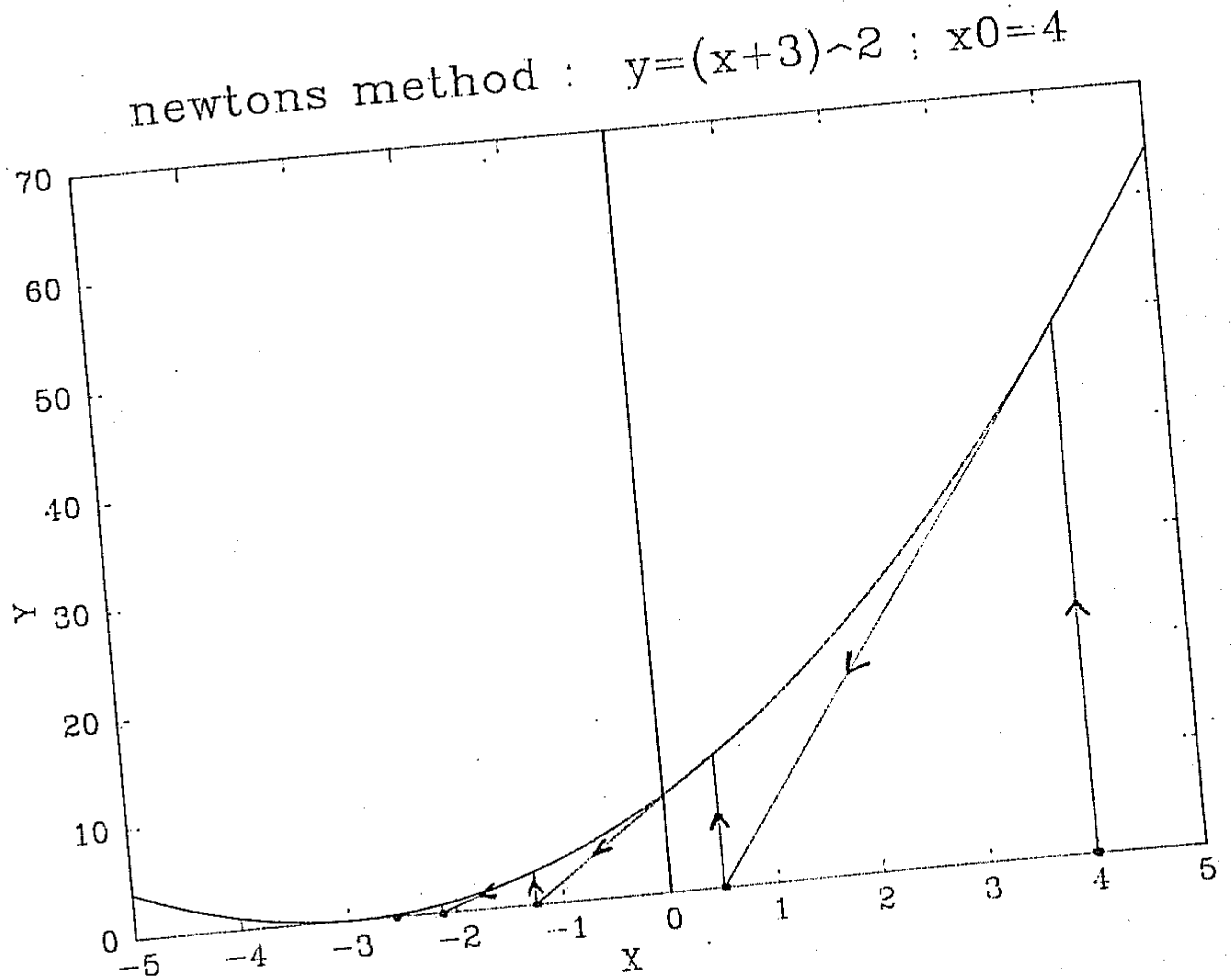
For example: If $f(x) := (x + 3)^2$ (where the actual root is $x = -3$), starting with the initial guess $x_0 = 4$ we have

$$x_1 = x_0 - f(x_0)/f'(x_0) = x_0 - (x_0 + 3)/2 = \frac{1}{2}$$

$$x_2 = \frac{1}{2} - (\frac{1}{2} + 3)/2 = -1\frac{1}{4}$$

$$x_3 = -2\frac{1}{8}$$

etc. (see diagram below)



We are interested in determining under what conditions the successive approximations derived from Newton's method converge to a zero of f.

Our basic result is the following.

<u>THEOREM:</u> *Provided* f' *is continuous at the zero* x_∞ *of* f *and* $f'(x_\infty) \neq 0$ *and provided the initial estimate* x_0 *is sufficiently near to* x_∞ (and this may have to be very near indeed), *then the successive iterates produced by Newton's Method will converge to* x_∞. *Indeed, given any positive number* $\varepsilon < 1$, *for* x_0 *sufficiently near to* x_∞ (the "nearness" required depending upon ε) *we have*

$$|x_n - x_\infty| \le \varepsilon^n |x_0 - x_\infty| \; .$$

Thus provided f satisfies the conditions of the theorem, choosing $\varepsilon = \frac{1}{10}$ and starting with x_1 appropriately near to x_0 we obtain an improvement in accuracy of one decimal place at each iteration.

<u>PROOF</u> (of the theorem)

From the continuity of f' at x_∞, for any x sufficiently near to x_∞ we have that $f'(x) \neq 0$ and so we may form $x - f(x)/f'(x)$. Now for such an x

$$\frac{[x - f(x)/f'(x)] - x_\infty}{x - x_\infty} = 1 - \frac{f(x)}{f'(x)(x - x_\infty)}$$

$$= \frac{1}{f'(x)}\left[f'(x) - \frac{f(x)}{x - x_\infty}\right]$$

$$= \frac{1}{f'(x)}\left[f'(x) - \frac{f(x) - f(x_\infty)}{x - x_\infty}\right]$$

(as $f(x_\infty) = 0$, by assumption)

In the limit as $x \to x_\infty$ this last expression becomes

$$\frac{1}{f'(x_\infty)}\left[f'(x_\infty) - \lim_{x \to x_\infty} \frac{f(x) - f(x_\infty)}{x - x_\infty}\right]$$

$$= \frac{1}{f'(x_\infty)}[f'(x_\infty) - f'(x_\infty)] \quad \text{(by the definition of a derivative)}$$

$$= 0$$

and so we conclude that

$$\lim_{x \to x_\infty} \frac{[x - f(x)/f'(x)] - x_\infty}{x - x_\infty} = 0.$$

In particular then, for all x sufficiently near x_∞ we have

$$\left|\frac{[x - f(x)/f'(x)] - x_\infty}{x - x_\infty}\right| < \varepsilon$$

or

$$|[x - f(x)/f'(x)] - x_\infty| < \varepsilon |x - x_\infty| .$$

Thus provided x_0 is chosen sufficiently near to x we have

$$|x_1 - x_\infty| = |[x_0 - f(x_0)/f'(x_0)] - x_\infty| < \varepsilon|x_0 - x_\infty|$$

while

$$|x_2 - x_\infty| = |[x_1 - f(x_1)/f'(x_1)] - x_\infty| < \varepsilon|x_1 - x_\infty|$$

$$< \varepsilon^2|x_0 - x_\infty| .$$

Similarly,

$$|x_3 - x_\infty| < \varepsilon|x_2 - x_\infty| < \varepsilon^3|x_0 - x_\infty| .$$

Repeating this argument n times establishes the result.

REMARK: If f satisfies the additional assumption that f'' exists and is bounded by M in a neighbourhood of x_∞, then inserting the "error estimate"

$$|f(x) - [f(x_\infty) + f'(x_\infty)(x - x_\infty)]| \le M|x-x_\infty|^2/2$$

into the above argument we obtain for x_0 sufficiently near x_∞ the faster (quadratic) rate of convergence

$$|x_{n+1} - x_\infty| \le K|x_n - x_\infty|^2$$

where K is any constant greater than $M/|f'(x_\infty)|$.

Without assumptions on f such as those of the theorem any of several possibilities can occur:

(a) The scheme may converge, but possibly at a slower rate than suggested by the theorem.

(b) The successive iterates may diverge away from the zero, no matter how near our initial approximation may be. They may, of course, eventually approach some other zero of f.

(c) The successive iterates may "oscillate" about the zero without either approaching or receding.

To illustrate these possibilities it is only necessary to consider

relatively simple functions of the form $y = x^\alpha$ $(\alpha > 0)$.

Each such function has as its only zero $x_\infty = 0$ and, as a simple calculation will show, for this class of functions Newton's Method becomes

$$x_{n+1} = \left(\frac{\alpha-1}{\alpha}\right) x_n .$$

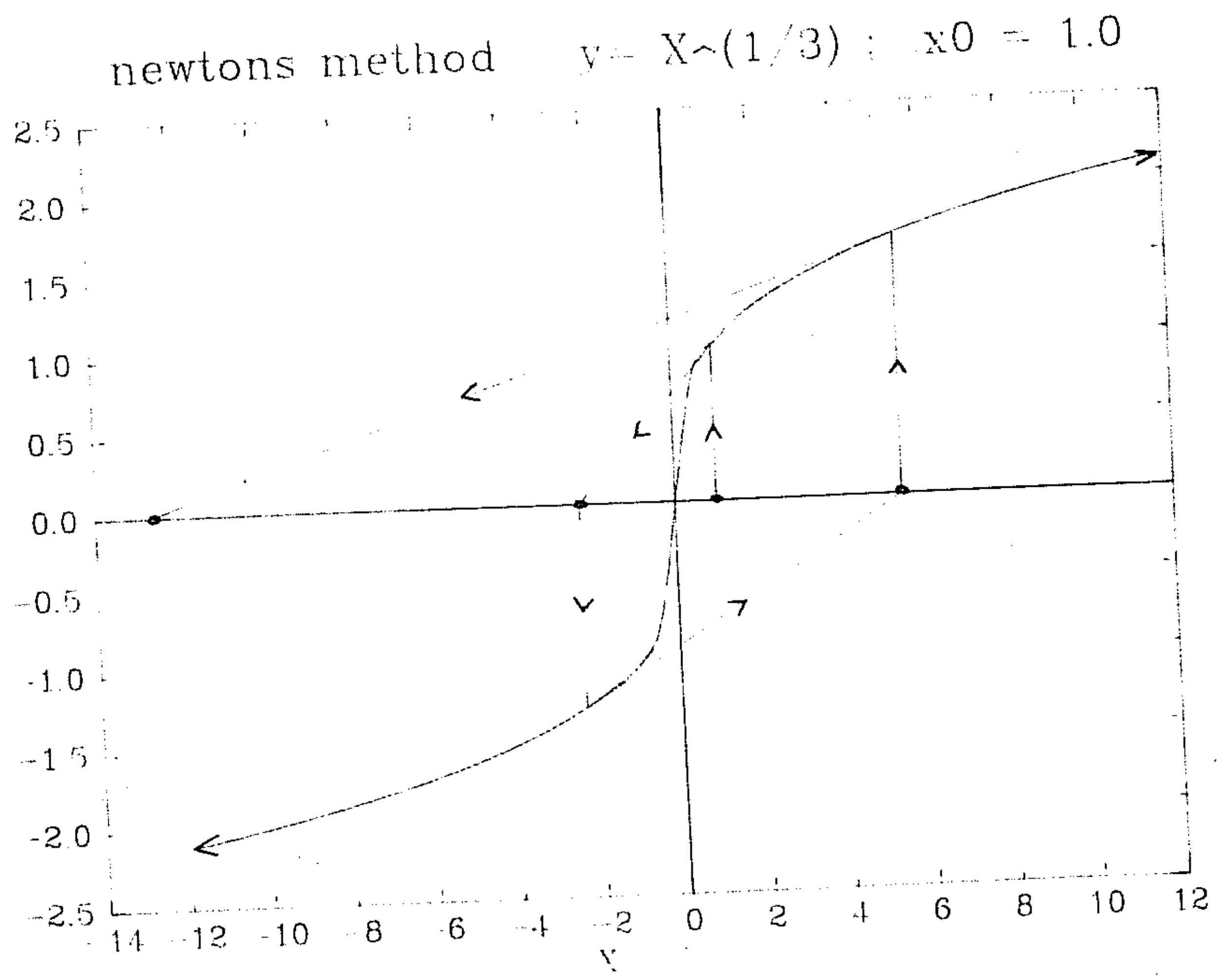
Thus starting with an initial approximation x_0 we have as the n'th iterate

$$x_n = \left(\frac{\alpha-1}{\alpha}\right)^n x_0$$

(a) For α a positive integer, $\frac{\alpha-1}{\alpha}$ is between 0 and 1, and so the successive iterates will converge to 0. The error after each iteration is $|\frac{\alpha-1}{\alpha}|$ that for the previous estimate. By choosing α large we may make $\frac{\alpha-1}{\alpha}$ as near to 1 as we please, thus decreasing the rate at which x_n converges to 0. (Note $|\frac{\alpha-1}{\alpha}|$ is fixed by the function and cannot be varied as the ε in the above theorem.) For example; no matter what the starting value, it takes 4 successive iteration to improve the accuracy of the estimate by one decimal place when $\alpha = 2$, while for $\alpha = 10$ it requires 23 iterations.

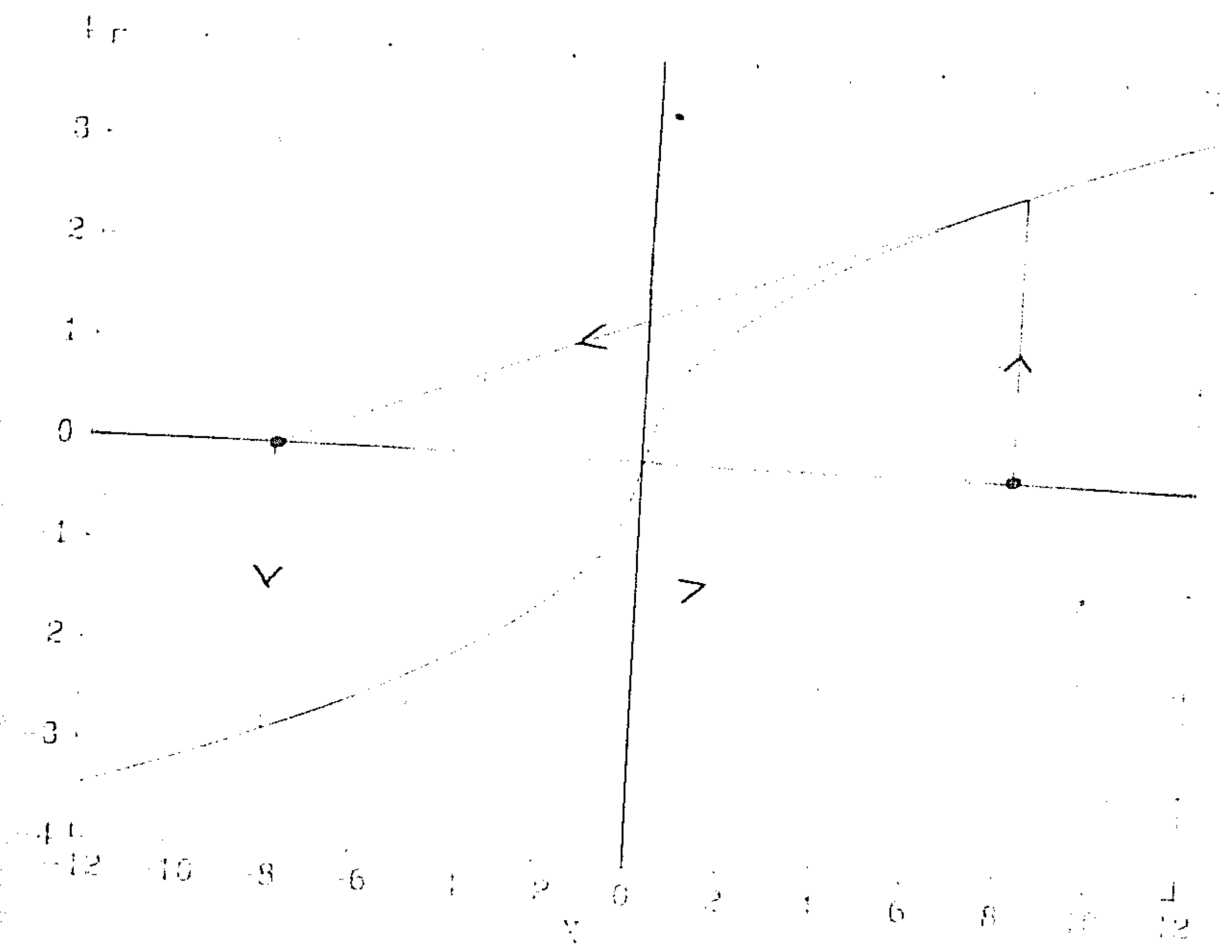
(b) Choosing $\alpha = \frac{1}{3}$ we have $\frac{\alpha-1}{\alpha} = -2$ and so the sequence of iterates, $x_n = (-2)^n x_0$ fails to converge to 0. For example with $x_0 = 1$ we have

$$x_1 = -2, \quad x_2 = 4, \quad x_3 = -8, \quad x_4 = 16, \quad x_5 = -32 \quad \text{etc.}$$



(c) To obtain a function which exhibits oscillatory behaviour we choose $\alpha = \frac{1}{2}$ and extend the function $y = \sqrt{x}$ as an odd function to the whole of the real line according to the formula

$$f(x) = \begin{cases} \sqrt{x} & \text{for } x \geq 0 \\ -\sqrt{-x} & \text{for } x \leq 0 \end{cases}$$



In this case $\frac{\alpha-1}{\alpha} = -1$ and so the iterates are alternatively $+x_0$ and $-x_0$. For example with $x_0 = 1$ we have as successive "approximations" to the zero 0; 1, -1, 1, -1, 1, -1, 1, -1, etc.

Newton's Method works well for a great many functions, however, as the above examples illustrate it is fallible and care must be exercised when applying it. For more complicated functions the method can "misbehave" in more diverse ways than those suggested above; for example, the sequence of iterates may first appear to approach a zero before eventually diverging.

A major problem with Newton's Method is the difficulty in obtaining useful error estimates. The "half-interval method" is a useful, though often slower, alternative procedure which overcomes this deficiency.

EXERCISES:

(1) By successive differentiation, find the 3rd degree MacLaurin polynomial for the following functions:

(a) $f(x) = \tan x$

(b) $f(x) = \tan^{-1} x$

(2) Determine the n'th degree MacLaurin polynomial for $f(x) = \cos x$.

(3) (a) Show that

$$\left| \left[x - \frac{x^3}{3!} + \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right] - \sin x \right| \le |x|^{2n+2}/(2n+2)!$$

(b) How many terms of its MacLaurin expansion will be sufficient to obtain a value for sin (0.5) which is accurate to five decimal places?

[Hint: To obtain the first n for which

$$\frac{1}{2^{2n+2}\,(2n+2)!} < 0.000005$$

I suggest you evaluate the expression for $n = 1, 2, \dots$ using a calculator.]

(c) Deduce that for each real x the series $\sum_{n=0}^{\infty} (-1)^n x^{2n+1}/(2n+1)!$ converges to $\sin x$.

(4) (a) Determine a number of terms in the MacLaurin expansion of $f(x) = \ln(1 - x)$ which will be sufficient to obtain a value for $\ln(1.5)$ accuarate to at least 3 decimal places.

(b) Compute approximations to $\ln(1\frac{1}{2})$ using the first 2 terms, 3 terms, ... up to the number of terms determined in (a) of its MacLaurin expansion.

(c) *(Improving the rate of convergence)*

By substituting $-x$ for x in the series for $\ln(1 - x)$ write down the series for $\ln(1 + x)$.

By combining these two series write down a series for

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1 + x) - \ln(1 - x).$$

Use the first two non-zero terms of this last series with $x = \frac{1}{5}$ to estimate the value of $\ln(1.5)$. Compare the result with those of (b).

(5) (i) Since $p_n(x) = 1 + x + x^2 + \dots + x^n$, the first n terms of the MacLaurin expansion for $\frac{1}{1-x}$, is a geometric progression, deduce that

$$p_n(x) = \frac{1 - x^{n+1}}{1 - x} \qquad (x \neq 1)$$

and hence that the remainder

$$\frac{1}{1-x} - p_n(x) = \frac{x^{n+1}}{1-x}.$$

Replacing x by $-x^2$ this gives:

$$\frac{1}{1+x^2} - [1 - x^2 + x^4 - x^6 + \dots + (-1)^n x^{2n}] = \frac{(-1)^{n+1} x^{2n+2}}{1+x^2}.$$

Noting that $\tan^{-1} x = \int_0^x \frac{1}{1+t^2}\,dt$

conclude that

$$\tan^{-1} x - \left[x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots + \frac{(-1)^n x^{2n+1}}{2n+1}\right]$$
$$= (-1)^{n+1} \int_0^x \frac{t^{2n+2}}{1+t^2}\,dt \qquad \dots\dots\dots\dots (***)$$

(ii) From $1 \le 1 + t^2 \le 2$ for $0 \le t \le 1$ deduce that for $0 \le x \le 1$

$$\frac{x^{2n+3}}{2(2n+3)} \le \left| \tan^{-1} x - \left[x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots + \frac{(-1)^n x^{2n+1}}{2n+1}\right] \right| \le \frac{x^{2n+3}}{2n+3}$$

(iii) Using (ii):

(a) Deduce that the series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots + (-1)^n/(2n+1) + \dots$$

converges to $\frac{\pi}{4}$,

(b) Determine the least number of terms necessary to obtain

(iv) *(Improving the rate of convergence)*

Using the identity $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$ show that

$$\frac{\pi}{4} = \tan^{-1} 1 = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}.$$

Using the result in (ii) to approximate both $\tan^{-1}(\frac{1}{5})$ and $\tan^{-1}(\frac{1}{239})$ find π accurate to 4 decimal places.

[All mathematics students should at some stage determine the first few digits of π for themselves.]

