SOLUTIONS 1

Page 4.

1. Verify axioms (M1)–(M4).

(M1) : Obvious.

(M2) : $d_1(\underset{\sim}{x},\underset{\sim}{y}) = 0 \iff |x_1 - y_1| = |x_2 - y_2| = 0 \iff x_1 = y_1, x_2 = y_2$
$\iff \underset{\sim}{x} = \underset{\sim}{y}$.

(M3) : Obvious since $|x-y| = |y-x|$ for $x, y \in \mathbb{R}$.

(M4) : Let $\underset{\sim}{x}, \underset{\sim}{y}, \underset{\sim}{z} \in \mathbb{R}^2$. Then

$$d(\underset{\sim}{x},\underset{\sim}{z}) = |x_1 - z_1| + |x_2 - z_2| \leq |x_1 - y_1| + |y_1 - z_1| + |x_2 - y_2| + |y_2 - z_2|$$
$$= d(\underset{\sim}{x},\underset{\sim}{y}) + d(\underset{\sim}{y},\underset{\sim}{z}).$$

2. (M1)–(M3) are clear. To prove (M4), let $x, y, z \in \mathbb{R}$. We want to prove

$$\min\{1, |x-z|\} \leq \min\{1, |x-y|\} + \min\{1, |y-z|\}.$$

Now the RHS is equal to one of the numbers $a = 2$, $b = 1 + |y-z|$, $c = 1 + |x-y|$, $d = |x-y| + |y-z|$. If $|x-z| \leq 1$, then the LHS $= |x-z| \leq a, b, c, d$, so LHS $\leq$ RHS. If $|x-z| > 1$, then LHS $= 1 \leq a, b, c$ and $1 < d$ since $1 < |x-z| \leq |x-y| + |y-z|$, so LHS $\leq$ RHS.

3. (M1) : Obvious.

(M2) : $d^*(x,y) = 0 \iff d(x,y) = 0 \iff x = y$.

(M3) : $d^*(x,y) = \dfrac{d(x,y)}{1 + d(x,y)} = \dfrac{d(y,x)}{1 + d(y,x)} = d^*(y,x)$.

(M4) : We first show that if $a \geq 0, b \geq 0, c \geq 0$ and $c \leq a + b$, then $\dfrac{c}{1+c} \leq \dfrac{a}{1+a} + \dfrac{b}{1+b}$. Now

$$\text{RHS} - \text{LHS} = \frac{a(1+b)(1+c) + b(1+a)(1+c) - c(1+a)(1+b)}{(1+a)(1+b)(1+c)}$$

$$= \frac{(a+b-c) + 2ab + abc}{(1+a)(1+b)(1+c)} \geq 0.$$

Substituting $a = d(x,y)$, $b = d(y,z)$, $c = d(x,z)$ gives (M4).

4\. (i) (M1): $d(\underset{\sim}{x},\underset{\sim}{y}) = \max\{d^{(1)}(x_1,y_1), d^{(2)}(x_2,y_2)\} \geq 0.$

(M2): $d(\underset{\sim}{x},\underset{\sim}{y}) = 0 \iff d^{(1)}(x_1,y_1) = d^{(2)}(x_2,y_2) = 0$
$\iff x_1 = y_1, x_2 = y_2 \iff \underset{\sim}{x} = \underset{\sim}{y}.$

(M3): $d(\underset{\sim}{x},\underset{\sim}{y}) = \max\{d^{(1)}(x_1,y_1), d^{(2)}(x_2,y_2)\}$
$= \max\{d^{(1)}(y_1,x_1), d^{(2)}(y_2,x_2)\} = d(\underset{\sim}{y},\underset{\sim}{x}).$

(M4): Let $\underset{\sim}{x},\underset{\sim}{y},\underset{\sim}{z} \in X^{(1)} \times X^{(2)}$. Then

$$\begin{aligned}
d(\underset{\sim}{x},\underset{\sim}{z}) &= \max\{d^{(1)}(x_1,z_1), d^{(2)}(x_2,z_2)\} \\
&\leq \max\{d^{(1)}(x_1,y_1) + d^{(1)}(y_1,z_1), d^{(2)}(x_2,y_2) + d^{(2)}(y_2,z_2)\} \\
&\leq \max\{d^{(1)}(x_1,y_1), d^{(2)}(x_2,y_2)\} \\
&\qquad + \max\{d^{(1)}(y_1,z_1), d^{(2)}(y_2,z_2)\} \\
&\qquad (\text{since } \max\{a+b, c+d\} \leq \max\{a,c\} + \max\{b,d\}) \\
&= d(\underset{\sim}{x},\underset{\sim}{y}) + d(\underset{\sim}{y},\underset{\sim}{z}).
\end{aligned}$$

(ii) (M1): $d(\underset{\sim}{x},\underset{\sim}{y}) = d^{(1)}(x_1,y_1) + d^{(2)}(x_2,y_2) \geq 0.$

(M2): $d(\underset{\sim}{x},\underset{\sim}{y}) = 0 \iff d^{(1)}(x_1,y_1) = d^{(2)}(x_2,y_2) = 0$
$\iff x_1 = y_1, x_2 = y_2 \iff \underset{\sim}{x} = \underset{\sim}{y}.$

(M3): As above, follows from symmetry of $d^{(1)}$ and $d^{(2)}$.

