Department of Mathematics
PURE MATHS 211/311

ANALYSIS IN METRIC SPACES

Assignment 1 Solutions

1. We define $d(x,y) = \begin{cases} 0, & \text{if } x = y \\ |x|+|y|, & \text{if } x \neq y \end{cases}$ $\quad (x,y, \in \mathbb{R})$

To show d is a metric on $\mathbb{R}$.

(M1) Clearly $d(x,y) \geq 0 \quad \forall x,y \in \mathbb{R}$

(M2) If $x = y$, then $d(x,y) = 0$ by definition. If $x \neq y$, then at least one of $|x|$ and $|y|$ is greater than zero; hence $d(x,y) = |x|+|y| > 0$.

$\therefore \; d(x,y) = 0 \Leftrightarrow x = y$.

(M3) If $x = y$, then $d(x,y) = 0 = d(y,x)$. If $x \neq y$, then $d(x,y) = |x|+|y| = |y|+|x| = d(y,x)$.

$\therefore \; d(x,y) = d(y,x) \quad \forall x,y \in \mathbb{R}$

(M4) If $x = y$, then for any $z \in \mathbb{R}$, we have $d(x,y) = 0 \leq d(x,z) + d(z,y)$, since $d(x,z)$ and $d(z,y)$ are both non-negative. Otherwise suppose $x \neq y$. If $z \neq x$ and $z \neq y$, then

$d(x,z) + d(z,y) = |x|+|z|+|z|+|y| \geq |x|+|y| = d(x,y)$

If $z = x$, then $z \neq y$, so

$d(x,z) + d(z,y) = |z|+|y| = |x|+|y| = d(x,y)$

Similarly if $z = y$.

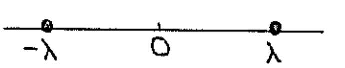
Overall, then, $d(x,y) \leq d(x,z) + d(z,y) \quad \forall x,y,z \in \mathbb{R}$

Since (M1) - (M4) hold, $(\mathbb{R},d)$ is a metric space.

(a) $d(x,0) = 1 \Leftrightarrow |x|=1 \Leftrightarrow x = \pm 1$.

$d(x,0) = 2 \Leftrightarrow |x|=2 \Leftrightarrow x = \pm 2$.

$d(x,0) = \lambda$: If $\lambda = 0$, then $x = 0$.



Otherwise, $|x|= \lambda$, i.e., $x = \pm\lambda$.

(b) $d(x,1) = 1 \Rightarrow |x|+1 = 1 \Rightarrow x = 0$

$d(x,1) = 2 \Rightarrow |x|+1 = 2 \Rightarrow |x|= 1$.

BUT, if $x = 1$, then $d(x,1) = 0$, as $d(x,y) = 0$ if $x = y$.

Hence $x = -1$ is the only solution to $d(x,1) = 2$.

$d(x,1) = \lambda$: If $\lambda = 0$, then $x = 1$

If $0 < \lambda < 1$, then $|x|+1= \lambda \Rightarrow |x|= \lambda -1 < 0$ - contradiction.

If $\lambda \geq 1$, then $|x|+1= \lambda \Rightarrow |x|= \lambda - 1 \Rightarrow x = \pm(\lambda-1)$.

$\therefore$ those x for which $d(x,1) = \lambda$ are given by:

$$\begin{cases} x = 1, & \text{if } \lambda = 0 \\ x = -1, & \text{if } \lambda = 2 \\ x = \pm(\lambda-1), & \text{if } \lambda \geq 1,\ \lambda \neq 2. \end{cases}$$

If $0 < \lambda < 1$, there is no solution to $d(x,1) = \lambda$.

(c) We fix y, and solve $d(x,y) = \lambda$:

If $\lambda = 0$, then $x = y$.

If $0 < \lambda < |y|$, then $|x|+|y| = \lambda \Rightarrow |x| = \lambda - |y| < 0$ – a contradiction.