(6) Replacing x by $-t^2$ in

$$\left|e^x - [1 + x + x^2/2! + \ldots + x^n/n!]\right| \le \frac{|x|^{n+1}}{(n+1)!} \cdot e$$

show that

$$\left|\int_0^x e^{-t^2}\,dt - \left[x - \frac{x^3}{3} + \frac{x^5}{5.2!} - \frac{x^7}{7.3!} + \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)n!}\right]\right| \le \frac{|x|^{2n+3}}{(2n+3)(n+1)!} \cdot e.$$

Hence, determine how many terms of the sum $1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \ldots + (-1)^n/(2n+1)n!$ would be sufficient to obtain a value for $\int_0^1 e^{-t^2}dt$ which is accurate to 4 decimal places.

[Remark: The integral $\int_0^x e^{-t^2}\,dt$ is closely related to the normal distribution which is of fundamental importance in statistics. It is also a fact of life that the integral cannot be expressed finitely in terms of the elementary functions.]

(7) From the results of exercise 8.1(5), deduce that the second degree MacLaurin polynomial for $f(x) = e^{-1/x^2}$ is

$$p_2(x) \equiv 0.$$

In fact one can show that $f^{(n)}(0) = 0$ for all n and so the MacLaurin expansion for $f(x)$ is identically zero. This provides a simple example of a function for which the MacLaurin expansion exists and converges, but not to the function value.

(8) (i) Determine the first three non-zero terms in the Taylor expansion of $y = \sin x$ about the point $x = \frac{\pi}{6}$.

(ii) Use the result of (i) to obtain an approximate value for $\sin(33^\circ)$.

(9) Starting with an initial approximation of $x_0 = 1$ use Newton's method to obtain 4 successive approximations to the value of x for which $e^{-x} = x$.

(10) By means of a sketch, or otherwise, deduce that the function $f(x) = \tan x - \frac{1}{2}x$ has a zero x_∞ in the interval $\pi < x < 3\pi/2$. Show that the convergence of Newton's method to x_∞ is ensured provided our initial guess is sufficiently near to x_∞.

For the intial guesses 4.3, 3.58 and 3.5 Newton's method produces the following iterates.

Initial guess	4.3	3.58	3.5
Iterates	4.27627	5.41544	5.64803
	4.27478	7.46999	9.06152
	4.27478	7.66016	16.68127
			19.26463
			32.50537
			35.94775
			36.35737
			37.54639
			73.68813

Briefly explain how these results arise (a sketch of the function $y = f(x)$ should help).

(11) By applying Newton's method to the function $f(x) = x^2 - \alpha$ (α a positive number) derive the iterative scheme (known to the ancient Babylonians) $x_{n+1} = \frac{1}{2}(x_n + \frac{\alpha}{x_n})$ for approximating $\sqrt{\alpha}$.

[Show that this scheme converges for all values $\alpha > 0$.]

CHAPTER 11

COMPLEX NUMBERS

11.1 The System of Complex Numbers

In classical Greek mathematics *numbers* were either whole numbers or their ratios (that is to say, "fractions" or rational numbers). The extension to include "irrationals" such as $\sqrt{2}$ or π which could be manipulated by the same rules as fractions was a major advance (due in part to the gradual infusion of arabic mathematics into western culture and in part to the slow erosion of Greek ideals about rigor). For example, it allowed equations such as $x^2 - 2 = 0$ to be systematically solved "algebraically" rather than "geometrically" as the Greeks had done.

In many situations a useful further extension is to complex numbers of the form $z = a + bi$, where a and b are real numbers, which satisfy the same rules (for addition and multiplication) as the *field* of real numbers with $i^2 = -1$. That is,

<u>addition</u> : $(a + bi) + (c + di) := (a + c) + (b + d)i$

<u>multiplication</u>: $(a + bi)(c + di) := (ac - bd) + (ad + bc)i$

Using complex numbers it is possible to completely factorize any quadratic (indeed any polynomial, as we shall see in the appendix); for example

$$x^2 + a^2 = (x + ai)(x - ai).$$

Historically complex numbers were first introduced in the 16th century as a tool for finding the (real) roots of arbitrary cubic equations. It was not however until the late eighteenth century that complex numbers were comfortably accepted into mathematics.

Terminology and Geometric Representation

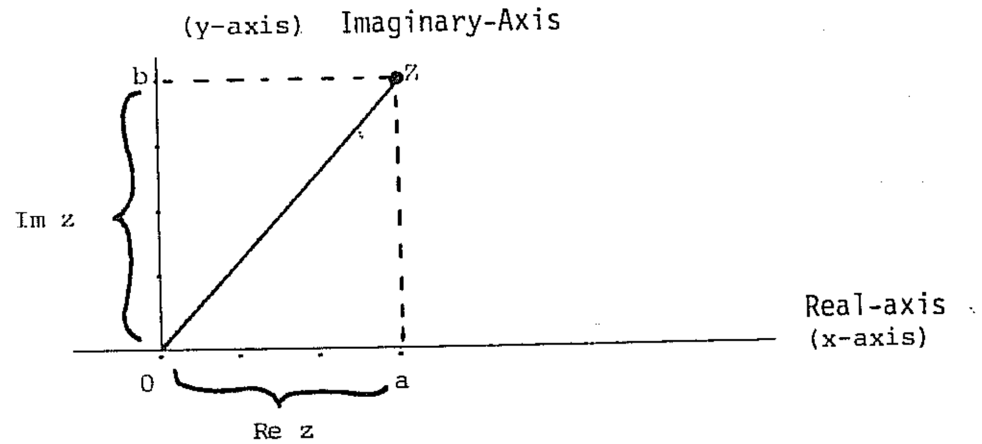
For the complex number $z = a + bi$ we will refer to a and b respectively as the real and imaginary parts of z and denote them by

Re z and Im z. For example; if $z = 3 + 4i$, then

$$\text{Re } z = 3 \qquad \text{and} \qquad \text{Im } z = 4.$$

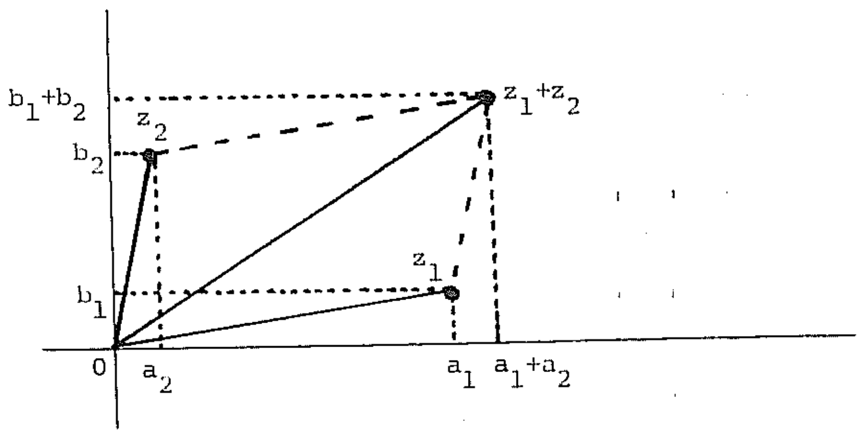
Note: We will not distinguish between the real number a and the complex number $a + 0i$. Note also, that the imaginary part of a complex number is the *real* number b, the "coefficient" of i; i itself is not included. We will say that the complex number $bi = 0 + bi$ is *purely imaginary*.

It is frequently useful to regard the complex number $z = a + bi$ as representing the point in the Cartesian plane with x-coordinate a, and y-coordinate b.



When points are interpreted in this way we will refer to the plane as the complex (or Argand) plane, the x-axis as the real-axis and the y-axis as the imaginary-axis.

Note that in the complex plane the addition of two complex numbers $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$ corresponds to "vector-addition".



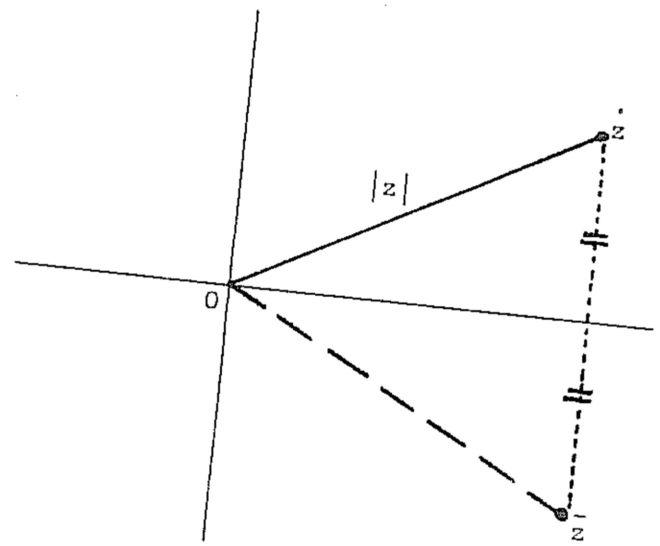
Given the complex number $z = a + bi$ we define its

$$\text{conjugate} \quad \bar{z} := a - bi$$

and

$$\text{modulus} \quad |z| := \sqrt{z\,\bar{z}} = \sqrt{a^2 + b^2}$$

Note: geometrically $\bar{z}$ is the reflection of the point z in the real axis, while $|z|$ is the distance of the point z from the origin



The Algebra of Complex Numbers

You should verify (see exercise 1), and familiarize yourself with, each of the following identities. They can greatly simplify computations with complex numbers by avoiding an epidemic of real and imaginary parts.

For z, z_1 and z_2 complex numbers we have:

(i) $z_1 + z_2 = z_2 + z_1$ *(complex addition is commutative)*

(ii) $z_1 z_2 = z_2 z_1$ *(complex multiplication is commutative)*

(iii) $z(z_1 + z_2) = zz_1 + zz_2$ *(distributive law)*

(iv) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(v) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

(vi) $|z_1 z_2| = |z_1||z_2|$

Also useful are the identities

(vii) $\text{Re } z = \frac{1}{2}(\bar{z} + z)$

(viii) $\text{Im } z = \frac{i}{2}(\bar{z} - z)$

and the inequalities

(ix) $|z_1 + z_2| \le |z_1| + |z_2|$ (*triangle inequality*)

(x) $|\text{Re } z| \le |z|$

(xi) $|\text{Im } z| \le |z|$

Given $z = a + bi$ the complex number $-z := -a + (-b)i$ and usually written $-a - bi$ is an *additive inverse* for z, in the sense that $z + (-z) = 0$.

We will write $z_1 - z_2$ for $z_1 + (-z_2)$.

If z is a non-zero complex number then from

$$\frac{z\bar{z}}{|z|^2} = 1$$

we see that the complex number

$$z^{-1} := \frac{\bar{z}}{|z|^2}$$

is a *multiplicative inverse* for z, in that $z^{-1}z = 1$.

We will take z_1/z_2 to mean $z_1 z_2^{-1} = \dfrac{z_1\bar{z}_2}{|z_2|^2}$.

For example: If $z_1 = 1 - i$, $z_2 = 3 + 4i$ then,

$$z_1 - z_2 = -2 - 5i = -(2 + 5i)$$

while

$$z_1/z_2 = \frac{1 - i}{3 + 4i} = (1 - i)\frac{3 - 4i}{3^2 + 4^2} = -\frac{1}{25}(1 + 7i)$$

(Note the close similarity between the last calculation and the procedure for rationalizing a denominator).

The Polar Form for Complex Numbers

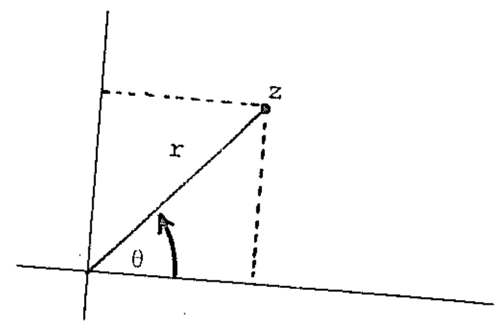
If (r,θ) are polar coordinates for the point $z = a + bi$ in the complex plane; that is

$$r = \sqrt{a^2 + b^2} = |z|$$

and θ is such that

$$\cos\theta = a/r = \text{Re } z/|z|$$

$$\sin\theta = b/r = \text{Im } z/|z|$$



then we have the representation

$$z = r(\cos\theta + i\sin\theta)$$

[Sometimes it is convenient to abbreviate this to $z = r \text{ cis } \theta$.]

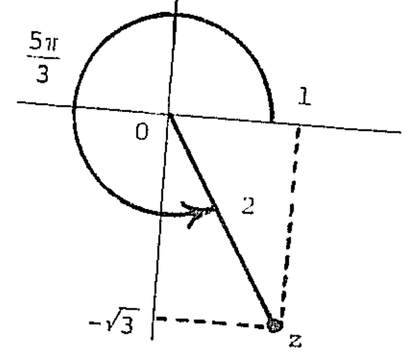
We refer to θ as an **argument** for the complex number z. Note that, for $n=1,2,3,\dots$ the angles $\theta \pm 2n\pi$ are also arguments for z, and we write

$$\arg z = \theta \pm 2n\pi \quad (\text{for any of the values } n=0,1,2,\dots).$$

When we wish to refer to the unique value of θ satisfying $0 \le \theta < 2\pi$ we will write $\text{Arg } z$ (note the capital).

For example; if $z = 1 - \sqrt{3}i$, then $r = |z| = 2$ and $\text{Arg } z = \frac{5\pi}{3}$, while $\arg z$ is any of the values

$$\dots, -\frac{7\pi}{3}, -\frac{\pi}{3}, \frac{5\pi}{3}, \frac{11\pi}{3}, \dots$$



Note that: the conjugate of z is characterized by

$$|\bar{z}| = |z| \quad \text{and} \quad \arg\bar{z} = -\text{Arg } z$$

and the inverse by

$$|z^{-1}| = |z|^{-1}, \quad \arg z^{-1} = -\text{Arg } z$$

Geometric Interpretation of Complex Multiplications

If the polar forms for two complex numbers are

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$

$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

then we have

$$z_1z_2 = r_1r_2\,[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \cos\theta_2\sin\theta_1)]$$

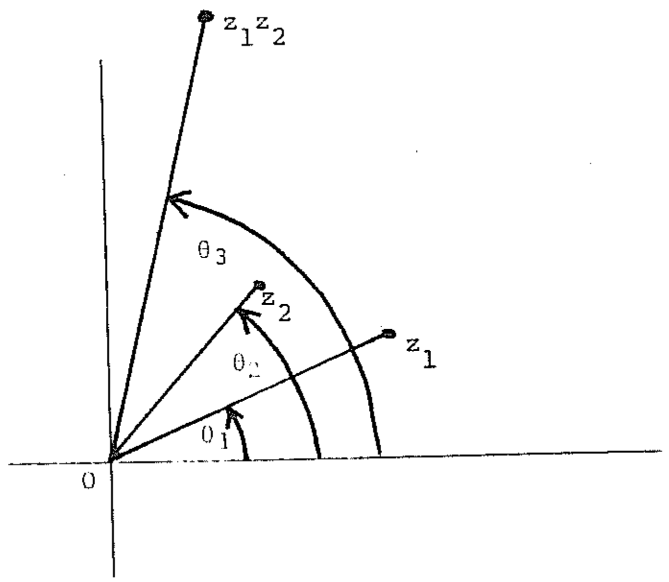
$$= r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2).]$$

Thus, for the product of two complex numbers we have:

$$|z_1z_2| = |z_1||z_2|$$

and

$$\arg(z_1z_2) = \arg(z_1) + \arg(z_2)$$



That is, when the complex number z_2 is multiplied by z_1, its modulus is scaled be a factor of $|z_1|$ and its "direction" is rotated through an angle $\theta_1 = \arg(z_1)$. Note in particular that multiplication by i effects a rotation through $\frac{\pi}{2}$.

EXERCISES:

(1) Let $z = a + bi$, $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$ be complex numbers. Verify each of the following:

(i) $z_1 + z_2 = z_2 + z_1$ (ii) $z_1z_2 = z_2z_1$

(iii) $z(z_1 + z_2) = zz_1 + zz_2$ (iv) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(v) $\overline{z_1z_2} = \bar{z}_1\,\bar{z}_2$ (vi) $|z_1z_2| = |z_1||z_2|$

(vii) $|z_1/z_2| = |z_1|/|z_2|$ (viii) $\arg(z_1/z_2) = \arg z_1 - \arg z_2$

(ix) $\text{Re } z = \frac{1}{2}(\bar{z} + z)$ (x) $\text{Im } z = \frac{i}{2}(\bar{z} - z)$

(2) Express in the form $a + bi$

(a) $\dfrac{(2 - 3i)}{1 + 2i}$ (b) $\dfrac{(2 + i)(2 + 3i)}{3 - 4i}$

(c) the solution z of the equation $\dfrac{1}{1 + i} + \dfrac{1}{z} = \dfrac{1}{1 - i}$

(3) For the complex numbers $u = \sqrt{3} - i$, $v = 1 + i\sqrt{3}$ find:

(i) $\bar{u}$, the conjugate of u; (ii) the modulus of v;

(iii) $\dfrac{u}{v}$; (iv) $\left|\dfrac{v^3}{u^2}\right|$;

(4) (i) Plot, in the complex plane, the numbers 1+i, -1+i, 1-i. Find their moduli and arguments. Find also the modulus and an argument of $\dfrac{1 + i}{1 - i}$.

(ii) Find the modulus and Argument of

$$\frac{5 - i}{2 - 3i}, \quad 1 - i\sqrt{3}$$

(iii) Prove, by induction, that $\arg(z^n) = n\arg(z)$ [n a positive integer]

(iv) Find the modulus and Argument of $z = (1 - \sqrt{3}\,i)^{23}$. Hence find Re z and Im z.

(5) For $z = \frac{\sqrt{3}}{2} + \frac{1}{2}i$, Find and plot, in the complex plane,

z, z^2, z^3, z^4, z^5

(6) Prove that if a and b are complex numbers then

$$|a + b|^2 + |a - b|^2 = 2(|a|^2 + |b|^2).$$

and state a geometrical theorem corresponding to this equality.

(7) Prove the inequalities

(i) $|z_1 + z_2| \leq |z_1| + |z_2|$

and

(ii) $\left||z_1| - |z_2|\right| \leq |z_1 - z_2|.$

Interpret these geometrically in terms of the side lengths of a triangle.

(8) Given that $z = \rho(\cos\theta + i\sin\theta)$ and $a = r(\cos\alpha + i\sin\alpha)$ are two complex numbers, find the value of $|z - a|^2$ in terms of the real quantities r, ρ, θ, α. Deduce that

$$|1 - \bar{a}z|^2 - |z - a|^2 = (1 - |z|^2)(1 - |a|^2).$$

(9) Prove the *Appolonius identity*:

$$|z - x|^2 + |z - y|^2 = \frac{1}{2}|x - y|^2 + 2\left|z - \frac{1}{2}(x + y)\right|^2,$$

for any three complex numbers z, x and y.

11.2 Curves in the Complex Plane

By imposing restrictions on Re z, Im z, $|z|$ or arg z an equation ("inequation") in the *complex variable* $z = x + iy$ may lead to a relation between the real part x and the imaginary part y of z and so define a curve (region) in the complex plane.

EXAMPLES:

(1) For fixed complex number $z_0 = a + bi$ and positive number r, the equation $|z - z_0| = r$ or equivalently $(z - z_0)(\bar{z} - \bar{z}_0) = r^2$ is satisfied if and only if

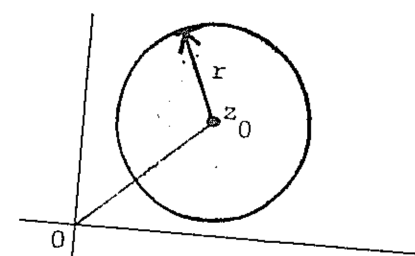
$$(x - a)^2 + (y - b)^2 = r^2$$

and so corresponds to the circle of radius r centred at z_0.

Similarly the *inequation*

$|z - z_0| \leq r$

corresponds to the "disk" of radius r centred at z_0.



Note that geometrically, $|z - z_0|$ represents the distance between z and z_0.

