THE UNIVERSITY OF NEW ENGLAND

DEPARTMENT OF MATHEMATICS

METRIC SPACES - TUTORIAL SHEET 1

1. Using $|x + y| \leqslant |x| + |y|$, show that $|x-y| \leqslant |x-z| + |z-y|$.
   Show that $d_1(x,y) = |x-y|$ is a metric on R.
   (i.e. Verify that d_1 satisfies $M_1, \ldots, M_4$)
   Sketch the following subsets of R:

   $$\{y: d_1(0,y) < 1\}, \quad \{y: d_1(1,y) = 1\}$$

   $$\{y: d_1(1,y) = 2\}$$

2. $d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$ is a metric on R.

   Sketch the following sets $\{y: d(0,y) < 1\}$
   $\{y: d(0,y) = 1\}$
   $\{y: d(0,y) = 2\}$

3. Let $\underset{\sim}{x} = (x_1, x_2)$ be an element of R^2

   $d(\underset{\sim}{x},\underset{\sim}{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$

   Sketch the sets $\{(1,y): d((0,0), (1,y)) = 1\}$
   $\{(x,y): d((0,0), (x,y)) = 1\}$
   $\{(x,y): d((1,1), (x,y)) = 1\}$
   $\{(x,y): d((1,1), (x,y)) = \frac{1}{2}\}$

   Sketch the region where: $d(\underset{\sim}{x},\underset{\sim}{y}) \leqslant 1$, when $\underset{\sim}{x}$ lies in the line segment
   $\underset{\sim}{x} = (0,a)$, $|a| < 1$

4. Repeat Q3 for the metrics

   $$d_1(\underset{\sim}{x},\underset{\sim}{y}) = |x_1 - y_1| + |x_2 - y_2|$$

   $$d_2(\underset{\sim}{x},\underset{\sim}{y}) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$$

METRIC SPACES - TUTORIAL SHEET 2

1. Define $\|\underset{\sim}{x}\| = \max\{|x_1|, |x_2|\}$ for $\underset{\sim}{x} = (x_1, x_2) \in R^2$.

   (a) Show that $\|.\|$ is a norm on R^2

   (b) Sketch the sets $\{\underset{\sim}{x}: \|\underset{\sim}{x}\| = \lambda\}$ for various values of λ.
   How are these sets similar?
   Is this the behaviour you would expect for every norm?
   (Consider n3.)

   (c) Sketch the sets $\{\underset{\sim}{x}: \|\underset{\sim}{x}-\underset{\sim}{y}_0\| = \lambda\}$ for a fixed $\underset{\sim}{y}_0$.
   How are these related to the sets in (b).

2. For the discrete metric, $d(\underset{\sim}{x},\underset{\sim}{y}) = \begin{cases} 0 & \text{if } \underset{\sim}{x} = \underset{\sim}{y} \\ 1 & \text{if } \underset{\sim}{x} \neq \underset{\sim}{y} \end{cases}$

   Here $\underset{\sim}{x} = (x_1, x_2) \in R^2$.

   Sketch $d(\underset{\sim}{x},0) = \lambda$, and $d(\underset{\sim}{x},\underset{\sim}{y}) = \lambda$ for various values of λ.
   Compare these sketches with Q1 (b) and (c).
   Prove that the discrete metric cannot be induced by a norm.

3. $P_1[0,1]$ is the linear space of all polynomials of degree 1, i.e. things like $p(x) = ax + b$, $x \in [0,1]$.

   Define $\|p(x)\|_\infty = \max_{x\in[0,1]} |p(x)|$.

   (a) Show that $\|.\|_\infty$ is a norm on $P_1[0,1]$ (sketch things in most cases).

   (b) Describe (by a sketch, or in words) the sets

   $$\|p(x)\|_\infty = 1, \quad \|p(x)\|_\infty \leqslant 1.$$
   $$\|p(x)-x\|_\infty = 1 .$$

4. Try a similar thing when the norm is

   $$\|p(x)\|_1 = \int_0^1 |p(x)|\,dx.$$

   Part (b) is more difficult to describe.

5. Define $d(x,y) = \begin{cases} 0 & \text{if } x = y \\ |x|+|y| & x \neq y \end{cases}$

   $(x,y \in R)$

   Show that it is a metric on R.