If $\lambda \geq |y|$, then $|x| = \lambda - |y| \Rightarrow x = \pm(\lambda - |y|)$

Note that if $\lambda = 2|y|$, then there is only one solution for x (ie, $x = -y$).

$\therefore$ those x for which $d(x,y) = \lambda$ are given by:

$$\begin{cases} x = y, \text{ if } \lambda = 0 \\ x = -y, \text{ if } \lambda = 2|y| \\ x = \pm(\lambda-|y|), \quad \text{if } \lambda \geq |y|,\ \lambda \neq 2|y|. \end{cases}$$

If $0 < \lambda < |y|$, there is no solution to $d(x,y) = \lambda$.

If we now draw the sets $d(x,y) \leq \lambda$ for fixed y, and various $\lambda \geq 0$, we obtain:

(1) $\lambda = 0$:

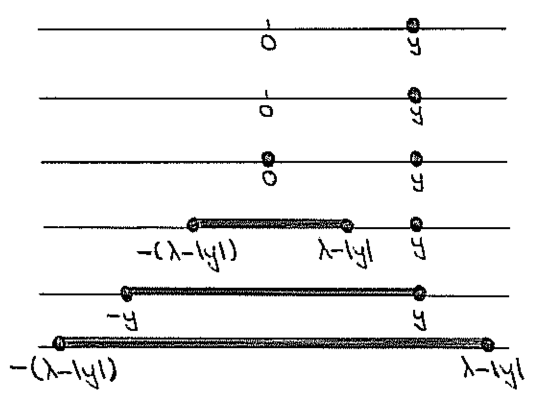
(2) $0 < \lambda < |y|$:

(3) $\lambda = |y|$:

(4) $|y| < \lambda < 2|y|$:

(5) $\lambda = 2|y|$:

(6) $\lambda > 2|y|$:



In Q1 of Tutorial Sheet 2, the sets $d(x,y) \leq \lambda$ were squares centred on y, and there were no isolated points like above. This irregular behaviour above suggests that d is not induced by any norm. This is in fact the case, although the proof is slightly tricky:

Suppose $\| \ \|$ is a norm on the vector space $\mathbb{R}$ with "addition" denoted by $\oplus$, and "subtraction" by $\ominus$, such that

$$d(x,y) = \|x \ominus y\| \text{ for all } x,y \in \mathbb{R}.$$

Put $k = 2 \ominus 1$

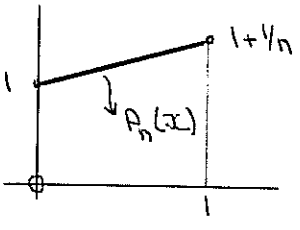
$$
\begin{aligned}
\text{Then } 1 = d(1,0) &= \|1 \ominus 0\| \\
&= \|(1 \oplus k) \ominus (0 \oplus k)\| \\
&= d(1 \oplus k, 0 \oplus k) \\
&= |1 \oplus k| + |0 \oplus k| \\
&= |1 \oplus (2 \ominus 1)| + |0 \oplus k| \\
&= 2 + |0 \oplus k| \\
&\geq 2
\end{aligned}
$$

i.e., $1 \geq 2$, which is a clear contradiction. Hence d is not induced by any norm.

2 We have $\|p(x)\|_\infty = \max\limits_{x \in [0,1]} |p(x)|$, for

$p(x) \in P_1[0,1]$, the linear space of all polynomials of degree 1.

Define $P_n(x) = 1 + \dfrac{x}{n}$.



(a) $$\|P_n(x)\|_\infty = \max_{x \in [0,1]} |P_n(x)| = 1 + \frac{1}{n} \quad \text{(from the diagram)}$$

$$\|P_n(x) - P_m(x)\|_\infty = \max_{x \in [0,1]} |P_n(x) - P_m(x)| = \max_{x \in [0,1]} \left|(1+\frac{x}{n}) - (1+\frac{x}{m})\right|$$

$$= \max_{x \in [0,1]} \left|x(\frac{1}{n} - \frac{1}{m})\right| = \max_{x \in [0,1]} \left(|x| \left|\frac{1}{n} - \frac{1}{m}\right|\right)$$

$$= \left|\frac{1}{n} - \frac{1}{m}\right| \left(\max_{x \in [0,1]} |x|\right) = \left|\frac{1}{n} - \frac{1}{m}\right|.$$

(b) Let $\varepsilon > 0$ be given, and choose N to be any integer greater than $2/\varepsilon$.