(M4): Let $\underset{\sim}{x},\underset{\sim}{y},\underset{\sim}{z} \in X^{(1)} \times X^{(2)}$. Then

$$\begin{aligned}
d(\underset{\sim}{x},\underset{\sim}{z}) &= d^{(1)}(x_1,z_1) + d^{(2)}(x_2,z_2) \\
&\leq d^{(1)}(x_1,y_1) + d^{(1)}(y_1,z_1) + d^{(2)}(x_2,y_2) + d^{(2)}(y_2,z_2) \\
&= d(\underset{\sim}{x},\underset{\sim}{y}) + d(\underset{\sim}{y},\underset{\sim}{z}).
\end{aligned}$$

5\. By the triangle inequality we have

$$d(x,z) \leq d(x,y) + d(y,z)$$

$$\text{i.e.} \quad d(x,z) - d(z,y) \leqslant d(x,y). \qquad (1)$$

Also by the triangle inequality we have

$$d(z,y) \leqslant d(z,x) + d(x,y)$$

$$\text{i.e.} \quad -d(x,y) \leqslant d(x,z) - d(z,y). \qquad (2).$$

Combining (1) and (2) we obtain the desired result.

6. We must verify (M1) - (M4).

(M1) : Let $x, z \in X$. Setting $y = x$ in (M2$'$) we obtain $d(x,x) \leqslant 2\,d(x,z)$. But $d(x,x) = 0$ by (M1$'$) so $d(x,z) \geqslant 0$ for any $x, z \in X$.

(M2) : This is (M1$'$).

(M3) : Set $z = x$ in (M2$'$). This gives $d(x,y) \leqslant d(y,x)$ for any $x, y \in X$, and interchanging x and y we have $d(y,x) \leqslant d(x,y)$. Thus $d(x,y) = d(y,x)$.

(M4) : This is just (M2$'$), bearing in mind that d has been shown to be symmetric.

SOLUTIONS 2

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1. Since $\|x-y\|$ defines a metric on X, the proof that $d(x,y) = \min\{1, \|x-y\|\}$ defines a metric on X is similar to the proof of Problem 1.2. For d to be induced by some norm $\|\cdot\|'$, then by (n3) we have $d(\lambda x, \lambda y) = \|\lambda x - \lambda y\|' = |\lambda| \cdot \|x-y\|' = |\lambda| d(x,y)$ for all $x, y \in X$ and all scalars λ. Let $x \in X$ such that $\|x\| = 1$. Then $d(0, 2x) = \min\{1, \|2x\|\} = 1$, but $2d(0,x) = 2$.

2. $\|x\| = \|(x-y) + y\| \leq \|x-y\| + \|y\|$

$$\Rightarrow \quad \|x\| - \|y\| \leq \|x-y\| \text{ for all } x, y \in X.$$

$\Rightarrow$ (by symmetry) $\|y\| - \|x\| \leq \|y-x\| = \|x-y\|$.

Putting ~~these~~ two inequalities ~~together~~ we obtain

$$-\|x-y\| \leq \|x\| - \|y\| \leq \|x-y\| \quad \text{as required.}$$

3. (i) By Taylor's Theorem, $f = p_n + R_n$ where

$$R_n(x) = \frac{1}{n!}\int_0^x (x-t)^n f^{(n+1)}(t)\,dt.$$

$$= \frac{1}{n!}\int_0^x (x-t)^n e^t\,dt.$$

Then
$$\begin{aligned} d_\infty(f, p_n) &= \|f - p_n\|_\infty \\ &= \|R_n\|_\infty \\ &= \sup_{0\leq x\leq 1} \left| \frac{1}{n!}\int_0^x (x-t)^n e^t\,dt \right| \\ &\leq \sup_{0\leq x \leq 1} \frac{1}{n!} x^{n+1} e^x = \frac{e}{n!}. \end{aligned}$$

Since $\|f\|_1 \leq \|f\|_\infty$ for all $f \in C[0,1]$, we have $d_1(f,g) \leq d_\infty(f,g)$ for all $f,g \in C[0,1]$, and so also $d_1(f,p_n) \leq e^{x}/n!$

(ii) Here $R_n(x) = \frac{1}{n!}\int_0^x (x-t)^n \sin^{(n+1)}(t)\, dt$.

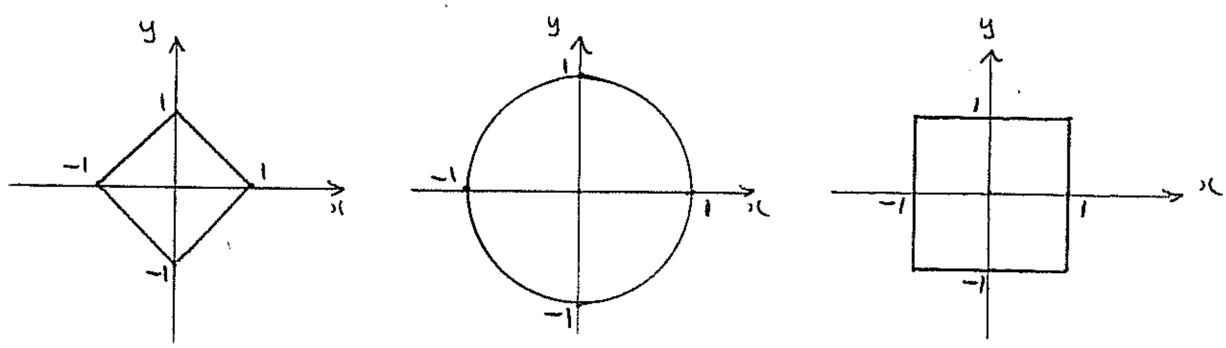
Then $d_\infty(f, p_n) = \|R_n\|_\infty$

$$= \sup_{0\leq x\leq 1} \left| \frac{1}{n!}\int_0^x (x-t)^n \sin^{(n+1)}(t)\,dt \right|$$

$$\leq \sup_{0\leq x\leq 1} \frac{1}{n!} x^{n+1} = \frac{1}{n!}$$

Thus also $d_1(f, p_n) \leq \frac{1}{n!}$.

4.



