PRINCIPLES OF MATHEMATICS 170–1

for Economics

by Brailey Sims

PRINCIPLES OF MATHEMATICS 171-1
For Economics

Principles of Mathematics 170-1, for Economics

PREFACE

The course Principles of Mathematics 170-1 is an introduction to more advanced mathematics specifically designed for Economics and business students. Wherever possible the mathematics has been evolved from an economic situation, developed and then illustrated by reapplying it to further economic situations. None-the-less, a prior knowledge of Economics is not assumed. The economic situations chosen for consideration are relatively self contained and are developed within the notes.

The course is normally taught at the rate of three lectures and one practical class a week for a thirteen week semester. To complete the course in this time the following *work schedule* should be adhered to.

Week 1 Chapter 1, 2.1, 2.2, 2.3, 2.4

Week 2 2.5, 2.6, Chapter 3

Week 3 Chapters 4 and 5

Week 4 Chapter 6

Week 5 Chapters 7 and 8

Week 6 Chapter 9, 10.1, 10.2

Week 7 10.3, 10.4, Chapter 11 to end of 11.1

Week 8 11.2, 11.3, Chapter 12

Week 9 Chapter 13

Week 10 Chapter 14, Chapter 15 to end of 15.1

Week 11 15.2, 15.3, 16.1, 16.2, 16.3

Week 12 16.4, 16.5, Chapter 17

Week 13 Chapter 18

The course will be *assessed* by one three hour examination held in the November examination period. The whole semester's work is examinable.

Submission of *assignments* is compulsory. Failure to submit at least eight of the ten assignments may lead to a failure in the course being recorded. Performance on assignments will be taken into consideration if your exam result is borderline. Late assignments will not be accepted unless a satisfactory reason, such as illness certified by a Doctor's certificate, is provided.

For internal students *attendance at practical classes* is also compulsory unless special exemption has been granted. Failure to attend at least nine of the practical classes without acceptable reasons, such as illness certified by a Doctor's certificate, may lead to a failure in the course being recorded.

The *reference book*

> MATHEMATICAL ANALYSIS *for Business and Economics*, by Jagdish Arya and Robin Lardner, Prentice-Hall

is a good source of additional problems, worked examples, and an alternative approach for much of the material in the course. Chapters 4, 5, 6, 9, 10, 11.1, 12, 13, 14, 15, and 18 are particularly relevant. This book is especially recommended to students with a weak mathematical background. For such students working through chapters 1, 2, and 3 would be good preparation for the course.

Students with a strong mathematical background who wish to pursue in greater detail topics introduced in these notes are referred to the book

> Mathematical methods in Accountancy, Economics and Finance, by Daniel Leonard, Prentice-Hall of Australia.

Remember, Mathematics is a *doing subject* and can only be mastered by practice and perseverance. If you can't do a problem (we all experience this) go back and look at any similar problems you have done or that are done in the notes, reread relevant sections of the notes, then try again. If you still can't succeed leave the problem aside and go on to something else. Come back to it at a later time. It is important that the study you do is effective. Without looking at notes, ask yourself "what did I study yesterday (last week), what type of problems did I do and what did I use to do them?" If your answer to any of these, or similar questions, is negative, revision is called for, otherwise the work you have done will be wasted. Learn from your mistakes, make sure you know what they were so you can avoid commiting them again.

If you have any specific questions concerning the work, or the course in general, please write directly to me at the Department of Mathematics, Statistics and Computing Science, or phone me on (067) 732119.

Wishing you enjoyment and success from your studies.

Dr Brailey Sims
May 1989

PRINCIPLES OF MATHEMATICS 170-1

ERRATA

Table of contents $\ell 1$ Principles of Mathematics 170-1

Table of contents $\ell 16$ Inverse Matrices

p8	$\ell 4$	$\cdots$ This cost may $\cdots$
p12	$\ell 15$	So $x_0 = 5$ and then $y_0 = \underline{1 \cdot 1} \times 5 + 2 = 7 - 5. \cdots$
p18	$\ell 17$	$\cdots$ to the general expression $ax^{\underline{2}} + bx + c, \cdots$
p18	$\ell 21$	$\cdots \; c - \frac{b^2}{4a} \; \cdots$
p19	$\ell 14$	parabola opens upward (is convex) with $\cdots$
	$\ell 21$	$\cdots$ opening downward (concave) with $\cdots$
p20	$\ell 9$	$\cdots$ parabola
p21	$\ell 2$	$a(x + \frac{b}{2a})^2 = \cdots$
p21	$\ell 25$	is such that the
p21	$\ell 29$	$Kp^2 - (M + cK)p + (F + c\underline{M}) = 0$
p24	$\ell 7$	$\cdots \; y = -x^2 + 6x - 5 \cdots$
p39	$\ell 21$	$R = p(\underline{M} - Kp)$
p46	$\ell 4$	$R(p) = 10p - 0.2p^2 \cdots$
p51	$\ell 7$	Value after 18 months is $V(1.5) = \underline{1000} e^{0.15} = \$1161.83$
p70		On the diagram q_{max} should be level with the dashed horizontal line and not the p-axis.
p73		On the second diagram C_1 and C_2 should be c_1 and c_2
p82		before $\ell 6$ add "which gives the parallelogramme rule for
p83		on the second diagram B should be the point $(0, y_2)$ not $(0, y_1)$.
p88	$\ell 12$	sometimes convenient to also regard it as the $n \times 1$ matrix
p105		In the tableau at the bottom of the page the zero's above the diagonal should be deleted.
p108	$\ell 7$	should read $A^{-1} = \frac{1}{ad - bc} = \begin{bmatrix} \underline{d} & -b \\ -c & a \end{bmatrix}$
p111	$\ell 4$	obtain a nearby point $\cdots$
p117	$\ell 21$	$= \frac{d}{dv}(v^m) \times \frac{dv}{d\underline{x}}$

PRINCIPLES OF MATHEMATICS 170-1

ERRATA - continued

p31 $\cdots$ we have $p(-1) = -6$, so $\cdots$

39 on the graph $y = (T + C)(p)$ should read $y = C(p) = (T + P)(p)$

$y = C(p)$ should read $y = P(p)$

p40 $\ell 4$ $= -Kp^2 + (M + cK)p - (F + cM)$

$\ell 6$ $\cdots (M + cK)/2K. \cdots$

p51 $\ell 17$ $(1 + \frac{x}{3})^3 = 1 + x + x^2/3 + x^3/27$

$\ell 19$ $(1 + \frac{x}{4})^4 = 1 + x + \frac{3x^2}{8} + \frac{x^3}{16} + \frac{x^4}{256}$

p53 Exercise (8) $S = S_0(1 - e^{-at})$

p65 $\ell 1$ $\cdots$ of approximately 3 $\cdots$

$\ell 3$ $y = 2x^3$.

p71 $\ell 25$ $4 = \frac{1}{\frac{5}{4}-1} \not\leq (\frac{5}{4})^2 + (\frac{5}{4}) - 1 = \frac{29}{16}$

p131 Third column second compartment of the table should read "The value of $f'(x)$ at a nearby point to the left of x_0 is strictly negative, while $\cdots$

p142 last line $\cdots \frac{\partial Q}{\partial L}$ is the marginal productivity of labour $\cdots$

p144 $\ell 19$ $\delta z = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y$

p145 $\ell 21$ $= 2900 + 10000(p - 2.9) + 2.9(q - 10000)$

p149 last line $L(x, y, \lambda) = 2\ell n(x - 5) + 3\ell n(y - 5) + \lambda(x + 2y - 99)$

p150 $\ell 5$ $\cdots \frac{2}{x-5} = -\lambda = \frac{3/2}{y-5}$ and so $4(y - 5) = 3(x - 5), \cdots$

$\ell 6$ $3x - 4y + 5 = 0$

$\ell 8$ $x = 193/5 = 38.6$

$\ell 10$ $y = 30.2$

$\ell 11$ $\cdots$ is $x = 38.6, y = 30.2$

$\ell 12$ $\cdots (38.6, 30.2) \cdots$

P86 input $P = (150, 100)$ → output $M = (600, 175$

P99 3rd ↔ 4th tableau

P100 Ex 2 ref. should be to Ex 12.5(1)

P108 $\ell 2$ $\left[\begin{array}{cc|} a & b \\ 0 & d - \frac{cb}{a} \end{array}\right.$ ....

P147 $\ell 1$ ... both be negative if

1. INTRODUCTION

1.1 Concept of a Function

Central to models of many economic situations is the concept of a function.

For each value of a quantity x, a **function** f assigns a unique *value* $f(x)$.

For Example: If q is the number of units of a certain commodity sold in any one day the net profit from one day's transactions may be a function P of q. For any day in which 10 units are sold the profit will be $\$P(10)$.

In mathematics functions are usually denoted by the letters $f,\ g,\ h,\ \ldots$, letters at the end of the alphabet; $x,\ y,\ z,\ u,\ v,\ w,\ ,t$ etc. are used for variables and the letters $a,\ b,\ c,\ \ldots$ are used to denote constants. In the particular context of an application we will often use more meaningful symbols; P or p for profit or for price, R or r for revenue, t for time, W or w for wheat production (in tonnes, say), etc.

Typically, for a general function f we will write $y = f(x)$ and refer to x as the **independent variable** and to y as the **dependent variable** (its value depends on those of x). The allowable values of x constitute the **domain** of the function f. Unless otherwise stated we will usually take the domain of such a function to be all real numbers. In applications, the domain may be smaller; for instance, in our example where q is the number of units sold in one day, the only economically meaningful values that can occur are positive numbers.

The domain of a function may consist of things that are not numbers. It might for example be a list of stock exchanges with the function assigning to each stock exchange the value of some appropriate daily trading index. In such a case the function would usually be specified by means of a table. In simple cases when the domain and function-values are numbers, a function may be specified by an algebraic formula.

For Example. The net profit function in our earlier example might be given by the expression

$$P(q) = 85q - 0.9q^2 - 10.$$

If for a particular day, $q = 10$, we can readily calculate that the net profit is

$$P(10) = 85 \times 10 - 0.9 \times 10^2 - 10 = 850 - 90 - 10 = 750.$$

1.2 Graph of a Function

Graphs provide a particularly effective way of displaying interesting features of a function. This is accomplished through the introduction of **Cartesian coordinates** into the plane. We select two perpendicular lines as axes and prescribe an appropriate scale on each axis. This enables us to establish a correspondence between points P in the plane and ordered pairs of real numbers (x_P, y_P), as illustrated in Figure 1.1.

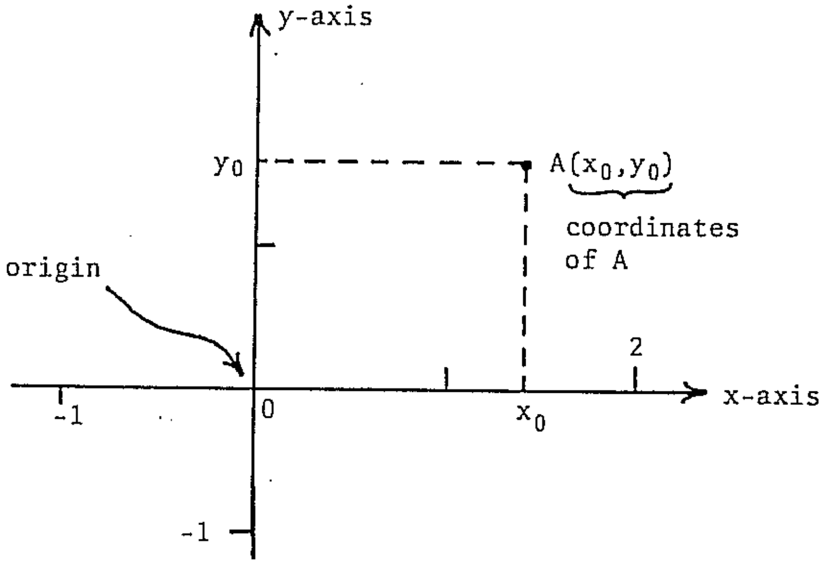


Figure 1.1

Let f be a function whose domain is an interval of real numbers. Each x in the domain determines a point (x, y) of the plane with $y = f(x)$. The set of all such points forms a *curve* which we call the **graph** of f. Since for a function there is a unique value $f(x)$ for each x, each vertical line cuts the graph of f at most once.

For Example. The function $P(q) = 85q - 0.9q^2 - 10$ has the graph illustrated in Figure 1.2.

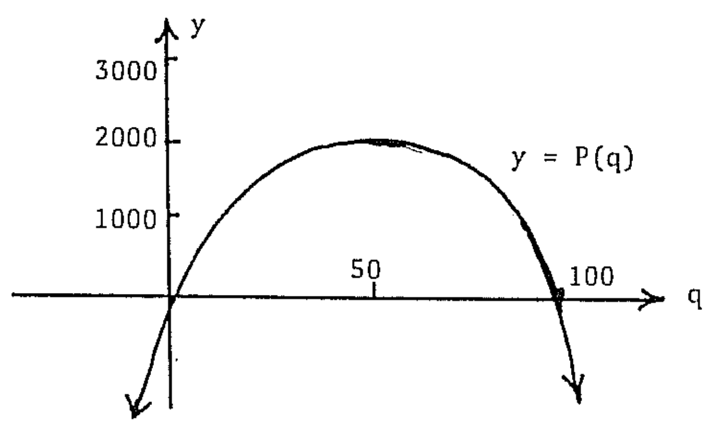


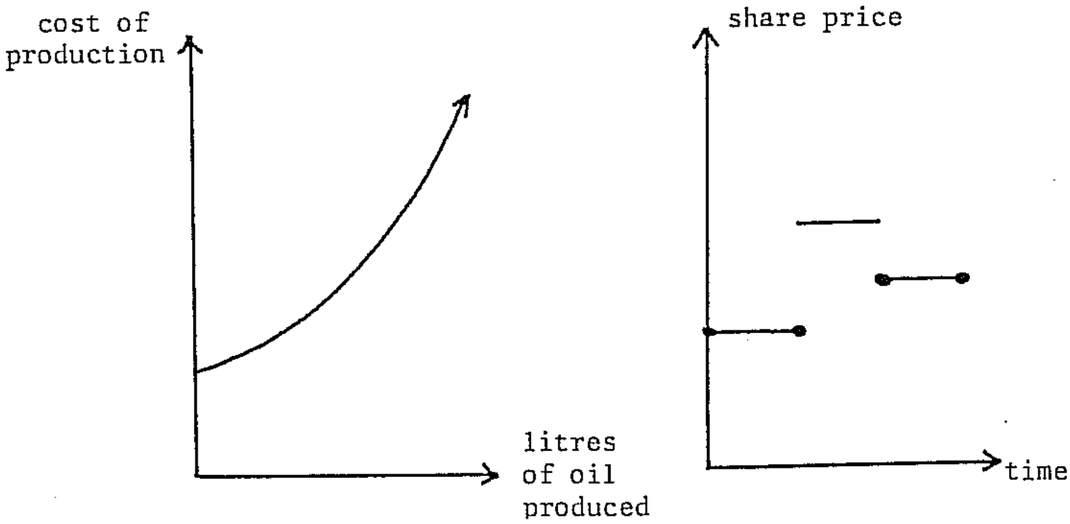
Figure 1.2. Graph of $y = 85p - 0.9p^2 - 10$

Being able to sketch the graphs of important types of functions is a major component of this course. In many cases, a sketch enables us to see "at a glance" important and economically significant features of a function. Much of this course is devoted to the analysis of such features, some of which we will now briefly review in a qualitative way.

1.3 Important Features of Functions

Continuity

A continuous function is one for which there are no sudden jumps in the value of the function. A function with jumps is said to be *discontinuous*.



A continuous function.

A discontinuous function

Figure 1.3

Intuitively a continuous function is one whose graph may be drawn without lifting our pen from the paper.

Smoothness

A function is smooth if there are no sharp corners on its graph, or more precisely, there is a unique tangent line at each point of the graph.

For Example. The function of figure 1.2 is smooth, while the function depicted in figure 1.4 is not.

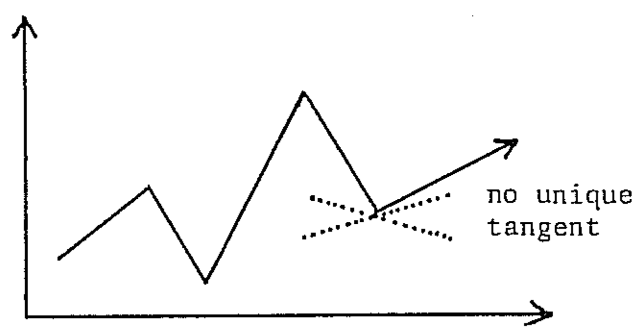
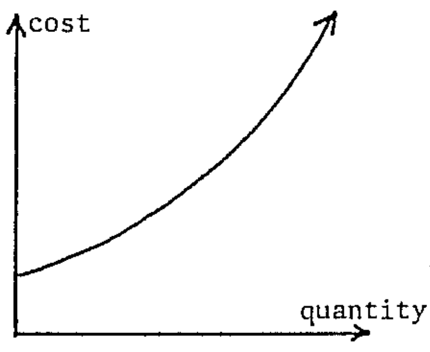


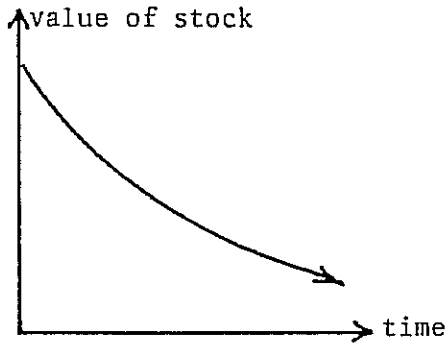
Figure 1.4

Monotonicity

A function $y = f(x)$ is **increasing** if its value is larger at larger values of x. A function is **decreasing** if its value is smaller at larger values of x. A function which is either decreasing or increasing is said to be **monotonic**.



An increasing function

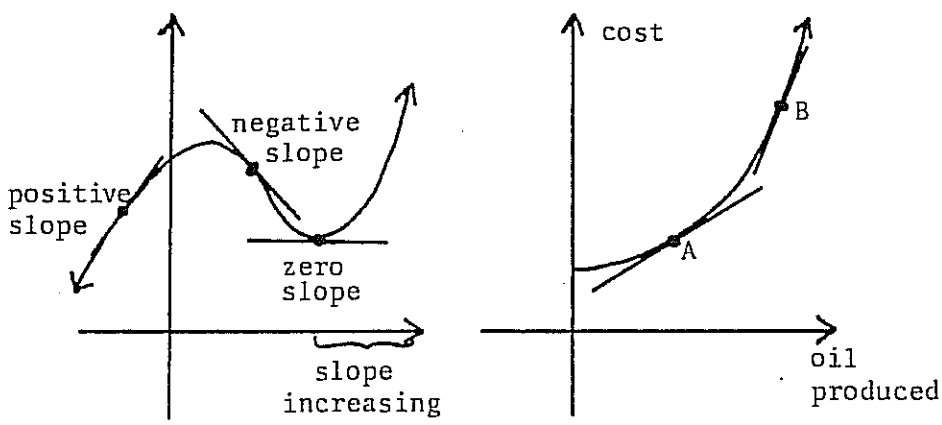


A decreasing function

Steepness

The 'steepness' or slope of a function measures how rapidly it is increasing (positive slope) or decreasing (negative slope). Economists often speak of slopes as *marginal quantities.* For example, if $y = f(x)$ represents the cost of production for x litres of oil, the marginal cost of production will be the slope of the graph. (Some books use 'gradient' in place of 'slope'.)