Then, if $n,m \geq N$, we have

$$\|P_n(x) - P_m(x)\|_\infty = \left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m} \leq \frac{1}{N} + \frac{1}{N} = \frac{2}{N} < 2 \cdot \frac{\varepsilon}{2} = \varepsilon$$

i.e., if $n,m \geq N$, then $\|P_n(x) - P_m(x)\|_\infty < \varepsilon$

Hence, by definition of a Cauchy sequence,

$\{P_n(x)\}$ is Cauchy.

(c) $\|P_n(x) - 1\|_\infty = \max\limits_{x \in [0,1]} \left|(1 + \frac{x}{n}) - 1\right| = \max\limits_{x \in [0,1]} \left|\frac{x}{n}\right|$

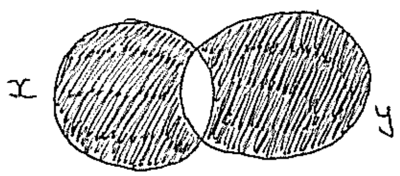
$= \frac{1}{n} \cdot \max\limits_{x \in [0,1]} |x| = \frac{1}{n}$

Since $\lim\limits_{n\to\infty} \frac{1}{n} = 0$, it is reasonable to assume that $P_n(x) \to 1$. To prove this, let $\varepsilon > 0$ be given, and let N be any integer greater than $\frac{1}{\varepsilon}$.

Then if $n > N$, we have $\|P_n(x) - 1\|_\infty = \frac{1}{n} < \frac{1}{N} < \varepsilon$

i.e., $\|P_n(x) - 1\|_\infty < \varepsilon$ whenever $n > N$. Thus, by definition, $\{P_n(x)\}$ converges to the constant function 1.

3. For $x,y \in X$, the class of all finite subsets of $\mathcal{R}$, we define $d_0(x,y)$ to be the number of elements in $x \,\Delta\, y = (x \cap y') \cup (y \cap x')$



The shaded part is $x \,\Delta\, y$.

To show d_0 is a metric:

(M1) Clearly $d_0(x,y) \geq 0$ for all $x,y \in X$.

(M2) If $x = y$, then $x \cap y' = \phi$, and $y \cap x' = \phi$, so $x \,\Delta\, y = \phi$.

$\therefore$ the no. of elements in $x \,\Delta\, y$ is 0, i.e., $d_0(x,y) = 0$.

Conversely, suppose $d_0(x,y) = 0$.

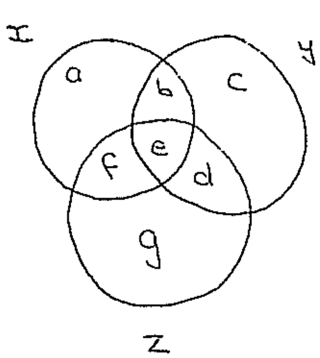
Then both $x \cap y'$ and $y \cap x'$ are empty sets.

Since $x \cap y' = \phi$, it follows that x must be contained in y i.e., $x \subseteq y$.

Similarly, since $y \cap x' = \phi$, it follows that $y \subseteq x$. Since $x \subseteq y$ and $y \subseteq x$, it follows that $x = y$ $\therefore d_0(x,y) = 0 \Leftrightarrow x = y$.

(M3) Since $x \,\Delta\, y = y \,\Delta\, x$, it follows that $d_0(x,y) = d_0(y,x)$

(M4) Let x,y,z be 3 elements of X as shown, and let a,b,c,... denote the number of elements in the regions shown.