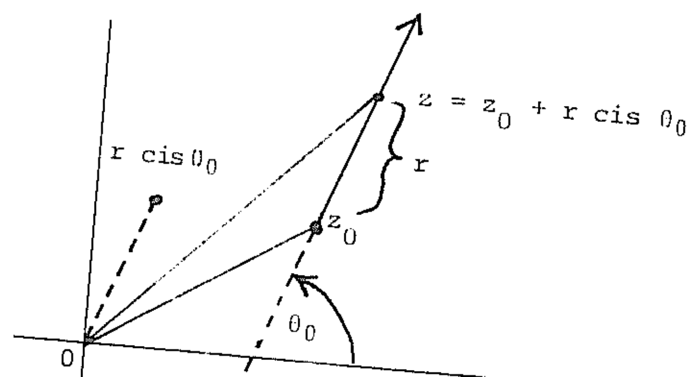
(2) For fixed complex number z_0 and real number θ_0 the equation

$$\arg(z - z_0) = \theta_0$$

is satisfied if and only if

$$z - z_0 = r\,\text{cis}\,\theta_0 \quad \text{or} \quad z = z_0 + r\,\text{cis}\,\theta_0,$$

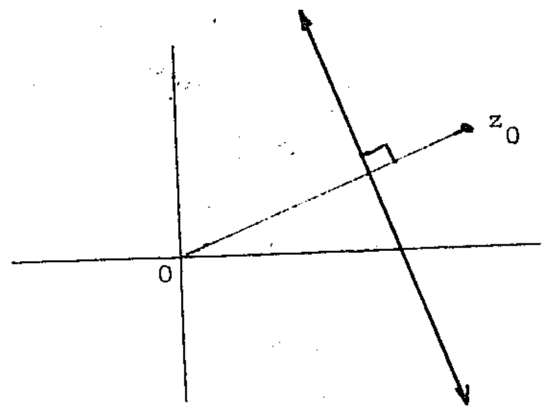
where r is any positive real number, and so corresponds to a half-line from z_0:



(3) For fixed complex number $z_0 = a + bi$ and real number c, the equation

$$\mathrm{Re}(z_0 z) = c$$

is satisfied if and only if $ax - by = c$ and so corresponds to a straight line, similarly the inequation $\mathrm{Re}(z_0 z) \le c$ defines a half-plane



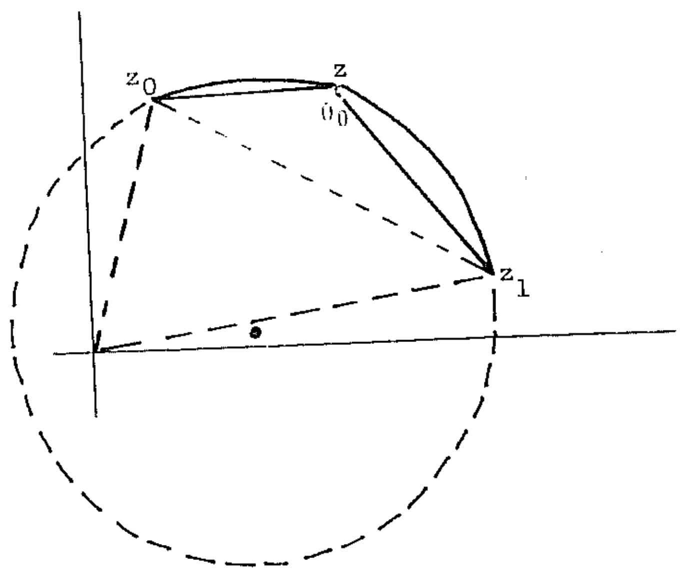
(4) For fixed complex numbers z_0 and z_1 and real number θ_0 the equation

$$\arg \frac{z - z_0}{z - z_1} = \theta_0$$

or

$$\arg (z - z_0) - \arg (z - z_1) = \theta_0$$

implies the lines from z_0 and z_1 to z meet at the fixed angle θ_0, and so by a well known theorem of geometry z lies either on the major or minor arc of a circle through z_0 and z_1:

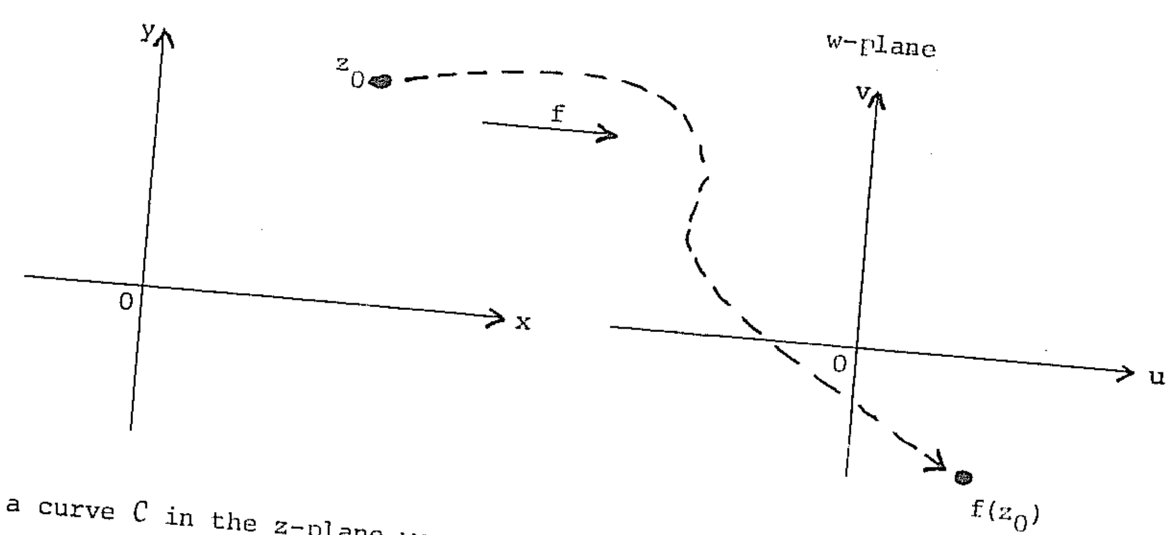


Another useful idea is that of a *complex transformation.* If $z = x + i$ and $w = u + iv$ are complex variables we may think of an expression

$$w = f(z)$$

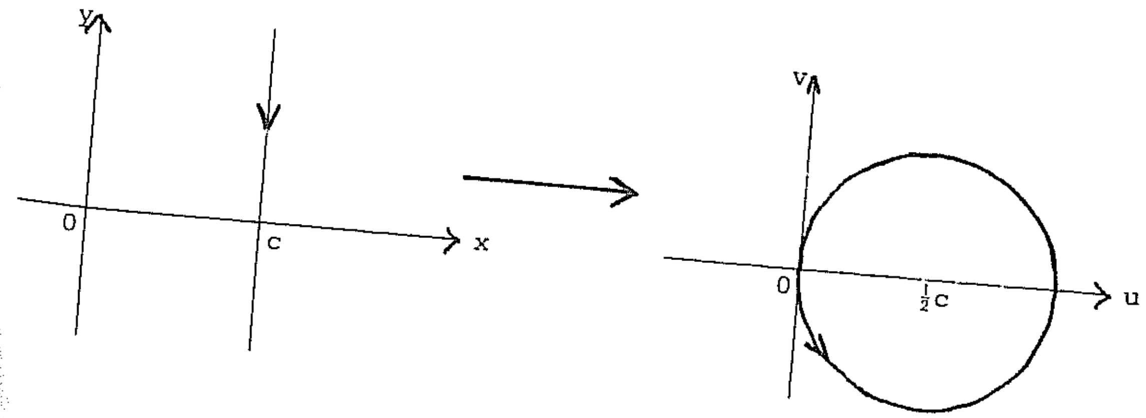
as transforming a point z_0 in the "z-plane" to a point $w_0 = f(z_0)$ in the "w-plane".

For example: If $w = f(z) := z^{-1}$, then the point $z_0 = 1 + i$ is transformed to the point $w_0 = \frac{1}{1 + i} = \frac{1}{2}(1 - i)$, pictorially:



For a curve C in the z-plane we can ask: What is the image in the w-plane of all the points on C under the transformation f?

For example: Under the *inversion* $w = z^{-1}$ the line $\mathrm{Re}\, z = c$ transforms to the circle, centre $(\frac{1}{2c}, 0)$ and radius $1/2|c|$



To see this we may proceed as follows:

$$c = \text{Re } z = \frac{1}{2}(\bar{z} + z)$$

$$= \frac{1}{2}(1/\bar{w} + 1/w), \quad \text{as} \quad z = 1/w$$

$$= \frac{1}{2}\left(\frac{w + \bar{w}}{\bar{w}\, w}\right)$$

So $w\bar{w} - \frac{1}{2c}(w + \bar{w}) = 0$

and "completing the square"

$$(w - \frac{1}{2c})(\bar{w} - \frac{1}{2c}) = \frac{1}{4c^2}$$

or

$$(w - \frac{1}{2c})\overline{(w - \frac{1}{2}c)} = \frac{1}{4c^2}$$

That is,

$$\left|w - \frac{1}{2c}\right|^2 = \frac{1}{4c^2}$$

or

$$\left|w - \frac{1}{2c}\right| = \frac{1}{2|c|} .$$

Thus w is constrained to lie on the specified circle [see Example (1) above].

The notion of complex transformations led to the development of a theory of complex valued functions of a complex variable - one of the deepest and richest branches of modern mathematics.

EXERCISES:

(1) Sketch the curves, or regions, described by the following.

(i) $|z + 3i|^2 - |z - 3i|^2 = 12$

(ii) $|z + i|^2 + |z - i|^2 = 4$

(iii) $\text{Im}[(3 + 2i)z] \geq 4$

(iv) $|(2z + 1)/(iz + 1)| = 2$

(v) $(z - i)/(z - 1)$ is purely imaginary.

(2) For distinct complex numbers z_0 and z_1 describe the curve specified by

$$|z - z_0| = |z - z_1| .$$

Show that for a z satisfying the above relationship we have

$$\arg[(2z - z_0 - z_1)/(z_0 - z_1)] = \pm \pi/2$$

(3) (i) Find the image of the line $\text{Im } z = k$ under the transformation $w = 1/z$. Show that this curve intersects the image of $\text{Re } z = c$ under $w = 1/z$ at right-angles.

Sketch these curves for several values of c and k.

(ii) Find the image of the circle $|z - 3| = 3$ under the transformation

$$w = 1/(z - 3) + 17/3.$$

11.3 Complex Sequences and Series

Let $z_n = a_n + b_n i$ (a_n, b_n real numbers) be a sequence of complex numbers. As with real sequences we will say z_n **converges to** $z = a + bi$ if the sequence of moduli $|z_n - z|$ converges to zero; that is, if given any $\varepsilon > 0$ we can find a number n_0 such that $n \geq n_0$ implies $|z_n - z| < \varepsilon$.

From the inequalities

$$|a_n - a|, |b_n - b| \leq |z_n - z| = \sqrt{(a_n - a)^2 + (b_n - b)^2}$$

we have:

THEOREM 11.3.1 *The complex sequence* z_n *converges to* $z = a + bi$ *if and only if the two real sequences* $a_n = \text{Re } z_n$ *and* $b_n = \text{Im } z_n$ *converge respectively to* a *and* b.

In particular then, since the real and imaginary parts of the partial sums of the complex series $z_0 + z_1 + z_2 + \ldots + z_n + \ldots$ are the partial sums of the sequences of real and imaginary parts we have:

COROLLARY 11.3.2 *The complex series*

$$(a_0 + b_0 i) + (a_1 + b_1 i) + (a_2 + b_2 i) + \ldots + (a_n + b_n i) + \ldots .$$

converges to $a + bi$ *if and only if* $\sum_{n=0}^{\infty} a_n$ converges to a *and* $\sum_{n=0}^{\infty} b_n$ converges to b.

As a further corollary we note that, since $|a_n|$, $|b_n| \le |a_n + b_n i|$, both the series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are absolutely convergent if $\sum_{n=0}^{\infty} |z_n|$ converges, and so we have

COROLLARY 11.3.3 *The complex series* $\sum_{n=0}^{\infty} z_n$ *converges (absolutely) if the* real *series* $\sum_{n=0}^{\infty} |z_n|$ *is convergent.*

In particular then, the **complex power series** $\sum_{n=0}^{\infty} a_n z^n$ converges if the real power series $\sum_{n=0}^{\infty} |a_n||z|^n$ does. The ratio test therefore assures us that $\sum_{n=0}^{\infty} a_n z^n$ will converge for all z with $|z| < \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$.

The Complex Exponential

From the above discussion we see that the complex power series $\sum_{n=0}^{\infty} z^n/n!$ converges for all z. We define the **complex exponential** e^z or $\exp(z)$ to be the sum of this series; that is

$$e^z := \sum_{n=0}^{\infty} z^n/n! \qquad \text{(for all complex numbers } z).$$

For real numbers r_1 and r_2 we know $e^{r_1} e^{r_2} = e^{r_1+r_2}$ and so by the uniqueness of power series representations (§10.3, p.208) we must have that the product

$$(1 + r_1 + r_1^2/2! + \ldots + r_1^n/n! + \ldots .)(1 + r_2 + r_2^2/2! + \ldots + r^n/n! + \ldots .)$$

upon term by term multiplication simplifies to

$$1 + (r_1 + r_2) + (r_1 + r_2)^2/2! + \ldots + (r_1 + r_2)^n/n! + \ldots .$$

[Alternatively, one could verify this by direct computation - see exercises.]

Since complex addition and multiplication satisfy the same rules as real addition and multiplication we must therfore also have the above equality when r_1 and r_2 are complex numbers, and so

$$e^{z_1+z_2} = e^{z_1}e^{z_2}, \quad \text{for any two complex numbers } z_1, z_2.$$

In particular, substituting $z = x + iy$ $(x, y$ real) we have

$$\begin{aligned} e^{x+iy} &= e^x e^{iy} \\ &= e^x(1 + iy + (iy)^2/2! + (iy)^3/3! + \ldots .) \\ &= e^x([1 - y^2/2! + y^4/4! - \ldots .] \\ &\qquad + i[y - y^3/3! + y^5/5! - \ldots .]) \end{aligned}$$

which, using our results on Maclaurin expansions, we recognise as

$$\boxed{e^{x+iy} = e^x(\cos y + i \sin y)}$$

Using this the polar form of a complex number can be re-expressed as

$$r(\cos\theta + i \sin\theta) = re^{i\theta}$$

Example; for the complex number $z = 1 - \sqrt{3}\,i$ we have

$$z = 2(\cos\frac{5\pi}{3} + i \sin\frac{5\pi}{3}) = 2e^{i5\pi/3}.$$

This last exponential form is extremely useful.

Demoivre's Theorem:

For any real number s we have

$$[r(\cos\theta + i\sin\theta)]^s = [r\,e^{i\theta}]^s = r^s e^{is\theta},$$

so $$[r(\cos\theta + i\sin\theta)]^s = r^s(\cos s\theta + i\sin s\theta)$$

For example;

$$(1-\sqrt{3}\,i)^8 = [2(\cos\tfrac{5\pi}{3} + i\sin\tfrac{5\pi}{3})]^8$$
$$= 2^8\left(\cos\tfrac{40\pi}{3} + i\sin\tfrac{40\pi}{3}\right)$$
$$= 256(\cos\tfrac{4\pi}{3} + i\sin\tfrac{4\pi}{3})$$

(removing multiples of 2π)

$$= 256(-\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i)$$
$$= -128(1+\sqrt{3}i)$$

As further applications of this last identity we present the following.

Trigonometric Identities

Using Demoivre's Theorem the multiple-angle formulae for sine and cosine, as well as many other trigonometric identities, are readily deduced. We illustrate this with the following example:

Express $\cos 6\theta$ as a polynomial in $\cos\theta$.

We have

$$\cos 6\theta = \mathrm{Re}(\cos 6\theta + i\sin 6\theta)$$
$$= \mathrm{Re}(\cos\theta + i\sin\theta)^6 \quad \text{(Demoivre's Threorem)}$$
$$= \mathrm{Re}\sum_{n=0}^{6}\binom{6}{n}\cos^{6-n}\theta\,(i\sin\theta)^n \quad \text{(Binomial Theorem)}$$
$$= \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta$$

(the odd powers of $i\sin\theta$ are imaginary)

$$= \cos^6\theta - 15\cos^4\theta(1-\cos^2\theta) + 15\cos^2\theta(1-\cos^2\theta)^2 - (1-\cos^2\theta)^3$$
$$= 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1 .$$

An expression for $\sin 6\theta$ could be found similarly by considering Imaginary parts.

[You might like to compare the work involved by calculating the above expression for $\cos 6\theta$ using the multiple angle formula $\cos 2x = 2\cos^2 x - 1$.]

The n n'th roots of a complex number

The same complex number z is represented by any of the polar forms

$$r(\cos[\theta + 2k\pi] + i\sin[\theta + 2k\pi]), \qquad k = 0, \pm1, \pm2, \ldots.$$

Applying Demoivre's Theorem with $s = \frac{1}{n}$ gives

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos\left[\frac{\theta + 2k\pi}{n}\right] + i\sin\left[\frac{\theta + 2k\pi}{n}\right]\right)$$

for $k = 0, \pm1, \pm2, \ldots.$

Here $r^{\frac{1}{n}}$ means the positive n'th root of the real number r.

Thus, any one of the complex numbers

$$\ldots,\ r^{\frac{1}{n}}\operatorname{cis}(\tfrac{\theta - 2\pi}{n}),\ r^{\frac{1}{n}}\operatorname{cis}(\tfrac{\theta}{n}),\ r^{\frac{1}{n}}\operatorname{cis}(\tfrac{\theta + 2\pi}{n}),\ r^{\frac{1}{n}}\operatorname{cis}(\tfrac{\theta + 4\pi}{n}),\ \ldots.$$

is an n'th root of z. This list contains exactly n distinct complex numbers:

$$r^{\frac{1}{n}}\operatorname{cis}(\tfrac{\theta}{n}),\ r^{\frac{1}{n}}\operatorname{cis}(\tfrac{\theta + 2\pi}{n}),\ \ldots,\ r^{\frac{1}{n}}\operatorname{cis}(\tfrac{\theta + 2[n-1]\pi}{n}),$$

the argument of any other number in the list differs by a multiple of 2π from the argument of one of the above n numbers.

Thus,

the n distinct n'th roots of $z = r(\cos\theta + i\sin\theta)$ are the numbers
$r^{\frac{1}{n}}\left(\cos\left[\frac{\theta + 2k\pi}{n}\right] + i\sin\left[\frac{\theta + 2k\pi}{n}\right]\right)$
for $k = 0, 1, 2, \ldots, n-1$.

For example:

The four fourth roots of $(8 + 8\sqrt{3}\,i) = 16(\cos\frac{\pi}{3} + i \sin\frac{\pi}{3})$

are:

$2 \text{ cis } (\frac{\pi}{12})$, $2 \text{ cis } (\frac{7\pi}{12})$, $2 \text{ cis } (\frac{13\pi}{12})$, $2 \text{ cis } (\frac{19\pi}{12})$

[Note, for example; the next number in sequence would be

$2 \text{ cis } (\frac{25\pi}{12}) = 2 \text{ cis } (\frac{\pi}{12} + 2\pi)$.]

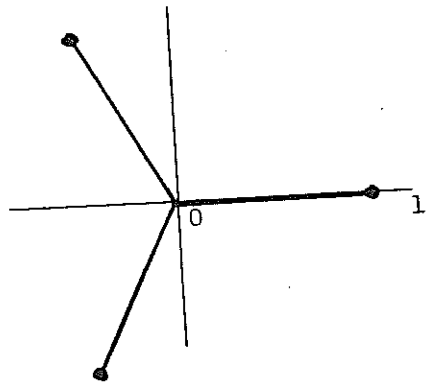
Of particular interest are the n'th roots of unity.

For example: Since $1 = \cos 0 + i \sin 0$, the 3 cube roots of unity are:

$\text{cis } 0$, $\text{cis } 2\pi/3$, $\text{cis } 4\pi/3$

or

1 , $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$, $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$



Remarks

1. The technique we used of defining the complex exponential in terms of the complex version of the power series representation for e^x may be used to define complex extensions of other functions; for example,

$\sin z = z - z^3/3! + z^5/5! - \ldots$

$\ln(1 + z) = z - z^2/2 + z^3/3 - \ldots \qquad |z| < 1,$

and provides one possible foundation for a theory for functions of a complex variable.

2. From $e^{i\theta} = \cos\theta + i \sin\theta$ we get

$e^{-i\theta} = \cos\theta - i \sin\theta$.

and so

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$$

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

which suggest that

$$\cos i\theta = \frac{1}{2}(e^{-\theta} + e^{\theta}) = \cosh\theta$$

and

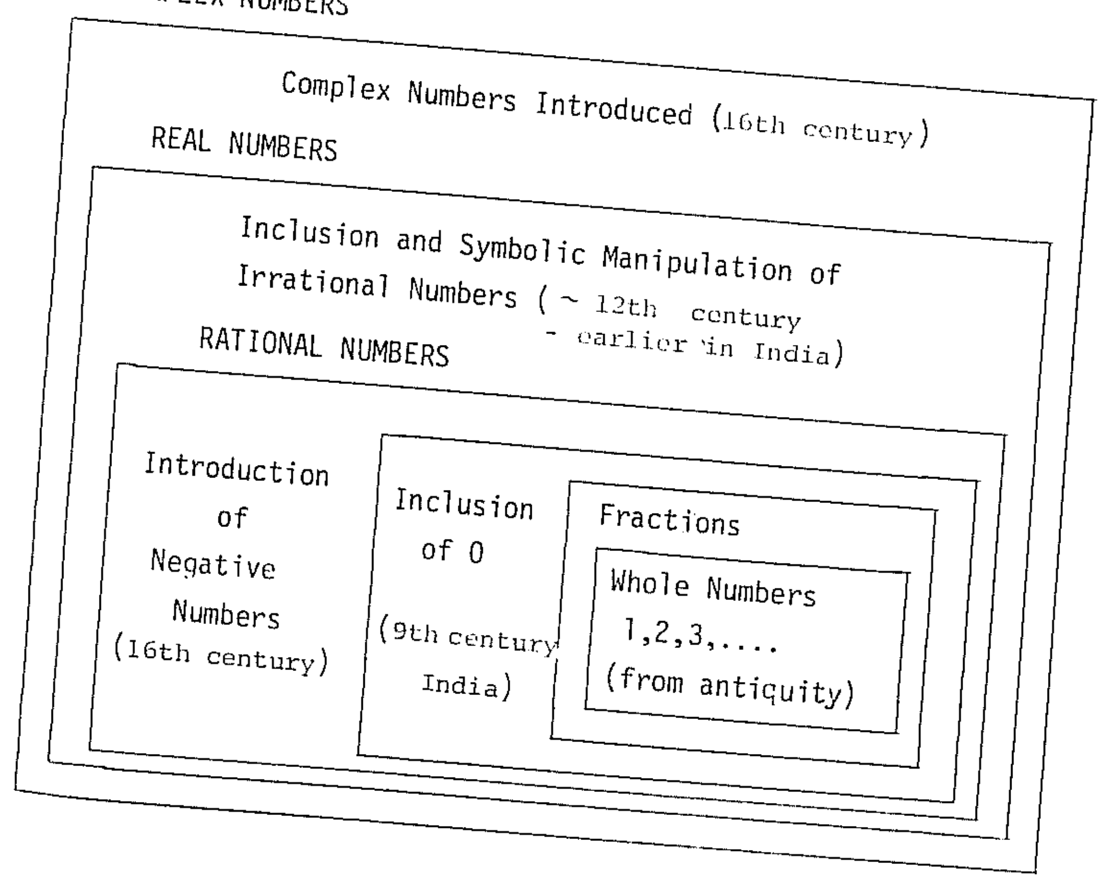
$$\sin i\theta = \frac{1}{2i}(e^{-\theta} - e^{\theta}) = i \sinh\theta$$

or

$$\cosh\theta = \cos i\theta , \quad \sinh\theta = -i \sin i\theta$$

This explains *Osborne's Rule* (Exercise 8.1.7, p.154) and the sign change for products of sinh's; $(-i)^2 = -1$.

COMPLEX NUMBERS



EXERCISES

(1) Show that the series $\sum_{n=1}^{\infty} (\frac{1}{2^n} + \frac{1}{3^n} i)$ is convergent and find its sum.

(2)(i) Show that the first four terms of the product

$$\left(\sum_{k=0}^{\infty} r_1^k/k!\right)\left(\sum_{k=0}^{\infty} r_2^k/k!\right)$$ are those expected

from the discussion on page 237.