For Example.



(a) (b)

Figure 1.5

In figure 1.5(b) the marginal cost of production at B is greater than at A.

Asymptotic Behaviour

It is sometimes important to know how a function behaves for large values of the variable x. We refer to this as the *asymptotic behaviour* of $f(x)$ as x tend to infinity, or in symbols, as $x \to \infty$. It is also sometimes useful to know the behaviour as x tends to minus infinity, or in symbols, as $x \to -\infty$.

For Example. For the function in figure 1.6 the quantity Q of a good sold appears to approach a (saturation) value s as the quantity of the good available for sale q tends to ∞.

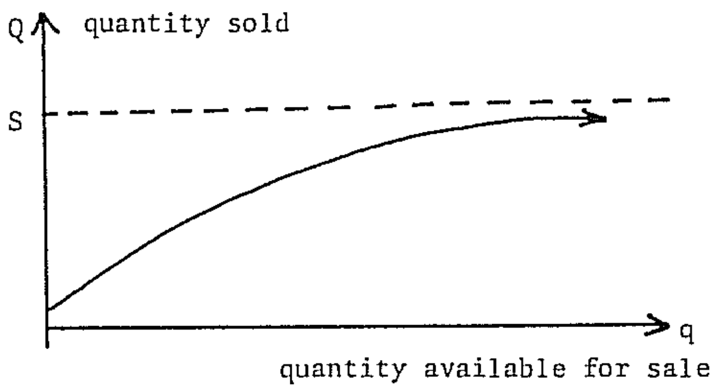


Figure 1.6

Maxima and minima

If at a particular point the value of a function is greater than or equal to the values at all nearby points, the function has a **local maximum** at that point. Similarly a **local minimum** is a point at which the value of the function is less than or equal to its value at all nearby points. If the value is greater than or equal to (less than or equal to) the value at any other point then we speak of a **global** maximum (minimum). Thus $f(x)$ has a global maximum at $x = x_0$ if for all x, $f(x)$ is less than or equal to $f(x_0)$.

If the function-values represent profits, we usually seek maxima; similarly, we usually aim to minimize losses. The most desirable value, whether maximum or minimum, is called the **optimum** value. Determining optimum values is obviously of considerable importance in economic theory.

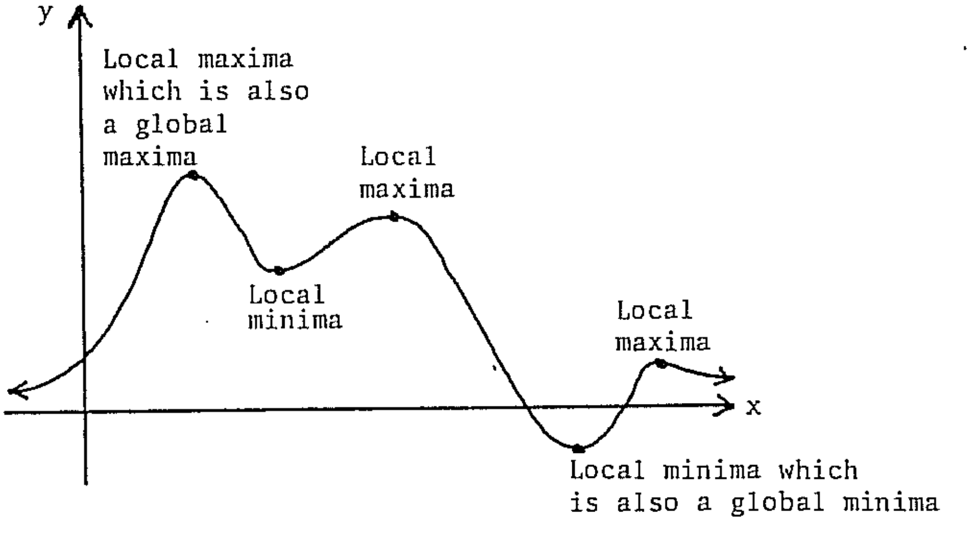


Figure 1.7

Convexity

Many functions considered in economics are **convex**. A function is convex if all chords (lines joining two point on the graph) lie above the graph; tangent lines then necessarily lie below the graph. For a convex function, any local minimum is a global minimum.

A function for which all chords lie below the graph is said to be **concave**.

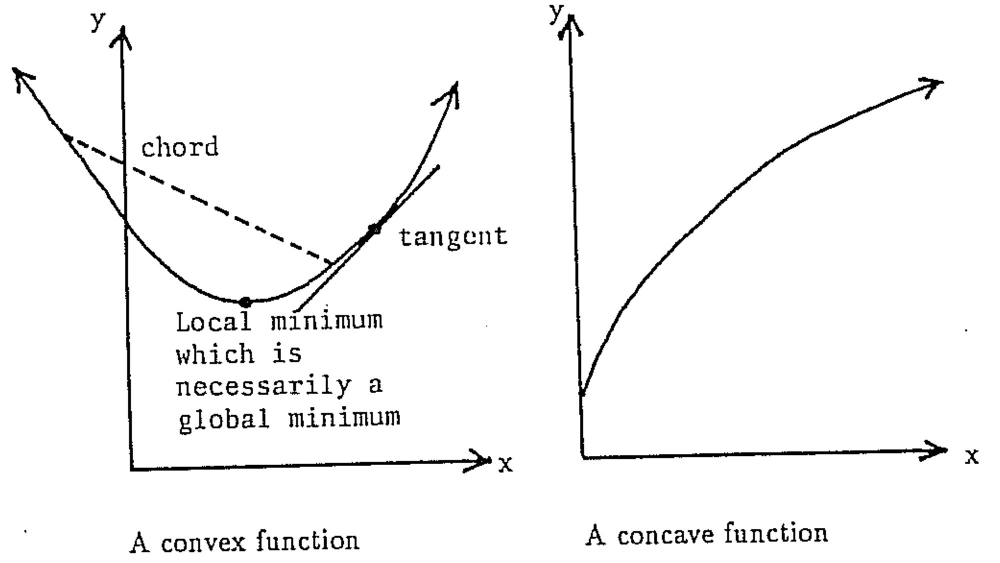


Figure 1.8

Roots

The roots of a function $f(x)$ are the points at which its graph cuts the x-axis. They correspond to solutions of the equation $f(x) = 0$. The location of roots is often important since they may give the boundaries of the regions where an enterprise operates at a profit, (If $f(x)$ represents a profit, positive values correspond to actual profits; negative values will represent losses.)

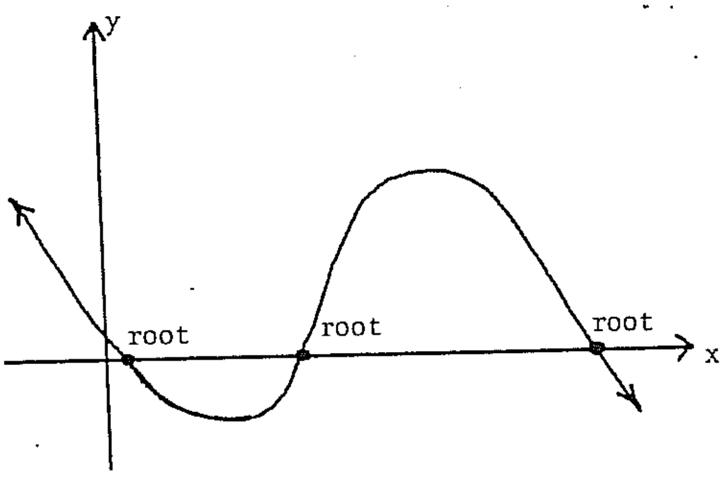


Figure 1.9

Intersections of Two Graphs

Often the points where two graphs intersect are particularly significant.

For Example. If $r(q)$ is the revenue and $c(q)$ the cost of production when q units are produced, the points of intersection of $y = r(q)$ and $y = c(q)$ give the break-even points.

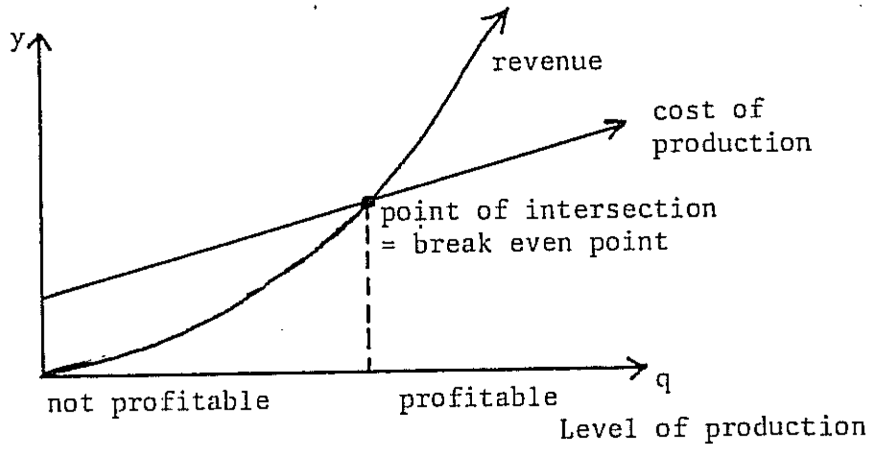


Figure 1.10

2. LINEAR FUNCTIONS

2.1 Linear Models

The following example is typical of those Economic situations which may admit the simplest type of mathematical description.

We wish to estimate the annual cost of supply for commodity, to be specific *electricity*, to community. This cost my be broken into two parts in the following way:

$$\begin{bmatrix}\text{annual cost}\\ \text{of supply}\end{bmatrix} = \begin{bmatrix}\text{fixed annual cost}\\ \text{for the maintenance}\\ \text{of buildings,}\\ \text{generating equipment,}\\ \text{power-lines, etc.,}\\ \text{including the cost of}\\ \text{labour.}\end{bmatrix} + \begin{bmatrix}\text{variable cost}\\ \text{depending on}\\ \text{the amount of}\\ \text{electricity}\\ \text{supplied in}\\ \text{the given year}\end{bmatrix}$$

The variable cost results from factors such as the cost of fuel needed to supply the generators and so is likely to be directly proportional to the amount of electricity produced (producing twice as much electricity will require twice as much fuel which will cost twice as much). If the cost of producing one unit of electricity is c (the marginal cost of electricity), then the cost of producing E units of electricity will be cE.

We therefore have

$$C = F + cE,$$

where:

C is the annual cost of supply,

F is the fixed annual cost, and

E is the number of units of electricity supplied in the given year.

This is one instance of a function $y = f(x)$ described by a formula of the form

$$y = mx + b$$

where m and b are constants. In our example $m = c$, $b = F$, and the variables x and y have been replaced by symbols E and C which are more meaningful in the particular context.

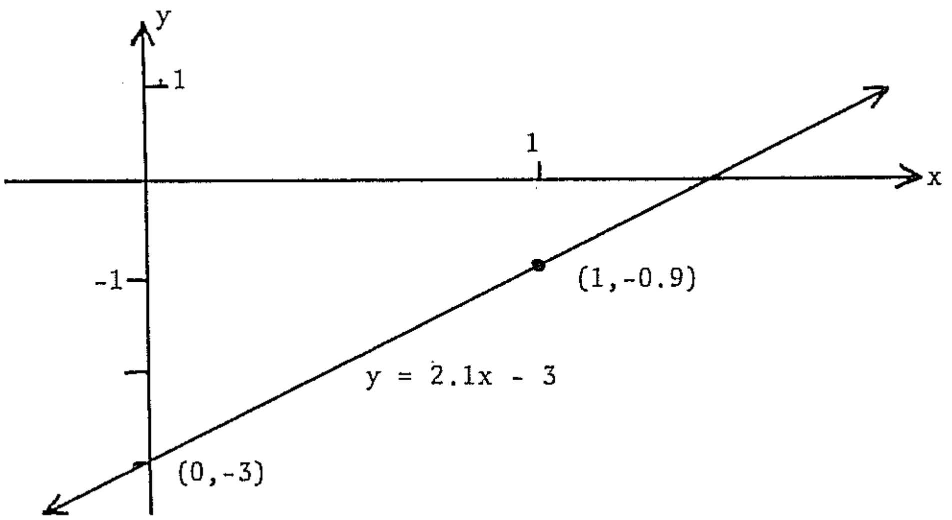
Functions of this form are referred to as *linear functions*. For example: $y = 2.1x - 3$, $y = x + 4$, $C = 2385000 + 0.87E$, $y = 7 - \frac{1}{2}x$ are all linear functions.

Fitting such functions to data is an important aspect of elementary econometrics. They are the simplest functions to work with. For example, if $y = 2.1x - 3$, then it is an easy calculation to see that when $x = 1.7$ we have $y = 2.1 \times 1.7 - 3 = 0.57$.

2.2 Graph of a Linear Function

The graph of a linear function is a straight line. Since a straight line is determined by any two points through which it passes, to draw the graph of a function of the form $y = mx + b$ we need only plot two points on it and then draw the straight line through them. Although any two points will do, the points with x-coordinates 0 and 1 are usually the easiest to calculate.

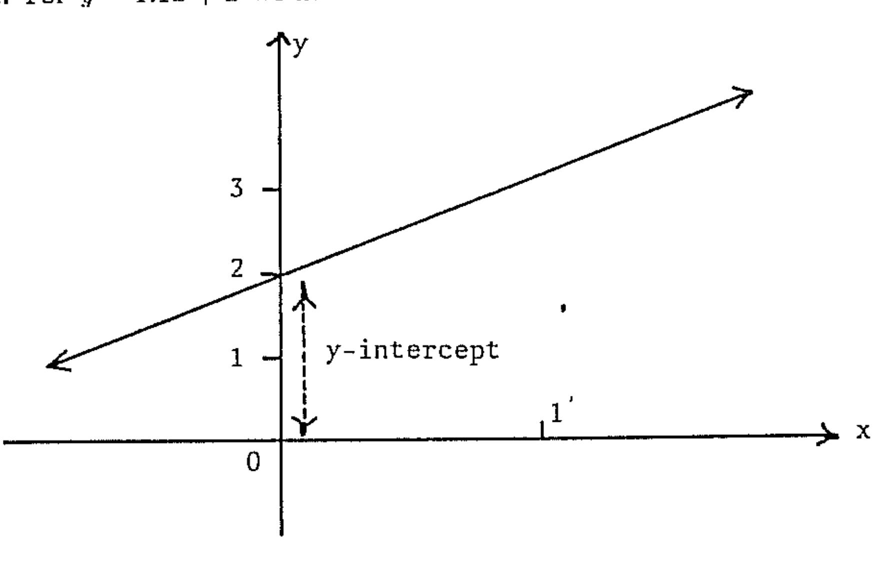
For example. To graph $y = 2.1x - 3$: substituting $x = 0$ we see that $y = -3$ so $(0, -3)$ is a point on the graph. Similarly, when $x = 1$, we have $y = 2.1 \times 1 - 3 = -0.9$, so the point $(1, -0.9)$ is also on the graph. Plotting these two points and drawing the line through them we obtain the graph.



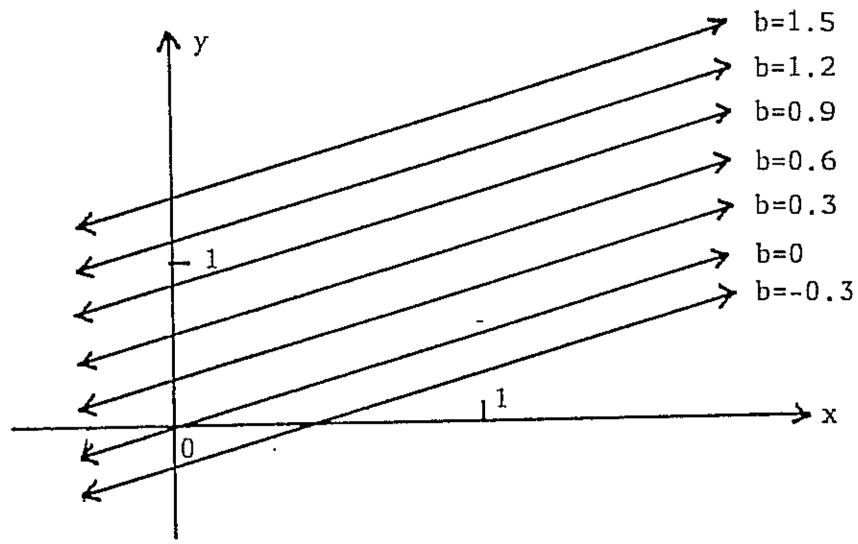
2.3 Vertical-Intercept

Putting $x = 0$ in $y = mx + b$ gives $y = b$; thus the point where the graph of $y = mx + b$ cuts the y-axis $(x = 0)$ has a y- coordinate of b. The value of b is referred to as the y-*intercept* of the line $y = mx + b$. The point is below the x-axis if b is negative, as in the graph above where $b = -3$, and above the x-axis when b is positive.

For example. For $y = 1.1x + 2$ we have



Keeping m fixed and changing b gives a family of parallel lines:



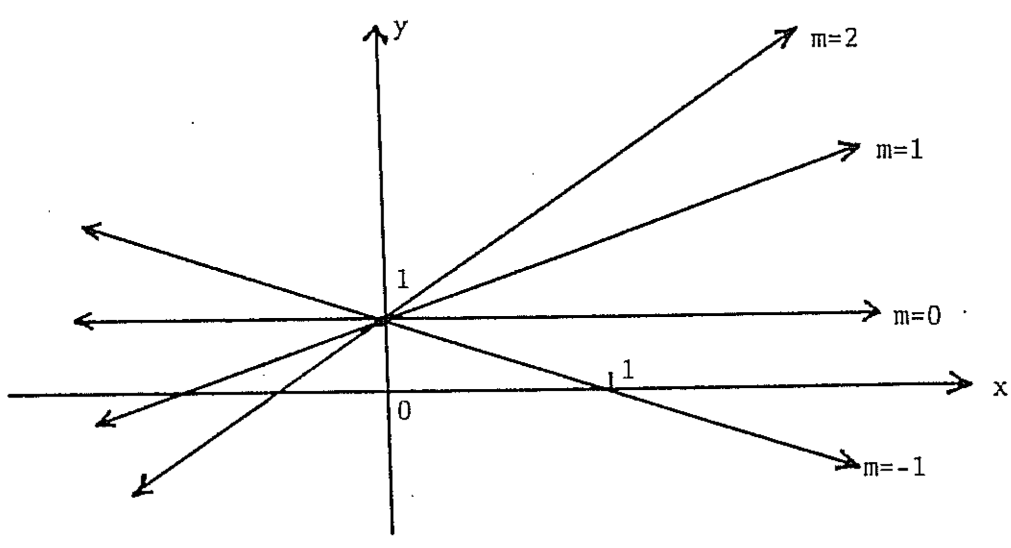
The family of lines $y = 0.6x + b$

The y- intercept often has an important interpretation, for example in our cost of supply model it represents the fixed cost F.

2.4 Slope of a Line

Varying m changes how steeply the line $y = mx + b$ slopes.