Then $d_0(x,y) = a + f + c + d$

$d_0(x,z) = a + b + d + g$

$d_0(z,y) = b + c + f + g$

Hence $d_0(x,z) + d_0(z,y) - d_0(x,y)$

$= 2(b+g) \geq 0$ (as b and g are non-negative integers)

$\therefore d_0(x,y) \le d_0(x,z) + d_0(z,y)$

Since (M1)- (M4) hold, (X, d_0) is a metric space.

(a) Suppose $\{x_n\}$ is a sequence in X that converges to $y \in X$. By definition, then, given $\varepsilon > 0$, there exists N such that $d_0(x_n,y) < \varepsilon$ for all $n \ge N$.

Now take $\varepsilon = 1$. Then there exists N such that $d_0(x_n,y) < 1$ for all $n \ge N$.

But $d_0(x_n,y)$ is the number of elements in $x_n \Delta y$, and must be a non-negative integer. Since $d_0(x_n,y) < 1$ for all $n \ge N$, we must have $d_0(x_n,y) = 0$ for all $n \ge N$, and so $x_n = y$ for all $n \ge N$ (by (M2)).

That is, the sequence $\{x_n\}$ becomes constant after a finite number of terms.

(b) If $x_1 = \{1\}$

$x_2 = \{\frac{1}{2},1\}$, ...

$x_n = \{\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}$, ...

then the sequence $\{x_n\}$ cannot converge, as it does not become constant after a finite number of terms.

(c) Let $\{x_n\}$ be a Cauchy sequence in (X, d_0). Then, given $\varepsilon > 0$, there exists N such that $d_0(x_n,x_m) < \varepsilon$ whenever $n,m \ge N$. Now, as before, take $\varepsilon = 1$. Then there exists N such that $d_0(x_n,x_m) < 1$ for $n,m \ge N$.

As previously, this means that $x_n = x_m$ for $n,m \ge N$ and (taking $m = N$), we see $x_n = x_N$ for $n \ge N$.

Therefore, $\{x_n\}$ becomes constant after a finite number of terms and hence is convergent i.e., Any Cauchy sequence in (X,d_0) is convergent, and so, be definition, (X,d_0) is complete.

METRIC SPACES - TUTORIAL SHEET 4

1. Show that for any set X with the discrete metric,

$$B_r(x_0) = \begin{cases} \{x_0\} & \text{if } r \leqslant 1 \\ X & \text{if } r > 1 \end{cases}$$

2. (a) In $\mathbb{R}$, with metric $d(x,y) = \begin{cases} 0 & \text{if } x = y \\ |x|+|y| & \text{if } x \neq y \end{cases}$

   Show that the set $A = \{x: 1 \geqslant x > -1\}$ is an open set with respect to the metric d. (Look at the open sets $d(x,1) < \lambda$ for various λ.)

   (b) Show that A is neither open nor closed with respect to the metric $d_1(x,y) = |x - y|$.

3. In the set $\mathbb{R}$ with the same metric as in 2(a), show that almost all single point sets are open ({0} is the only single point set that is not open).
   Which single point sets are closed?
   Show that any set not containing {0} is open.
   Find a set (other than {0}) that is not open.
   What are the only possible non-open sets? Are they closed?

4. Is the set $\underset{\sim}{Q}$ of rational numbers an open subset of $\mathbb{R}$ (use the usual metric $d(x,y) = |x-y|$). What is Int $\underset{\sim}{Q}$?

5. In the space C[0,1] with metric $\|\cdot\|_\infty$ as defined on Sheet 3, Q2, is the set P of polynomials an open set?
   Is the complement of P an open set? (Q1 on p19 of notes may help.)

6. Show that every single element subset of (X,d_0) (defined on Sheet 3, Q3) is an open set. (Be careful to remember that elements of X are finite subsets of $\mathbb{R}$, and that d_0 is a distance measure between such finite subsets.)

   Describe the set $\{x: d_0(x,x_0) \leqslant 1\}$ where $x_0 = \{a_1,a_2,\ldots,a_n\} \in X$. Is it open?