(ii) Prove by an algebraic argument the general result

$$\left(\sum_{k=0}^{\infty} r_1^k/k!\right)\left(\sum_{k=0}^{\infty} r_2^k/k!\right) = \sum_{k=0}^{\infty} (r_1 + r_2)^k / k!$$

(3) Compute $(1 + i)^7$

(4) Express $\sin 9\theta$ in terms of $\sin\theta$ and $\cos\theta$.

(5) Show that $\sin\theta + \sin 2\theta + \ldots + \sin n\theta = \frac{\sin[(n+1)\theta/2]\sin[n\theta/2]}{\sin[\theta/2]}$

[Hint: Consider the Imaginary part of an appropriate "Geometric Progression".]

(6) Find an expression for $\tan 5\theta$ in terms of $\tan\theta$.

(7) Find four distinct values for $(1 + \sqrt{3}\, i)^{3/4}$

(8) (i) Find the 5th roots of unity.

(ii) Show that there is an n'th root of unity w such that $1, w, w^2, w^3, \ldots, w^{n-1}$ are the n distinct roots of unity and prove $1 + w + w^2 + w^3 + \ldots + w^{n-1} = 0$.

APPENDIX 11.4

The Fundamental Theorem of Algebra

Given any (non-constant) polynomial

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

of degree n $(n \geq 1, a_n \neq 0)$ we aim to show *there exists a complex root* z_0 *of* p; that is substituting $x = z_0$ we have $p(z_0) = 0$.

From this it readily follows that $p(x) = (x - z_0)q(x)$, where $q(x)$ is a polynomial of degree $n - 1$. Repeated application of this result establishes the *factorization:*

$$p(x) = a_n(x - z_0)(x - z_1)\ldots(x - z_n)$$

where $z_0, z_1, \ldots, z_n$ are the n roots of p, not all of which need be distinct.

[Note: in what follows it is not essential that the coefficients $a_0, a_1, a_2, \ldots, a_n$ are real. They could be complex. When the coefficients are real we see that $p(\bar{z}) = 0$ if $p(z) = 0$ and so it follows that if z is a root, then so too is $\bar{z}$; that is complex roots occur in conjugate pairs.]

Proof

Since $a_n \neq 0$ (otherwise p would not be of degree n) it is convenient to "divide it out", and so assume that p has the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + x^n$$

Our strategy is to show that the function $|p(z)|$ has a minimum value at some point z_0 and that this minimum value is 0, for then $p(z_0) = 0$ as required.

Since

$$\begin{aligned}|p(z)| &= |z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0| \\ &\geq |z^n| - |a_{n-1} z^{n-1} + \ldots + a_1 z + a_0| \\ &\geq |z^n| - |a_{n-1}|\,|z^{n-1}| - \ldots - |a_1||z| - |a_0| \\ &= |z^n| \ldots\end{aligned}$$

we see that $|p(z)| > |p(0)|$ for all z with $|z| > R$, for some sufficiently large R.

Now let z_0 be a point where $|p(z)|$ has a minimum value in the disk $|z| \leq R$. That such a point exists follows from the continuity of p, although it will not be proved in this course. (As the function $1/z$ shows, the existence of such a point need not follow without some restriction such as $|z| \leq R$.) Since 0 belongs to the disk $|z| \leq R$ we have, by its nature $|p(z_0)| \leq |p(z)|$ for all $|z| \leq R$ and $|p(z_0)| \leq |p(0)| \leq p(z)$ for all $|z| > R$.

Thus $|p(z)|$ is a minimum at $z = z_0$.

We shall prove that $p(z_0) = 0$ by assuming that $|p(z_0)| \gneq 0$ (the only other possibility) and deriving a contradiction, thereby showing that our assumption must have been false and so $p(z_0) = 0$ as required.

Thus, assume $p(z_0) \neq 0$, then we may define a new polynomial of degree n,

$$q(z) = \frac{p(z + z_0)}{p(z_0)} .$$

If $q(z) = b_0 + b_1 z + b_2 z^2 + \ldots + b_n z^n$, then since $q(0) = 1$ we have $b_0 = 1$ and so

$$q(z) = 1 + b_1 z + \ldots + b_n z^n.$$

Also, since p is non-constant so too is q, and at least one of the coefficients $b_1, b_2, \ldots, b_n$ must be non-zero. Let $m \geq 1$ be the smallest index for which $b_m \neq 0$, then

$$q(z) = 1 + b_m z^m + \ldots + b_n z^n.$$

From the choice of z_0 we see that $|q(z)|$ has a minimum value of 1 at $z = 0$. We now derive a contradiction by proving the existence of numbers z_1 with $|q(z_1)| \lneq 1$ thereby completing the proof.

Let $z_1 = \alpha w$ where α is a positive real number and w is an m'th

$b_m w^m = -1$.

Then

$$|q(\alpha w)| = |1 + b_m \alpha^m w^m + b_{m+1} \alpha^{m+1} w^{m+1} + \ldots + b_n \alpha^n w^n |$$

$$= |1 - \alpha^m + b_{m+1} \alpha^{m+1} w^{m+1} + \ldots + b_n \alpha^n w^n|$$

$$\leq 1 - \alpha^m + |b_{m+1} \alpha^{m+1} w^{m+1} + \ldots + b_n \alpha^n w^n|$$

(provided $\alpha \leq 1$)

$$\leq 1 - \alpha^m + |b_m| \alpha^{m+1} |w|^{m+1} + \ldots + |b_n| \alpha^n |w|^n$$

$$\leq 1 - \alpha^m + (|b_m| + \ldots + |b_n|)\, \alpha^{m+1} |w|^{m+1}$$

(provided α is sufficiently small so that $\alpha|w| \leq 1$; that is $\alpha \leq |w|^{-1}$)

$$= 1 - \alpha^m [1 - (|b_m| + \ldots + |b_n|) |w|^{m+1} \alpha]$$

But, then for α sufficiently small we see that the quantity in [...] is positive (its limit as $\alpha \to 0$ is 1) and so

$$|q(\alpha w)| \leq 1 - \alpha^m \times \textit{positive}$$

$$\lneq 1.$$

Giving the desired contradiction and so proving our result.

CHAPTER 12

DIFFERENTIAL EQUATIONS

12.0 Introduction

In §8.1 we saw how a number of simple situations led to the equation $y' = ky$.

"Models" for many other situations produce a relationship between a function y and its derivatives:

$$F(x, y, y', y'', \dots) = 0 .$$

For example, a quantum mechanical model of the hydrogen atom leads to

$$xy'' + (2 - x)y' + \left(\frac{e^2}{\hbar}\sqrt{\frac{m}{2E}}\right) y = 0 \qquad \dots\dots\dots \qquad (0.1)$$

while Van der Plank's model for the progress of an epidemic is

$$\frac{dy}{dt} = \alpha y(p - y). \qquad \dots\dots\dots \qquad (0.2)$$

Such relationships are referred to as **differential equations**. We say the differential equation is of **order** n if the highest derivative invloved is $y^{(n)}$. For example; (0.1) is a second order differential equation while (0.2) is first order.

12.1 First Order Differential Equations

We are interested in finding explicit solutions to equations of the form

$$\frac{dy}{dx} = f(x,y).$$

Only the merest handful of such equations can be solved explicitly (even the equation $\frac{dy}{dx} = ky$ required us to "invent" a new function, the exponential function, in order to "solve" it.) For example: it is not possible to find an elementary solution for the spread of an epidemic in an exponentially growing population, described by

$$\frac{dy}{dt} = \alpha y(pe^{\beta t} - y).$$

In the great majority of cases, where no explicit solution is possible, the key questions which lie at the heart of the "theory of differential equations" are:

(1) Does a solution exist? It may happen that any function satisfying the equation would have to have self-contradictory properties, in which case there can be no solution.

(2) If there is a solution, how can we determine its properties (that is, build up a "picture" of it) directly from the differential equation? Here questions such as *'is the solution unique?'*, *'when does the solution vanish?'* are important.

A partial answer to (1) is provided by the

EXISTENCE UNIQUENESS THEOREM: *If for values of* x *and* y *near* x_0 *and* y_0 *we have* $f(x,y)$ *is a continuous function of* x *and* $|f(x,y_1) - f(x,y_2)| \le K|y_1 - y_2|$, *for some constant* K, *then in a neighbourhood of* x_0 *there exists a unique solution of*

$$\frac{dy}{dx} = f(x, y)$$

satisfying $y(x_0) = y_0$.

Note: Since solving $\frac{dy}{dx} = f(x,y)$ involves, in some sense, an integration we would expect the solution to contain an arbitrary constant. An *initial condition* such as $y(x_0) = y_0$, is therefore necessary in order to pick out a specific solution from among the resulting family of solutions.

We will not concern ourselves further with these general questions. Instead we look at a couple of techniques whereby explicit solutions may be obtained in a few special cases.

12.1(a) Separable Equations

When it is possible to rearrange the equation into the form

$$Y(y)\frac{dy}{dx} = X(x)$$

where Y is a function of y only and X is a function of x only, then we say the equation is *separable* and may proceed to solve it by integrating both sides with respect to x to obtain

$$\int Y(y)\frac{dy}{dx}\,dx = \int X(x)\,dx,$$

or using the substitution formula

$$\int Y(y)\,dy = \int X(x)\,dx.$$

Assuming both integrations can be performed in elementary terms we obtain an implicit relationship between y and x which we may (or may not) be able to solve for y as a function of x. A solution will then be of this form.

For example:

If $$\frac{dy}{dx} = -e^x y$$

we have $$y^{-1}\frac{dy}{dx} = -e^x$$

so $$\int y^{-1}\,dy = -\int e^x\,dx$$

or $$\ln y = -e^x + C$$

Thus

$$y = e^{C-e^x}$$

is a solution for any value of C, the constant of integration.

To determine the particular solution satisfying the initial condition $y(0) = 1$ we have

$$1 = y(0) = e^{C-e^0}$$

so $$1 = e^{C-1}$$

or $$C - 1 = \ln 1 = 0$$

giving $C = 1$ and

$$y = e^{1-e^x}.$$

[This equation, known as Gompertz' Formula, has been used to model the demise of a group of equally aged individuals - can you see why?]

As a second example consider equation (0.2) of §12.0;

$$\frac{dy}{dx} = \alpha y(p - y) .$$

Separating variables leads to

$$\int \frac{dy}{y(p-y)} = \alpha \int dx = \alpha x + C$$

or, using partial fractions

$$\int \frac{1}{p}[\frac{1}{y} + \frac{1}{p-y}]\, dy = \alpha x + C$$

from which we obtain

$$\ln y - \ln (p - y) = p(\alpha x + C)$$

or

$$\ln \frac{y}{p-y} = p\alpha x + C_1 \qquad \text{(where } C_1 = pC)$$

so

$$\frac{y}{p-y} = e^{p\alpha x + C_1} = e^{C_1} e^{p\alpha x}$$

$$= Ke^{p\alpha x} \qquad \text{(where } K = e^{C_1}).$$

Solving this implicit relationship for y we have

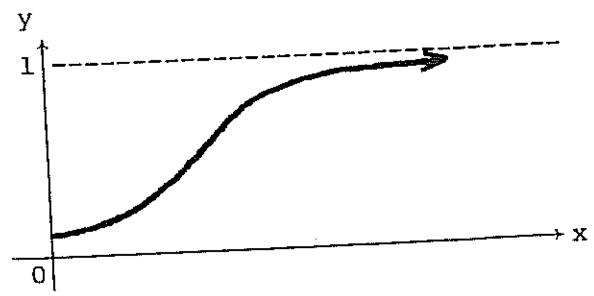
$$y = \frac{p}{1 + \frac{1}{K}e^{-p\alpha x}} .$$

In case $p = \alpha = 1$ we have

$$y = \frac{1}{1 + \frac{1}{K}e^{-x}}$$

and if $y(0) = 0.1$ we may determine the arbitrary constant K to obtain the solution

$$y = \frac{1}{1 + 9e^{-x}}$$



12.1(b) First Order Linear Equations

A first order equation of the form

$$y' + p(x)y = f(x) \qquad \ldots\ldots \quad \text{(b1)}$$

is termed linear and may be solved using an *integrating factor* $\mu(x)$. The idea is to multiply (b1) through by μ to obtain

$$\mu y' + \mu py = \mu f \qquad \ldots\ldots \quad \text{(b2)}$$

Then observe that if μ is chosen so that $\mu(x) \neq 0$ for any x, (b1) and (b2) have precisely the same solutions. Further if μ can also be chosen so that

$$\mu' = \mu p \qquad \ldots\ldots \quad \text{(b3)}$$

then (b2) may be written as

$$\mu y' + \mu' y = \mu f$$

or, by the product rule for differentiation,

$$(\mu y)' = \mu f .$$

An equation which may be readily integrated to obtain

$\mu y = \int \mu f$ and so we have the solution

$$y = \frac{1}{\mu(x)}\int \mu(x)\, f(x)\, dx.$$

To see that this strategy is viable it is only necessary to show that (b3) has a solution μ which is never zero. Now (b3) is a separable first order equation. Indeed (b3) becomes

$$\frac{1}{\mu}\frac{d\mu}{dx} = p(x) \quad \text{so} \quad \ln \mu = \int p(x)\, dx$$

or

$$\mu(x) = e^{\int p(x) dx}$$

which, as required, is a non-vanishing function provided p is integrable.

Note: Rather than remember these formulae it is advisable to learn the strategy and apply it in any particular example.

For example: To find the solution of

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{\cos x}{x^2} \qquad \text{satisfying} \quad y(\pi) = 0,$$

multiplying through by $\mu(x)$ we have

$$\mu\frac{dy}{dx} + \frac{2}{x}\mu y = \frac{\mu \cos x}{x^2}$$

and choosing μ to satisfy

$$\mu' = \frac{2}{x}\mu$$

that is, $\frac{\mu'}{\mu} = \frac{2}{x}$ so

$\ln\mu = \ln x^2$ or $\mu = x^2$

we have

$$\mu\frac{dy}{dx} + \frac{2}{x}\mu y = \frac{d}{dx}(\mu y) = \frac{d}{dx}(x^2 y) = \cos x.$$

Thus, integrating, $x^2 y = \int \cos x \, dx = \sin x + C$ and so $y = \frac{1}{x^2}(\sin x + C)$.

To complete the problem we choose the constant of integration C so that

$$0 = y(\pi) = \frac{1}{\pi^2}(\sin \pi + C) = \frac{C}{\pi^2}$$

that is $C = 0$ and so the solution is

$$y = \frac{\sin x}{x^2}.$$

EXERCISES

(1) For each of the following differential equations find the solution satisfying the given initial condition.

(i) $\frac{dy}{dx} = 1 - y$, $y(0) = \frac{1}{4}$

(ii) $\frac{dy}{dx} = -(e^x + 1)y$, $y(0) = 1$

(iii) $\frac{dy}{dx} = y^2 + 5y + 6$, $y(1) = 0$

(iv) $\frac{dy}{dx} = \frac{y(1-y)}{x(2-x)}$, $y(1) = \frac{1}{2}$

(v) $\frac{dy}{dx} = \frac{3x^2}{2(y-1)}$, $y(0) = -1$

(vi) $\frac{dy}{dx} = x + y$, $y(0) = 1$

(vii) $\frac{dy}{dx} = \frac{xy - y}{x - xy}$, $y(1) = 2$

... to find y as an explicit function of x,

(2) It has been found that the growth of human population in developing countries is described by $\frac{dN}{dt} = \lambda N^r$ for some constant $r > 1$. Find N as a function of t. [This same equation with $r = \frac{2}{3}$ has been used to describe the mass of a growing cell.]

If $N(0) = 1$, $\lambda = 1$ and $r = 2$ plot the solution.

(3) Starting with an initial blood alcohol level of zero, if one commences to imbibe at the constant rate of r middies per hour, to a first approximation the blood alcohol level $\ell(t)$ ml/ℓ satisfies

$$\frac{d\ell}{dt} = 0.02r - 0.4\ell; \quad \ell(0) = 0.$$

Find an expression for ℓ as a function of t and r.

At what rate must one drink if at the end of the first hour $\ell = 0.05$?

What is the rate if ℓ is to be 0.05 after 6 hours?

Show that if one continued to drink indefinitely the blood alcohol level would approach a constant value. What is the rate of drinking if this asymptotic level is to be 0.05?

(4) The equation $\frac{dy}{dx} = F(x,y)$ is said to be homogeneous if the right hand side depends only on the ratio of y to x; that is

$$\frac{dy}{dx} = f(y/x).$$

Using the substitution $u = y/x$ show that this may be re-written as the <u>separable</u> equation

$$x\frac{du}{dx} + u = f(u).$$

Hence solve

$$\frac{dy}{dx} = (2y^2 - x^2)/xy, \quad y(1) = \sqrt{2}$$

(5) For each of the following differential equations find the solution satisfying the given initial conditions.

(i) $\frac{dy}{dx} = e^x - y$, $y(0) = \frac{1}{2}$

(ii) $\frac{dy}{dx} + (\tan x)y = \sin x \cos x$, $y(0) = 0$

(iii) $\frac{dy}{dx} + \frac{y}{x} = 3 \cos 2x$, $y(\pi) = 1$

(iv) $y' - 2xy = x$, $y(0) = 1$

(v) $y' - 2xy = 1$, $y(0) = 1$

(If all goes well you will have to leave your answer to (v) as an integral.)

(6) (i) If $y = \phi(x)$ is a solution of

$$y' + p(x)y = 0$$

show that $y = c\phi(x)$ is also a solution for any value of the constant c.

(ii) Let y_0 be a solution of

$$y' + p(x)y = 0$$

and let y_p be a solution of

$$y' + p(x)y = f(x) \qquad \text{........ } (*)$$

show that $y = y_0 + y_p$ is also a solution of (*).

12.2 Second Order Differential Equations

We will only consider second order equations of the form

$$ay'' + by' + cy = f(x) \qquad \text{-----------} \qquad (2.1)$$

where a, b and c are constants and f is a known function of x. Such an equation is known as a constant coefficient second order linear differential equation.

Since the solution in some sense involves integrating twice we would expect the general solution to involve two arbitrary constants of integration. To uniquely specify a solution will therefore require two initial conditions, usually of the form $y(x_0) = y_0$ and $y'(x_0) = y_0'$, where x_0, y_0 and y_0' are known constants. The need to satisfy two such initial conditions is also reasonable from the physical situations modelled by equations of this type.

For example: *Newton's Second Law* for the motion of a particle on a straight line relates the acceleration, $\frac{d^2x}{dt^2}$, to the forces acting on the particle and so leads naturally to a second order equation of the form

$$m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + cx = f(t)$$

where the first order term $b \frac{dx}{dt}$ represents a resistive force (such as air drag) related to the velocity, the term cx corresponds to a force proportional to position, such as that due to the stretching of a spring and $f(t)$ is a time varying externally applied force.

To determine the subsequent motion of such a particle we must know both its initial position $x(0)$ and velocity $\dot{x}(0)$.

12.2(a) Constant Coefficient Second Order Linear Homogeneous Equations

When $f(x)$ in (2.1) is identically zero we say the equation is homogeneous.

A fundamental property of the second order linear homogeneous equation

$$ay'' + by' + cy = 0$$

is that if $y = \phi_1(x)$ and $y = \phi_2(x)$ are two solutions then $y = A\phi_1(x) + B\phi_2(x)$ is also a solution for any values of the constants A and B (you are asked to prove this in the exercises). This corresponds to the physical "principle of superposition" and it is because of this property that the equation is termed "linear".

Enough general theory (at least for the moment). We now consider the question of finding the solution of

$$ay'' + by' + cy = 0$$

satisfying the initial conditions $y(x_0) = y_0$, $y'(x_0) = y_0'$.

Since the "corresponding" first order equation

$$by' = -cy \quad \text{has} \quad y = Ae^{-cx/b}$$

as its solution, it is reasonable to try for a solution of the form

$$y = Ae^{\lambda x}.$$

We seek values of A and λ for which this will indeed be a solution. Substituting into the differential equation we see that this requires A and λ to be such that

$$a\lambda^2 Ae^{\lambda x} + b\lambda Ae^{\lambda x} + cAe^{\lambda x} = 0$$

or

$$A[a\lambda^2 + b\lambda + c]e^{\lambda x} = 0$$

for all values of x.

Since $e^{\lambda x}$ is not zero for all values of x (even when λ is complex), the only two possibilities are:

(1) to choose $A = 0$, which leads to the *trivial solution* $y = 0$ for all x,

or

(2) to choose λ to be a root of the quadratic equation $a\lambda^2 + b\lambda + c = 0$.

> Thus, $y = Ae^{\lambda x}$ is a non-trivial solution of
>
> $$ay'' + by' + cy = 0$$
>
> if and only if λ is a root of the characteristic equation:
>
> $$a\lambda^2 + b\lambda + c = 0.$$

Three possibilities arise:

(1) <u>The two roots λ_1, λ_2 are real and unequal</u>

In this case

$$y = Ae^{\lambda_1 x} \quad \text{and} \quad y = Be^{\lambda_2 x} \quad \text{are}$$

two solutions for any values of the constants A and B.

From our earlier discussion their sum

$$y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$$

is also a solution containing two arbitrary constants A and B to which values may be assigned so that the two initial conditions are satisfied.

<u>For example</u>: The problem

$$y'' - 5y' + 6y = 0\ ; \quad y(0) = 1, \quad y'(0) = 0$$

has characteristic equation

$$\lambda^2 - 5\lambda + 6 = 0$$

with roots

$$\lambda = 2, 3.$$

Thus, the general solution is

$$y = Ae^{2x} + Be^{3x}.$$

Substituting this into the initial conditions yields

$$1 = y(0) = A + B$$

and

$$0 = y'(0) = 2A + 3B$$

from which we have $A = 3$ and $B = -2$ so the solution to our problem is

$$y = 3e^{2x} - 2e^{3x}.$$

(II) The characteristic equation has equal roots

Let λ_0 (necessarily real) be the common root, then $y = Ae^{\lambda_0 x}$ is a solution for any value of A. However, the one arbitrary constant A does not give us enough freedom to satisfy a pair of initial conditions and we are led to expect that there is a second solution with a form other than $Be^{\lambda x}$. In this case it can be shown (see exercises) that

$$y = Bxe^{\lambda_0 x} \text{ is also a solution.}$$

Thus the general solution is

$$y = (A + Bx)e^{\lambda_0 x}$$

and the constants A and B can be chosen to satisfy the initial conditions.