For example:

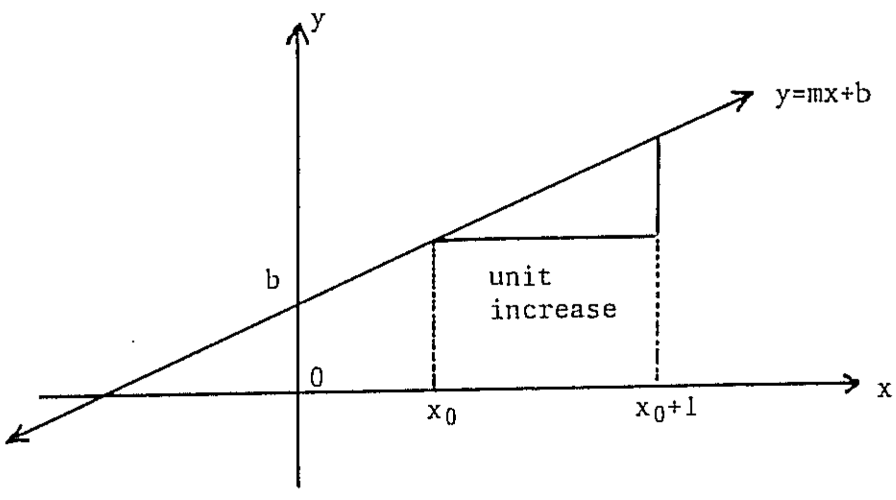


The effect of changing m in $y = mx + 1$

We call m the slope of the line $y = mx + b$.

Lines which slope upward from left to right have a positive slope. Sloping downward from left to right corresponds to a negative slope. Thus a linear function $y = mx + b$ is **increasing** if m is positive and **decreasing** if m is negative.

If x is increased by 1, from x_0 to $x_0 + 1$, y changes from $mx_0 + b$ to $m(x_0 + 1) + b = (mx_0 + b) + m$ and so the change in y corresponding to a unit increase in x is equal to the slope m.



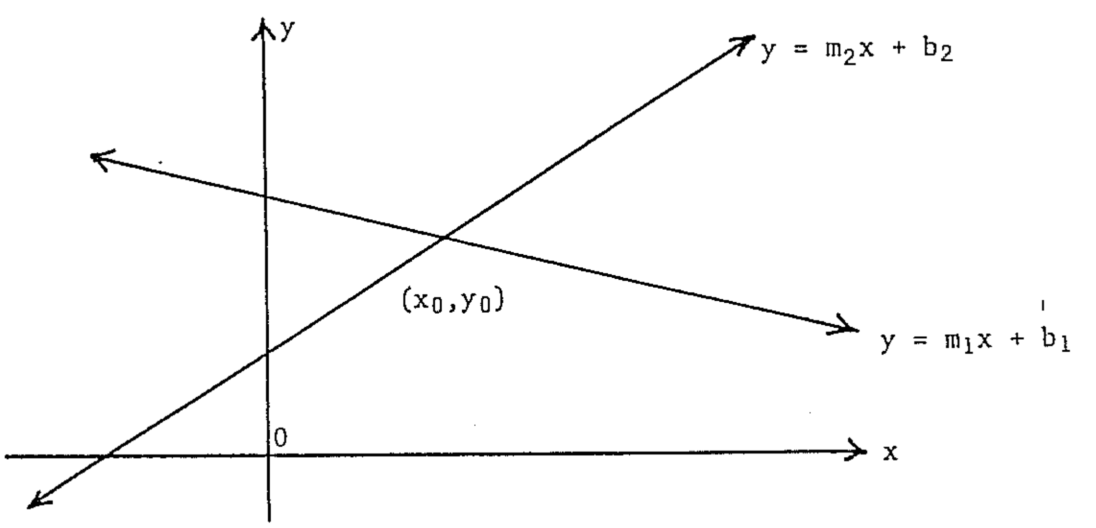
In Economics the slope therefore represents a marginal quantity. For example, in our cost of supply model the slope c was the marginal cost of electricity.

2.5 Intersection of Lines

It is frequently important to determine the point where the line $y = mx + b$ cuts the x-axis. This may be found by putting $y = 0$ and solving $mx + b = 0$ for x.

For example. For $y = 2.1x - 3$, when $y = 0$ we have $2.1x - 3 = 0$ and so $x = 3/2.1 = \frac{10}{7}$. Thus $y = 2.1x - 3$ cuts the x-axis at $x = 1\frac{3}{7}$.

More generally it is often important to determine the point (x_0, y_0) where the graphs of two linear functions, $y = m_1x + b_1$ and $y = m_2x + b_2$, intersect.



Since (x_0, y_0) lies on both the lines, we have that

$$y_0 = m_1x_0 + b_1 \quad \text{and} \quad y_0 = m_2x_0 + b_2 \ .$$

Subtracting these two equations:

$$\begin{array}{rrcrcl} & y_0 & = & m_1x_0 & + & b_1 \\ - & y_0 & = & m_2x_0 & + & b_2 \\ \hline & 0 & = & (m_1-m_2)x_0 & + & b_1-b_2 \end{array}$$

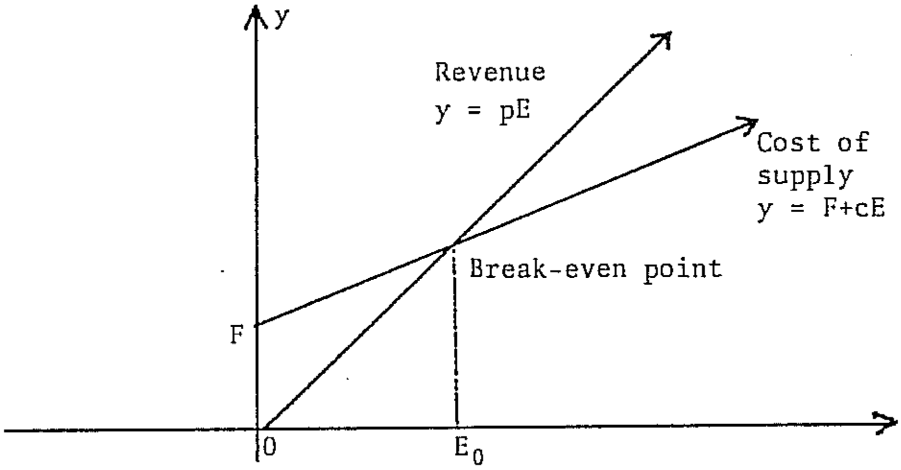
we obtain an equation which may be solved for x_0. The corresponding value y_0 of y may then be found by substituting the value x_0 for x into either of the two equations $y = m_1x + b_1$ or $y = m_2x + b_2$.

For example: to find the point of intersection (x_0, y_0) of the two lines $y = 1.1x + 2$ and $y = 2.1x - 3$, we have

$$\begin{array}{rrcrcl} & y_0 & = & 1.1x_0 & + & 2 \\ - & y_0 & = & 2.1x_0 & - & 3 \\ \hline & 0 & = & -x_0 & + & 5 \end{array}$$

so $x_0 = 5$ and then $y_0 = 1.5 \times 5 + 2 = 7.5$. Thus the point of intersection is $(5, 7.5)$.

To see how such a calculation might arise in practice, suppose our electricity supplier sells each unit of electricity to the community for a price p. If in a given year E units are sold, then the electricity supplier's revenue for the year is pE. Combining this with our model for the annual cost of supply we see that the **break-even point**, where revenue equals costs, corresponds to the point of intersection of the two lines $y = pE$ and $y = F + cE$.



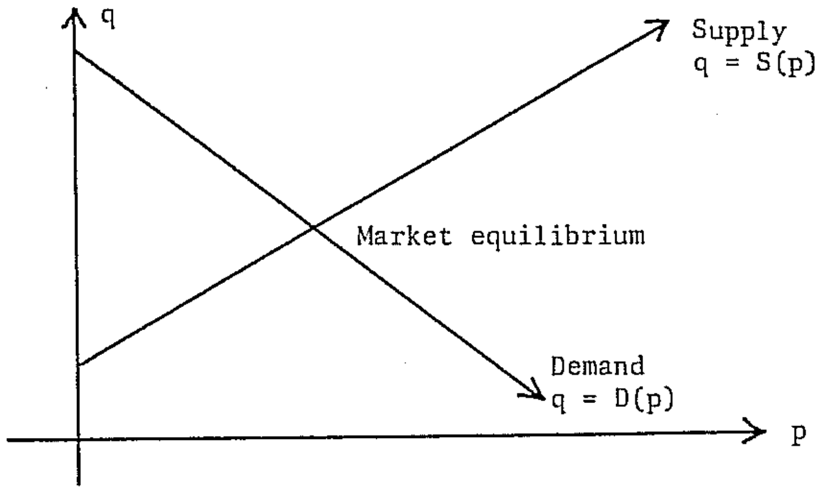
Subtracting the two equations gives $F + (c - p)E_0 = 0$ and so the break-even point is at

$$E_0 = \frac{F}{p - c}.$$

Supply and demand provides another example.

For a given commodity the demand function $D(p)$ equals the quantity of the commodity which will be sold when the price per unit is p. Typically it is a decreasing function of p.

The supply function $S(p)$ equals the quantity which supplies are prepared to sell at a price per unit of p. In general, it will be an increasing function of p. **Market Equilibrium** occurs when supply equals demand. This corresponds to the point of intersection of the two graphs $q = D(p)$ and $q = S(p)$, and so to the point of intersection of two lines when $D(p)$ and $S(p)$ are both linear functions.



2.6 Equation of a Line Given Two Points on it.

If we know the coordinates (x_1, y_1) and (x_2, y_2) of two points on a line we have enough information to determine its equation $y = mx + b$, that is, to find m and b. For the points to lie on the line we must have

$$y_1 = mx_1 + b \quad \text{and} \quad y_2 = mx_2 + b\ .$$

Subtracting these we have

$$\begin{array}{rrcrl} & y_1 & = & mx_1 & + \ b \\ - & y_2 & = & mx_2 & + \ b \\ \hline & y_1 - y_2 & = & m(x_1 - x_2) & \end{array}$$

and so

$$m = \frac{y_1 - y_2}{x_1 - x_2}\ .$$

(We may assume that $x_1 \neq x_2$ as otherwise the line is vertical and so is not the graph of a function.)

Substituting this value for m in either of the equations $y_1 = mx_1 + b$ or $y_2 = mx_2 + b$ allows us to find b, and hence the equation of the line.

For example. To find the equation of the line passing through the points $(-1, 3)$ and $(4, -2)$, we have

$$3 = m \times (-1) + b \quad \text{and} \quad -2 = m \times 4 + b\ .$$

Subtracting

$$\begin{array}{rrcrcl} & 3 & = & -m & + & b \\ - & -2 & = & 4m & + & b \\ \hline & 5 & = & -5m & & \end{array}$$

so $m = -1$ and then $3 = -(-1) + b = 1 + b$. giving $b = 2$. Thus the line has equation

$$y = 2 - x\ .$$

EXERCISES 2.7

(1) The cost $\$C$ of hiring a car is given by

$$C = 20 + (0.1)d$$

where d is the distance travelled in kilometres. Find C when $d = 300$ and when $d = 450$. Draw a graph of this relation, including the above values of d.

(2) Sketch a graph for each of the following:

(a) $y = 2x - 5$

(b) $y = 0.9x + 0.5$

(c) $y = 4 - 0.8x$

For each case state the slope and y-intercept of the line.

(3) (a) Find the slope and y-intercept for each of the following

(i) $y = -x + 4$

(ii) $y = 3.2x - 0.1$

(b) If $3x + 2y = 6$, express y as a function of x and hence find the slope and y-intercept.

(4) Find the equations of the straight lines satisfying

(a) slope 1/5 and y-intercept 7

(b) slope -8 and y-intercept 0

(5) (a) Find the slope of the line joining the points $(2,7)$ and $(4,3)$;

(b) Find the equation of this line;

(c) Find its intercept on the y-axis.

(6) Sketch the graphs of the following demand relations where p denotes the price per unit and q is the quantity demanded:

(a) $q = \frac{3}{2} - \frac{1}{2}p$

(b) $q = 5 - \frac{5}{2}p$.

(7) (**Break-even Analysis**) The fixed costs of producing a certain product are \$1000 per month and the variable cost is \$4 per unit. If the product sells for \$6.50 per unit, find

(a) The break- even point,

(b) The number of units that must be produced and sold each month to obtain a profit of \$5000 per month.

(8) Find the point of intersection of the two lines

$$y = 1 - \tfrac{2}{5}x$$
$$y = \tfrac{7}{3} - \tfrac{1}{3}x.$$

(9) **(Demand Analysis)** A manufacturer can sell 3000 units per month at \$5 each but only 2000 units per month at \$6 each. Determine the demand law, assuming it is linear.

(10) **(Market Equilibrium)** At a price of \$150 per tonne, the demand for a certain commodity is 2500 tonnes whereas the supply is 2000 tonnes. If the price is increased by \$50 per tonne the demand and supply will be 2200 tonnes and 3000 tonnes respectively.

(a) Assuming linearity, determine the laws of supply and demand.

(b) Find the equilibrium prices and quantity

(c) If an additional tax of \$5 per tonne is imposed on the supplier, find the increase in equilibrium price and decrease in equilibrium quantity.

(d) What subsidy should be given to the supplier so that the equilibrium quantity increases by 100 tonnes?

3. QUADRATIC FUNCTIONS

3.1 Quadratic Models

A manufacturer finds that if a certain commodity is offered to the market at a price per unit of p then the quantity q sold each month is given by the decreasing linear demand function

$$q = M - Kp,$$

where M and K are positive constants.

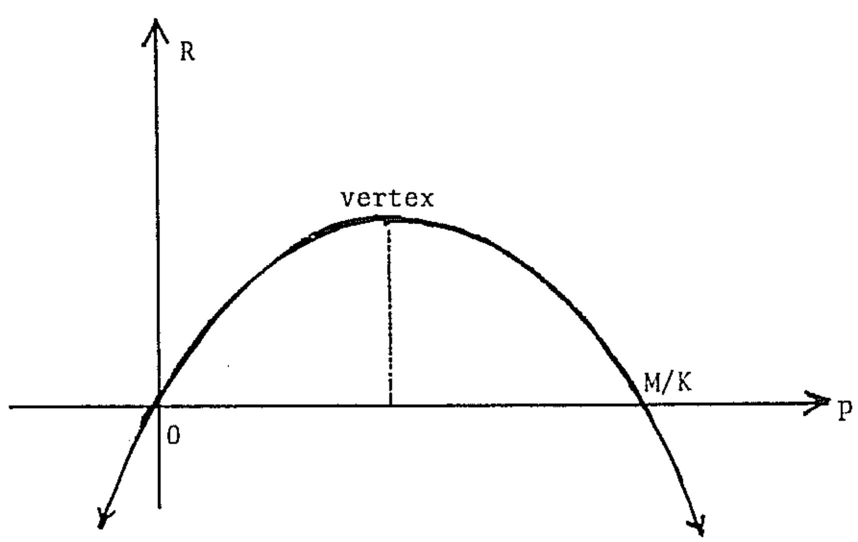
When the commodity is sold at a price per unit of p the monthly sales revenue will therefore be

$$\begin{aligned} R &= (\text{price per unit}) \times (\text{number of units sold}) \\ &= p \times q, \end{aligned}$$

so

$$\begin{aligned} R &= p(M - Kp) \\ &= Mp - Kp^2 . \end{aligned}$$

The graph of R as a function of p is a **parabola**, which cuts the p-axis at the points where $p(M - Kp) = 0$; that is, when $p = 0$ or $M - Kp = 0$ and so $p = M/K$.



The value of p at the vertex is important. It is the value of p at which the function is a *maximum* and so corresponds to the selling price at which the sales revenue is maximized.

The function $R = Mp - Kp^2$ is particular case of the general quadratic function

$$y = ax^2 + bx + c,$$

where the variables x and y have been replaced by the more meaningful symbols p and R, and $a = -K$, $b = M$, $c = 0$.

The graph of $y = ax^2 + bx + c$ is also a parabola. Our principal concerns will be to:

(i) locate its vertex; that is, the location of the function's maximum or minimum value;

(ii) sketch its graph;

(iii) determine its **roots**; that is, the points where the graph cuts the x-axis, which correspond to the solutions of the quadratic equation $ax^2 + bx + c = 0$.

3.2 Completing the Square

The single most important computation we can perform on the expression $ax^2 + bx + c$ is that of **completing the square.**

Multiplying out $(x + \alpha)^2$, we have

$$(x + \alpha)^2 = x^2 + 2\alpha x + \alpha^2$$

which we can rearrange as

$$x^2 + 2\alpha x = (x + \alpha)^2 - \alpha^2.$$

Replacing 2α by β, so that $\alpha = \beta/2$, we have

$$x^2 + \beta x = (x + \beta/2)^2 - \beta^2/4.$$

For example: $x^2 - 3x = (x - 3/2)^2 - 9/4$.

Applying this to the general expression $ax^2 + bx + c$, we obtain

$$\begin{aligned} ax^2 + bx + c &= a(x^2 + bx/a) + c \\ &= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right] + c \end{aligned}$$

and so

$$\boxed{ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a^2}\right)}$$

When $ax^2 + bx + c$ is expressed in this last form we say that we have *completed the square.*

For example:

$$\begin{aligned} 2x^2 - 6x + 5 &= 2(x^2 - 3x) + 5 \\ &= 2[(x - 3/2)^2 - 9/4] + 5 \\ &= 2(x - 3/2)^2 + 5 - 9/2 \\ &= 2(x - 3/2)^2 + 1/2 \end{aligned}$$

3.3 Locating the Vertex

By completing the square we have

$$\begin{aligned} y &= ax^2 + bx + c \\ &= a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right). \end{aligned}$$

When $x = -b/2a$, the first term vanishes leaving $y = c - b^2/4a$. Thus

$$\left(-\frac{b}{2a},\ c - \frac{b^2}{4a}\right)$$

is a point on the graph. It is in fact the vertex. To verify this we distinguish two cases:

(I) **When a is positive.** For $x \neq -b/2a$, we have

$$\begin{aligned} y &= a(x + b/2a)^2 + (c - b^2/4a) \\ &= (\textit{strictly positive}) \times (\textit{non-zero square}) + (c - b^2/4a) \\ &= (\textit{strictly positive}) + (c - b^2/4a), \end{aligned}$$

and so for every value of x other than $-b/2a$, we see that y is greater than $c - b^2/4a$. Thus $(-b/2a,\ c - b^2/4a)$ is the lowest point on the graph and we conclude that the parabola opens upward with vertex at the point $x = -b/2a$. The y-coordinate of the vertex is given by $c - b^2/4a$. This also may be calculated by substituting $x = -b/2a$ into the equation $y = ax^2 + bx + c$.

For example: $y = 2x^2 - 6x + 5$ has vertex at $x = -\dfrac{-6}{2 \times 2} = \dfrac{3}{2}$, at which $y = 2 \times \dfrac{9}{4} - 6 \times \dfrac{3}{2} + 5 = \dfrac{1}{2}$.

(II) **When a is negative,** a similar analysis shows that

$$y = ax^2 + bx + c$$

is a parabola opening downward with vertex at $x = -b/2a$.

Example. For the parabola of our revenue model

$$R = Mp - Kp^2$$

the vertex is at $p = M/2K$, at which value $R = M^2/2K - M^2/4K = M^2/4K$. Thus the maximum sales revenue of $M^2/4K$ is achieved by selling at a price per unit of $M/2K$.