$\|\underset{\sim}{x}\|_1 = |x|+|y| = 1$ $\quad$ $\|\underset{\sim}{x}\|_2 = \sqrt{x^2+y^2} = 1$ $\quad$ $\|\underset{\sim}{x}\|_\infty = \max\{|x|,|y|\} = 1$.

5. We show that l_2 is a vector space by showing that it is closed under addition and scalar multiplication.

Let $\underset{\sim}{x}, \underset{\sim}{y} \in l_2$. To show $\sum_{n=1}^{\infty}(x_n+y_n)^2$ converges.

By Minkowski's Inequality we have for each N,

$$\sum_{n=1}^{N}(x_n+y_n)^2 \leq \left\{\left(\sum_{n=1}^{N} x_n^2\right)^{\frac{1}{2}} + \left(\sum_{n=1}^{N} y_n^2\right)^{\frac{1}{2}}\right\}^2$$

Denoting the LHS by A_N and the RHS by B_N, we have that $\{B_N\}_{N\geq 1}$ is an increasing, convergent sequence. Its limit is therefore an upper bound for the increasing sequence $\{A_N\}$, which thus converges itself. But this means that the series $\sum_{n=1}^{\infty} (x_n+y_n)^2$ converges, and hence $\underset{\sim}{x}+\underset{\sim}{y} \in \ell_2$.

Also, if $\underset{\sim}{x} \in \ell_2$ and $\lambda \in \mathbb{R}$, then the series $\sum_{n=1}^{\infty} (\lambda x_n)^2$ is clearly convergent, so $\lambda \underset{\sim}{x} \in \ell_2$.

To show that $\|\underset{\sim}{x}\|_2 = \left(\sum_{n=1}^{\infty} x_n^2\right)^{\frac{1}{2}}$ defines a norm on ℓ_2. Axioms (n1), (n2) and (n3) are trivial to verify, so we just prove (n4) (the triangle inequality).

Let $\underset{\sim}{x}, \underset{\sim}{y} \in \ell_2$. By Minkowski's Inequality we have for each N,

$$\left(\sum_{n=1}^{N} (x_n+y_n)^2\right)^{\frac{1}{2}} \leq \left(\sum_{n=1}^{N} x_n^2\right)^{\frac{1}{2}} + \left(\sum_{n=1}^{N} y_n^2\right)^{\frac{1}{2}}.$$

Letting $N \to \infty$ on both sides we obtain

$$\|x+y\|_2 \leq \|x\|_2 + \|y\|_2 .$$

SOLUTIONS 3

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1. We saw in problem 2.3 that $d_\infty(\exp, p_n) \leqslant e/n!$ Thus $d_\infty(\exp, p_n) \to 0$ as $n \to \infty$ so $p_n \to \exp$ in $(C[0,1], d_\infty)$. Since $d_1(f,g) \leqslant d_\infty(f,g)$ for all $f, g \in C[0,1]$, it is also true that $p_n \to \exp$ in $(C[0,1], d_1)$.

2. By the triangle inequality we have
$$d(x_n, y_n) \leq d(x_n, x) + d(x, y) + d(y, y_n)$$
and also
$$d(x,y) \leqslant d(x, x_n) + d(x_n, y_n) + d(y_n, y).$$
Thus
$$|d(x_n, y_n) - d(x,y)| \leqslant d(x_n, x) + d(y_n, y)$$
$$\to 0 \text{ as } n \to \infty,$$
which implies that $d(x_n, y_n) \to d(x,y)$ in $(\mathbb{R}, d_1)$.

3. Let $x_n \to x$ in (X, d) where d is the discrete metric. By the definition of convergence, there exists N such that $d(x_n, x) < \frac{1}{2}$ for $n > N$. But since d takes only the values 0 and 1, this implies that $d(x_n, x) = 0$ hence $x_n = x$ for $n > N$. Thus there are at most $N+1$ distinct points in the sequence $\{x_n\}$.

4. Let $\{x_n\}$ be a Cauchy sequence in (X, d). Then there exists N such that $d(x_n, x_m) < \frac{1}{2}$ for $n, m > N$. Thus $x_n = x_m = x$, say, for $n, m > N$, so clearly $x_n \to x$.

5. Example (one of many): In $(\mathbb{R}, d_1)$, $\frac{1}{n} \to 0$ as $n \to \infty$. But if d is the discrete metric on $\mathbb{R}$, then $d(\frac{1}{n}, 0) = 1$ for all n, so $\frac{1}{n} \not\to 0$ in $(\mathbb{R}, d)$.

6. Let $\{x_n\}$ be a Cauchy sequence in (X, d). The "only if" part is trivial, for if $\{x_n\}$ is convergent, then $\{x_n\}$ itself is a convergent subsequence. To prove the "if" part, let $\{x_{n_k} : k = 1, 2, \ldots\}$ be a convergent subsequence with limit x, and let $\epsilon > 0$. Since $x_{n_k} \to x$ there exists K such that

$$k > K \Rightarrow d(x, x_{n_k}) < \epsilon/2.$$

Since $\{x_n\}$ is Cauchy there exists N such that

$$n, m > N \Rightarrow d(x_n, x_m) < \epsilon/2.$$

Let $N_1 = \max(n_K, N)$. Then if $n > N_1$, choosing k such that $k > K$ and $n_k > N$, we have

$$\begin{aligned} d(x, x_n) &\leq d(x, x_{n_k}) + d(x_{n_k}, x_n) \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon, \end{aligned}$$

so $x_n \to x$ as $n \to \infty$, proving $\{x_n\}$ is itself convergent.

7. Consider $\mathbb{R}$ with usual metric d_1, and let $x_n = y_n = n$. Then $d(x_n, y_n) = 0$ for all n

so $\{d(x_n, y_n)\}$ is trivially convergent. However, $\{x_n\}$ is clearly not a Cauchy sequence.