For example: The problem

$$y'' + 2y' + y = 0 \quad ; \quad y(0) = 1, \quad y'(0) = 0$$

has characteristic equation $\lambda^2 + 2\lambda + 1 = 0$ with equal roots of -1. Thus, $y = (A + Bx)e^{-x}$ is the general solution. Substituting into the initial conditions we have

$$1 = A \;, \quad -A + B = 0 \quad \text{and so}$$

$$y = (1 + x)e^{-x}$$

is the solution. [Verify this by direct subsititution into the equation.]

(III) The roots are complex, and so of the form $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$. In this case we have a general solution, at least formally, of the form

$$y = A\,e^{(\alpha+i\beta)x} + B\,e^{(\alpha-i\beta)x}$$

$$= e^{\alpha x}(A\,e^{i\beta x} + B\,e^{-i\beta x}).$$

Since $e^{-i\beta x}$ is the conjugate of $e^{i\beta x}$, if we choose the two constants A and B to be the complex conjugates,

$$A = \tfrac{1}{2}(C - iD) \;, \quad B = \tfrac{1}{2}(C + iD) \;.$$

[Note we have still retained two arbitrary constants, namely C and D.]

then this last expression may be written as

$$y = e^{\alpha x}(Ae^{i\beta x} + \overline{Ae^{i\beta x}})$$

$$= e^{\alpha x} \times 2\ \mathrm{Re}[\tfrac{1}{2}(C - iD)e^{i\beta x}]$$

$$= e^{\alpha x}(C\cos\beta x + D\sin\beta x)$$

[by the Euler formula $e^{i\beta x} = \cos\beta x + i\sin\beta x$].

Thus, $y = e^{\alpha x}(C\cos\beta x + D\sin\beta x)$ is a purely real solution involving two arbitrary constants which may be determined to satisfy the initial conditions.

Note: In any given problem, once it has been found that the characteristic equation has complex roots we jump to assuming a solution of the form

$$y = e^{\alpha x}(C\cos\beta x + D\sin\beta x)$$

where α and β are respectively the real and imaginary parts of a root and then use the initial conditions to determine values for C and D.

For example: The equation

$$y'' - 6y' + 13y = 0; \quad y(0) = 1, \quad y'(0) = 0$$

has characteristic equation $\lambda^2 - 6\lambda + 13 = 0$ with roots $\lambda = 3 + 2i$ and $\lambda = 3 - 2i$.

Since the roots are complex, the solution is of the form

$$y = e^{3x}(C\cos 2x + D\sin 2x).$$

Substituting into the initial conditions gives:

$$1 = y(0) = C$$

so

$$y = e^{3x}(\cos 2x + D\sin 2x)$$

and

$$y' = e^{3x}(3\cos 2x + 2D\cos 2x + \textit{sine terms})$$

whence

$$0 = y'(0) = 3 + 2D \quad \text{requires} \quad D = -\tfrac{3}{2}$$

and the solution is

$$y = e^{3x}(\cos 2x - \tfrac{3}{2}\sin 2x).$$

12.2(b) The non-homogeneous case

We now turn to the solution of the non-homogeneous problem

$$ay'' + by' + cy = f(x); \quad y(x_0) = y_0, \; y'(x_0) = y_0' .$$

We begin by noting that if $y = y_p(x)$ is any *particular solution* of

$$ay'' + by' + cy = f$$

not necessarily satisfying the initial conditions, and if $y_h(x)$ is the general solution of the *corresponding homogeneous problem*

$$ay'' + by' + cy = 0,$$

then $y = y_p(x) + y_h(x)$ is also a solution of $ay'' + by' + cy = f$ (see exercises) involving the two arbitrary constants of $y_h(x)$ which may be chosen to satisfy the initial conditions. Thus to solve the problem completely it is only necessary to find some particular solution $y_p(x)$.

The approach we shall adopt, formally known as **the method of undetermined coefficients** is best described as "inspired" guessing. Consideration of the form of f often suggests the possible form of a function $y_p(x)$ which when combined with its derivatives into $ay_p'' + by_p' + cy_p$ will simplify to f. The method is best explained in terms of an example.

PROBLEM: Solve

$$y'' - 2y' + 2y = xe^x \; ; \; y(0) = 0 \; , \; y'(0) = 0.$$

Since the only way derivatives get to contain e^x is if we start with it we can infer any particular solution $y_p(x)$ will involve e^x. Also we will obtain terms like xe^x in the first and second derivatives if we started with

$$y_p(x) = (ax^2 + bx + c)\, e^x$$

Thus, it seems plausible to try for a solution of this form. (Clearly it would not help to have $y_p(x)$ contain expressions such as sin x or ln x, though these might be appropriate if f(x) involved sin x, cos x or 1/x.)

We aim to see if the coefficients a, b and c can be determined so that

$$y_p(x) = (ax^2 + bx + c)\, e^x$$

satisfies $y_p'' - 2y_p' + 2y_p = xe^x$.

Substituting, we have

$$\begin{aligned} y_p &= ax^2e^x + bxe^x + ce^x \\ y_p' &= ax^2e^x + (2a + b)xe^x + (b + c)\, e^x \\ y_p'' &= ax^2e^x + (4a + b)xe^x + (2a + 2b + c)\, e^x \\ \hline y_p'' - 2y_p' + 2y_p &= ax\, e^x + bxe^x + (2a + c)e^x . \end{aligned}$$

For this to equal xe^x, equating coefficients, we must have

$$a = 0, \; b = 1 \quad \text{and } 2a + c = 0 \quad \text{so} \quad c = 0 .$$

That is, $y_p = xe^x$ is a particular solution.

A general solution is therefore

$$y = xe^x + e^x (C \cos x + D \sin x),$$

where the second term is the general solution of the corresponding homoegeneous equation $y'' - 2y' + 2y = 0$ (characteristic equation $\lambda^2 - 2\lambda + 2 = 0$ has roots $1 + i$ and $1 - i$).

To satisfy the initial conditions we require:

$$0 = y(0) = 0 + 1.(C.1 + D.0) = C$$

So $y = xe^x + De^x \sin x$

and then $y' = xe^x + e^x + De^x \sin x + De^x \cos x$

so

$$0 = y'(0) = 0 + 1 + D.1.0 + D.1.1 = 1 + D$$

giving

$$D = -1$$

Thus

$$y = xe^x - e^x \sin x$$

is the required solution.

EXERCISES

(1) For each of the following differential equations find the solution satisfying the given initial conditions.

(i) $y'' + 2y' - 3y = 0$; $y(0) = 1$, $y'(0) = 0$

(ii) $y'' + y' - 2y = 0$; $y(0) = 1$, $y'(0) = 1$

(iii) $y'' - 6y' + 9y = 0$; $y(0) = 0$, $y'(0) = 2$

(iv) $y'' + y' + y = 0$; $y(0) = 0$, $y'(0) = 1$

(v) $y'' + 4y' + 5y = 0$; $y(0) = 1$, $y'(0) = 0$

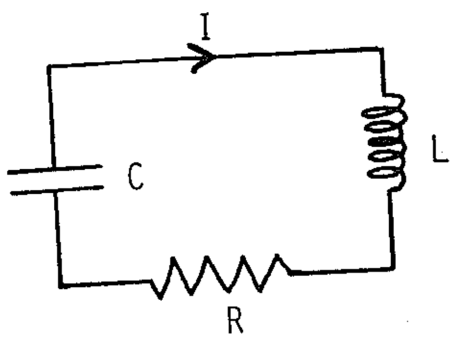
(vi) $y'' + 2y' + 2y = e^x \cos 2x$; $y(0) = y'(0) = 0$

(2) Show that the general solution of

$y'' + y = 0$ can be written as $y = A \sin(x + \phi)$ where A and ϕ are constants.

(3) The current I in an L-R-C circuit is governed by the differential equation

$$L \frac{d^2I}{dt^2} + R \frac{dI}{dt} + \frac{I}{C} = 0$$



Under what circumstance will the circuit oscillate (that is, have a solution I(t) with an infinite number of sign changes)?

(4) If $y = \phi_1(x)$ and $y = \phi_2(x)$ are two solutions of $ay'' + by' + cy = 0$ show that

$y = A\phi_1(x) + B\phi_2(x)$ is also a solution for any values of the constants A and B.

(5) If the characteristic equation of

$$ay'' + by' + cy = 0$$

has equal roots of λ_0, then it is of the form $a(\lambda - \lambda_0)^2 = 0$.

Hence conclude that the original equation was of the form

$$ay'' - 2a\lambda_0 y' + a\lambda_0^2 y = 0.$$

Show that

$$y = xe^{\lambda_0 x}$$

is a solution of this equation and so conclude that the general solution is

$$y = (A + Bx)e^{\lambda_0 x}.$$

(6) If $y = y_p(x)$ satisfies the non-homogeneous equation

$$ay'' + by' + cy = f \qquad \text{-----------} \quad (*)$$

and $y = y_h(x)$ is a solution of the corresponding homogeneous equation, show that

$y = y_p(x) + y_h(x)$ is also a solution of (*).

DEPARTMENT OF MATHEMATICS, STATISTICS AND COMPUTING SCIENCE

Pure Mathematics 111-2

Supplementary Notes

Page 127–128. In Theorem 7.2.2 it is shown that any primitive of a continuous function f is an antiderivative of f. I feel that the converse, mentioned on p.128, requires a little clarification:

Theorem 7.2.3. Let f be continuous on the interval (a,b), and suppose that f has an antiderivative G. Then any primitive of f differs from G by a constant.

Proof. Let F be any primitive of f. Then $F' = f$ by 7.2.2, while $G' = f$ by definition. Hence $(F - G)' = 0$ and by Corollary 1, p.105, $F - G$ is constant on (a,b). This is the stated result.

Thus it is almost (see exercise) true to say that antiderivatives and primitives coincide in the case of a continuous function.

Exercise. Recall that by definition a primitive of f is a function $F(x) = \int_c^x f$ for some c. Find all primitives of the function $f(x) = 1/(1+x^2)$; and confirm that $G(x) = \tan^{-1} x + \pi$ is an antiderivative but not a primitive. Check, however, that Theorem 7.2.3 is valid for G.

Page 136. The integral $\int x\sqrt{1-x^2}\,dx$ may also be found by the substitutions $u = (1-x^2)^{1/2}$, $u = (1-x^2)^{3/2}$, $u = (1-x^2)^{5/2}$, etc., and quite possibly others.

Page 142. Another reason for using numerical integration is that even if we can get an exact answer for the antiderivative it may be extremely complicated – see p.169, for example. In such cases it may in fact be easier to accurately evaluate a definite integral by numerical methods, such as Simpson's rule, rather than by using the exact formula.

Page 152. An addition to exercise 4(i): the limits can be found by L'Hôpital's Rule since $\lim_{x\to\infty} e^x = \lim_{x\to\infty} x^n = \infty$. That $\lim_{x\to\infty} e^x = \infty$ is "obvious" – but can you prove it using only the material given in the notes? (However, you may assume it for the purposes of exercise 4, and for the corresponding assignment question.)

Page 158. Here it is finally shown that the exponential function exists. (This isn't as silly as it may sound – what is actually meant is that without any prior knowledge of the exponential function, we have shown that there is a function g such that $g' = g$ and $g(0) = 1$.) The logical sequence of pp.146–158 is therefore as follows.

(i) The function $f(t) = 1/t$ is continuous on $(0, \infty)$. Therefore by p.123 it has a primitive for $x > 0$, which we denote by ln:

$$\ln x = \int_1^x \frac{dt}{t}, \quad x > 0 .$$

(ii) Since $\ln'(x) = 1/x > 0$, the function ln, with domain $(0, \infty)$, is increasing (Corollary 2, p.105) and hence invertible. We denote its inverse by exp.

(iii) By the inverse function rule,

$$\exp' = \frac{1}{\ln' \circ \exp} = \frac{1}{1/\exp} = \exp ;$$

also $\ln(1) = 0$, so $\exp(0) = 1$.

(iv) Therefore exp is a solution of $g' = g$, $g(0) = 1$; by Lemma 4, p.148, it is the only solution, and the remaining properties follow as in the text.

Page 162–169. As the topic of partial fractions will be treated more lightly this year, some simplified notes follow.

We wish to integrate functions of the form $p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomials. We illustrate the procedure by examples, as the general results which prompt the method are beyond the scope of the present course.

Example 1. Find $\displaystyle\int \frac{x^3 - 4x^2 + 3x + 11}{x^2 - x - 2}\, dx$.

The first step is to divide out as much as possible and then factorise the denominator. Since

$$x^3 - 4x^2 + 3x + 11 = (x - 3)(x^2 - x - 2) + (2x + 5) \qquad (1)$$

we have

$$\frac{x^3 - 4x^2 + 3x + 11}{x^2 - x - 2} = x - 3 + \frac{2x + 5}{(x - 2)(x + 1)} .$$

We now apply the method of *partial fractions* to simplify the last term. Suppose that a and b are constants such that

$$\frac{2x + 5}{(x - 2)(x + 1)} = \frac{a}{x - 2} + \frac{b}{x + 1} . \qquad (2)$$

Multiplying out the denominators we have

$$2x + 5 = a(x + 1) + b(x - 2) = (a + b)x + (a - 2b) . \qquad (3)$$

Equating coefficients we obtain

$$a + b = 2$$
$$a - 2b = 5 .$$

which equations are easily solved to yield $a = 3$, $b = -1$. Thus

$$\int \frac{x^3 - 4x^2 + 3x + 11}{x^2 - x - 2}\, dx = \int \left(x - 3 + \frac{3}{x - 2} - \frac{1}{x + 1} \right) dx$$
$$= \tfrac{1}{2}x^2 - 3x + 3\ln(x - 2) - \ln(x + 1) + C , \quad x > 2 .$$

Here we have specified $x > 2$ in order that the logarithmic terms exist. (The restriction can be weakened by more careful work.)

So we have integrated one rational function.* What about other rational functions? Many work by exactly the same method, but some are more difficult. Let us examine the above procedure.

Division of a polynomial $p(x)$ by another polynomial $q(x)$ to obtain a polynomial quotient, leaving a remainder with degree less than that of $q(x)$, as in (1), is always possible. You should be familiar with this fact, and you should also be able to perform the calculation by long division, or (with practice!) by writing the answer down one term at a time.

Secondly, a polynomial can always be factorised into a product of linear and quadratic terms. You probably know various ways of attacking such a problem; you probably also know that it may be a very difficult task indeed. This is in fact the only part of the present procedure which cannot be approached in a purely mechanical fashion.

Thirdly, an expression of the form (2) can always be found if the denominator is a product of any number of linear factors, no two of which are the same. Before expanding on this point we shall give another example; and before giving the example we shall mention a method of finding a and b which is often easier than that given above. Since (3) is true for all x, we simply choose the most convenient values for x. Letting $x = 2$,

$$9 = 3a + 0b \quad \text{so} \quad a = 3 ;$$

and letting $x = -1$,

$$3 = 0a - 3b \quad \text{so} \quad b = -1 .$$

This method will be useful also in harder examples, but may not give *all* the required constants: in such cases we will have to fall back on equating coefficients.

Example 2. Find $\displaystyle\int \frac{8\, dx}{x^4 - 2x}$.

Here no division is necessary, and the factorisation is easy. The partial fraction expression will be of the form

$$\frac{8}{x^3(x - 2)} = \frac{a}{x^3} + \frac{b}{x^2} + \frac{c}{x} + \frac{d}{x - 2} . \qquad (4)$$

* A *rational function* means the quotient of two polynomials.

Multiplying out,

$$8 = a(x-2) + bx(x-2) + cx^2(x-2) + dx^3 .$$

Putting $x = 0$ we get $a = -4$; putting $x = 2$ we get $d = 1$. We find b and c by equating coefficients. The coefficients of x and x^2 yield respectively

$$0 = a - 2b , \quad 0 = b - 2c ;$$

hence $b = -2$, $c = -1$. Thus

$$\int \frac{8\,dx}{x^4 - 2x^3} = \int \left(\frac{-4}{x^3} - \frac{2}{x^2} - \frac{1}{x} + \frac{1}{x-2} \right) dx$$
$$= \frac{2}{x^2} + \frac{2}{x} - \ln x + \ln(x-2) + C , \quad x > 2 .$$

Example 3. Find $\displaystyle\int \frac{x^3 + 2x^2 + x + 6}{x^4 - x^2 - 2x + 2}\,dx$.

It is easily found that $(x-1)^2$ is a factor of the denominator, and we obtain

$$x^4 - x^2 - 2x + 2 = (x-1)^2(x^2 + 2x + 2) ,$$

where the last factor cannot be split up any further. In this case the partial fractions will be given by

$$\frac{x^3 + 2x^2 + x + 6}{x^4 - x^2 - 2x + 2} = \frac{a}{(x-1)^2} + \frac{b}{x-1} + \frac{cx + d}{x^2 + 2x + 2} ; \tag{5}$$

that is, the numerator above the quadratic will be a linear polynomial and not just a constant (though it may turn out that $c = 0$). We have

$$x^3 + 2x^2 + x + 6 = a(x^2 + 2x + 2) + b(x-1)(x^2 + 2x + 2) + (cx + d)(x-1)^2 ,$$

and letting $x = 1$ yields $a = 2$. Examining the coefficients of $1, x, x^2$ we have

$$\begin{aligned} 2a - 2b \qquad + d &= 6 \\ 2a \qquad + c - 2d &= 1 \\ a + b - 2c + d &= 2 . \end{aligned}$$

which (since $a = 2$) has the solution $b = 0$, $c = 1$, $d = 2$. Hence

$$\int \frac{x^3 + 2x^2 + x + 6}{x^4 - x^2 - 2x + 2}\,dx = \frac{-2}{x-1} + \int \frac{x+2}{x^2 + 2x + 2}\,dx .$$

To tackle the final integral we relate the numerator to the derivative of the denominator: $\frac{d}{dx}(x^2 + 2x + 2) = 2x + 2$ and $x + 2 = \frac{1}{2}(2x + 2) + 1$, so

$$\int \frac{x+2}{x^2 + 2x + 2}\,dx = \frac{1}{2}\int \frac{2x + 2}{x^2 + 2x + 2}\,dx + \int \frac{dx}{(x+1)^2 + 1}$$
(completing the square)
$$= \frac{1}{2}\ln(x^2 + 2x + 2) + \tan^{-1}(x+1) + C$$

where we have solved the first integral by the substitution $u = x^2 + 2x + 2$ and the second by $v = x + 1$. Check these for yourself.

It remains to give full details of the partial fraction expression for a rational function. In fact it can be shown that if the denominator contains a factor $(x - \alpha)^m$, then the partial fraction expansion contains terms

$$\frac{a_m}{(x-\alpha)^m} , \quad \frac{a_{m-1}}{(x-\alpha)^{m-1}} , \quad \ldots , \quad \frac{a_1}{x - \alpha}$$

(example: equations (2), (4), (5)), while if the denominator contains an irreducible term $(x^2 + \beta x + \gamma)^n$ then the expansion includes terms

$$\frac{b_n x + c_n}{(x^2 + \beta x + \gamma)^n} , \quad \frac{b_{n-1}x + c_{n-1}}{(x^2 + \beta x + \gamma)^{n-1}} , \quad \ldots , \quad \frac{b_1 x + c_1}{x^2 + \beta x + \gamma}$$

(example: (5)). For instance, we have

$$\frac{1}{x^2(x-1)(x^2+1)^2} = \frac{a}{x^2} + \frac{b}{x} + \frac{c}{x-1} + \frac{dx + e}{(x^2+1)^2} + \frac{fx + g}{x^2 + 1} ,$$

where the constants $a, b, \ldots, g$ would take a long time to find.

Exercise. The only problem remaining is to calculate integrals such as

$$\int \frac{dx}{(x^2+1)^2} .$$

Find this integral by writing

$$\int \frac{dx}{(x^2+1)^2} = \int \frac{(x^2+1) - x^2}{(x^2+1)^2}\,dx = \int \frac{dx}{x^2+1} + \frac{1}{2}\int x\,\frac{-2x}{(x^2+1)^2}\,dx$$

and then using integration by parts. Use a similar method to calculate

$$\int \frac{dx}{(x^2+1)^3} .$$

Page 205. A slightly simplified proof of the irrationality of e.

As on p.205, assume that $e = a/b$ where a, b are integers; let n be an integer such that $n \geq b + 1$ and $n \geq 4$; and write

$$k = n!\left(\frac{a}{b} - \left(1 + \frac{1}{1!} + \cdots + \frac{1}{(n-2)!} + \frac{1}{(n-1)!} + \frac{1}{n!}\right)\right). \tag{1}$$

Then

$$|k| < n!\frac{4}{(n+1)!} = \frac{4}{n+1} < 1$$

from (**) on p.204. Now since b is one of the numbers $1, 2, 3, \ldots, n-1$ we see that $(n-1)!/b$ is an integer; therefore

$$k = na\frac{(n-1)!}{b} - n! - n! - \cdots - n(n-1) - n - 1$$

is an integer. Clearly dividing k by n will leave a remainder of -1, and therefore k *cannot be zero*. Combining this fact with (1), $|k|$ is an integer between 0 and 1, which is clearly impossible. Thus e is not, as we first supposed, rational.

Page 207. A sketch of a proof that for any $x > 0$,

$$\lim_{n \to \infty} \frac{x^n}{n!} = 0 .$$

Trying an example, say $x = 4$, the sequence under consideration is

$$4,\ 8,\ 10\tfrac{2}{3},\ 10\tfrac{2}{3},\ 8\tfrac{8}{15},\ 5\tfrac{31}{45},\ 3\tfrac{79}{315},\ 1\tfrac{197}{315},\ \cdots$$

and we see that from a certain point onwards the sequence will decrease rapidly. Indeed, let k be an integer such that $K \leq 2x < K + 1$, and let $M = x^K/K!$. Then

$$\frac{x^{K+1}}{(K+1)!} = M\frac{x}{K+1} < \tfrac{1}{2}M$$

$$\frac{x^{K+2}}{(K+2)!} = \frac{x^{K+1}}{(K+1)!}\frac{x}{K+2} < \tfrac{1}{2}M \cdot \tfrac{1}{2} = \left(\tfrac{1}{2}\right)^2 M$$

$$\vdots$$

$$\frac{x^{K+k}}{(K+k)!} < \left(\tfrac{1}{2}\right)^k M .$$

Hence $x^n/n! \to 0$ as $n \to \infty$. The reader is invited to fill in the details of this proof.