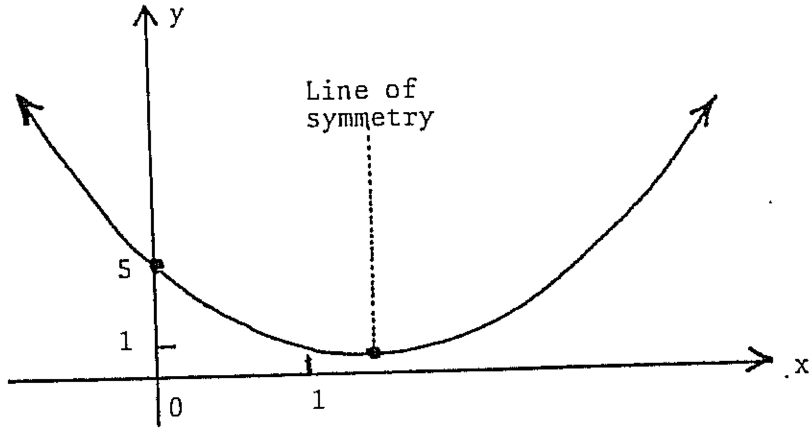
Thus for both cases we have:

> The parabola
>
> $$y = ax^2 + bx + c$$
>
> has vertex at the point with x-coordinate
>
> $$x = \frac{-b}{2a}.$$

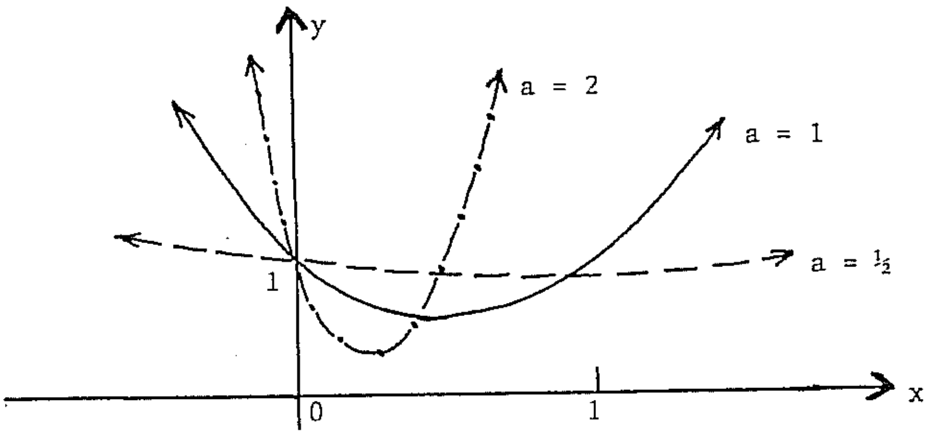
3.4 Graphing $y = ax^2 + bx + c$

Knowing the general shape of a parabola and the fact that it has a vertical line of symmetry through the vertex, to sketch $y = ax^2 + bx + c$ we need only locate the vertex and one other point on it. The calculations are particularly simple if we choose to locate the point where $x = 0$ (or $x = 1$, if the vertex is at $x = 0$).

For example, we saw in the previous section that the parabola $y = 2x^2 - 6x + 5$ has vertex at $(3/2, 1/2)$. When $x = 0$, $y = 5$, so $(0,5)$ is also a point on the graph and we can now draw:



As a check it is worth noting that the prbola should open upward if a is positive and downward if a is negtive. It is also sometimes useful to observe that increasing the magnitude of a makes the parabola *sharper*.



The effect of changing a in $y = ax^2 - x + 1$

3.5 The Roots of $y = ax^2 + bx + c$

We seek the value(s) of x at which the graph cuts the x-axis. That is we seek those values of x for which $ax^2 + bx + c = 0$. Completing the square, we have

$$a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} = 0$$

and so

$$a\left(x+b\frac{b}{2a}\right)^2 = \frac{b^2}{4a} - c$$

$$\left(x+\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$= \frac{b^2-4ac}{4a^2} .$$

Taking square-roots,

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2-4ac}}{2a}$$

and so

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} .$$

Thus, if $b^2 - 4ac$ is positive, there will be two roots x_1 and x_2 given by

$$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a} \qquad \text{and} \qquad x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a} .$$

For example: The parabola $y = 2x^2 - 2x - 4$ has roots

$$x_1 = \frac{2+\sqrt{4+32}}{4} \qquad \text{and} \qquad x_2 = \frac{2-\sqrt{4+32}}{4} ,$$

that is, $x_1 = 2$ and $x_2 = -1$.

If $b^2 - 4ac = 0$ the two roots of $y = ax^2 + bx + c$ coincide. This corresponds to the parabola having its vertex on the x-axis.

If $b^2 - 4ac$ is negative, our expressions for the two roots involve the square root of negative quantity and so there are no real roots. This corresponds to the graph lying entirely above or below the x-axis, and so failing to cut it.

In practice the need to find roots for a quadratic function arises when we seek the points where a straight line intersects a parabola, or where two parabolas intersect.

For example. We saw that if a given commodity is sold at a price per unit of p, then a quantity $q = M - Kp$ will be sold each month, yielding a monthly sales revenue of $R = Mp - Kp^2$. If the cost of supplying q units of the commodity is

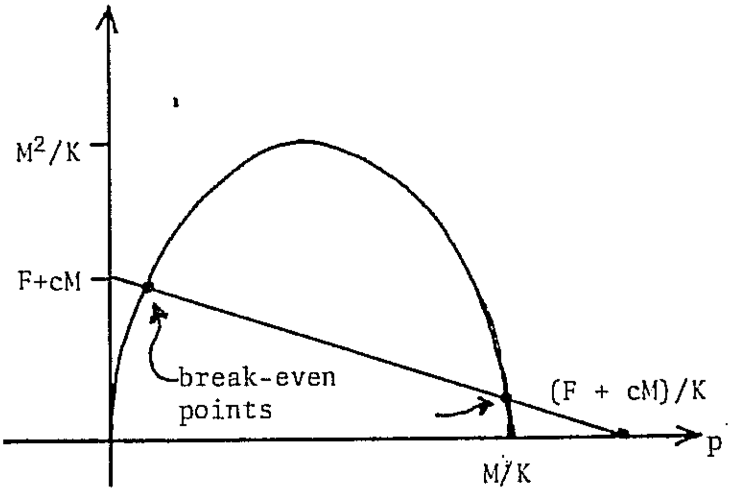
$$C = F + cq = F + c(M - Kp)$$

then a break-even point occurs when the selling price per unit is such the cost of supply equals the revenue received. That is, at a value of p for which $C = R$ and so

$$F + c(M - Kp) = Mp - Kp^2 .$$

Rearranging, we see that a break-even point occurs at a root of the quadratic

$$Kp^2 - (M + cK)p + (F + cm) = 0 .$$



Only roots at which p and $R = C = F + c(M - Kp)$ are positive correspond to meaningful break-even points. Depending on the values of the parameters M, K, F and c there may be one, two, or no break-even points in this model.

3.6 Factorizing a Quadratic

If x_1 nd x_2 are the real roots of $y = ax^2 + bx + c$ then we have

$$\begin{aligned} y &= ax^2 + bx + c \\ &= a(x - x_1)(x - x_2) . \end{aligned}$$

[This may be verified by multiplying out the last expression, using the formulas for x_1 and x_2 obtained earlier.]

When $ax^2 + bx + c$ is expressed in the form $a(x - x_1)(x - x_2)$ we say it has been **factorized** and refer to $(x - x_1)$ and $(x - x_2)$ as **factors**. When expressed in this way, it is easy to see that x_1 and x_2 are the two values of x for which y is zero.

For example. We saw that the quadratic $y = 2x^2 - 2x - 4$ has roots $x_1 = 2$ and $x_2 = -1$. Thus we have

$$\begin{aligned} y &= 2x^2 - 2x - 4 \\ &= 2(x - 2)(x + 1) . \end{aligned}$$

EXERCISES 3.7

(1) (a) Sketch the graph of the given quadratic function after finding the vertex.

(i) $y = x^2 - 4$

(ii) $y = -x^2 + 4x - 3$

(iii) $y = x^2 - x - 1$

(iv) $y = 2x^2 - 4x + 5$

(b) Find the roots (where possible) of the above quadratics.

*(2) For the quadratic function $y = ax^2 + bx + c$ verify that for any value of t the function has the same value at $x = -\dfrac{b}{2a} + t$ and $x = -\dfrac{b}{2a} - t$. (Note that this proves the vertical line $x = -b/2a$ is indeed a line of symmetry for the function.)

Hint: The calculations may be easier if you first complete the square.

*(3) In the notes it is stated that a similar argument to that for the case when a is positive shows that when a is negative, the parabola $y = ax^2 + bx + c$ opens downward and has vertex at $x = -b/2a$. Give the argument in this case.

(4) Factorize each of the following quadratics:

(i) $y = x^2 + 4x - 5$

(ii) $y = 2x^2 - 2x - 12$

(iii) $y = -2x^2 + 3x - 1$

(iv) $y = 12x^2 + x - 1$.

*(5) Given that $b^2 - 4ac$ is positive verify that

$$ax^2 + bx + c = a\left(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right).$$

(6) The fixed costs at the Armidale Widget Manufacturing Company are \$500 per day, and the variable cost \$$v$ per widget is a linear function of the number q of widgets produced. When $q = 100$, $v = 4.00$; and when $q = 500$, $v = 2.50$. Find v in terms of q, and then find the total daily cost C as a function of q.

(7) The daily demand for widgets q is a linear function of the selling price \$$p$ per widget. When $p = 5$, $q = 300$; and when $p = 7$, $q = 200$.

(a) Find q in terms of p, and then find the revenue R as a function of p.

(b) What are the maximum revenue and the corresponding value of p?

(8) From the first to the tenth day of the month it was observed that the Dow-Jones index on the tth day was given approximately by the formula

$$2t^2 - 56t + 2442.$$

Assuming this model continues to apply, at which day of the month will the index have its minimum value?

(9) Find where the graph of the quadratic function $y = -x^2 + 6x = 5$ intersects the graph of

(a) the linear function $y = x + 1$, and

(b) the quadratic function $y = x^2 - 6x + 11$.

In both cases sketch the graphs of the two functions involved.

(10) If a certain beverage is sold at a price p per litre, then $q = -10p + 120$ litres will be sold each day. This gives a daily sales revenue of $R = -10p^2 + 120p$. If the cost of supplying q litres of the beverage is $C = 160 + 2q$, find the break-even selling price per litre, i.e. at what price (or prices) the cost of supply equals the revenue received.

4. POWER FUNCTIONS

An important class of functions sometimes encountered in economics are *power functions* of the form $y = Ax^\alpha$. In the expression Ax^α, α is referred to as the *exponent* (or *power*).

For example: $y = x^{-1}$, $y = x^{1/3}$, $y = x^{1/2}$, $y = x$ (a straight line), $y = x^2$ (a parabola) and $y = x^3$ are all power functions.

When working with these functions it is important to keep in mind the conventions:

$$x^1 = x$$
$$x^0 = 1$$
$$x^{-\alpha} = \frac{1}{x^\alpha}$$

and for p and q integers with q greater than 0,

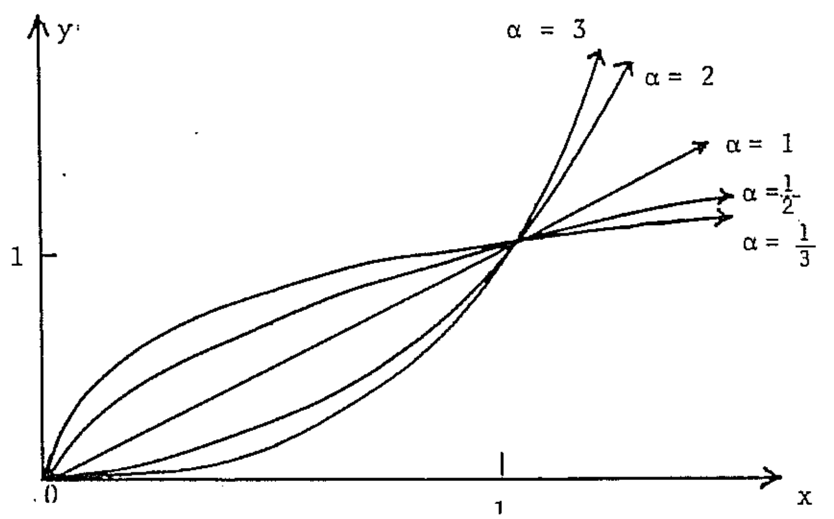
$$x^{p/q} = \left(\sqrt[q]{x}\right)^p.$$

For example, $y = x^{-3/2} = \dfrac{1}{\left(\sqrt{x}\right)^3}$. So when $x = 4$, $y = \dfrac{1}{\left(\sqrt{4}\right)^3} = \dfrac{1}{8}$ and when $x = 9$, $y = 1/27$.

Negative values of x are rarely meaningful in applications; further, for such values, some of the above expressions may not be defined. Consequently, in what follows we will concentrate on positive values of x.

For all values of A and α the graph of $y = Ax^\alpha$ passes through the point $(1, A)$.

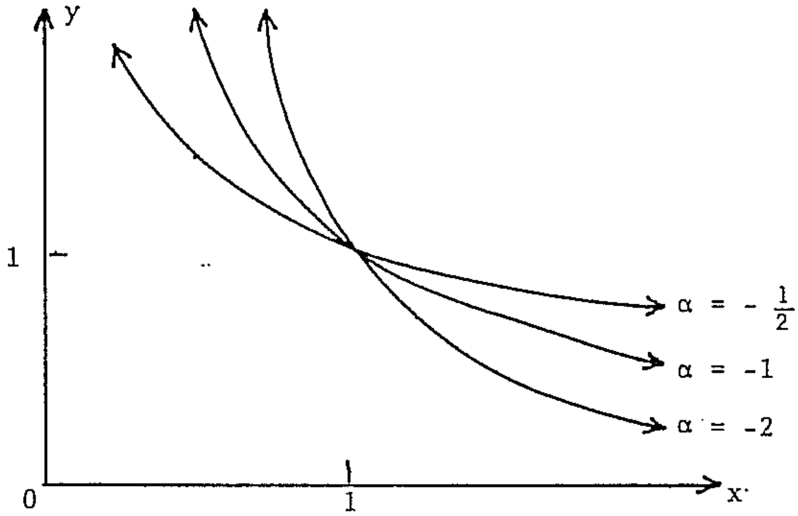
Typical graphs of power functions with both A and α positive are sketched below.



$y = x^\alpha$ for $\alpha = 1/3$, $1/2$, 1, 2 and 3

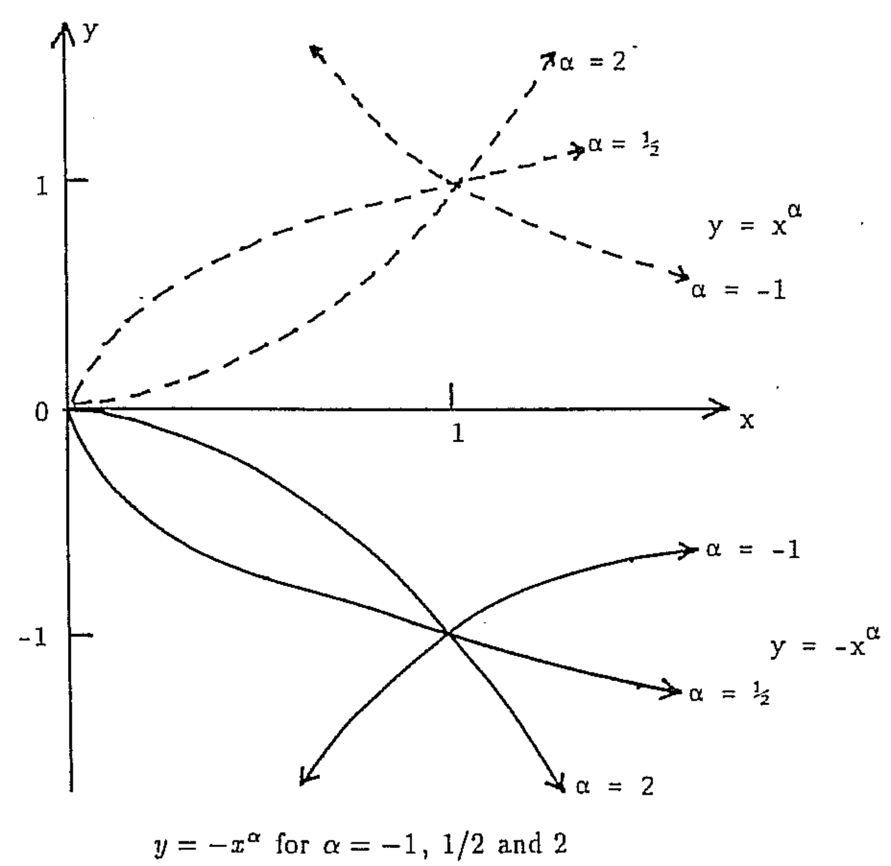
In this case each of the functions is increasing. For α greater than one the slope increases as x does, while for α between zero and one it decreases. It should also be noted that for x greater than one the graphs for larger α lie above those with smaller α. For x between zero and one, the reverse is the case; that is, graphs for larger α lie below those with smaller α.

When A is positive and α is negative, the function $y = Ax^\alpha$ is decreasing. When $x = 0$ the function is undefined with the y-axis as a vertical *asymptote*. Some typical examples are sketched below:



$y = x^\alpha$ for $\alpha = -1/2$, -1 and -2

The graphs with A negative are the reflections in the x-axis of those for A positive:



$y = -x^\alpha$ for $\alpha = -1$, $1/2$ and 2

Of particular importance is the case when $\alpha = -1$; that is, when $y = x^{-1} = \frac{1}{x}$ and so y is *inversely proportional* to x.

For example:

1) In our electricity supply model of Section 2.1, with a fixed annual cost of supply of F, the fixed cost f per unit of electricity varies inversely with the number of units supplied:

$$f = \frac{F}{E}.$$

2) Because $D = A/p$ is a decreasing function of p it is sometimes used as a model of the demand function for the sale of a commodity at a price per unit of p. One consequence of such a model is that the sales revenue

$$\begin{aligned} R &= [\text{price per unit}] \times [\text{number of units sold}] \\ &= pD(p) \\ &= p\frac{A}{p} \\ &= A \end{aligned}$$

is constant regardless of the selling price. Such a deduction may be used to check the reasonableness of the model in any given situation.

EXERCISES 4.1

(1) Express in the form x^α for appropriate exponent α:

$$\sqrt{x}, \quad \frac{1}{\sqrt{x}}, \quad \frac{1}{x\sqrt{x}}, \quad \frac{x^3}{x\sqrt{x}}, \quad \sqrt[5]{x^2}, \quad \frac{1}{x^3}, \quad x^{1/3}x^3, \quad \left(x^{1/3}\right)^3.$$

(2) Sketch the graph of $y = x^\alpha$ for $\alpha = \frac{1}{2}$, $\alpha = 2$, $\alpha = -2$.

If α is increased from 2 to 10, how does the graph of $y = x^\alpha$ change?

(3) Sketch the graph of $y = 1/x$ and find its intersection with the straight line $y = -2x + 1$.

(4) **(Revenue)** A firm has a total revenue of \$2000 per day regardless of the price of its product. If p denotes the price per unit of the product and x the number of units that can be sold at the price p, express p in terms of x.

Complete the following table:

x	50	100	200	250	500	1000
p						

Plot these points on a graph and join them with a smooth curve.

(5) By considering graphs of the power function $y = x^\alpha$ for x positive and for various values of α, determine for which values of α the function is convex and for which values of α it is concave.

5. GENERAL POLYNOMIAL FUNCTIONS

Linear functions, quadratic functions and power functions with whole number exponents are all special cases of the general **polynomial function**

$$y = p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n .$$

The constants a_0, a_1, ..., a_n are referred to as **coefficients**; thus a_1 is the coefficient of x; a_0 which may be described as the coefficient of x^0 is usually called the **constant term**.