   (a) Sketch the sets

   $$d(x,0) \leqslant 1, \quad d(x,0) \leqslant 2, \ldots$$

   and describe $d(x,0) = \lambda$.

5. (cont'd)

(b) Sketch $d(x,1) = 1$, $d(x,1) \leqslant \frac{3}{2}$, 2, ...
and describe $d(x,1) \leqslant \lambda$

(d) Finally describe $d(x,y) \leqslant \lambda$.

In view of the properties of a norm (e.g. the pictures in Q1), do you expect this metric can be induced from a norm? Prove your conjecture.

6. Define
$$d(p(x),q(x)) = \frac{\|p(x)-q(x)\|_\infty}{1 + \|p(x)-q(x)\|_\infty}$$

where $\|p(x)\|_\infty$ is the norm from Q3.

Show that $d(p(x),(q(x))$ is a metric on $P_1[0,1]$.

Sketch the sets
$$d(p(x),0) < 1$$
$$d(p(x),0) = \tfrac{1}{2}$$
$$d(p(x),0) \leqslant \tfrac{1}{2}$$
$$d(p(x),0) = 2$$

Show that $d(p(x),q(x))$ is not induced by any norm.

METRIC SPACES - TUTORIAL SHEET 3

1. In the space $P_1[0,1]$ with norm $\|p(x)\|_\infty$ defined as in Q.3 of Sheet 2, define

$$p_n(x) = 1 + \frac{x}{n} .$$

(a) Compute $\|p_n(x)\|_\infty$

$$\|p_n(x) - p_m(x)\|_\infty$$

(b) Show that $\{p_n(x)\}$ is a Cauchy Sequence

(e) Compute $\|p_n(x)-1\|_\infty$ and show that $\{p_n(x)\}$ is convergent. What is the limit of $\{p_n(x)\}$?

2. $C[0,1]$ is the set of continuous functions on the closed interval $[0,1]$. $\|f(x)\|_\infty = \max\limits_{x \in [0,1]} |f(x)|$ is a norm on this space.

Find sequences of elements $q_n(x)$ in $C[0,1]$ such that $\|q_n(x)\|_\infty < 1$ and $q_n(x)$ converges to

(a) $q(x) = -1 + x$

(b) $q(x) = x^2$

(c) $q(x) = \sin \pi x$.

(Note that every element of $P_1[0,1]$ is in $C[0,1]$.

3. Let X be the class of all finite subsets of R. (The empty set is one of these subsets.) For any x, y $\in$ X define $d_0(x,y)$ to be the number of elements in the symmetric difference $x \Delta y$. $[x \Delta y = (x \cap y') \cup (y \cap x')]$ d_0 is a metric. (Show this.)

(a) Take a sequence of elements $x_n \in X$ convergent to $y \in X$. Can x_n differ from y for all values of n? (Notice that $d_0(x,y)$ can only take integer values.)

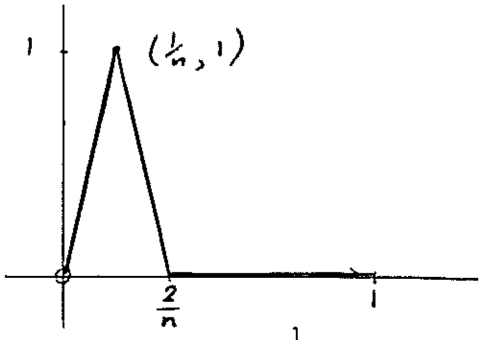
(b) Does the sequence

$$x_1 = \{1\},\ x_2 = \{\tfrac{1}{2},\ 1\},\ \ldots,\ x_n = \left\{\frac{1}{n},\ \frac{2}{n},\ \ldots,\ \frac{n-1}{n},\ 1\right\},\ \ldots$$

converge?

(c) Let $\{x_n\}$ be a Cauchy sequence in X. Does it converge? (Refer to (a).) Is the space X with this norm, complete?

4. Consider the set of continuous functions on [0,1] that look like



(a) Using $\|f(x)\| = \int_0^1 |f(x)|\,dx$, does the sequence $f_n(x)$ converge?

(b) Using $\|f(x)\| = \max_{x \in [0,1]} |f(x)|$, does $\{f_n(x)\}$ converge?