8. Let $\{\underset{\sim}{x}_n\} = \{(x_{1n}, x_{2n})\}$ be a Cauchy sequence in $X^{(1)} \times X^{(2)}$ under the metric d of problem 1.4(i). Thus $d(\underset{\sim}{x}_n, \underset{\sim}{x}_m) = \max\{d_1(x_{1n}, x_{1m}), d_2(x_{2n}, x_{2m})\} \longrightarrow 0$ as $m, n \to \infty$. This implies that $\{x_{1n}\}$ and $\{x_{2n}\}$ are Cauchy sequences in the complete spaces $(X^{(1)}, d_1)$ and $(X^{(2)}, d_2)$ respectively, hence converge to limits x_1, x_2 respectively. Then letting $\underset{\sim}{x} = (x_1, x_2)$ we have

$$d(\underset{\sim}{x}_n, \underset{\sim}{x}) = \max\{d_1(x_{1n}, x_1), d_2(x_{2n}, x_2)\}$$
$$\longrightarrow 0 \text{ as } n \to \infty,$$

hence $\underset{\sim}{x}_n \to \underset{\sim}{x}$ in $(X^{(1)} \times X^{(2)}, d)$, proving $(X^{(1)} \times X^{(2)}, d)$ is complete.

The proof for the metric d of problem 1.4(ii) is exactly similar.

9.

9. Let $\{\underset{\sim}{x}_n\}$ be a Cauchy sequence in l_2, where $\underset{\sim}{x}_n = (x_{n1}, x_{n2}, \ldots\ldots)$; ie. $\|\underset{\sim}{x}_n - \underset{\sim}{x}_m\|_2 \to 0$ as $m, n \to \infty$;

ie. $\left(\sum_{i=1}^{\infty} (x_{ni} - x_{mi})^2\right)^{1/2} \to 0$ as $m, n \to \infty$.

$\Rightarrow$ for each i, $\{x_{ni} : n = 1, 2, \ldots\}$ is Cauchy in $\mathbb{R}$.
Let $x_i = \lim_{n\to\infty} x_{ni}$ and $\underset{\sim}{x} = (x_1, x_2, \ldots)$.

We claim that $\underset{\sim}{x} \in l_2$ and that $\underset{\sim}{x}_n \to \underset{\sim}{x}$.
Let $\varepsilon > 0$. By the Cauchy property of $\{\underset{\sim}{x}_n\}$, there exists N such that

$$m, n > N \Rightarrow \left(\sum_{i=1}^{\infty} (x_{mi} - x_{ni})^2\right)^{1/2} < \varepsilon.$$

$$\Rightarrow \left(\sum_{i=1}^{k} (x_{mi} - x_{ni})^2\right)^{1/2} < \varepsilon \text{ for each } k.$$

Letting $n \to \infty$ we have that

$$m > N \Rightarrow \left(\sum_{i=1}^{k} (x_{mi} - x_i)^2\right)^{1/2} \leq \varepsilon \text{ for each } k. \quad (*)$$

Then for $m > N$ and each k we have by Minkowski's Inequality,

$$\left(\sum_{i=1}^{k} x_i^2\right)^{1/2} \leq \left(\sum_{i=1}^{k} (x_{mi} - x_i)^2\right)^{1/2} + \left(\sum_{i=1}^{k} x_{mi}^2\right)^{1/2}$$

$$\leq \varepsilon + \|\underset{\sim}{x}_m\|_2.$$

Thus the series $\sum_{i=1}^{\infty} x_i^2$ converges, so $\underset{\sim}{x} \in l_2$, and it follows from $(*)$ that $\|\underset{\sim}{x}_m - \underset{\sim}{x}\|_2 \leq \varepsilon$ for $m > N$, proving $\underset{\sim}{x}_m \to \underset{\sim}{x}$.

SOLUTIONS 4

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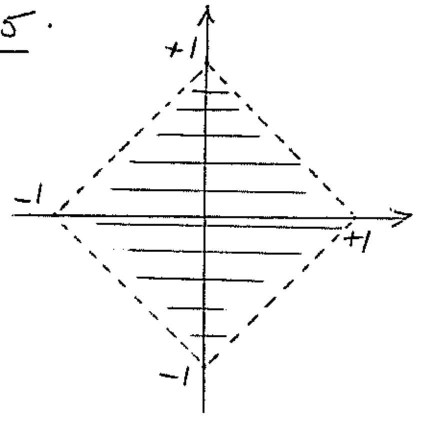
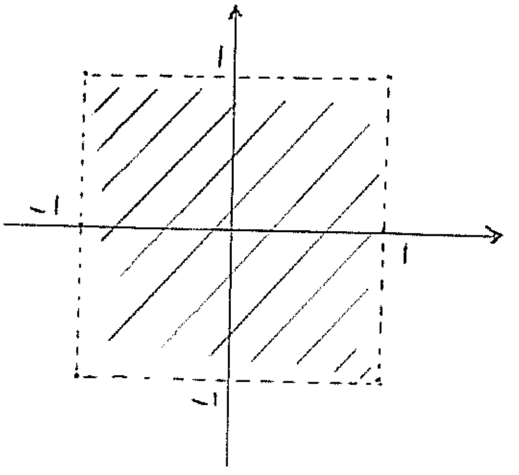
1. (i)



(ii)