The **degree** of a polynomial is the highest power of the independent variable which occurs in it. Thus $p(x)$ is a polynomial of degree n, assuming $a_n \neq 0$.

For example. Linear functions are of degree 1, quadratics have degree 2; $y = 13x^8$ has degree 8; and $y = x^3 - 3x^2 + 2x$ is a polynomial of degree 3 (sometimes referred to as a **cubic**).

Higher degree polynomials arise as approximations in more advanced economic models. We will concentrate on some general features which help in sketching their graphs.

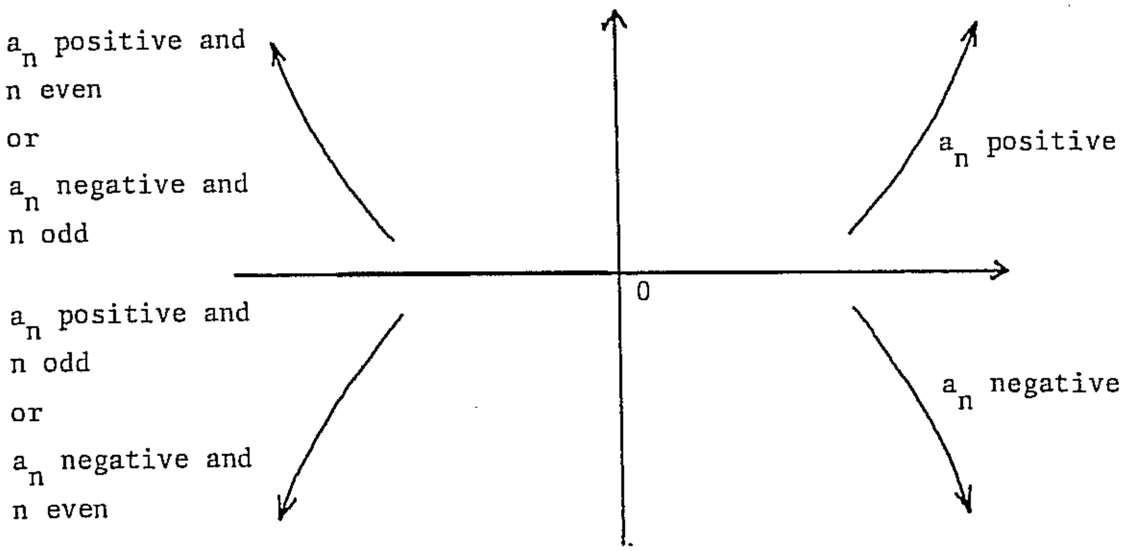
5.1 Asymptotic Behaviour

When the magnitude of x is large, the value of $p(x)$ is dominated by the highest degree term $a_n x^n$. Indeed as the magnitude of x increases, the ratio $\frac{p(x)}{a_n x^n}$ tends to 1.

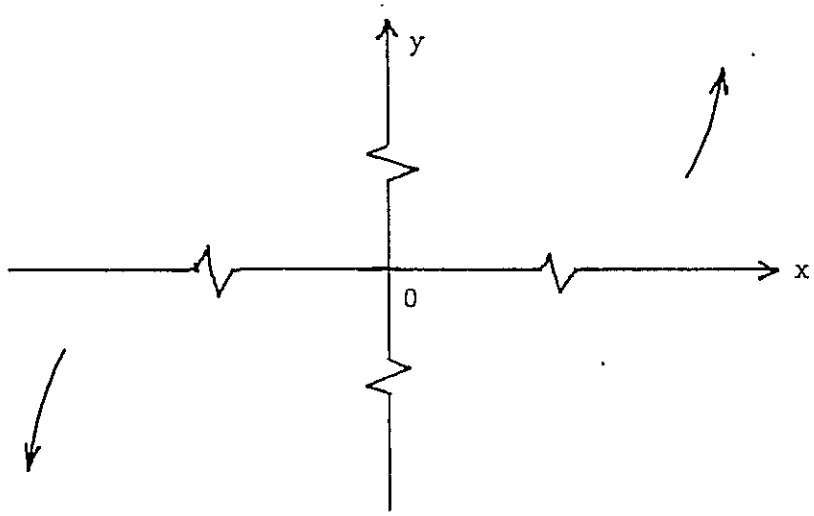
For example. When $p(x) = x^3 - 3x^2 + 2x$, we have:

x	x^3	$p(x)$	$p(x)/x^3$
10	1000	720	0.7200
100	1000000	970200	0.9702
1000	1000000000	997002000	0.9970
10000	1000000000000	999700020000	0.9997

Thus the larger the magnitude of x, the more nearly the graph of $y = p(x)$ looks like that of the power function $y = a_n x^n$. The following diagram shows the different kinds of behaviour that can arise.



For example. When $p(x) = x^3 - 3x^2 + 2x$, we have the following behaviour:



5.2 Roots

A root of the polynomial $y = p(x) = a_0 + a_1 x + \cdots + a_n x^n$ is a value x_0 of x for which $p(x) = 0$. That is

$$a_0 + a_1 x_0 + \cdots + a_n x_0^n = 0 .$$

x_0 is a root of the polynomial $p(x)$ precisely when the graph of $y = p(x)$ intersects the x-axis at $x = x_0$.

If x_0 is a root of $p(x)$ then the polynomial may be factorized as

$$p(x) = (x - x_0)q(x), \qquad (*)$$

where $q(x)$ is a polynomial of degree $n-1$ of the form

$$q(x) = a_n x^{n-1} + \cdots + b_1 x + b_0 .$$

The polynomial $q(x)$ may be found by long division of $p(x)$ by $(x - x_0)$, or by multiplying out the RHS of $(*)$ and equating coefficients with those of the LHS.

We have already seen how to find roots for linear and quadratic functions. In the case of quadratics, the roots were given by the formula

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

More complicated formulas exist for the roots of cubics and quartics (polynomials of degree 4). For polynomials of degree 5 and more it has been proved that no general formula for the roots exists.

A polynomial of degree n can have up to n roots, and finding them can be quite difficult.

Remember when finding a root we are trying to find a value of x at which the function vanishes. In simple cases by trying values of x such as $0, 1, -1, 2, -2, \ldots$, it is sometimes possible to locate a root by trial and error.

For example. When $p(x) = x^3 - 3x^2 + 2x$:

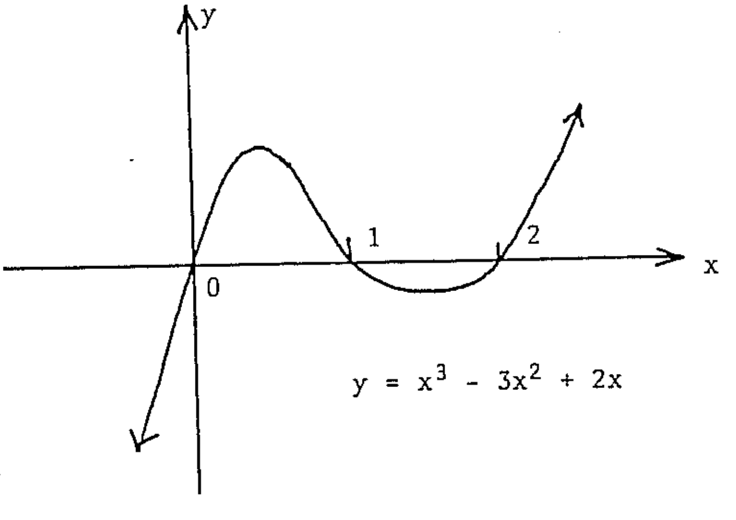
for $x = 0$, we have $p(0) = 0$, so 0 is a root,

for $x = 1$, we have $p(1) = 0$, so 1 is a root,

for $x = -1$, we have $p(-1) = -2$, so -1 is not a root,

for $x = 2$, we have $p(2) = 0$, so 2 is a root.

Since $p(x)$ is a cubic and so can have at most 3 roots, we conclude that the roots are 0, 1 and 2. Combining this with the asymptotic behaviour noted above, we see that the graph of $y = x^3 - 3x^2 + 2x$ looks like



As this example illustrates, between any two consecutive roots of a polynomial, we have either at least one local maximum or at least one local minimum. Another useful observation is that if the values of p(x) at two points x_1 and x_2 are of opposite sign, then p(x) has a root between x_1 and x_2.

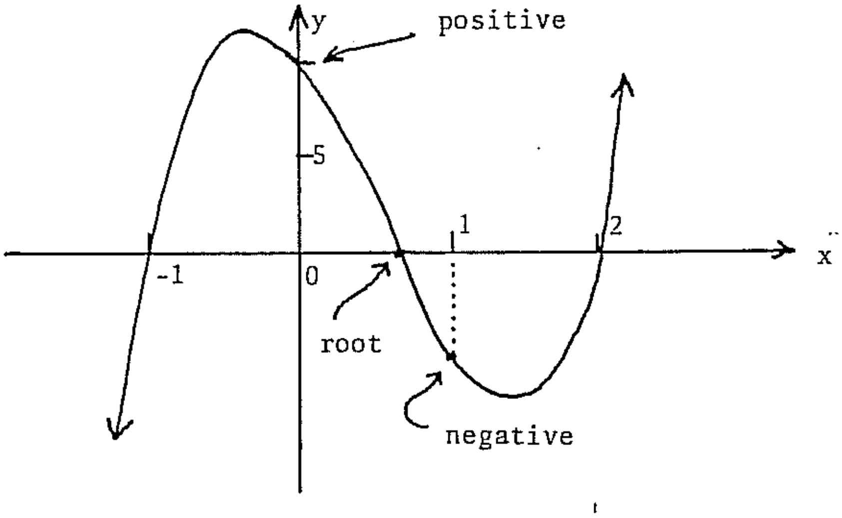
For example. If $y = 8x^3 - 13x^2 - 11x + 10$, we have

$\left.\begin{array}{l} p(0) = 10 \\ p(1) = -6 \end{array}\right\}$ of opposite sign, so there is a root between 0 and 1;

$p(-1) = 0$, so -1 is a root,

$p(2) = 0$, so 2 is a root.

The graph will therefore look like



Graph of $y = 8x^3 - 13x^2 - 11x + 10$

Provided we have enough information about the roots, it is usually possible to sketch the graph of a polynomial.

For example. If $y = p(x) = x^3 - x^2 - 2x + 2$, we find by substituting values of x that 1 is a root. Thus $p(x)$ may be factorized as

$$p(x) = (x - 1)(x^2 + ax + b)$$

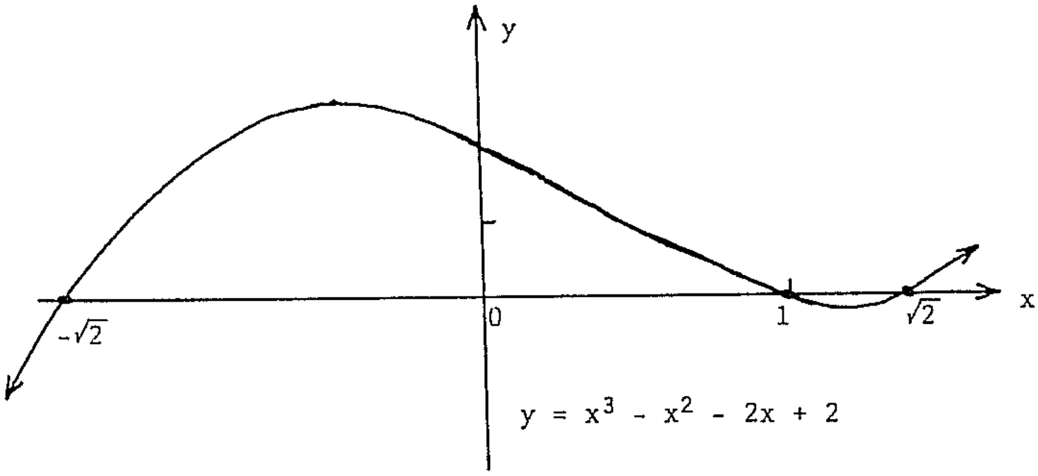
for some constants a and b. Multiplying this out we have that

$$x^3 - x^2 - 2x + 2 = x^3 + (a - 1)x^2 + (b - a)x - b \ ,$$

from which we see that we must have $b = -2$ and $a = 0$, so

$$p(x) = (x - 1)(x^2 - 2) \ .$$

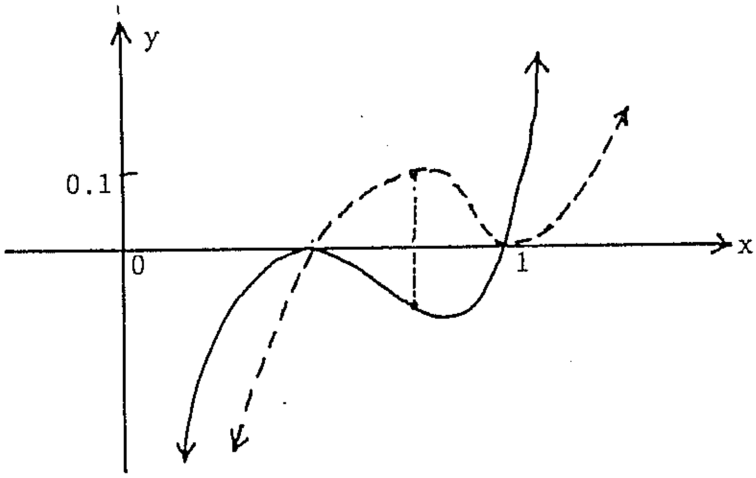
Since the quadratic $x^2 - 2$ has roots at $x = \pm\sqrt{2}$, we conclude that the roots of $p(x)$ are $-\sqrt{2}$, 1 and $\sqrt{2}$, and so the graph looks like:



For example. If $y = p(x) = 4x^3 - 8x^2 + 5x - 1$, we find by substituting values for x that $x = 1$ is a root. Factorizing gives

$$\begin{aligned} y &= (x-1)(4x^2 - 4x + 1) \\ &= (x-1)4(x^2 - x + 1/4) \\ &= 4(x-1)(x-1/2)^2 \end{aligned}$$

and so in this case $p(x)$ has roots at $x = 1$ and $x = 1/2$, with the root at $x = 1/2$ being **repeated.** Since these are the only points where the graph can cut the x-axis, we are able to infer from the asymptotic behaviour that the graph must look like one of the following possibilities:

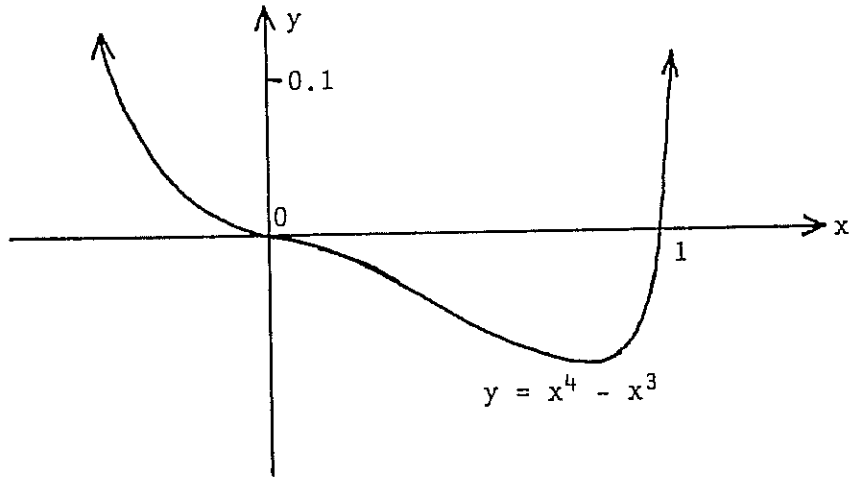


That it is indeed the heavy curve and not the broken one is easily confirmed by noting that $p(3/4) = -1/16$ is negative.

In general, at a root which is repeated an even number of times, a polynomial will have a local maximum or minimum and the graph will intersect the x-axis without crossing it, as illustrated in the last example.

In our work on differentiation, we will also see that at a root which is repeated an odd number of times, the graph crosses the x- axis, but has horizontal *slope* at the root.

For example. $y = x^4 - x^3 = x^3(x-1)$ has roots at 0 and 1, with the root at 0 being repeated 3 times. The graph looks like



When the roots are difficult to find, or provide us with insufficient information, we must seek alternative methods, such as locating the local maxima and minima. This will be the subject of some of our subsequent work on differentiation.

For example, the polynomial $y = x^3 + x$ is easily seen to have only one real root, at $x = 0$, and so its graph cannot readily be sketched using the above methods.

5.3 Expanding $(1+x)^n$

Expanding $(1+x)^n$ by multiplying $1+x$ with itself n times, we obtain a polynomial of degree n:

$$(1+x)^n = c_0 + c_1 x + \cdots + c_n x^n .$$

For example,

$(1+x)^0 = 1,$

$(1+x)^1 = 1+x,$

$(1+x)^2 = (1+x)(1+x) = 1+2x+x^2,$

$(1+x)^3 = (1+x)(1+x)^2 = 1+3x+3x^2+x^3,$

$(1+x)^4 = (1+x)(1+x)^3 = 1+4x+6x^2+4x^3+x^3.$

The coefficients in these expansions can be conveniently found, at least for relatively small values of n, by using Pascal's triangle:

$n=0$	1							
$n=1$	1	1						
$n=2$	1	2	1					
$n=3$	1	3	3	1				
$n=4$	1	4	6	4	1			
$n=5$	1	5	10	10	5	1		
$n=6$	1	6	15	20	15	6	1	
$n=7$	1	7	21	35	35	21	7	1

etc.

The numbers in the nth row are the coefficients in the expansion of $(1+x)^n$.

For example.

$$(1+x)^6 = 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6 .$$

Except for the bordering 1's, each number in Pascal's triangle is the sum of the two numbers immediately to the right and left of it in the row above. Using this pattern, the triangle may be extended indefinitely allowing the coefficients to be found for any given value of n.

Although you will not be expected to know it, for larger values of n, the formula

$$\frac{\overbrace{n(n-1)(n-2)\cdots(n-m+1)}^{m \text{ factors}}}{m(m-1)\cdots 2\cdot 1}$$

for the coefficient of x^m in $(1+x)^n$ may also be used.

For example. The coefficient of x^4 in $(1+x)^6$ is

$$\frac{6\cdot 5\cdot 4\cdot 3}{4\cdot 3\cdot 2\cdot 1} = 15 .$$

EXERCISES 5.4

(1) Express the following functions in polynomial form, i.e. as sums of multiples of powers of x:

(a) $(x-1)(x-2)(x-3)$

(b) $(x-3)(x-2)(x-1)$

(c) $((x-6)x+11)x-6$

(d) $(x-1)^3-3(x-1)^2+2(x-1)$

(2) For each of the following polynomials $p(x)$, sketch the graph of $y=p(x)$:

(a) $p(x)=(x-1)(x-2)(x-3)$

(b) $p(x)=(x+1)(x-2)^2$

(c) $p(x)=x^2-2x-3$

(3) Expand $(x-2)^4$ in powers of x and check by evaluations at $x=1$ and $x=3$.

(4) Factorize the following polynomials:

(a) $6x^2-19x+15$

(b) x^3-x^2-3x-1

(c) x^4-2x^3+2x-1

(d) $x^4-3x^3-7x^2+27x-18$

(e) $x^4-12x^3+51x^2-92x+60$

*(5) Let $f(x)=x^3-3x^2+2x=x(1-x)(2-x)$ and $g(x)=2x(1-x)$. Show that for $0<x<1$, $f(x)<g(x)$ and deduce that the maximum value of $f(x)$ for $0<x<1$ is less than $\frac{1}{2}$. Illustrate by sketching the graphs of $y=f(x)$ and $y=g(x)$ on the same coordinate system.