Page 208, Example (1). The point here is that we have found the MacLaurin series for e^{-x^2} with much less work than if we had calculated

$$\begin{aligned} f(x) &= e^{-x^2}, & & \\ f'(x) &= -2xe^{-x^2}, & & f(0) = 1 ; \\ f''(x) &= 4x^2e^{-x^2} - 2e^{-x^2} = (4x^2 - 2)e^{-x^2}, & & f'(0) = 0 ; \\ f'''(x) &= -2x(4x^2 - 2)e^{-x^2} + 8xe^{-x^2} = (-8x^3 + 12x)e^{-x^2}, & & f''(0) = -2 ; \\ & & & f'''(0) = 0 \end{aligned}$$

and so on. Indeed, to make this method work we would need to find a general formula for $f^{(n)}(x)$, which would be very difficult.

Note, however, that the method on p.208 gives the MacLaurin expansion of $f(x)$ *if such an expansion exists at all*; to check that it exists, we consider the remainder term. From the box on p.202 we have

$$e^x - \left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}\right) = \frac{1}{n!}\int_0^x t^n e^{x-t}\,dt$$

and so

$$e^{-x^2} - \left(1 - x^2 + \frac{x^4}{2!} - \cdots \pm \frac{x^{2n}}{n!}\right) = \frac{1}{n!}\int_0^{-x^2} t^n e^{-x^2-t}\,dt .$$

Now for t between $-x^2$ and 0 we have

$$t^n e^{-x^2-t} \leq |t|^n e^{-x^2-t} \leq x^{2n}e^0 = (x^2)^n ;$$

and therefore (Theorem 7.1.1) the remainder is at most $x^2(x^2)^n/n!$, which tends to zero as $n \to \infty$. This proves that e^{-x^2} has MacLaurin series as stated on p.208. Similar comments apply to example (2): for the remainder, see exercise (5), p.219.

Page 209. An example of a Taylor series: let $f(x) = x^{-2}$, $x_0 = 1$. we have

$$\begin{aligned} f'(x) &= -2x^{-3}, & & \\ f''(x) &= 6x^{-4}, & f'(1) &= -2 ; \\ & & f''(1) &= 6 ; \\ &\vdots & &\vdots \\ f^{(n)}(x) &= (-1)^n(n+1)!\,x^{-(n+2)}, & f^{(n)}(1) &= (-1)^n(n+1)! . \end{aligned}$$

Therefore the Taylor expansion of $f(x)$ about $x = 1$ is

$$f(x) = \frac{1}{x^2} \sim 1 - \frac{2!}{1!}(x-1) + \frac{3!}{2!}(x-1)^2 - \cdots + (-1)^n\frac{(n+1)!}{n!}(x-1)^n + \cdots ,$$

which simplifies to

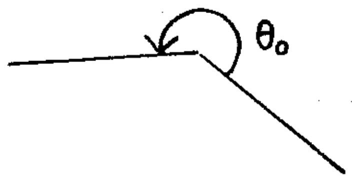
$$f(x) = \frac{1}{x^2} \sim 1 - 2(x-1) + 3(x-1)^2 - \cdots + (-1)^n(n+1)(x-1)^n + \cdots .$$

The remainder for a Taylor polynomial approximation can be expressed by an integral in much the same way as for a MacLaurin polynomial.

Page 225. We may add to the list of properties the *associative* laws for addition and multiplication: for any complex numbers z_1, z_2 and z_3,

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)\,,$$
$$(z_1 z_2) z_3 = z_1 (z_2 z_3)\,.$$

Exercise. Check these.



Page 232. Note that θ_0 as drawn in the diagram on p.232 is to be considered as a negative (i.e., clockwise) angle. Alternatively we could take θ_0 to be the positive angle shown at right.

Page 241. On page 238 we saw how to use De Moivre's Theorem in order to express the cosine of a multiple angle as a sum of powers of $\cos\theta$. The formulæ

$$\cos\theta = \tfrac{1}{2}(\epsilon^{i\theta} + \epsilon^{-i\theta})\,, \qquad \sin\theta = \tfrac{1}{2i}(\epsilon^{i\theta} - \epsilon^{-i\theta})$$

on p.241 enable us, conversely, to express a power of $\cos\theta$ or $\sin\theta$ as a sum of the cosines or sines of multiple angles. For example,

$$\begin{aligned}
\cos^6\theta &= \left(\tfrac{1}{2}(\epsilon^{i\theta} + \epsilon^{-i\theta})\right)^6 \\
&= \tfrac{1}{2^6}(\epsilon^{6i\theta} + 6\epsilon^{5i\theta}\epsilon^{-i\theta} + 15\epsilon^{4i\theta}\epsilon^{-2i\theta} \\
&\qquad + 20\epsilon^{3i\theta}\epsilon^{-3i\theta} + 15\epsilon^{2i\theta}\epsilon^{-4i\theta} + 6\epsilon^{i\theta}\epsilon^{-5i\theta} + \epsilon^{-6i\theta}) \\
&= \tfrac{1}{2^6}\left((\epsilon^{6i\theta} + \epsilon^{-6i\theta}) + 6(\epsilon^{4i\theta} + \epsilon^{-4i\theta}) + 15(\epsilon^{2i\theta} + \epsilon^{-2i\theta}) + 20\right) \\
&= \tfrac{1}{32}\cos 6\theta + \tfrac{3}{16}\cos 4\theta + \tfrac{15}{32}\cos 2\theta + \tfrac{5}{16}\,.
\end{aligned}$$

FUNCTIONS OF TWO VARIABLES.

1. Introduction. In Chapter 5 we introduced the concept of a function: roughly speaking, a function is a "rule" associating with each number from some set a number from some other (possibly the same) set. For example the function

$$A_1 : [0,\infty) \to \mathbb{R}\,, \quad A_1(x) = \pi x^2$$

can be interpreted as giving the area of a circle with radius x, for $x \geq 0$. Often some quantity will be determined not by one variable but by two or more. For example, consider the function

$$A_2 : [0,\infty) \times [0,\infty) \to \mathbb{R}\,, \quad A_2(x,y) = \pi x y\,,$$

which can be thought of as giving the area of an ellipse with semi-axes x and y $(x, y \geq 0)$. Here the notation $[0,\infty) \times [0,\infty)$ denotes the *Cartesian product* of two sets. In general, $S \times T$ is the set of all ordered pairs whose first element lies in S and whose second lies in T:

$$S \times T = \{(s,t) : s \in S, t \in T\}\,.$$

If $T = S$ it is common to write S^2 instead of $S \times S$. In particular, $\mathbb{R}^2$ is the set of all ordered pairs of real numbers (and similarly $\mathbb{R}^3$ is the set of all ordered triples of real numbers, and so on). It is clear in the above example that $A_2(x,y) = A_2(y,x)$; but this is usually not true. For example, we may choose

$$f : [-1,1] \times \mathbb{R} \to \mathbb{R}\,, \quad f(x,y) = (y+2)\sin^{-1} x\,;$$

clearly $f(x,y) \neq f(y,x)$.

As with functions of one variable, the domain of a function, if not explicitly stated, is understood to be the largest possible. It is not necessary to write (it is not even necessary that it be possible to write) the domain in the form $S \times T$. Consider, for example, the function

$$g : D \to \mathbb{R}\,, \quad g(x,y) = \sqrt{1 - xy}$$

where D is some subset of $\mathbb{R}^2$. The maximum possible domain of g is

$$D = \{(x,y) : xy \leq 1\}\,. \tag{1}$$

which cannot be written as $S \times T$ for any $S \subseteq \mathbb{R}$, $T \subseteq \mathbb{R}$.

The extension of the ideas of this section to functions of three or more variables involves no essentially new ideas, and we shall generally restrict ourselves to two variables. We can talk about the sum, difference, product and quotient of two functions as in Chapter 5. It is not possible to have an identity function: the equation $\mathrm{id}(x,y) = (x,y)$ is meaningless since the left hand side is a number while the right is a *pair* of numbers. In its place we have the *projection* function p_1 which simply picks out the first element of an ordered pair:

$$p_1(x,y) = x ;$$

and similarly there is the projection $p_2(x,y) = y$. Therefore the *ordered pair of functions* (p_1, p_2) in some sense gives us an identity function; but we shall not pursue this idea here.

Composition of functions involves a similar problem; if f and g are functions of two variables then $f \circ g$ does not make sense, as we are attempting to apply f to a number instead of to a pair of numbers. However, if h is also a function of two variables we may consider the function

$$f(g(x,y), h(x,y)) .$$

(There is no commonly accepted "circle" notation for such a construction, though $f \circ (g,h)$ would make sense.) For example, if

$$f(x,y) = \tfrac{2}{5}x^2 - \tfrac{4}{5}xy - \tfrac{1}{5}y^2 , \quad g(x,y) = x + 2y , \quad h(x,y) = 2x - y$$

then

$$f(g(x,y), h(x,y)) = \tfrac{2}{5}(x+2y)^2 - \tfrac{4}{5}(x+2y)(2x-y) - \tfrac{1}{5}(2x-y)^2$$
$$= -2x^2 + 3y^2 .$$

Indeed, it is not necessary that g and h be functions of two variables: if

$$f(x,y) = x^2 - y^2 , \quad g(t) = \cos t , \quad h(t) = \sin t .$$

then

$$f(g(t), h(t)) = \cos^2 t - \sin^2 t = \cos 2t .$$

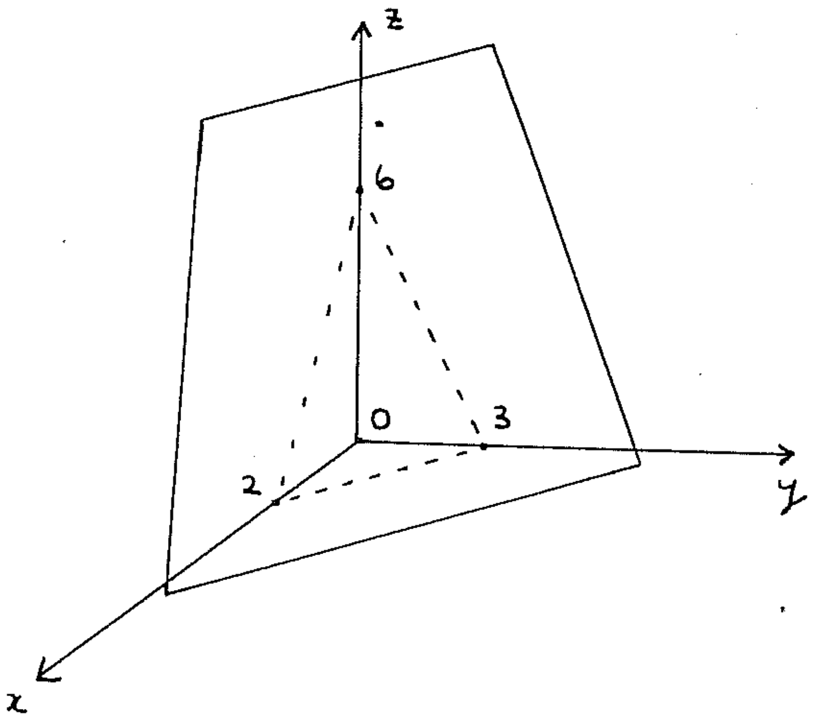
Observe that here, although $f(x,y)$ is a function of two variables, $f(g(t), h(t))$ is a function of only one variable.

We may think of the graph of a function $f(x,y)$ with domain D in similar terms to those we used for functions of one variable: it is the set of points (x,y,z) in three-dimensional space such that (x,y) is in D and $z = f(x,y)$. This "graph" will then be not a curve but a surface. Being three-dimensional, such graphs are often hard to draw on two-dimensional paper.

Consider the function

$$f : \mathbb{R}^2 \to \mathbb{R} , \quad f(x,y) = 6 - 3x - 2y .$$

Since $3x + 2y + z = 6$ represents a plane, the graph of f is a plane passing through the points $(2,0,0)$, $(0,3,0)$ and $(0,0,6)$.



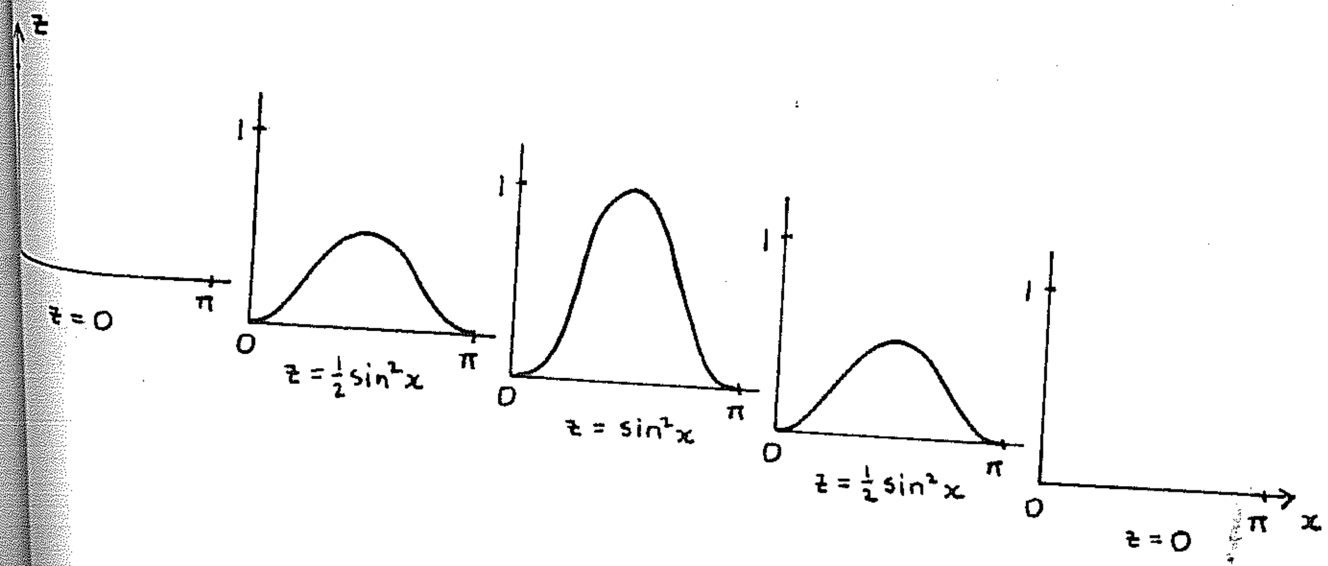
For a rather harder example, consider

$$g : [0,\pi]^2 \to \mathbb{R} , \quad g(x,y) = \sin^2 x \sin^2 y .$$

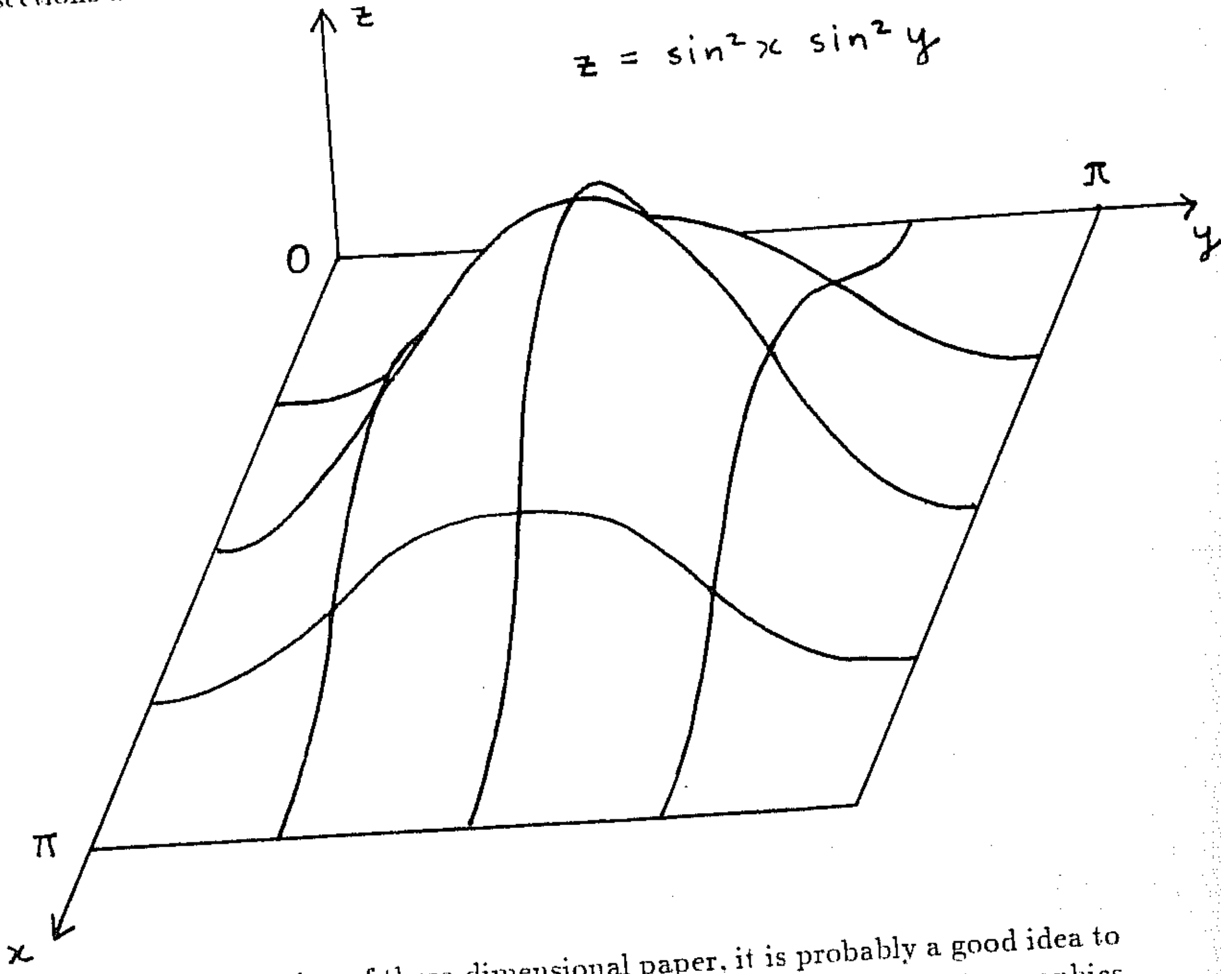
We may sketch the graph of g by considering its cross-sections: that is, we fix one of the variables x and y and consider what happens as the other one ranges over all possible values. Doing this for various different values of the fixed variable enables us to build up a picture of the graph. We may choose, for example, the values $y = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$; we have

$$g(x,0) = 0 , \quad g(x,\tfrac{\pi}{4}) = \tfrac{1}{2}\sin^2 x , \quad g(x,\tfrac{\pi}{2}) = \sin^2 x ,$$
$$g(x,\tfrac{3\pi}{4}) = \tfrac{1}{2}\sin^2 x , \quad g(x,\pi) = 0 .$$

and therefore the five cross-sections are as follows.



We can now put these cross-sections together to create a picture of the graph. It is useful to also take cross-sections in the y direction, with x fixed; in the present case, since the function $z = \sin^2 x \sin^2 y$ is symmetric with respect to x and y, these cross-sections are the same as the ones we have just done.



Pending the invention of three-dimensional paper, it is probably a good idea to leave anything especially complicated in this field to artists or to computer graphics programs.

Exercises.

1. Find the domains of the functions

$$f(x,y) = \frac{y^2}{\sqrt{4 - x^2}}\ ; \quad g(x,y) = \frac{1}{x^2 + y^2 - 1}\ ; \quad h(x,y) = \ln(y - x^2 + 1)\ .$$

Sketch the domains as regions in the x-y plane.

2. Sketch the region D in equation (1).

3. Let $S = \{1,2,3\}$, $T = \{1,2\}$. Write down all the elements of $S \times T$. Give an alternative interpretation of a function with domain $S \times T$.

4. Let

$$f(x,y) = \frac{y}{x - x^2}\ , \quad g(x,y) = xy^2\ , \quad h(x) = x^2\ .$$

Write explicit formulae for the functions fg, $f(g(x,y), h(x))$, $g(h(x), f(x,y))$, $h(f(x,y), g(x,y))$, $h(g(h(x), h(x)))$ where possible.

5. Draw the graph of the function

$$f : \mathbb{R}^2 \to \mathbb{R}\ , \ f(x,y) = -2y + 2\ .$$

6. Give a geometric interpretation of the function

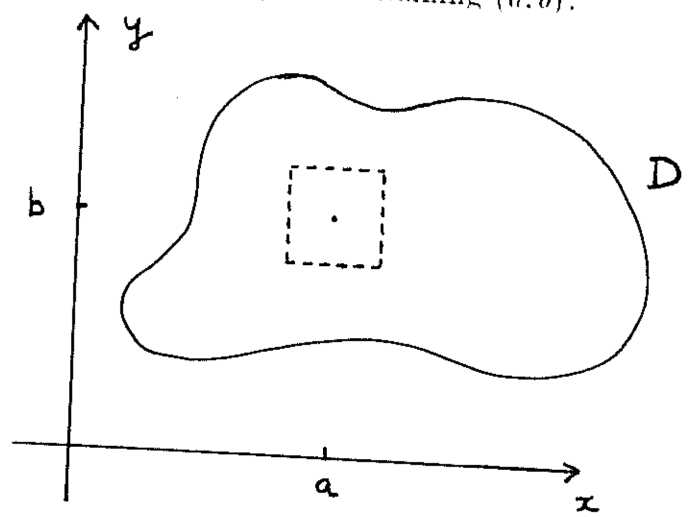
$$z = \sqrt{4 - x^2 - y^2}\ ,$$

and draw the graph of the function in three-dimensional space.

2. Limits and continuity. Here again the treatment is very similar to the one-variable case. Let $f(x,y)$ be a function of two variables. Intuitively,

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

means that as (x,y) gets closer to (a,b), $f(x,y)$ gets closer to L. In order for this to make sense, $f(x,y)$ must be defined near (a,b). (But not necessarily at (a,b) itself – compare p.83.) As we are talking about two variables, this is to say that the domain D of f contains a square containing (a,b).