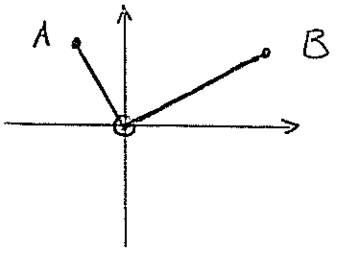


2 (a) As usual, (M1)-(M3) are clear, and only the triangle inequality (M4) needs checking; i.e. for any $\underset{\sim}{x}, \underset{\sim}{y}, \underset{\sim}{z} \in \mathbb{R}^2$,

$$d(\underset{\sim}{x}, \underset{\sim}{z}) \leq d(\underset{\sim}{x}, \underset{\sim}{y}) + d(\underset{\sim}{y}, \underset{\sim}{z}).$$

If $\underset{\sim}{x} = \underset{\sim}{z}$, $d(\underset{\sim}{x}, \underset{\sim}{z}) = 0$ so the inequality is obvious. If either $\underset{\sim}{x} = \underset{\sim}{y}$ or $\underset{\sim}{y} = \underset{\sim}{z}$, then LHS = RHS, so the inequality is again obvious. The only other possibility is that $\underset{\sim}{x}, \underset{\sim}{y}, \underset{\sim}{z}$ are all distinct, in which case
LHS $= \|\underset{\sim}{x}\|_2 + \|\underset{\sim}{z}\|_2 \leq \|\underset{\sim}{x}\|_2 + 2\|\underset{\sim}{y}\|_2 + \|\underset{\sim}{z}\|_2 =$ RHS.

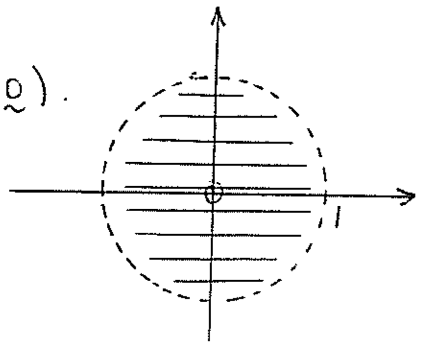
[Geometric interpretation: The distance between two distinct points is the sum of their "usual" (Euclidean) distances from the origin:



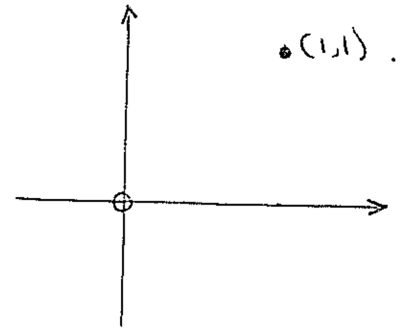
Thus $d(A, B) = AO + OB$.
(Mail from A to B goes via the "mail exchange" O, hence the name "Post Office metric")].

2 (b)

$B_1(\underset{\sim}{0})$.



$B_{\frac{1}{2}}((1,1))$.



(The only point whose distance from $(1,1)$ is less than $\frac{1}{2}$ is $(1,1)$ itself, since the distance of any other point from $(1,1)$ is at least $\|(1,1)\|_2 = \sqrt{2}$).

3. Let $y \in A$. We want to show that $d(x,y) < 2r$.
Let $z \in A \cap B_r(x)$, i.e. $z \in A$ and $d(x,z) < r$.
By the triangle inequality, $d(x,y) \leq d(x,z) + d(z,y)$.
Since $z, y \in A$ and A has diameter $< r$, $d(z,y) < r$.
Thus $d(x,y) < 2r$; (i.e. $y \in B_{2r}(x)$).

4. (i) $x \in \text{Int } A \Rightarrow$ there exists $r > 0$ such that $B_r(x) \subseteq A$; since $A \subseteq B$ we have $B_r(x) \subseteq B$, so $x \in \text{Int } B$.

(ii) $x \in \text{Int}(A \cap B) \iff$ there exists $r > 0$ such that $B_r(x) \subseteq A \cap B \iff$ there exists $r > 0$ such that $B_r(x) \subseteq A$ and $B_r(x) \subseteq B \iff x \in (\text{Int } A) \cap (\text{Int } B)$.

(iii) Since both $A \subseteq A \cup B$ and $B \subseteq A \cup B$, we have by (i) that $\text{Int } A \subseteq \text{Int}(A \cup B)$ and $\text{Int } B \subseteq \text{Int}(A \cup B)$, so $(\text{Int } A) \cup (\text{Int } B) \subseteq \text{Int}(A \cup B)$.

(iv). In $(\mathbb{R}, d_1)$, let $A = [0,1]$, $B = [-1,0]$. Then $0 \in \text{Int}(A \cup B) = (-1,1)$, but neither $0 \in \text{Int } A$ nor $0 \in \text{Int } B$.

5. Let (X,d) be a metric space, $x \in X$. To show $X\setminus\{x\}$ is open. Given $y \in X\setminus\{x\}$, let $r = d(x,y)$. Then the open ball $B_r(y)$ does not contain x, hence lies within $X\setminus\{x\}$. By Theorem 4.1(i), $X\setminus\{x\}$ is open.

Now let $\{x_1,\ldots,x_n\}$ be any finite set in X. Then

$$X\setminus\{x_1,\ldots,x_n\} = \bigcap_{i=1}^{n} X\setminus\{x_i\}.$$

Thus $X\setminus\{x_1,\ldots,x_n\}$ is a finite intersection of open sets by the first part, so by Theorem 4.2(iii) is open. This is the "hence" method of proof.

An "otherwise" method: let $y \in X\setminus\{x_1,\ldots,x_n\}$ and let $r_i = d(x_i,y)$, $i=1,\ldots,n$ and let $r = \min_i r_i$. Then the open ball $B_r(y)$ does not contain any of the points $x_1,\ldots,x_n$, hence lies within $X\setminus\{x_1,\ldots,x_n\}$, which is therefore open by Theorem 4.1(i).

6. The "only if" part is trivial, since if every subset of X is open, then in particular each singleton set is open. For the "if" part, observe that every subset is a union of singleton sets, and by Theorem 4.2(ii), any union of open sets is open.