(6) A revenue function has the form Aq^2+Bq where A and B are constants. If $q=2$ the revenue is 36, and when $q=3$ the revenue is 51. Find A and B and hence determine the revenue for $q=4$.

6. COMBINING FUNCTIONS

The description of many economic situations requires us to build a new function by modifying and combining known functions in appropriate ways. In this section we look at some of the more commonly occurring ways in which this can happen. We will also see that the task of graphing a complicated function is often made easier if we recognise the function as a combination of simpler functions whose graphs are known or readily found.

6.1 Multiples of a Function

Given a function f and a number c we can form a new function, which we denote by cf, whose value at x is c times the value of f at x; that is,

$$(cf)(x) = c \cdot f(x). \qquad (*)$$

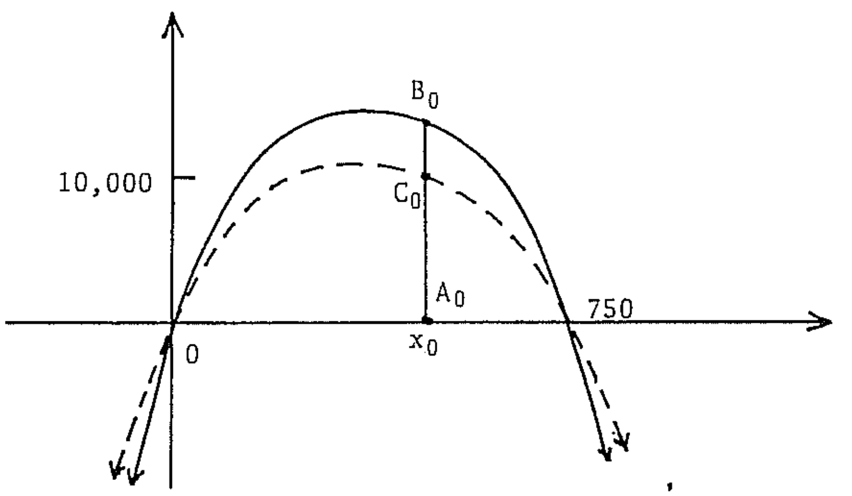
For Example. If the demand function for a certain commodity is $q(p) = 150 - 0.2p$, where $q(p)$ is the number of units of the commodity which will be sold when the price per unit is p, then the revenue from sales will be $R(p) = p(150 - 0.2p)$. If a sales tax of 30% is imposed on the commodity, then the tax payable will be $\frac{30}{100}R(p)$ and the net income from sales will be

$$\begin{aligned} I(p) &= R(p) - \tfrac{30}{100}R(p) \\ &= 0.7R(p) \\ &= 0.7p(150 - 0.2p)\,. \end{aligned}$$

Thus $I = 0.7R$, which is a particular instance of $(*)$ with $f = R$ and $c = 0.7$.

If we already know the graph of f it is relatively straightforward to construct the graph of cf from it by scaling vertical distances from the x-axis to the graph of $y = f(x)$ by a factor c.

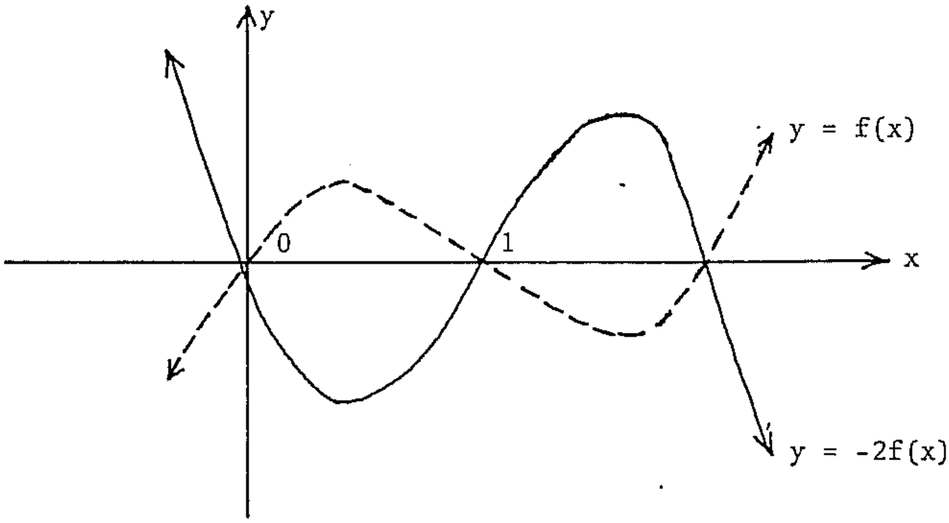
For Example. If $I(p) = 0.7R(p)$ where $R(p) = p(150 - 0.2p)$ we have



where for any point x_0 on the x-axis we have the distance $A_0C_0 = 0.7A_0B_0$. Usually it is adequate to determine the point C_0 approximately. In this example it would be sufficient to locate C_0 as "a little under three-quarters of the way from A_0 to B_0".

When c is negative the point on the graph of cf corresponding to a point above the x-axis on the graph of f will be an appropriately scaled distance below the x-axis, and vice versa. In the case when $c = -1$ the graph of $y = -f(x)$ is the reflection of the graph for $y = f(x)$ in the x-axis.

For Example. When f is the function $f(x) = x^3 - 3x^2 + 2x$, the graphs of $y = f(x)$ and $y = -2f(x) = -2x^3 + 6x^2 - 4x$ are as illustrated below.



6.2 Sums of Functions

Given two functions f and g, we can form a new function, the sum of f and g, denoted by $f + g$, whose domain consists of those values of x common to the domain of f and the domain of g, and whose value at such an x is the sum of the values of f and g at x; that is,

$$(f + g)(x) = f(x) + g(x) .$$

For Example. A retailer is prepared to purchase all the stock he can at a price of $\$p$ per item. If the supplier is prepared to sell a quantity q at this price, where q is given by the supply function $q = 5p - 10$, then the purchase of stock from the supplier will cost the retailer an amount $P = pq = p(5p - 10)$. If in addition each item of stock purchased involves the retailer in a transport cost of \$3, and hence a total transport cost of $T = 3(5p - 10) = 15p - 30$, then C, the total cost of purchase for the supplier, is the sum of the two functions P and T:

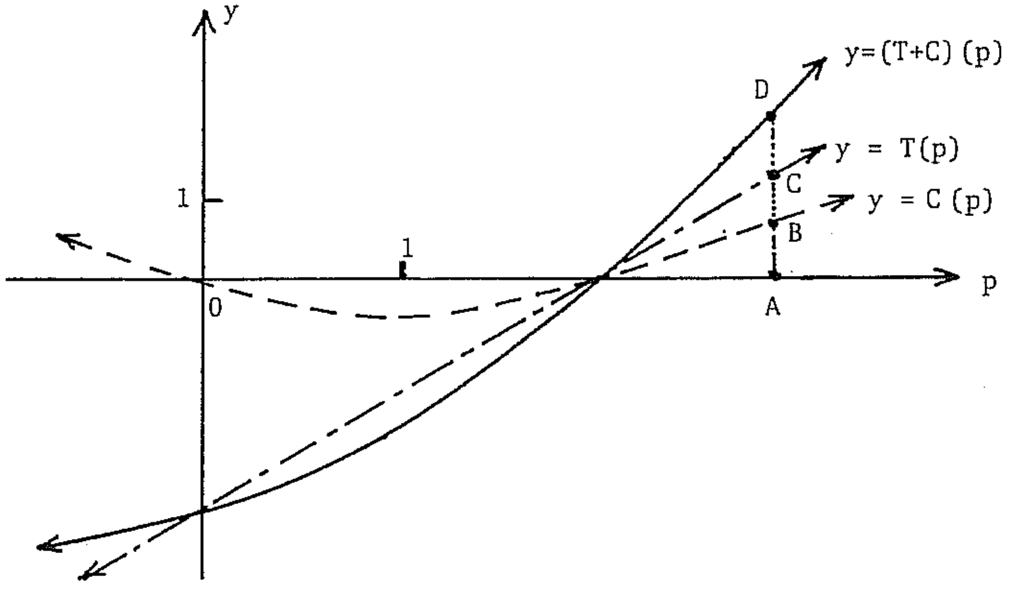
$$C = P + T .$$

Thus

$$\begin{aligned} C(p) &= P(p) + T(p) \\ &= p(5p - 10) + (15p - 30) \\ &= 5p^2 + 5p - 30 . \end{aligned}$$

The graph of a sum $f+g$ may be constructed from those for f and g. The vertical distances from the x-axis to a point on the graph of $y=(f+g)(x)$ is the sum of the vertical distance from the x-axis to the graph $y=f(x)$ and the vertical distance from the x-axis to the graph of $y=g(x)$. Thus, in the illustration below the distance $AD=AB+AC$. The addition of these distances can be done in many ways, for example; by "eye", by marking them on the edge of a piece of paper, by using a pair of dividers (or your fingers as dividers), or by using a ruler. You should aim to become practised in doing this. Particular care should be taken at points where the graph of one or other of the functions $y=f(x)$ and $y=g(x)$ lies below the x-axis and so the corresponding vertical distance is negative.

For Example, graphs of the functions $y=P(p)$ and $y=T(p)$ considered above together with the graph of $y=C(p)$ constructed from these are illustrated below:



The **difference** of two functions f and g, $f-g$, can be similarly defined by $(f-g)(x)=f(x)-g(x)$. Alternatively, it may be regarded as the sum of f and $-g$. The graph may be constructed by taking away rather than adding the appropriate vertical distances. Again, special care should be taken when one or other of the graphs is below the x-axis.

For Example, by offering a certain product at a price per unit of p, a manufacturer sells a quantity q each month according to the demand function $q=M-Kp$, and so receives a monthly sales revenue of

$$R=p(R-Kp)\ .$$

If the total cost of production for q units is $F+cq$, then the monthly cost of production is

$$C=F+c(M-Kp)\ .$$

The monthly profit P is therefore the difference $R - C$, and so

$$\begin{aligned} P(p) &= R(p) - C(p) \\ &= p(M - Kp) - (F + c(M - Kp)) \\ &= -Kp^2 + (M + K)p - (F + cM)\,. \end{aligned}$$

Thus the monthly profit is a quadratic function of the selling price whose maximum value occurs at a price per unit of $(M + K)/2K$. It is interesting to note that this is different from the selling price for which the sales revenue is maximized.

6.3 Products

Given two functions f and g, at those values of x common to both the domain of f and the domain of g, we can form their **product** fg whose value at x is the product of the values of f and g at x, that is

$$(fg)(x) = f(x)g(x)\,.$$

For Example: At time t (years) the population of a developing country is given by

$$P(t) = \frac{100,000,000}{(1 - 0.1t)}\,.$$

If the per capita income varies linearly with time according to the formula

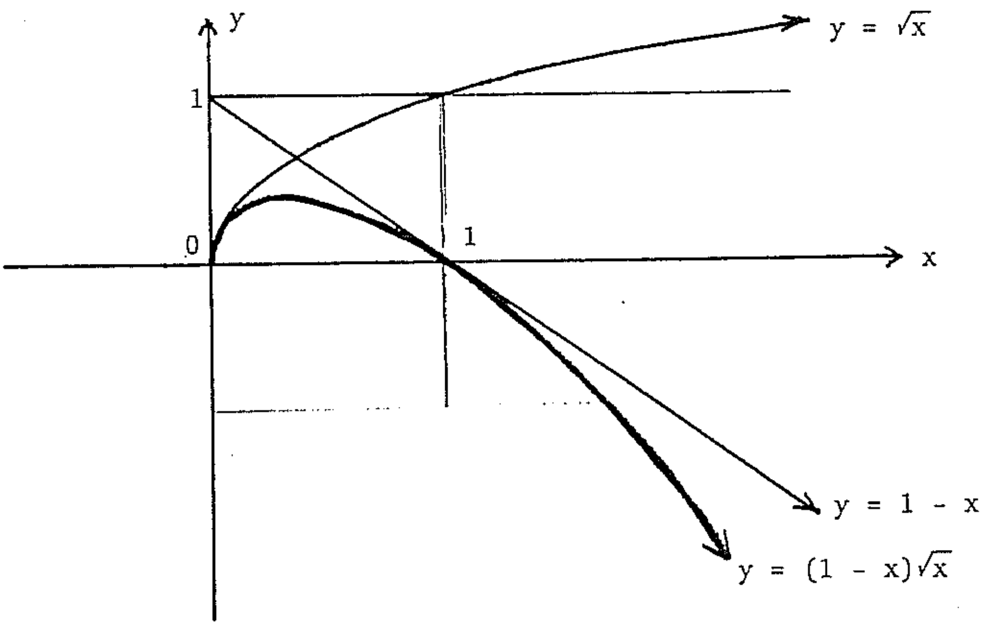
$$I(t) = 1000 + 40t\,,$$

then the national income is given by the product $N = PI$, thus

$$\begin{aligned} N(t) &= P(t)I(t) \\ &= \frac{100,000,000(1000 + 40t)}{1 - 0.1t}\,. \end{aligned}$$

To graph the product fg from graphs of f and g, it is necessary to perform "approximate" multiplications (mentally, or otherwise) of $f(x)$ and $g(x)$. In doing this use should be made of simple observations such as: the product of two numbers larger than 1 is bigger than either of them; the product of a number larger than 1 and a positive number less than 1 lies between them; the product of two positive numbers smaller than 1 is less than either of them; the product of a positive number and a negative number is negative.

For Example, from graphs of $y = 1 - x$ and $y = \sqrt{x}$, we see that $y = (1 - x)\sqrt{x}$ has the graph shown below:



6.4 Quotients

Given two functions f and g, at those values of x common to the domains of f and g for which $f(x) \neq 0$, we can define the **quotient** $\frac{g}{f}$ or g/f by

$$\frac{g}{f}(x) = \frac{g(x)}{f(x)},$$

that is, the value of g/f at any x is obtained by dividing the value of g at x by the value of f at x.

For Example. If at time t, the total number of primary school students in N.S.W. is $S(t)$ and the number of primary teachers is $T(t)$, then at time t the average class size in N.S.W. primary schools is

$$A(t) = \frac{S(t)}{T(t)} = \frac{S}{T}(t).$$

The quotient g/f can be regarded as the product of g and the **reciprocal** $1/f$. When constructing a graph of g/f from graphs of f and g, it is often convenient to draw, as an intermediate step, the graph of $1/f$. To do this, it is necessary (mentally, or otherwise) to divide 1 by $f(x)$. Remember: taking the reciprocal of a smaller number gives a larger number, and vice versa. In particular, if the values of $f(x)$ are approaching 0 as x approaches x_0, then, as x approaches x_0, the values of $f(x)$ will tend to $+\infty$ or $-\infty$ depending on whether $f(x)$ is positive or negative.

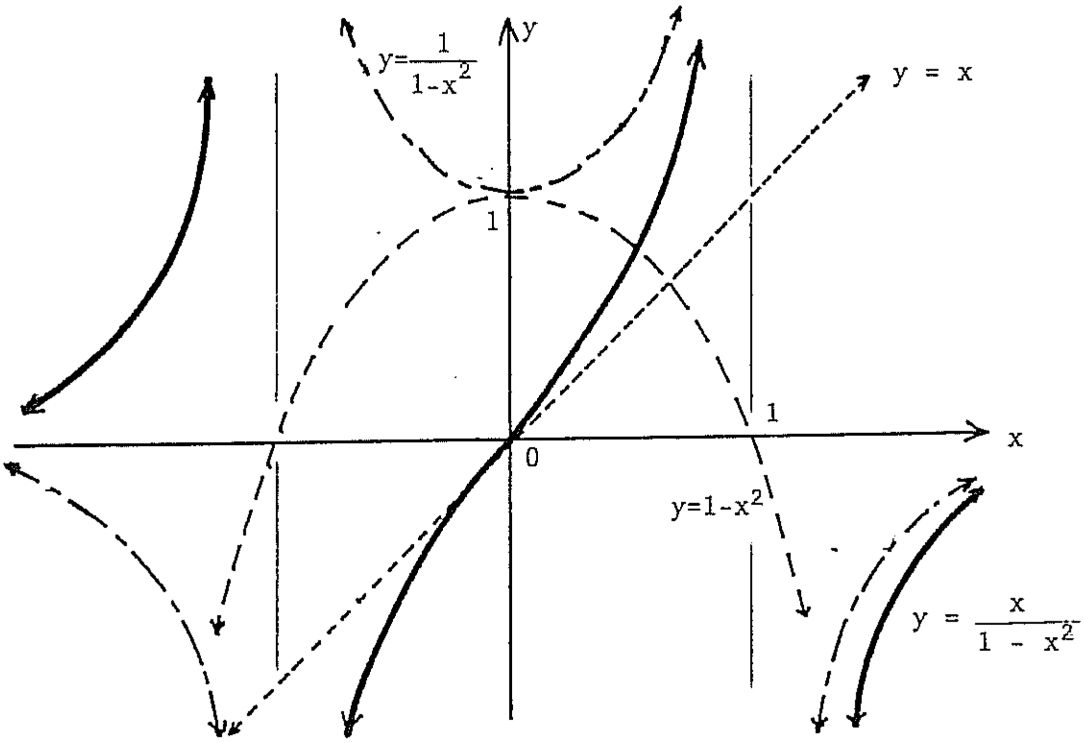
For Example. To graph

$$y = \frac{x}{1 - x^2},$$

we may first graph $y = 1 - x^2 = (1-x)(1+x)$ and construct from this the graph of

$$y = \frac{1}{1-x^2},$$

and finally multiply this by x to obtain the desired graph. This is illustrated below:



Note; for large x, $1-x^2$ behaves like $-x^2$ and so $x/(1-x^2)$ is like $x/(-x^2) = -1/x$. Thus, as x becomes large in magnitude, $x/(1-x^2)$ approaches zero.

6.5 Composites

Given two functions f and g, we can form the composite function $g \circ f$, read as "g of f", by

$$(g \circ f)(x) = g\big(f(x)\big).$$

The domain of $g \circ f$ consists of those x in the domain of f for which $f(x)$ is in the domain of g.

Note: The composite of $g \circ f$ is sometimes referred to as a *function of a function*; its value at x may be described as g of f of x.

For Example. A retailer finds that the quantity q of a certain item sold per month changes according to the demand function

$$q(p) = 500 - 5p.$$

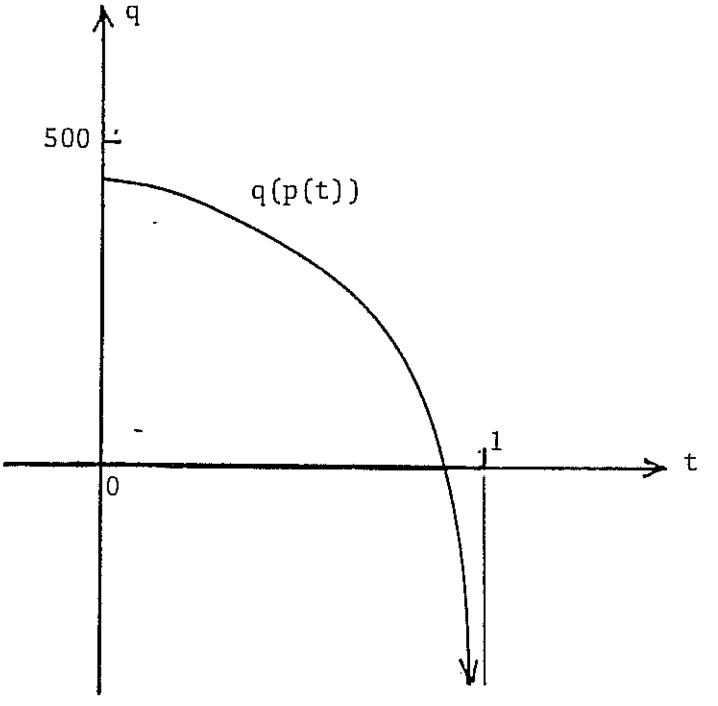
A sales tax of $100t\%$ applies to sales of the item and the retailer decides to adjust his prices so as to receive after tax a fixed amount \$10 per item sold.