(A rectangle, a circle, or even an arbitrary shape would do just as well, but the discussion would become more complicated.) This leads to a definition like that of Chapter 5:

Definition S.2.1. Let $f : D \to \mathbb{R}$ be a function of two variables, and let (a,b) be a point such that D contains a square region around (a,b). Then f has limit L as (x,y) approaches (a,b), written

$$\lim_{(x,y)\to(a,b)} f(x,y) = L \qquad \text{or} \qquad f(x,y) \to L \quad \text{as} \quad (x,y) \to (a,b) \,,$$

if for each $\varepsilon > 0$ there exists $\delta > 0$ such that if

$$|x-a| < \delta \,, \quad |y-b| < \delta \,, \quad (x,y) \neq (a,b)$$

then

$$|f(x,y) - L| < \varepsilon \,.$$

All the results that you would expect hold: for example

Theorem S.2.1. If $f(x,y) \to L$ and $g(x,y) \to M$ as $(x,y) \to (a,b)$, then $(f+g)(x,y) \to L+M$ as $(x,y) \to (a,b)$.

Proof. Let $\varepsilon > 0$. Since $f(x,y) \to L$ and $g(x,y) \to M$, there exist $\delta_1, \delta_2 > 0$ such that

$$|f(x,y) - L| < \tfrac{1}{2}\varepsilon$$

whenever $|x-a| < \delta_1$, $|y-b| < \delta_1$, $(x,y) \neq (a,b)$, and

$$|g(x,y) - M| < \tfrac{1}{2}\varepsilon$$

whenever $|x-a| < \delta_2$, $|y-b| < \delta_2$, $(x,y) \neq (a,b)$. Let $\delta = \min(\delta_1,\delta_2)$. Then $\delta > 0$; if $|x-a| < \delta$, $|y-b| < \delta$, $(x,y) \neq (a,b)$ we have

$$\begin{aligned} |(f+g)(x,y) - (L+M)| &\le |f(x,y) - L| + |g(x,y) - M| \\ &\qquad \text{(by the triangle inequality)} \\ &< \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon \\ &= \varepsilon \,. \end{aligned}$$

Therefore $\displaystyle\lim_{(x,y)\to(a,b)} (f+g)(x,y) = L + M$.

Note that this is a perfectly standard proof of a limits theorem; any such result should be proved in a similar manner (at least until you understand the topic well enough to take short cuts). The proof follows the definition step by step: the definition says "for each $\varepsilon > 0$", so we begin with "Let $\varepsilon > 0$". Next, "there exists $\delta > 0$", so by using the given information about f and g we find a suitable δ. The definition continues with "if $|x-a| < \delta$, $|y-b| < \delta$, $(x,y) \neq (a,b)$..." so we assume that this is the case; "then $|(f+g)(x,y) - (L+M)| < \varepsilon$", and we finish the proof by showing that this inequality holds.

We use the addition theorem, and the corresponding results for subtraction, multiplication and division, together with the basic results (compare Lemma, p.84)

$$\lim_{(x,y)\to(a,b)} C(x,y) = C \tag{1}$$

for the constant function $C(x,y) = C$, and

$$\lim_{(x,y)\to(a,b)} p_1(x,y) = a \,, \qquad \lim_{(x,y)\to(a,b)} p_2(x,y) = b \tag{2}$$

for the projections p_1 and p_2, to build up limits as in Chapters 5 and 6. For example,

$$\lim_{(x,y)\to(1,1)} \frac{3xy+1}{x+y} = 2 \,.$$

An interesting example is provided by the function

$$f : D \to \mathbb{R} \,, \quad f(x,y) = \frac{x^2 - y^2}{x^2 + y^2} \,, \tag{3}$$

where D contains all points in $\mathbb{R}^2$ except the origin. If we take $x = 0$ and let $y \to 0$ we have $f(x,y) \to -1$; if on the other hand we take $y = 0$ and let $x \to 0$ then $f(x,y) \to 1$. Since we get different results as (x,y) approaches $(0,0)$ by different paths, the limit of f at $(0,0)$ does not exist.

Definition S.2.2. The function $f : D \to \mathbb{R}$ is said to be continuous at the point (a,b) of its domain if

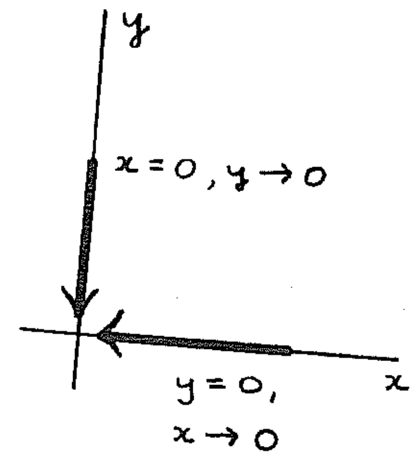
$$\lim_{(x,y)\to(a,b)} f(x,y)$$

exists and is equal to $f(a,b)$.

From our limit theorems, the sum, difference, product and quotient of continuous functions are again continuous.

Exercises.

1. Prove results (1) and (2). (*Hint:* you can follow p.84–85 almost word for word.)
2. Consider the function f of equation (3). We have seen that if $x = 0$ and $y \to 0$ then $f(x,y) \to -1$, while if $y = 0$ and $x \to 0$ then $f(x,y) \to 1$. We can view these limits as letting (x,y) approach $(0,0)$ along different paths in the x-y plane, as at right. Investigate what happens to $f(x,y)$ as (x,y) tends to the origin along other paths.

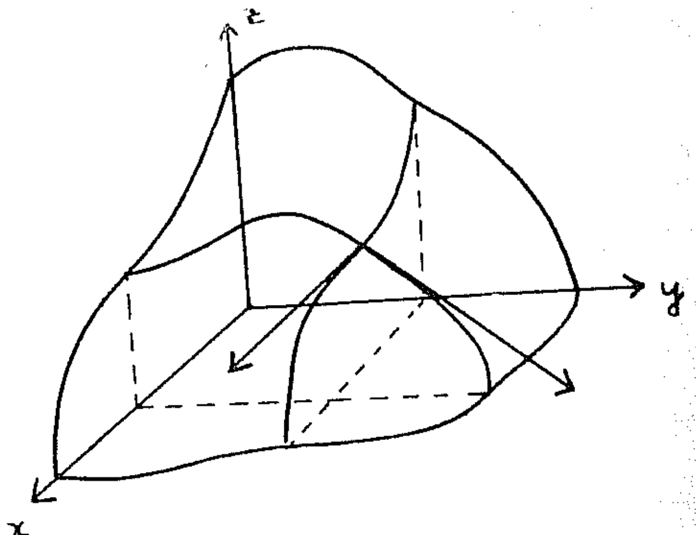


3. Define

$$f : \mathbb{R}^2 \to \mathbb{R} \,, \quad f(x,y) = \begin{cases} \dfrac{x^4 + y^4}{x^2 + y^2} \,, & (x,y) \neq (0,0) \,; \\ 0 \,, & (x,y) = (0,0) \,. \end{cases}$$

Is f continuous? (*Hint:* $x^4 + y^4 \le x^4 + 2x^2y^2 + y^4$.)

3. Partial differentiation. Consider a function f of x and y. It can be seen from the diagram that the rate of change of f at a point (x,y) depends not only on the point under consideration but also on the *direction* of change. In this section we shall treat the case where the change is purely in the x direction, y remaining constant, or purely in the y direction, with x remaining constant.



Suppose, therefore, that y is fixed. Then $f(x,y)$ is a function of x alone, and the rate of change is given by the derivative

$$\lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

if it exists. We define the *partial derivative* of f *with respect to* x, notated

$$f_x \qquad \text{or} \qquad f_x(x,y) \qquad \text{or} \qquad \frac{\partial f}{\partial x},$$

to be the above limit. That is, the partial derivative with respect to x of a function $f(x,y)$ is obtained by holding y constant and differentiating $f(x,y)$ as if it were a function of x only. Note that the ∂ in the last notation is a special symbol, and is not a Greek delta "δ" or an English "d".

In the same way, the rate of change of $f(x,y)$ in the y direction, x fixed, is given by the partial derivative of f with respect to y

$$f_y \quad = \quad f_y(x,y) \quad = \quad \frac{\partial f}{\partial y},$$

defined as

$$\lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}.$$

Partial derivatives of a function of two (or more) variables are therefore calculated by exactly the same rules as ordinary derivatives, only remembering which variable is to be regarded as changing and which is (are) to be kept constant. In particular we have

$$\frac{\partial}{\partial x}(f+g) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial x},$$

$$\frac{\partial}{\partial x}(fg) = f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x},$$

$$\frac{\partial}{\partial x}\left(\frac{f}{g}\right) = \frac{g\frac{\partial f}{\partial x} - f\frac{\partial g}{\partial x}}{g^2}.$$

Also, if $g(x)$ is a function of x only and $h(y)$ is a function of y only, then the derivative of $f(g(x),h(y))$ is given by the chain rule

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial g}\frac{dg}{dx}. \tag{1}$$

Warning: this is no longer true if g and h are functions of both x and y (see section 5). Observe also that in (1) we wrote $\frac{dg}{dx}$ rather than $\frac{\partial g}{\partial x}$ since g is a function of x alone. However it would not be incorrect to write $\frac{\partial g}{\partial x}$.

The above rules are of course also valid for partial differentiation with respect to y.

Examples. Let $f(x,y) = \frac{1}{2}x^3 - \frac{1}{2}xy^2$. Thinking of y as a constant we have

$$\frac{\partial f}{\partial x} = \frac{3}{2}x^2 - \frac{1}{2}y^2; \tag{2}$$

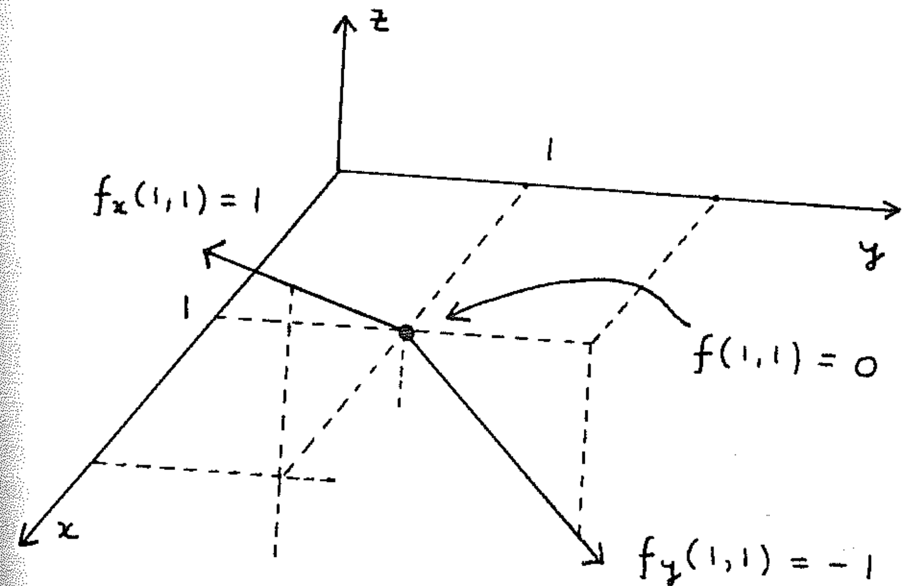
thinking of x as a constant we have

$$\frac{\partial f}{\partial y} = -xy \tag{3}$$

(since $\frac{\partial}{\partial y}(x^3) = 0$). Evaluating these at the point $(1,1)$ we have

$$f_x(1,1) = 1, \quad f_y(1,1) = -1.$$

Therefore, noting that $f(1,1) = 0$, the direction of the surface $z = f(x,y)$ at the point $(1,1)$ can be illustrated as below.



Similarly if $g(x,y) = \sin x \sin y$ we have

$$g_x = \cos x \sin y, \quad g_y = \sin x \cos y$$

and so, for example, $g_x(\frac{\pi}{3},\frac{\pi}{2}) = \frac{1}{2}$, $g_y(\frac{\pi}{3},\frac{\pi}{2}) = 0$.

As in the case of ordinary derivatives we can find second and higher partial derivatives. We define f_{xx} to be the partial derivative of f_x with respect to x; in the ∂ notation,

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right).$$

Similarly

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right),$$

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right),$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right).$$

Note the two so-called "mixed" derivatives f_{xy} and f_{yx}, which correspond to differentiating first with respect to x and then with respect to y (for f_{xy}), and in the reverse order for f_{yx}. There is no *a priori* reason why these two derivatives should be equal; however it turns out that under fairly mild restrictions on f, we do indeed have $f_{xy} = f_{yx}$.

Example. Let $f(x,y) = \frac{1}{2}x^3 - \frac{1}{2}xy^2$ as above. Using (2) and (3) we have

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(\tfrac{3}{2}x^2 - \tfrac{1}{2}y^2) = 3x; \qquad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(\tfrac{3}{2}x^2 - \tfrac{1}{2}y^2) = -y;$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(-xy) = -y; \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(-xy) = -x.$$

Observe that here $f_{xy} = f_{yx}$.

Exercises.

1. Find the partial derivatives f_x and f_y, where $f(x,y)$ is

$$\text{(a) } \ln(x^2+y^2)^2; \quad \text{(b) } \frac{xy+1}{y+1}; \quad \text{(c) } \frac{x(x+y)}{\sqrt{1-y^2}};$$

$$\text{(d) } e^{y/x}; \quad \text{(e) } x\sin xy.$$

2. Find the rate of change of $xy + (1/x)$ at the point $(1, -2)$, if x varies while y remains fixed.

3. Find f_x, f_y, f_{xx}, f_{xy}, f_{yx} and f_{yy}, where

$$\text{(a) } f(x,y) = (2x - 3y)^4; \quad \text{(b) } f(x,y) = xe^{y^2};$$

$$\text{(c) } f(x,y) = y\ln(x+y); \quad \text{(d) } f(x,y) = \cos x\cos y - \sin x \sin y.$$

4. If $g(t)$ is a function of one variable, with derivative $g'(t)$, find

$$\frac{\partial}{\partial x}\left(g\left(\frac{y}{x}\right)\right); \quad \frac{\partial}{\partial y}\left(g\left(\frac{y}{x}\right)\right); \quad \frac{\partial}{\partial x}g(x^2y); \quad \frac{\partial}{\partial y}g(x^2y).$$

5. Let $f(x,t) = g(x - at)$, where a is a constant and $g(z)$ is a function of one variable such that $g''(z)$ exists. Show that

$$\frac{\partial^2 f}{\partial t^2} - a^2\frac{\partial^2 f}{\partial x^2} = 0.$$

(This equation describes the amplitude x at time t of a wave travelling with velocity a.)

4. The directional derivative. We can now, generally, find the slope of a surface $z = f(x,y)$ as (x,y) changes in any direction. Let the required direction be given by the unit vector $\mathbf{u} = u\mathbf{i} + v\mathbf{j}$. Moving a distance h in this direction takes us from (x,y) to $(x+uh, y+vh)$;* therefore the rate of change of f at (x,y) in the direction of $\mathbf{u}$ is

$$\lim_{h\to 0}\frac{f(x+uh, y+vh) - f(x,y)}{h},$$

provided the limit exists. We can write this fraction as

$$u\,\frac{f(x+uh, y+vh) - f(x, y+vh)}{uh} + v\,\frac{f(x, y+vh) - f(x,y)}{vh}$$

which has limit

$$uf_x(x,y) + vf_y(x,y) \tag{1}$$

as $h \to 0$, provided that f_x and f_y exist and are continuous at (x,y). A careful proof of this is given in section 6.

We introduce the notation ∇f (pronounced "del f" or "grad f") for the *gradient vector* of f, that is,

$$\nabla f = (f_x, f_y).$$

Then the expression (1) is simply the scalar product $\mathbf{u}\cdot\nabla f$. We have shown

Theorem S.4.1. If $f : D \to \mathbb{R}$ is a function whose partial derivatives f_x and f_y exist and are continuous at a point (a,b), and if $\mathbf{u}$ is any unit vector, then the directional derivative of f at (a,b) in the direction of $\mathbf{u}$ exists and is equal to $\mathbf{u}\cdot\nabla f(a,b)$.

* Note that by using Pythagoras' Theorem, the distance between these points is found to be $\sqrt{(uh)^2 + (vh)^2} = |h|\sqrt{u^2+v^2} = |h||\mathbf{u}| = |h|$. This is why $\mathbf{u}$ must be a unit vector.

Example. Let $f(x,y)=\sqrt{9-x^2-y^2}$. Then $z=f(x,y)$ represents the top half of a sphere of radius 3, centre $(0,0,0)$. Consider the point $(2,1,2)$ which lies on this surface. Find the slope of the surface at this point in the direction of (i) $\mathbf{u}=(1,1)$; (ii) $\mathbf{v}=(1,-2)$; (iii) $\mathbf{w}=(-1,0)$. Illustrate these directions in the x-y plane.

Solution. First we calculate

$$\frac{\partial f}{\partial x}=\frac{-x}{\sqrt{9-x^2-y^2}}=-\frac{x}{z}\,,\qquad \frac{\partial f}{\partial y}=\frac{-y}{\sqrt{9-x^2-y^2}}=-\frac{y}{z}\,;$$

substituting values at $(2,1,2)$ we have

$$\nabla f(2,1)=(-1,-\tfrac{1}{2})\,. \tag{2}$$

(i) Since $|\mathbf{u}|=\sqrt{2}$, a unit vector in the same direction as $\mathbf{u}$ is $\frac{1}{\sqrt{2}}\mathbf{u}$. Therefore the slope of the surface in this direction is

$$\frac{1}{\sqrt{2}}(1,1)\cdot(-1,-\tfrac{1}{2})=-\frac{3}{2\sqrt{2}}\,.$$

(ii) Similarly $|\mathbf{v}|=\sqrt{5}$ and the slope in the direction of $\mathbf{v}$ is

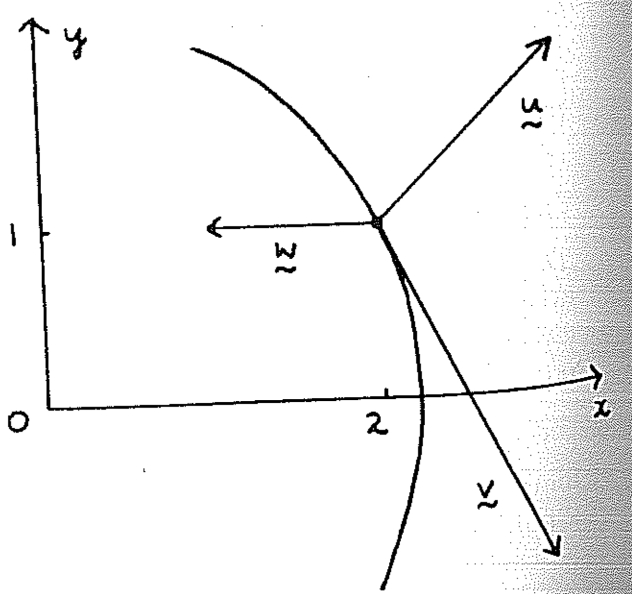
$$\frac{1}{\sqrt{5}}(1,-2)\cdot(-1,-\tfrac{1}{2})=0\,,$$

and

(iii) the slope in the direction of $\mathbf{w}$ is

$$(-1,0)\cdot(-1,-\frac{1}{2})=1\,.$$

The three vectors are illustrated as shown. Recall that the surface is the upper half of a sphere centred at the origin. Since $\mathbf{u}$ is heading (more or less) away from the origin, it is clear that the surface is falling in this direction; this fits in with our negative result for the directional derivative. Similarly the derivative in the direction of $\mathbf{w}$ is positive, and that in the direction of $\mathbf{v}$ is zero: here we are heading around a contour, as may be seen from the circle drawn in the diagram.



The preceding work can be used to quickly find the *direction of steepest increase* of a function at a given point. For the directional derivative of f at (x,y) in the direction of $\mathbf{u}$ is

$$\mathbf{u}\cdot\nabla f(x,y)=|\mathbf{u}|\,|\nabla f(x,y)|\cos\theta=|\nabla f(x,y)|\cos\theta\,, \tag{3}$$

where θ is the angle between $\mathbf{u}$ and $\nabla f(x,y)$. Suppose that $|\nabla f(x,y)|\neq 0$. Since $\nabla f(x,y)$ is some fixed number, (3) is greatest when $\cos\theta=1$, i.e., $\theta=0$; it is least when $\cos\theta=-1$, i.e., $\theta=\pi$; and is zero when $\theta=\frac{1}{2}\pi$. Thus we have

Theorem S.4.2. Let f be a function of two variables, and (x,y) a point at which $\nabla f\neq(0,0)$. Then f increases most rapidly when (x,y) changes in the direction of ∇f; decreases most rapidly in the opposite direction to ∇f; and remains constant when (x,y) moves in a direction perpendicular to ∇f.

Example. Let $f(x,y)=\sqrt{9-x^2-y^2}$. Then as in (2),

$$\nabla f(2,1)=(-1,-\tfrac{1}{2})=\tfrac{1}{2}(-2,-1)\,.$$

Therefore the direction of steepest increase from $(2,1)$ is $(-2,-1)$, that is, directly towards the origin. This is geometrically obvious since the surface $z=f(x,y)$ is a (half) sphere with its centre at the origin. Similarly, the direction of steepest decrease is $(2,1)$, or directly away from the origin; and the direction in which f remains constant is $\pm(1,-2)$, as found above.

Example. Let $g(x,y)=e^{2x}\cos(y^2)$. Find the direction of steepest increase at the point $(1,\sqrt{\pi})$, and the rate of steepest increase.

Solution. We have

$$\nabla f=(f_x,f_y)=(2e^{2x}\cos(y^2),\ 2ye^{2x}\sin(y^2))$$

and so $\nabla f(1,\sqrt{\pi})=(-2e^2,0)$. Therefore the direction of steepest increase is $(-2e^2,0)$, or more simply $(-1,0)$, which lies in the same direction. The rate of increase is just $|\nabla f(1,\sqrt{\pi})|=2e^2$.

Exercises.

1. Find ∇f, where $f(x,y)$ is the function

   (a) x^3y^4 ; (b) $x(\sin x+\sin y)$; (c) $e^{2x}(1+x+y)$.