7. (a) Let $(X, \|\cdot\|)$ be a normed linear space. Let $B_r(x)$ be any open ball in X. Then besides the point x, $B_r(x)$ contains the point $\left(1+\frac{r}{2\|x\|}\right)x$ since the distance between these two points is the norm of their difference, which is $r/2 < r$. Thus an open ball always contains more than one point, hence so

does any nonempty open set.

(b) Let $\underset{\sim}{x} \in \mathbb{R}^2$, $\underset{\sim}{x} \neq \underset{\sim}{0}$. Then $\|\underset{\sim}{x}\|_2 \neq 0$ and the distance of any other point from x is at least $\|\underset{\sim}{x}\|_2$. Letting $r = \|\underset{\sim}{x}\|_2$, we thus have that $B_r(\underset{\sim}{x}) = \{\underset{\sim}{x}\}$, so that the singleton $\{\underset{\sim}{x}\}$ is an open ball and hence an open set. $\{\underset{\sim}{0}\}$ is not an open set, since an open ball around $\underset{\sim}{0}$ in the Post Office metric is the same as the corresponding open ball in the usual metric, which is a nonempty circular disc.

It is thus immediate by (a) that no norm on $\mathbb{R}^2$ induces the Post Office metric.

8. Referring to Theorem 4.1(i), it is sufficient to show that each open ball in one metric, with a given centre, contains an open ball in the other metric with the same centre.

So let $B_r(x)$ be any open ball in X with respect to the metric d. We want $r' > 0$ such that $B^*_{r'}(x) \subseteq B_r(x)$ (where B^* denotes open ball with respect to d^*), i.e. if $d^*(x,y) < r'$, then $d(x,y) < r$. Now $d^*(x,y) < r' \Rightarrow d(x,y) < r'/(1-r')$, so setting $r'/(1-r') = r$ gives $r' = r/(1+r)$.

Now let $B^*_r(x)$ be any open ball in X with respect to the metric d^*. We want $r' > 0$ such that $B_{r'}(x) \subseteq B^*_r(x)$, i.e. if $d(x,y) < r'$ then $d^*(x,y) < r$.

Observing that $d^*(x,y) \leq d(x,y)$, we can just take $r' = r$.

SOLUTIONS 5.

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1. (i) Let y be any cluster point of $B_r[x_0]$. Thus for each $\varepsilon > 0$ there exists $x \in B_r[x_0]$ such that $d(x,y) < \varepsilon$. Then $d(x_0,y) \leq d(x_0,x) + d(x,y) < r+\varepsilon$. Since ε can be arbitrarily small, we deduce that $d(x_0,y) \leq r$; i.e. $y \in B_r[x_0]$, proving $B_r[x_0]$ is closed.

(ii). Let (X,d) be a discrete metric space. Then for $x_0 \in X$, $B_1(x_0) = \{x_0\}$ which is closed, hence $\overline{B_1(x_0)} = \{x_0\}$. However, $B_1[x_0] = X$, so provided X has more than one point, we have $\overline{B_1(x_0)} \neq B_1[x_0]$.

2. (i) Since every cluster point of A is a cluster point of B if $A \subseteq B$, we have $A' \subseteq B'$, and hence $\bar{A} = A \cup A' \subseteq B \cup B' = \bar{B}$.

(ii) The inclusion $\supseteq$ follows by (i), since $A \subseteq A \cup B \Rightarrow \bar{A} \subseteq \overline{A \cup B}$ and similarly $\bar{B} \subseteq \overline{A \cup B}$, so $\bar{A} \cup \bar{B} \subseteq \overline{A \cup B}$.
For the reverse inclusion, note that $\bar{A} \cup \bar{B}$ is a closed set (by Thm. 5.4 (iii)) containing $A \cup B$, and hence contains $\overline{A \cup B}$, the smallest closed set containing $A \cup B$ by Thm. 5.5.

(iii) Either: $A \cap B \subseteq A$ and $A \cap B \subseteq B$, so by (i) $\overline{A \cap B} \subseteq \bar{A}$ and $\overline{A \cap B} \subseteq \bar{B}$, so $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$.
Or: $\bar{A} \cap \bar{B}$ is a closed set (by Thm. 5.4 (ii)) containing $A \cap B$, hence contains $\overline{A \cap B}$ by Thm. 5.5.

The reverse inclusion does not hold; e.g. let $A = (-1,0)$, $B = (0,1)$ in $(\mathbb{R}, d_1)$. Then $\overline{A \cap B} = \bar{\phi} = \phi$, but $\bar{A} \cap \bar{B} = [-1,0] \cap [0,1] = \{0\}$.

The inclusion does hold if either $A \subseteq B$ or $B \subseteq A$.

3. (i) If $x \in \mathrm{Int}\, A$, then some open ball $B_r(x)$ lies within A. Thus $x \notin \overline{X \setminus A}$, since neither $x \in X \setminus A$ nor is x a cluster point of $X \setminus A$ (otherwise $B_r(x)$ would contain some point of $X \setminus A$). Thus $x \notin \mathrm{bdry}\, A$, so $\mathrm{Int}\, A \cap \mathrm{bdry}\, A = \phi$.

(ii). The inclusion $\supseteq$ is immediate since both $\mathrm{Int}\, A \subseteq \bar{A}$ and $\mathrm{bdry}\, A \subseteq \bar{A}$ from their definitions. To show that every point of $\bar{A}$ belongs to either $\mathrm{Int}\, A$ or $\mathrm{bdry}\, A$: let $x \in \bar{A}$ such that $x \notin \mathrm{Int}\, A$. Since $x \notin \mathrm{Int}\, A$, every open ball $B_r(x)$ contains a point of $X \setminus A$ so $x \in \overline{X \setminus A}$. Thus $x \in \bar{A} \cap \overline{X \setminus A} = \mathrm{bdry}\, A$.