To achieve this he must adopt a selling price per item of

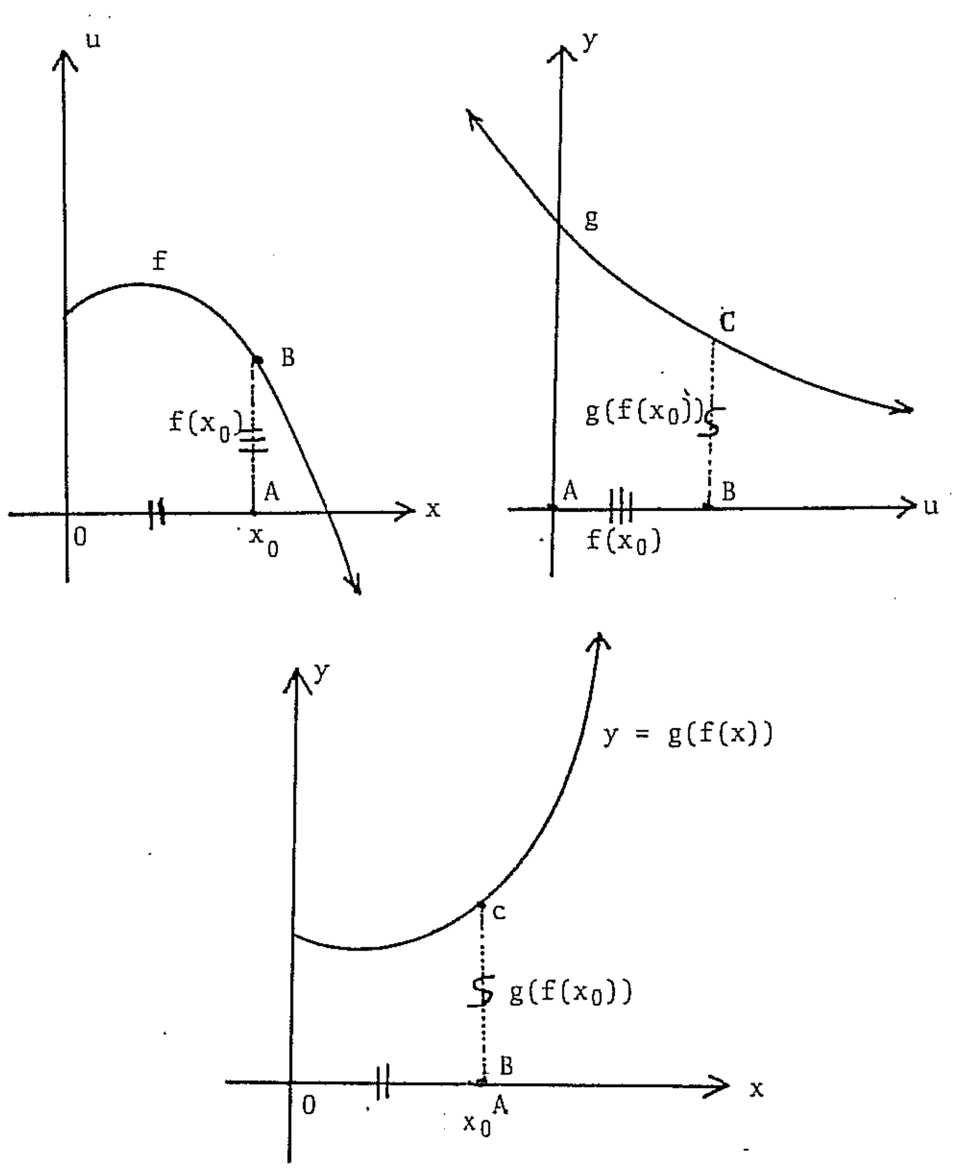
$$p(t) = \frac{10}{1-t} \, .$$

As a function of the sales-tax rate t, the quantity sold per month is given by the composite function

$$\begin{aligned} q\big(p(t)\big) &= q\Big(\frac{10}{1-t}\Big) \\ &= 500 - \frac{50}{1-t} \, . \end{aligned}$$



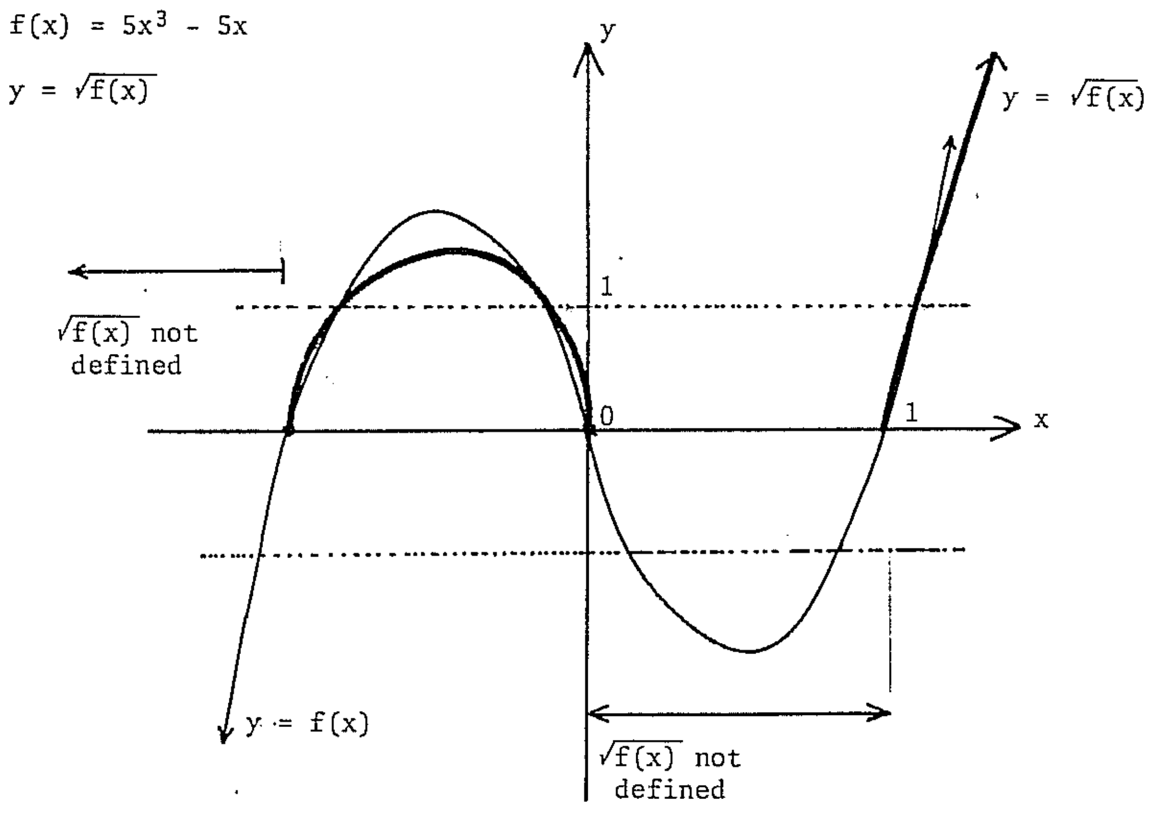
A graph of the composite function $g \circ f$ can be constructed from graphs of f and g. However, unlike our previous cases, it is better to use a separate set of axes for each of the graphs, as illustrated below. In many cases, it is easier to dermine an expression for the function $g \circ f$ and graph it directly from this (as was done in the case above).



It is important to note that in general $g \circ f$ and $f \circ g$ are different functions. For example, for $f(x) = x + 1$ and $g(x) = 2x$, we find $g\big(f(x)\big) = 2x + 2$, while $f\big(g(x)\big) = 2x + 1$.

If g is particularly simple, for example, $g(x) = x^2$ or $g(x) = \sqrt{x}$, so that we can "mentally" estimate $g(x)$, then the graph of $g\big(f(x)\big)$ may be obtained from that of $f(x)$.

For Example: The following illustration shows how we may sketch the graph of $y = \sqrt{f(x)}$ for $f(x) = 5x^3 - 5x = 5(x+1)x(x-1)$.



Here we have used the observation that

$$\text{for} \quad 0 < a < 1 \quad \text{we have} \quad 0 < \sqrt{a} < 1$$

while

$$\text{for} \quad a > 1 \quad \text{we have} \quad 1 < \sqrt{a} < a\,.$$

EXERCISES 6.6

(1) Let $f(x) = x^3 - 4x^2 + 5x - 2$. Sketch, on a common set of axes, graphs of $y = f(x)$, $y = \frac{1}{2}f(x)$, $y = -f(x)$, and $y = -2f(x)$.

(2) When a certain commodity is sold at a price per unit of \$$p$, the sales revenue is $R(p) = 10p = 0.2p^2$. Draw graphs of

(i) the sales revenue $R(p)$;

(ii) the net income from sales if a 35% sales tax applies;

(iii) the net income if a subsidy of 30 cents in the dollar applies.

(3) For the functions $f(x) = x^3 - 4x^2 + 5x - 2$ and $g(x) = 3x - x^2$, sketch graphs of f, g, $f + g$, and fg on a common set of axes.

(4) A manufacturer finds he sells $q(p) = 100 - 0.2p$ units of a given product per week when a price of \$$p$ per unit is charged. If the cost of manufacturing q units per week is $C(q) = 1000 + 60q$, write down expressions for the weekly sales revenue R and the weekly net profit $P = R - C$. Graph these as functions of p and find the prices at which maximum sales revenue and maximum net profit occur.

(5) In each of (a), (b) and (c) graph the functions indicated on a common set of axes.

(a) (i) $y = 1 + x^2$

(ii) $y = 1/(1 + x^2)$

(iii) $y = x/(1 + x^2)$

(b) (i) $y = x^2 - 4x + 3$

(ii) $y = 1/(x^2 - 4x + 3)$

(iii) $y = x - 2$

(iv) $y = \dfrac{x - 2}{x^2 - 4x + 3}$

(c) (i) $y = f(x)$, where $f(x) = x^3 - 3x^2 + 2x$

(ii) $y = f(x)^2$

(iii) $y = \sqrt{f(x)}$

(6) Monthly sales for a given commodity varies inversely with the square root of the price being charged per unit; that is,

$$q(p) = Ap^{-1/2},$$

where q is the number of units sold per month and $\$p$ is the price per unit. Determine A if 150 units were sold during a month when the price was $4 per unit. If the price per unit increases with time according to the formula

$$p(t) = 4(1.003)^t ,$$

where t is the time in months, find the quantity $Q(t)$ sold during the t^{th} month.

Draw graphs of $Q(t)$ and the monthly sales revenue

$$R(t) = p(t)Q(t) .$$

7. EXPONENTIAL FUNCTION

7.1 Interest on an Investment

We begin by examining the value V of an investment after one year when a sum of $\$P$ is invested at a *nominal interest rate* of $R\%$ per annum with the interest compounded n times per year. This means that the interest rate at each compounding is $R/n\%$. Let $x = R/100$ then we have:

n = 1 (Interest calculated annually)

$$\begin{aligned} V &= P + (\text{interest}) \\ &= P + Px \\ &= P(1+x) \end{aligned}$$

n = 2 (Interest calculated semiannually)

$$\begin{aligned} V &= \underbrace{P\left(1+\frac{x}{2}\right)}_{\substack{\text{value at} \\ \text{half year}}}\left(1+\frac{x}{2}\right) \\ &= P\left(1+\frac{x}{2}\right)^2 \end{aligned}$$

n = 4 (Interest calculated quarterly)

$$\begin{aligned} V &= \underbrace{\underbrace{\underbrace{\underbrace{P\left(1+\frac{x}{4}\right)}_{\substack{\text{value at} \\ \text{first quarter}}}\left(1+\frac{x}{4}\right)}_{\text{value at mid year}}\left(1+\frac{x}{4}\right)}_{\text{value at third quarter}}\left(1+\frac{x}{4}\right)}_{\text{value at end of year}} \\ &= P\left(1+\frac{x}{4}\right)^4 \end{aligned}$$

Similarly for

n = 12 (Interest calculated monthly)

$$V = P\left(1+\frac{x}{12}\right)^{12}$$

n = 365 (Interest calculated daily)

$$V = P\left(1+\frac{x}{365}\right)^{365}$$

and in general

$$V = P\left(1+\frac{x}{n}\right)^n.$$

To get a feel for this let us tabulate the case when $P = 1$ and $x = 1$

n	$(1+1/n)^n$
1	2
2	2.25
4	2.44
12	2.61
52	2.69
365	2.714
8760	2.718
525600	2.7182

We see that as n tends to ∞ the value of the investment appears to approach a limiting value e. Indeed continuing the process we would find $e \doteq 2.718281828\ldots$.

That is,

$$e = \operatorname*{Limit}_{n\to\infty} (1 + \frac{1}{n})^n$$

To see what happens in the general case as n tends to infinity it is convenient to let $m = n/x$ as then

$$V = P\left(1 + \frac{x}{n}\right)^n = P\left(1 + \frac{1}{m}\right)^{mx} = P\left[\left(1 + \frac{1}{m}\right)^m\right]^x$$

and so, since m tends to infinity as n does, we see that V approaches the limit

$$V = Pe^x.$$

Thus when interest is compounded "continuously" the value of our investment after 1 year is

$$V = Pe^x.$$

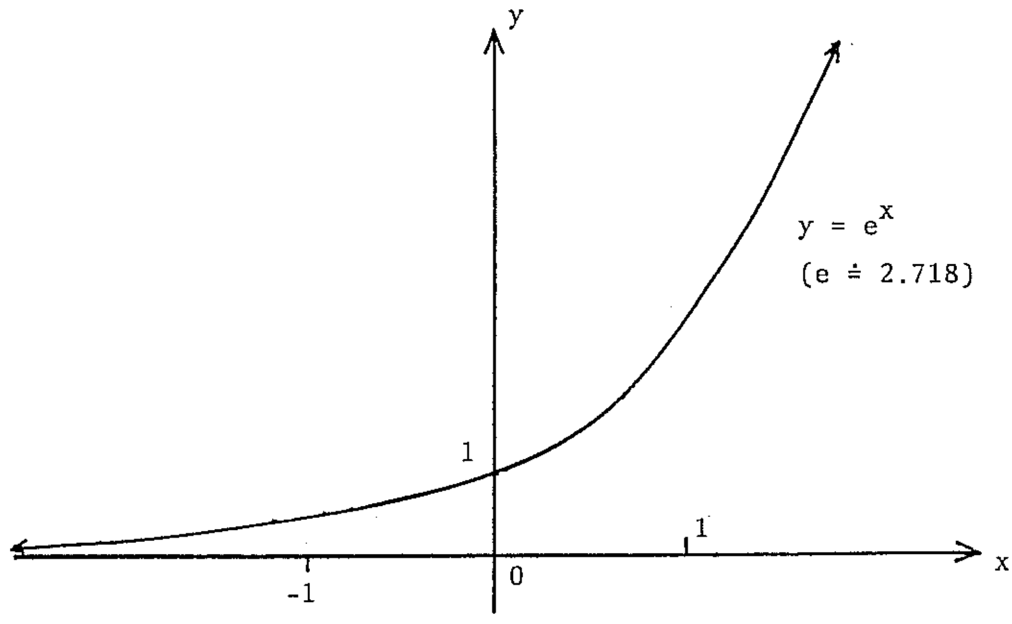
This also provides a useful approximation to the value of an investment when interest is compounded a large number of times per year. For example; after 1 year the value of an investment of $1000 at a nominal annual interest rate of 9% compounded daily is

$$\begin{aligned} V &= 1000\left(1 + \frac{0.09}{365}\right)^{365} \\ &= \$1094.16, \end{aligned}$$

while the approximation $V = 1000e^{0.09}$ gives $V = \$1094.17$.

7.2 The function $y = e^x$

The function $y = e^x$ sometimes also written as $y = \exp(x)$ is known as the *exponential function*. As we shall subsequently see it is one of the more commonly occurring functions in economic and business considerations.



A graph of the exponential function $y = exp(x) = e^x$.

Note: $y = e^x$ is a strictly positive, rapidly increasing, convex function. From the basic rules of exponents we see that

$$e^{-x} = 1/e^x$$
$$e^{x_1}e^{x_2} = e^{x_1+x_2}$$
$$(e^x)^\alpha = e^{\alpha x}$$

For example; if we want the value of our investment of a principal \$P at an interest rate of $100x\%$ compounded continuously after a period other than one year we have:

Value after 1 year $= Pe^x$

Value after 2 years $= \underbrace{Pe^x}_{\text{effective principal at start of second year}} \times e^x = Pe^{2x}$

Value after 3 years $= \underbrace{Pe^{2x}}_{\text{effective principal at start of third year}} \times e^x = Pe^{3x}$

and in general:

Value after t years $= Pe^{tx}$,

where t may be any positive number.

Thus if $P = 1000$ and the rate of interest is 10% we have

Value after t years $= V(t) = 1000e^{0.1t}$.

So (using a calculator to evaluate $e^{0.1t}$):

Value after 18 months is $V(1.5) = 10000e^{0.15} = \$1161.83$

and

Value after 30 years is $V(30) = 1000e^{3} = \$20085.54$.

7.3 Other representations of e^x.

We have seen that

$$\begin{aligned} e^x &= \underset{m\to\infty}{\text{Limit}}\left(1+\frac{1}{m}\right)^{mx} \\ &= \underset{n\to\infty}{\text{Limit}}\left(1+\frac{x}{n}\right)^{n}. \end{aligned}$$

If we expand $(1+x/n)^n$ we obtain a polynomial of degree n.

For Example.

$$\begin{aligned} \left(1+\frac{x}{2}\right)^2 &= 1+x+\frac{x^2}{4} \\ \left(1+\frac{x}{3}\right)^3 &= 1+x+3x^2+\frac{x^3}{27} \\ \left(1+\frac{x}{4}\right)^4 &= 1+x+\frac{2x^2}{3}+\frac{4x^3}{27}+\frac{x^4}{81} \end{aligned}$$

and in general

$$\left(1+\frac{x}{n}\right)^n = 1+x+\frac{n(n-1)}{2n^2}x^2+\frac{n(n-1)(n-2)}{3\times 2\times n^3}x^3+\cdots+\frac{x^n}{n^n}$$

Letting n tend to ∞ we obtain the infinite series

$$e^x = 1+x+\frac{x^2}{2}+\frac{x^3}{3\times 2}+\frac{x^4}{4\times 3\times 2}+\cdots$$

For any given x the higher order terms become vanishingly small quite quickly and by ignoring them we are able to obtain useful approximations to e^x. Indeed it is in this way that computers (and many calculators) evaluate the exponential function.