2. Find the derivative of $f(x,y)$ at the point (a,b) in the direction of $\mathbf{u}$, where

   (a) $f(x,y)=x^3y^4$, $(a,b)=(\frac{1}{2},-1)$, $\mathbf{u}=4\mathbf{i}+2\mathbf{j}$;

   (b) $f(x,y)=x(\sin x+\sin y)$, $(a,b)=(\frac{\pi}{2},\pi)$, $\mathbf{u}=3\mathbf{i}-4\mathbf{j}$;

   (c) $f(x,y)=e^{2x}(1-x+y)$, $(a,b)=(0,1)$, $\mathbf{u}=12\mathbf{i}+5\mathbf{j}$.

3. Find the direction of steepest increase of f at the point (a,b), and the rate of steepest increase, if

   (a) $f(x,y) = x^3 - y^5$, $(a,b) = (2,1)$;

   (b) $f(x,y) = \tan^{-1}(y/x)$, $(a,b) = (1,3)$;

   (c) $f(x,y) = x - \sqrt{xy}$, $(a,b) = (8,2)$.

4. Let $g(t)$ be a differentiable function of one variable and let $f(x,y) = g(x^2+y^2)$. Find the direction of steepest increase of f, and the direction in which f is constant, for any point (x,y). Interpret your results geometrically.

5. Repeat exercise 4 if $f(x,y)$ is defined as $g(y/x)$.

5. The chain rule. Suppose that x and y are differentiable functions of t. Then $f(x(t),y(t))$ is a function of t, and we can find $\frac{df}{dt}$ by means of the chain rule. We can reason in virtually the same way as we did for the chain rule in one variable on p.93. We consider the fraction

$$\frac{f(x(t+h),y(t+h)) - f(x(t),y(t))}{h}$$
$$= \frac{f(x(t+h),y(t+h)) - f(x(t),y(t+h))}{h} + \frac{f(x(t),y(t+h)) - f(x(t),y(t))}{h}$$

and investigate what happens as $h \to 0$. We can apply the Mean Value Theorem to the second fraction: we have, for any X, Y, H

$$f(X, Y + H) - f(X, Y) = H f_y(X, Y + C) \tag{1}$$

for some C between 0 and H. Here X is considered as fixed. Using the Mean Value Theorem again,

$$y(t+h) - y(t) = hy'(c) \tag{2}$$

for some c between 0 and h. Now write

$$X = x(t), \quad Y = y(t), \quad H = y(t+h) - y(t),$$

substitute (2) into (1), and divide by h:

$$\frac{f(x(t),y(t+h)) - f(x(t),y(t))}{h} = f_y(x(t),y(t) + C)\frac{y(t+h) - y(t)}{h}.$$

As $h \to 0$, $H \to 0$ since y is continuous, and so $C \to 0$; therefore the last expression tends to $f_y(x(t),y(t))y'(t)$, provided that f_y is continuous at (x,y). Treating the first fraction above in the same way we obtain

$$\frac{d}{dt}f(x(t),y(t)) = f_x(x(t),y(t))x'(t) + f_y(x(t),y(t))y'(t).$$

If we write this in the more easily remembered ∂ notation, we have

Theorem S.5.1. Suppose that the functions x and y are differentiable at t, and that f_x and f_y exist and are continuous at $(x(t),y(t))$. Then $f(x(t),y(t))$ is differentiable at t, and

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}. \tag{3}$$

Roughly speaking, we simply apply the ordinary chain rule; however we must remember that f may depend on t through both x and y.

Example. Let $f(x,y) = xy + y^2$, $x = t^2$, $y = e^{2t}$. Find $\frac{df}{dt}$.

Solution. We have

$$\frac{\partial f}{\partial x} = y, \quad \frac{\partial f}{\partial y} = x + 2y, \quad \frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 2e^{2t}$$

and therefore

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$
$$= 2ty + 2(x + 2y)e^{2t}$$
$$= 2(t + t^2)e^{2t} + 4e^{4t}.$$

Check. Alternatively, $f(x,y) = t^2e^{2t} + e^{4t}$, so

$$\frac{df}{dt} = 2te^{2t} + 2t^2e^{2t} + 4e^{4t}.$$

Now suppose that x and y are functions of two variables s and t. Then f is a function of s and t, and we can find $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$. Since we calculate $\frac{\partial f}{\partial s}$ by holding t constant and treating f, x and y as functions of s alone, (3) instantly gives

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

(note that we now have *partial* derivatives of x and y — contrast (3)), and similarly

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}.$$

Example. Let $f(x,y) = x^2 + 2y^2$, $x = r\cos\theta$, $y = r\sin\theta$. Then

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$$
$$= 2x\cos\theta + 4y\sin\theta$$
$$= 2r(\cos^2\theta + 2\sin^2\theta) = 2r(1 + \sin^2\theta)$$

and similarly

$$\frac{\partial f}{\partial \theta} = -2xr\sin\theta + 4yr\cos\theta = 2r^2\cos\theta\sin\theta = r^2\sin 2\theta\ .$$

A slightly more complicated situation arises in the case of a function $f(x,y)$ where y is a function of x. In actual fact the methods used are just the same, and quite easy to apply if you keep a cool head. We have

$$\begin{aligned}\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx}\\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}\ .\end{aligned}$$

In this sort of case it is more than ever necessary to carefully distinguish between $\frac{\partial f}{\partial x}$ and $\frac{df}{dx}$.

Example. Let $f(x,y) = 3xy - 2y^2$, $y = 1 + e^x$. Then

$$\begin{aligned}\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}\\ &= 3y + (3x-4y)e^x\\ &= 3 + (3x-1)e^x - 4e^{2x}\ .\end{aligned}$$

Note that $\frac{\partial f}{\partial x} = 3y = 3 + 3e^x \neq \frac{df}{dx}$. Checking the above result, $f(x,y) = (3x-2) + (3x-4)e^x - 2e^{2x}$, so

$$\frac{df}{dx} = 3 + (3x-1)e^x - 4e^{2x}$$

once again.

We may also consider the case where y is assumed to be an *unknown* function of x, and a given function $f(x,y)$ is known to be constant. Take, for example,

$$f(x,y) = (x+y)e^x = 1\ .$$

Then

$$\frac{df}{dx} = e^x(1+x+y) + e^x\frac{dy}{dx} = e^x\left(1+x+y+\frac{dy}{dx}\right);$$

but also $\frac{df}{dx} = 0$ since $f(x,y) = 1$ is constant. Therefore

$$\frac{dy}{dx} = -y - x - 1\ .$$

Equations such as this, giving a relationship between x, y and $\frac{dy}{dx}$, are known as differential equations, and form the subject of Chapter 12.

Exercises.

1. Find $\frac{df}{dt}$ if
   (a) $f(x,y) = \ln xy$, $x = 2\sinh t$, $y = \cosh t$;
   (b) $f(x,y) = \cos(x+y)\sin y$, $x = e^t$, $y = t^2$;
   (c) $f(x,y) = x^2y^3$, $x = e^t + e^{2t}$, $y = e^{-t} + e^{-2t}$.
2. Find f_s and f_t if
   (a) $f(x,y) = x^2 + xy$, $x = \cos s\cos t$, $y = \sin s\sin t$;
   (b) $f(x,y) = x^2e^{2y}$, $x = 2t - s$, $y = 2t + s^2$.
3. Let $f(x,y) = \sqrt{x^2+y^2}$, $g(x,y) = \tan^{-1}(y/x)$, $x = r\cos\theta$, $y = r\sin\theta$. Use the chain rule to find f_r, f_θ, g_r and g_θ: explain your results.
4. Suppose that y is a function of x. For the following functions $f(x,y)$, find $\frac{df}{dx}$: and if $f(x,y)$ is constant, find $\frac{dy}{dx}$ in terms of x and y.
   (a) $f(x,y) = xy + \cos x$;
   (b) $f(x,y) = x\ln y + y\ln x$;
   (c) $f(x,y) = x - \tan^{-1}(y/x)$.

6. Proof of the formula for the directional derivative. As in section 4,

$$\begin{aligned}&\frac{f(x+uh,y+vh) - f(x,y)}{h}\\ &\quad = u\,\frac{f(x+uh,y+vh) - f(x,y+vh)}{uh} + v\,\frac{f(x,y+vh) - f(x,y)}{vh}\end{aligned} \qquad (1)$$

provided $u, v \neq 0$. (If either u or v is zero, the directional derivative reduces to the partial derivative f_y or f_x respectively, and Theorem S.4.1 is clearly true.) Write $k = uh$, $l = vh$. Then by the Mean Value Theorem, the first fraction in (1) is

$$\frac{f(x+k,y+l) - f(x,y+l)}{k} = f_x(x+c,y+l) \qquad (2)$$

for some c between 0 and k. Now as $h \to 0$, $k \to 0$ so $c \to 0$; also $l \to 0$, and therefore

$$(x+c,y+l) \to (x,y)\ .$$

If f_x is continuous we then have

$$f_x(x+c,y+l) \to f_x(x,y) \quad \text{as} \quad h \to 0\ . \qquad (3)$$

The second fraction in (1) is

$$\frac{f(x,y+l) - f(x,y)}{l}$$

which tends to $f_y(x,y)$ as $l \to 0$, and therefore also as $h \to 0$. Using (1), (2), (3), the derivative of f at (x,y) in the direction of $\mathbf{u}$ is

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+uh,y+vh)-f(x,y)}{h} &= uf_x(x,y)+vf_y(x,y)\\ &= \mathbf{u}\cdot\nabla f(x,y)\,.\end{aligned}$$

Page 261. Further notes on finding a particular solution to a non-homogeneous differential equation.

The method rests on the assumption that the derivative of a function is somehow "like" the original function. For example, the derivative of a polynomial is a polynomial, and the derivative of an exponential is an exponential. Similarly, the derivative of xe^x is $xe^x + e^x$ which is still of much the same form. (However the derivative of $\ln x$ is $1/x$.) So if we consider the equation

$$y'' - 3y' + 2y = 4e^{3x}\,, \tag{1}$$

it would seem reasonable to try a particular solution of the form $y_p(x) = ae^{3x}$ for some constant a. Substituting into (1),

$$9ae^{3x} - 3\times 3ae^{3x} + 2ae^{3x} = 4e^{3x}\,,$$

whence $2a = 4$ and $a = 2$. Thus $2e^{3x}$ is a particular solution of (1), and the general solution is

$$y = Ae^x + Be^{2x} + 2e^{3x}$$

(check this). Similarly, consider

$$y'' - 3y' + 2y = \sin x\,. \tag{2}$$

If we try a particular solution $y_p(x) = a\sin x$, we have

$$-a\sin x - 3a\cos x + 2a\sin x = \sin x\,;$$

here there is no cos term on the right hand side, so we must have $a = 0$ which clearly does not work. For a second attempt we could therefore include $\cos x$ in our particular solution: that is, try $y_p(x) = a\sin x + b\cos x$. This leads to the equations

$$a + 3b = 1\,, \quad b - 3a = 0$$

which have the solution $a = \frac{1}{10}$, $b = \frac{3}{10}$. The general solution of (2) is thus

$$y = Ae^x + Be^{2x} + \tfrac{1}{10}\sin x + \tfrac{3}{10}\cos x\,.$$

Instead of making our first (failed) attempt, we could have anticipated the problem as follows. The right hand side of (2) is $\sin x$; the derivative of $\sin x$ is $\cos x$; so we take $y_p(x) = a\sin x + b\cos x$, where a and b are constants. Considering whether any further problems will arise, the derivative of $\sin x$ is $\cos x$ and the derivative of $\cos x$ is $-\sin x$; we already have terms of both these types in $y_p(x)$, so it ought to work. (As we have already seen, it does.)

For another example of this approach, consider the equation

$$y'' - 3y' + 2y = 5x\cos x\,. \tag{3}$$

Here $y_p(x)$ will contain an $x\cos x$ term; moreover,

(i) $\frac{d}{dx}(x\cos x) = \cos x - x\sin x$, so we should put a $\cos x$ term and an $x\sin x$ term into $y_p(x)$;

(ii) $\frac{d}{dx}\cos x = -\sin x$, so we add a $\sin x$ term, and $\frac{d}{dx}(x\sin x) = x\cos x + \sin x$ – we have already got terms like these;

(iii) $\frac{d}{dx}\sin x = \cos x$ and a $\sin x$ term has already been included.

Therefore we will try

$$y_p(x) = ax\cos x + b\cos x + cx\sin x + d\sin x\,.$$

It is now routine (though, be it admitted, rather messy) to show that in fact

$$\tfrac{1}{2}x\cos x - \tfrac{3}{5}\cos x - \tfrac{3}{2}x\sin x - \tfrac{17}{10}\sin x$$

is a particular solution of (3).

This method will not always work. Consider

$$y'' - 3y' + 2y = \frac{1}{x}\,. \qquad (4)$$

Here we would like $y_p(x)$ to involve $1/x$, its derivative $-1/x^2$, its derivative $2/x^3$ and so on. In this case it is clear that the process will never stop, and we will not be able to write a (finite) expression for $y_p(x)$. In fact (4) can be solved by *variation of parameters*, a method which you may learn in more advanced courses.

One further problem which may arise is illustrated by the example

$$y'' - 3y' + 2y = 2e^{2x}\,. \qquad (5)$$

Trying (according to the above ideas) a particular solution $y_p(x) = ae^{2x}$, we find

$$4ae^{2x} - 3\times 2ae^{2x} + 2ae^{2x} = 2e^{2x}\,.$$

which has no solution since the left hand side simplifies to zero. It is seen that the reason for this phenomenon is that our choice for $y_p(x)$ is already a solution of the homogeneous equation

$$y'' - 3y' + 2y = 0\,.$$

The rule of thumb in this case is to *multiply our previous attempt by* x. Trying $y_p(x) = axe^{2x}$, we have

$$y_p'(x) = a(1+2x)e^{2x}\,, \qquad y_p''(x) = a(4+4x)e^{2x}$$

and so

$$a(4+4x) - 3a(1+2x) + 2ax = 2\,.$$

On multiplying out it is found that the terms in x cancel and we are left with $a = 2$. Hence the general solution of (5) is

$$y = Ae^{x} + Be^{2x} + 2xe^{2x}\,.$$

We conclude with a fully worked example: solve

$$y'' + 6y' + 9y = e^{-3x}\,, \qquad (6)$$
$$y(0) = \tfrac{1}{2}\,, \quad y'(0) = -\tfrac{1}{2}\,. \qquad (7)$$

First we solve the characteristic equation:

$$\lambda^2 + 6\lambda + 9 = 0 \implies \lambda = -3\,;$$

therefore by p.258 the general solution of $y'' + 6y' + 9y = 0$ is

$$y = Ae^{-3x} + Bxe^{-3x}\,.$$

Our first thought for a particular solution of (6) would be $y_p(x) = ae^{-3x}$; but as this is already a solution of the homogeneous equation we try $y_p(x) = axe^{-3x}$. However this is still a solution of the homogeneous equation, so we multiply once more by x and try $y_p(x) = ax^2e^{-3x}$. Then

$$y_p'(x) = a(2x - 3x^2)e^{-3x}\,, \qquad y_p''(x) = a(2 - 12x + 9x^2)e^{-3x}\,;$$

hence

$$a(2 - 12x + 9x^2)e^{-3x} + 6a(2x - 3x^2)e^{-3x} + 9ax^2e^{-3x} = e^{-3x}\,,$$

which simplifies to give $a = \frac{1}{2}$. Thus the general solution of (6) is

$$y = (A + Bx + \tfrac{1}{2}x^2)e^{-3x}\,.$$

To find A and B subject to the initial conditions (7), we calculate the derivative $y' = ((-3A + B) + (1 - 3B)x - \frac{3}{2}x^2)e^{-3x}$, and then we have

$$A = y(0) = \tfrac{1}{2}\,, \qquad -3A + B = y'(0) = -\tfrac{1}{2}\,.$$

Thus the solution of (6) satisfying the initial conditions (7) is

$$y = (\tfrac{1}{2} + x + \tfrac{1}{2}x^2)e^{-3x} = \tfrac{1}{2}(1 + x)^2e^{-3x}\,.$$

Exercise. Check explicitly that trying (i) $y_p(x) = ae^{-3x}$, or (ii) $y_p(x) = axe^{-3x}$, does not give a particular solution of (6).

EXACT DIFFERENTIAL EQUATIONS. Another method of solving differential equations is based on our previous work on partial differentiation. Suppose that $f(x,y)$ is a function of two variables, and that y is an (unknown) function of x. Then

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} .$$

Now suppose that we have a differential equation of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 . \qquad (1)$$

If we can identify a function $f(x,y)$ such that $\frac{\partial f}{\partial x} = M$, $\frac{\partial f}{\partial y} = N$, then (1) reads

$$\frac{df}{dx} = 0 ,$$

and the solution is just $f(x,y) = C$, where C is an arbitrary constant.

Example. Consider the differential equation

$$y + (x + 4y^3)\frac{dy}{dx} = 0 . \qquad (2)$$

Let $f(x,y) = xy + y^4$. (How would we find f if we weren't told? — see below.) Then

$$\frac{\partial f}{\partial x} = y , \quad \frac{\partial f}{\partial y} = x + 4y^3 .$$

and so equation (2) is simply $\frac{df}{dx} = 0$, which has solution $f(x,y) = C$. The solution of (2) is thus

$$xy + y^4 = C ,$$

an equation which cannot easily be solved to give y in terms of x.

Example. If the right hand side, instead of being zero, were a function of x, the solution would be almost identical. For instance, the equation

$$y + (x + 4y^3)\frac{dy}{dx} = 3x^2 \qquad (3)$$

can be written

$$\frac{df}{dx} = 3x^2 ,$$

where $f(x,y)$ is as in the previous example. The solution of (3) is therefore $f(x,y) = x^3 + C$, that is,

$$xy + y^4 = x^3 + C .$$

Notation. It is common to write (1) as

$$M(x,y)\,dx + N(x,y)\,dy = 0 ; \qquad (4)$$

the method of solution given above then consists of finding a function f whose *exact differential*, that is, the expression

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy ,$$

is equal to the left hand side of (4). Then, as above, the solution of the equation $df = 0$ is $f = C$. For this reason, equations of the form (1) or (4) are called *exact differential equations.*

Consider an equation

$$M\,dx + N\,dy = 0 .$$

For this to be an exact differential equation, there must be a function f such that

$$\frac{\partial f}{\partial x} = M , \quad \frac{\partial f}{\partial y} = N . \qquad (5)$$

But to find f from (5) all we have to do is to integrate.

Example. Solve

$$y\cos x\,dx + (2y + \sin x)\,dy = 0 . \qquad (6)$$

Solution. We seek a function $f(x,y)$ such that

$$\frac{\partial f}{\partial x} = y\cos x , \quad \frac{\partial f}{\partial y} = 2y + \sin x . \qquad (7)$$

We can integrate the first equation with respect to x; note that since we have a *partial* derivative, y is to be regarded as a constant. Therefore

$$f(x,y) = y\sin x + C .$$

It must be observed that since y has been taken to be constant, an expression involving y may occur in the "constant" C. To emphasize this possibility we normally write the previous equation as

$$f(x,y) = y\sin x + C(y) .$$

We can now find $C(y)$ by differentiating f with respect to y and matching up the answer with the known value of $\frac{\partial f}{\partial y}$ from (7):

$$\frac{\partial f}{\partial y} = \sin x + C'(y) = 2y + \sin x ,$$

which has a solution $C(y) = y^2$. Thus a suitable function is

$$f(x,y) = y\sin x + y^2 .$$

and the solution of (6) is

$$y \sin x - y^2 = \text{constant} .$$

Example. Solve

$$(x + y^2)\,dx + (x + y)\,dy = 0 . \tag{8}$$

Solution. Integrating $\frac{\partial f}{\partial x} = x + y^2$ gives

$$f(x,y) = \tfrac{1}{2}x^2 + xy^2 + C(y) :$$

therefore

$$\frac{\partial f}{\partial y} = 2xy + C'(y)$$

which should equal $x + y$ in order for us to find $C(y)$. But this yields

$$C'(y) = x(1 - 2y) - y ,$$

which is impossible since $C'(y)$ does not depend on x. Therefore (8) is not an exact differential equation.

We can test an expression $M\,dx + N\,dy$ to see if it is an exact differential, without actually having to attempt the above process: for if the expression is the exact differential of a function f, then*

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

and therefore

$$\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial N}{\partial x} .$$

Hence

Theorem. If $M\,dx + N\,dy$ is the exact differential of a (suitable) function f, then

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} .$$

Example. Solve

$$\cos y\,dx - (1 + x)\sin y\,dy = 0 \tag{9}$$

with the initial condition $x = 0$ when $y = 0$.
Solution. We have

$$M = \cos y , \quad N = -(1 + x)\sin y .$$

* under certain mild conditions on f: see page S18.

$$\frac{\partial M}{\partial y} = -\sin y = \frac{\partial N}{\partial x} ,$$

and the left hand side of (9) is therefore exact. Putting

$$\frac{\partial f}{\partial x} = \cos y , \quad \frac{\partial f}{\partial y} = -(1 + x)\sin y ,$$

we have

$$f(x,y) = x \cos y + C(y)$$

and so

$$\frac{\partial f}{\partial y} = -x \sin y + C'(y) = -(1 + x)\sin y ;$$

therefore $C(y) = \cos y$. Hence $f(x,y) = (1 + x)\cos y$ and the solution of (9) is

$$(1 + x)\cos y = C .$$

Substituting the initial condition $x = 0$, $y = 0$ gives $C = 1$ and so the required solution is

$$(1 + x)\cos y = 1 .$$

Exercises. Determine whether the following equations are exact, and if they are, solve them (subject to initial conditions if given). Solve the equations also by other methods if possible.

(a) $(\sin 2x - \cos 2y + 2x\cos 2x)\,dx + 2x \sin 2y\,dy = 0$, $y = 0$ when $x = \pi$;

(b) $xe^y\,dx - ye^x\,dy = 0$;

(c) $e^{y^2}\,dx - 2xye^{y^2}\,dy = 2xe^{x^2}\,dx$;

(d) $2xy\,dx + (x^2 + 3y^2)dy = 0$, $y = 1$ when $x = 1$.