4. (i) $\Rightarrow$ (ii): Let Y be any closed superset of A. Then Y contains $\bar{A} = X$ by Thm. 5.5, so $Y = X$.

(ii) $\Rightarrow$ (iii): Let U be an open set such that $U \cap A = \phi$. Thus $A \subseteq X \setminus U$ so $\bar{A} \subseteq \overline{X \setminus U}$. But $\bar{A} = X$ and $\overline{X \setminus U} = X \setminus U$ (since $X \setminus U$ is closed). Thus $X = X \setminus U$, hence $U = \phi$.

(iii) $\Rightarrow$ (iv): Immediate (in fact (iv) is just the contrapositive of (iii)).

(iv) $\Rightarrow$ (i): Let $x \in X$, and let $B_r(x)$ be any open ball centred at x. Then $B_r(x)$ is a nonempty open set so there exists $y \in A \cap B_r(x)$. Thus $x \in \bar{A}$, so $\bar{A} = X$.

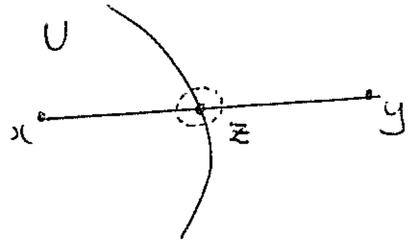
5.

6. Let X be a normed linear space, and let U be an open subset of X such that $U \neq \emptyset, X$. We show that U cannot be closed.

<u>NOTATION</u>: For $a, b \in X$, $[a,b]$ denotes the "closed line segment" joining a and b; i.e.

$$[a,b] = \{ a + t(b-a) : 0 \leq t \leq 1 \}.$$

Choose $x \in U$, $y \notin U$.



Define $t_0 = \sup \{ t \geq 0 : [x, t(y-x)] \subseteq U \}$. Then $0 \leq t_0 \leq 1$. Define $z = x + t_0(y-x)$. Intuitively it is obvious that $z \notin U$, since no open ball around z lies within U. We prove this rigorously.

If $t_0 = 1$, then $z = y$ and $y \notin U$, so we suppose $t_0 < 1$. Let $r > 0$ be arbitrary, $r \leq 1 - t_0$. Let $r' = \min\{r, r/\|y-x\|\}$. By definition of t_0, the interval $[z, z + r'(y-x)]$ does not lie within U, so there exists r_0, $0 \leq r_0 < r'$, such that $w = z + r_0(y-x) \notin U$. Then

$$d(w,z) = \|w - z\| = r_0\|y-x\| < r.$$

Thus the open ball $B_r(z)$ does not lie within U; since U is open and r was arbitrary, we conclude that $z \notin U$.

However, z is clearly a cluster point of U, since if $x_n = x + (t_0 - \frac{1}{n})(y-x)$, then $x_n \in U$ and $d(x_n, z) = \|x_n - z\| = \|x-y\|/n \longrightarrow 0$ as $n \longrightarrow \infty$. Thus U is not closed. (Here we have used the fact that $t_0 > 0$; if $t_0 = 0$, then $z = x \in U$, which contradicts $z \notin U$).

7. Since (X, d) is complete, there exists $a \in X$ such that $a_n \to a$. Thus a is a cluster point of A which is closed, hence $a \in A$.

SOLUTIONS 6.

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1. We use Thm. 6.1. Let $x \in X$, and let $x_n \to x$. Thus by definition $d(x_n, x) \to 0$. Now

$$d(x_n, x_0) \leq d(x_0, x) + d(x, x_n)$$

and $$d(x_0, x) \leq d(x_0, x_n) + d(x, x_n),$$

so $|d(x_n, x_0) - d(x_0, x)| \leq d(x, x_n) \to 0$. Thus $d(x_n, x_0) \to d(x, x_0)$; i.e. $f(x_n) \to f(x)$ as required.

2. Observing that the mapping F is linear, by Thm. 6.3 we need only show that it is bounded as a mapping from the normed linear space $(C[a,b], \|\cdot\|_\infty)$ to the normed linear space $(\mathbb{R}, |\cdot|)$.

Now for any $f \in C[a,b]$,

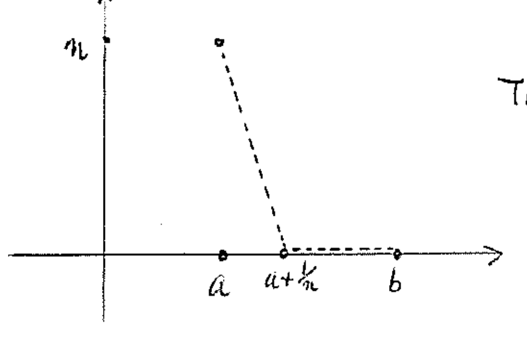
$$\begin{aligned} |F(f)| &= |f(x_0)| \\ &\leq \sup_{x \in [a,b]} |f(x)| = \|f\|_\infty, \end{aligned}$$

so F is bounded, hence continuous on $(C[a,b], d_\infty)$.

However, F is not bounded on $(C[a,b], \|\cdot\|_1)$.

e.g. if $x_0 = a$, define $f_n \in C[a,b]$ by

$$f_n(x) = \begin{cases} -n^2x + (n + n^2 a), & a \leq x \leq a + 1/n. \\ 0, & a + 1/n \leq x \leq b \end{cases}$$



Then $\|f_n\|_1 = \int_a^b |f_n| = 1/2$ for all n

But $|F(f_n)| = |f_n(a)| = n$.

Thus there cannot exist a constant $M > 0$ such that $|F(f)| \leq M\|f\|_1$ for all $f \in C[a,b]$, so F is not bounded hence not continuous on $(C[a,b], d_1)$.