For example:

$$\begin{aligned} e = e^1 &\simeq 1+1+\frac{1}{2}+\frac{1}{3\times 2}+\frac{1}{4\times 3\times 2}+\frac{1}{5\times 4\times 3\times 2}+\frac{1}{6\times 5\times 4\times 3\times 2}. \\ &\doteq 2.718 \end{aligned}$$

In particular, for x very small we have the *linear* approximation

$$e^x \simeq 1 + x.$$

EXERCISES 7.4

(1) A sum of \$5000 is invested at a nominal annual interest rate of 13% compounded monthly.

(a) What is the value of the investment after 1 year?

(b) What is the value of the investment after 3 years?

(c) Draw a graph showing the value of the investment over the first 10 years.

(2) Sketch graphs for each of the functions $y = 3e^{0.2x}$, $y = 3e^{0.5x}$, $y = 3e^{-0.5x}$, $y = 3e^{0.2x} - e^{0.5x}$.

(3) The same as (1), but with the interest compounded continuously.

(4) Which is better for the investor, continuous compounding at a nominal annual interest rate of 10%, or quarterly compounding at a nominal annual interest rate of 10.25%?

(5) The U.N. Trust Worthy Bank of Australia compounds interest weekly at a nominal annual interest rate of 12%. What *effective annual rate of interest* could it use in its advertisements. [Note: the effective annual rate of interest applied at the end of a year should give the same value for an investment as the 12% nominal annual interest rate compounded weekly throughout the year.]

(6) For 2 years an initial investment of \$2000 attracts interest at a nominal annual rate of 12% compounded continuously, and then for a further 3 years at a nominal annual rate of 15% compounded continuously. What is the value of the investment at the end of the five year period?

(7) When a fixed nominal annual interest rate is compounded continuously it is found that the value of an investment doubles after the first T years. Show that for any period of T years the value at the end of the period is double the value at the start of the period. Use this to graph the value of the investment when $T = 10$ and the sum invested is \$1000.

(8) A function of the form

$$S = S_0(1 - e^{at})$$

is sometimes used to describe the growth of sales toward a saturation level S_0. Draw a graph of S as a function time t when $S_0 = 10$ and $a = 0.5$.

(9) The spread of new technology through a community may be described by a function of the form

$$A = \frac{1}{1 + Ce^{-at}}.$$

Graph this function when $C = 1$ and $a = 0.5$.

[Note: A function of this form is known as a *Logistic function* and is also used to describe the growth of populations and the spread of diseases.]

(10) The fraction of individuals born at a particular time still alive by age t is given approximately by *Gompertz' function*

$$F = e^{a(1-e^{bt})}.$$

Graph F as a function of t when $a = 0.1$ and $b = 0.08$.

[*Gompertz'* function plays a role in actuarial studies.]

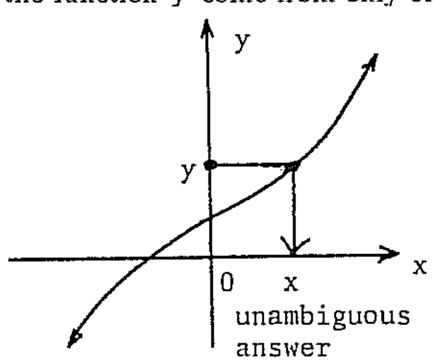
8. INVERSE FUNCTIONS

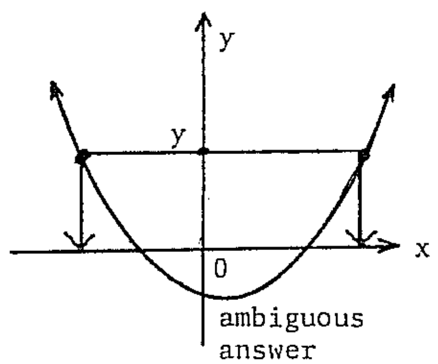
We have described the *demand* for a commodity by a function D giving the quantity q which will be consumed by the market as a function of p the price per unit at which the commodity is sold;

$$q = D(p).$$

Often we are interested in answering the "inverse" question: at what price per unit must the commodity be offered in order that a quantity q be sold. That is, we wish to express p as a function of q. When this can be done the resulting function is referred to as the inverse of D and denoted by D^{-1}; that is, $p = D^{-1}(q)$.

In general given a function $y = f(x)$ we are asking, for what value of x does $f(x)$ have a given value y? In order that this has an unambiguous answer it is necessary that each value y of the function f come from only one value of x.





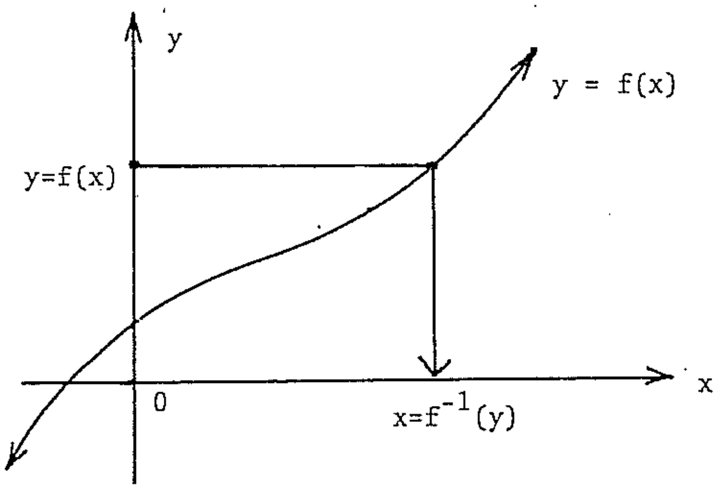
A function f for which each value y comes from only one x is said to be one-to-one; each x gives only one y (since f is a function), and each y comes from only one x.

For a function to be one-to-one each horizontal line must cut its graph only once (just as each vertical line must only cut it once in order that f is a function). In any region where the function is continuous this means it must be either strictly increasing or strictly decreasing.

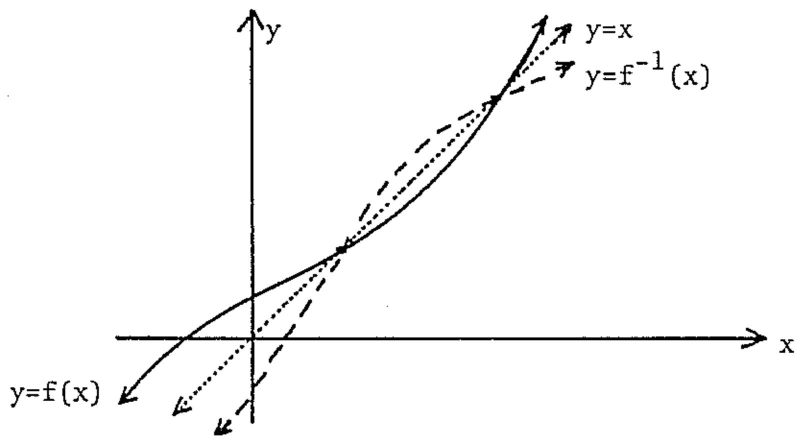
When f is one-to-one it is *invertible* and we can define the *inverse function*

$$x = f^{-1}(y)$$

which assigns to each value y of f the unique x for which $y = f(x)$.



By reflecting this diagram about the line $y = x$ so that the y-axis becomes horizontal and the x-axis vertical we obtain a graph of the inverse function $x = f^{-1}(y)$.



If $y = f(x)$ is given by a formula we may obtain an expression for $x = f^{-1}(y)$ by rearranging the formula so that x becomes the subject.

For example, if $y = f(x) = \frac{1}{2}x - 3$ we have $x = 2(y + 3)$ and so

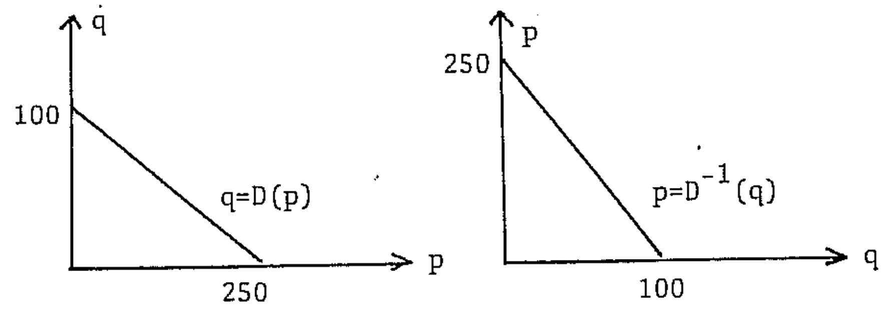
$$x = f^{-1}(y) = 2y + 6.$$

Similarly, for the demand function

$$q = D(p) = 100 - 0.4p$$

we have the inverse demand function

$$\begin{aligned} p = D^{-1}(q) &= \frac{10}{4}(100 - q) \\ &= 250 - 2.5q \end{aligned}$$

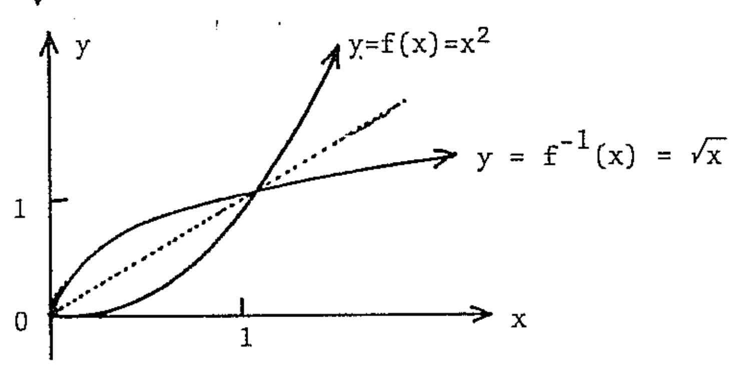


When the variables have meaningful names, such as p and q above, this is the end of the story. However in the general case when $y = f(x)$ is an invertible function these procedures give the graph or an expression for f^{-1} as a function of the variable y. If the variables do not have meaningful names which we would wish to preserve, it is conventional to use x and not y for the independent variable. To achieve this for f^{-1} it is necessary to replace y by x (and x by y) throughout the expression $x = f^{-1}(y)$.

For example: for positive x the function $y = f(x) = x^2$ is invertible with inverse

$$x = f^{-1}(y) = y^{\frac{1}{2}} = \sqrt{y} \quad \text{(obtained by solving } y = x^2 \text{ for } x\text{)}$$

To express this conventionally we swap the roles of x and y to obtain that the inverse of $y = f(x) = x^2$ is $y = f^{-1}(x) = \sqrt{x}$.



Note: The functions defined by

$$f^{-1}(y) = \sqrt{y}$$

and

$$f^{-1}(x) = \sqrt{x}$$

are of course the same function; only the name of the variable has been changed to protect the convention.

An important property of inverse functions is that

$$f^{-1}(f(x)) = x \text{ and } f(f^{-1}(y)) = y$$

EXERCISES 8.1

(1) For a particular commodity the quantity q demanded by the market per month is given by the demand function

$$q = D(p) = 1500 - 0.3p,$$

where p is the price asked per unit.

(i) Find an expression for, and graph, the *inverse* demand function $p = D^{-1}(q)$.

(ii) If demand is satisfied, find an expression for the monthly sales revenue $R = qp$ as a function of q. For what quantity of the commodity is the sales revenue R maximized?

(2) For each of the following functions determine whether or not the function is invertible. When the function is invertible find an expression for the the inverse function using x to represent the independent variable, and draw graphs of both the function and its inverse on the same set of axes.

(i) $y = f(x) = 5x - 3$.

(ii) $y = f(x) = x^2 + 1$, where x is positive.

(iii) $y = f(x) = 1/(x + 1)$, $x \neq -1$.

(iv) $y = f(x) = x^3 - x$.

*(3) If f and g are invertible functions show that $(f^{-1})^{-1} = f$ and $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

9. THE NATURAL LOGARITHM

9.1 The function $\ell n\, x$

We have seen that the function

$$y = f(x) = e^x$$

is strictly increasing. It is therefore invertible, with inverse $f^{-1}(y)$ satisfying

$$y = f(f^{-1}(y)) = e^{f^{-1}(y)}.$$

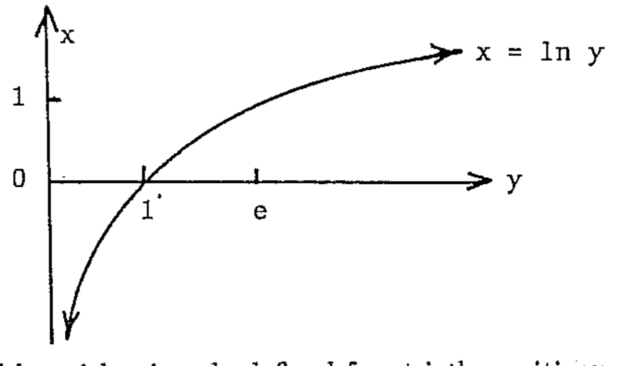
Thus $f^{-1}(y)$ *is the power to which* e *must be raised to obtain* y. Consequently we refer to $f^{-1}(y)$ as the logarithm of y to the base e, or the **natural logarithm** of y and denote it by

$$\log_e y$$

or

$$\ell n\, y.$$

By reflecting the graph of $y = e^x$ in the line $y = x$ we obtain the graph of $\ell n\, y$,



Note that the natural logarithm is only defined for strictly positive values of y, where it is a strictly increasing concave function. For this reason it is sometimes used as a *utility* function.

We also see that

$$\boxed{\ell n(1) = 0}$$

and

$$\ell n(e) = 1$$

Other properties of $\ell n\, y$ follow from those of the exponential function.

For example

$$\begin{aligned} \ell n(uv) &= \ell n(e^{\ell n\, u} e^{\ell n\, v}) \\ &= \ell n(e^{\ell n\, u + \ell n\, v}) \\ &= \ell n\, u + \ell n\, v \end{aligned}$$

Thus

$$\boxed{\ell n(uv) = \ell n\, u + \ell n\, v}$$

Similarly,

$$\begin{aligned} \ell n(u^\alpha) &= \ell n((e^{\ell n\, u})^\alpha) \\ &= \ell n\, e^{\alpha\, \ell n\, u} \\ &= \alpha\, \ell n\, u \end{aligned}$$

So,

$$\boxed{\ell n(u^\alpha) = \alpha\, \ell n\, u.}$$

In particular, taking $\alpha = -1$

$$\boxed{\ell n(1/u) = -\,\ell n\, u}$$

and it follows that

$$\ell n(u/v) = \ell n(u \times \frac{1}{v}) = \ell n\, u + \ell n\, \frac{1}{v}$$

so

$$\boxed{\ell n(u/v) = \ell n\, u - \ell n\, v}$$

9.2 Applications

1) A sum of \$1000 is invested at a nominal annual rate of interest of 12% compounded continuously. How long will it take for the investment to reach a value of \$1500?

The value of the investment after t years is

$$V = 1000e^{0.12t}$$

Thus we want t such that

$$1000e^{0.12t} = 1500$$

or

$$e^{0.12t} = 1.5$$

That is, $0.12t = \ell n\, 1.5$ and so

$$t = \frac{\ell n\, 1.5}{0.12} \doteq 3.38 \text{ years}$$
$$\doteq 3 \text{ years } 4\frac{1}{2} \text{ months}$$

2) If the value of an investment is to be doubled after 5 years at what nominal annual interest rate must it be invested if interest is calculated continuously? If the nominal annual interest rate is 100i% then after 5 years the value of the investment is

$$V = Pe^{5i}$$

where P is the sum invested. If V is to be $2P$ we must have

$$2P = Pe^{5i}$$

so

$$2 = e^{5i}$$

and

$$5i = \ell n\, 2.$$

Thus

$$i = \frac{\ell n\, 2}{5} = 0.1386$$

and so we must invest at a nominal annual interest rate of 13.86%.

Remark You may already be familiar with the logarithm of y to the base 10, usually denoted by $\log_{10} y$ or simply $\log\, y$. It is the power to which 10 must be raised to obtain y. That is

$$10^{\log\, y} = y.$$

Since $10 = e^{\ell n\, 10}$ we therefore have $e^{\ell n\, y} = y = 10^{\log\, y} = (e^{\ell n\, 10})^{\log\, y} = e^{\ell n\, 10 \log\, y}$ and so

$$\ell n\, y = \ell n\, 10 \log\, y.$$

Thus

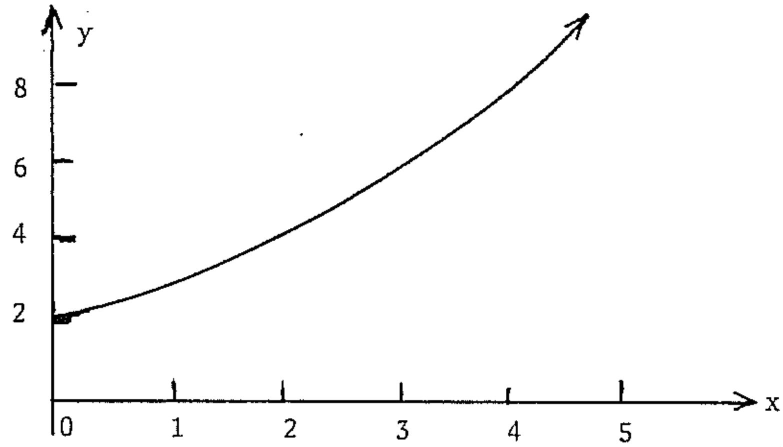
$$\ell n\, y \doteq 2.303 \log\, y$$

and

$$\log y \doteq 0.434\, \ell n\, y.$$

9.3 Transforming variables and Fitting Curves to data

The function $y = Ae^{\alpha x}$ may be presented graphically as illustrated below, with both x and y plotted on *linear scales.*



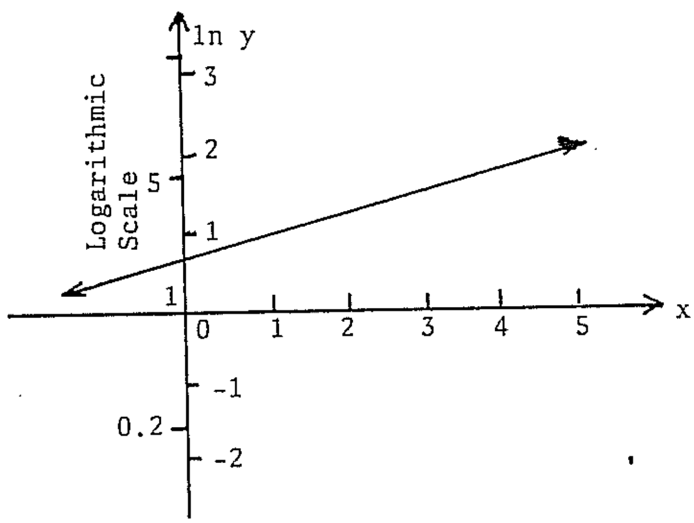
Alternatively, taking logs of both sides we have that

$$\ell n\, y = \ell n(Ae^{\alpha x})$$
$$= \ell n\, A + \ell n(e^{\alpha x})$$

so

$$\ell n\, y = \ell n\, A + \alpha x$$

Thus, we see that $\ell n\, y$ is linearly related to x, and so the same relationship between y and x may be presented graphically by plotting $\ell n\, y$ versus x to obtain a *straight line* of slope α and vertical intercept $\ell n\, A$.



Note: If y were linearly related to x, say $y = 2x + 1$, then plotting $\ell n\, y$ versus x would not give a straight line, but rather a graph like the following.