3. We use Thm. 6.1. Let $x_n \to x$ in X. Then $f(x_n) \to f(x)$ in Y since f is continuous; then $g(f(x_n)) \to g(f(x))$ in Z since g is continuous; i.e. $(g \circ f)(x_n) \to (g \circ f)(x)$ as required.

4. The sequence $\{1/n\}$ is clearly Cauchy in $(0, \infty)$, since $|1/n - 1/m| \to 0$ as $m, n \to \infty$. However, the sequence $\{f(1/n)\} = \{n\}$ is clearly not Cauchy. (We are assuming $f : x \mapsto 1/x$ is continuous on $(0,\infty)$).

5. Let A be any open subset of $\mathbb{R}$. Then a constant mapping $f : x \mapsto c$ is continuous and $f(A) = \{c\}$, which is not an open subset of $\mathbb{R}$, thus f is not open. ($\{c\}$ is not open by problem 4.7(a)).

6. Define $f : (-\pi/2, \pi/2) \to \mathbb{R}$ by $f(x) = \tan x$. Then f is continuous, one-one, and onto $\mathbb{R}$.

7. The verification that d'' is indeed a metric is routine.

To show that d', d'' are equivalent metrics, it is sufficient to show that if $y_n \to y$ in Y in one metric, then $y_n \to y$ in the other metric. (Proof: this condition is (by Thm. 6.1) equivalent to continuity of the identity mappings $I : (Y, d') \to (Y, d'')$ and $I : (Y, d'') \to (Y, d')$. If U is any open set in (Y, d''), then by continuity of the first mapping, $I^{-1}(U) = U$ is open in (Y, d') — Thm. 6.2. Similarly any open set U in (Y, d') is open in (Y, d''), hence d', d'' give rise to the same open sets, and are thus equivalent.)

So suppose $y_n \to y$ in (Y, d'); $\Rightarrow f^{-1}(y_n) \to f^{-1}(y)$ in (X, d) since f^{-1} is continuous $\Rightarrow d(f^{-1}(y_n), f^{-1}(y)) \to 0$ $\Rightarrow d''(y_n, y) \to 0 \Rightarrow y_n \to y$ in (Y, d'').

Conversely, suppose $y_n \to y$ in (Y, d''); $\Rightarrow$ $d''(y_n, y) \to 0 \Rightarrow d(f^{-1}(y_n), f^{-1}(y)) \to 0 \Rightarrow$ $f^{-1}(y_n) \to f^{-1}(y)$ in (X, d) $\Rightarrow y_n = f(f^{-1}(y_n)) \to f(f^{-1}(y)) =$ y in (Y, d') since f is continuous.

8. For $\underset{\sim}{x} = (x_1, \dots, x_n) \in V^n$, we have

$$\begin{aligned}
\| T(\underset{\sim}{x}) \|_1 &= \sum_{j=1}^{m} \left| \sum_{i=1}^{n} t_{ji} x_i \right| \\
&\leq \sum_{j=1}^{m} \sum_{i=1}^{n} |t_{ji}| . |x_i| \\
&\leq \sum_{j=1}^{m} \max_{1 \leq i \leq n} |t_{ji}| \sum_{i=1}^{n} |x_i| \\
&= \left(\sum_{j=1}^{m} \max_{1 \leq i \leq n} |t_{ji}| \right) \| x \|_1 ,
\end{aligned}$$

so T is bounded from ℓ_1^n to ℓ_1^m.

9. It follows from Thm. 6.2, on considering complements, that a mapping is continuous if and only if the inverse image of every closed set is a closed set. Now by problem 4.5, $\{0\}$ is a closed set in any normed linear space. Thus if T is a continuous linear mapping, $\text{Ker } T = T^{-1}(\{0\})$ is closed.

10. The condition $\|T(x)\|' \leq M\|x\|$ implies that T is bounded hence continuous. We require T^{-1} to exist and be continuous. Now $T(x) = T(y) \Rightarrow \|T(x) - T(y)\|' = \|T(x-y)\|' = 0$, so $m\|x-y\| \leq \|T(x-y)\|' = 0$, hence $\|x-y\| = 0$; i.e. $x = y$. Thus T is one-one; if $y \in T(X)$, $y = T(x)$, then $m\|x\| \leq \|T(x)\|'$; i.e. $m\|T^{-1}(y)\| \leq \|y\|'$; i.e. $\|T^{-1}(y)\| \leq m^{-1}\|y\|'$, proving T^{-1} is bounded hence continuous.

11. (i) Let $\sim$ denote "is homeomorphic to". For any metric space (X, d), the identity mapping $I: (X,d) \to (X,d)$ is clearly a homeomorphism, so $(X,d) \sim (X,d)$. i.e. $\sim$ is <u>reflexive</u>.

(ii) If $(X,d) \sim (Y,d')$, then there exists a homeomorphism $f: (X,d) \xrightarrow{\text{onto}} (Y,d')$; then clearly $f^{-1}: (Y,d') \xrightarrow{\text{onto}} (X,d)$ is a homeomorphism, so $(Y,d') \sim (X,d)$; i.e. $\sim$ is <u>symmetric</u>.

(iii). If $(X,d) \sim (Y,d')$ and $(Y,d') \sim (Z,d'')$ under homeomorphisms $f: (X,d) \xrightarrow{\text{onto}} (Y,d')$ and $g: (Y,d') \xrightarrow{\text{onto}} (Z,d'')$, then clearly $g \circ f: (X,d) \xrightarrow{\text{onto}} (Z,d'')$ is a homeomorphism, so $(X,d) \sim (Z,d'')$; i.e. $\sim$ is <u>transitive</u>.

Thus $\sim$ is an equivalence relation.