Banach Algebras

References:

Rudin "F^nl Analysis" (Part 3)

Introductory
- Wilanski "F^nl anal." (Ch 14)
- Simmons "Intro' to topology & Modern anal." (Ch 12–14)
- Packel "F^nl anal. — a short Course" Ch 6.

Bonsall & Duncan "Complete Normed algs"

Rickart "Banach algs"

Dixmier "C*-algebras"

Naimark "Normed Rings"

Gelfand, Raikov & Shilov "Comm. normed Rings"

Goodearl "Notes on Real and Complex C*-algs"

§1.

$\mathcal{A} \equiv \mathcal{A}(+, \cdot, \times) \ni a, b, c, \ldots$ is a complex (Real) algebra if $(\mathcal{A}\, +, \cdot)$ is a vec. sp. over $\mathbb{C}$ (or $\mathbb{R}$) & $\times$ is lft & rt dist over $+$ and

scalar mult. λa — alg product ab

$\lambda(ab) = (\lambda a)b = a(\lambda b)$.

$\mathcal{A}$ has idenity if $\exists$ (nec. !) $e \in \mathcal{A}$ s.t. $ae = ea = a$ $\forall a \in \mathcal{A}$.

$\mathcal{A} \equiv (\mathcal{A}, \|\cdot\|) \equiv \mathcal{A}(+, \cdot, \times, \|\cdot\|)$ is a normed (Banach) algebra if $\mathcal{A}(+, \cdot, \|\cdot\|)$ is a n.l. (Banach) sp. and $\underbrace{\|ab\| \le \|a\|\,\|b\| \quad \forall a, b \in \mathcal{A}.}$

$\Downarrow$

$\times$ is jointly continuous (ie $a_n \xrightarrow{\|\cdot\|} a,\ b_n \xrightarrow{\|\cdot\|} b \Rightarrow a_n b_n \xrightarrow{\|\cdot\|} ab$)

[consider $a_n b_n - ab = a_n(b_n - b) + (a_n - a)b$.]

A Normed (Banach) algebra with idenity e is unital, with unit e, if $\|e\| = 1$.

Some Examples

n. l. sp. X into itself, with

$\|T\| = \sup_{\|x\|=1} \|Tx\|$ and "×" composition (unital)

N.B. $\mathcal{B}(X)$ is a Banach algebra if X is complete (Banach)

2) For any set E & n. alg $\mathcal{A}$

$\ell^\infty(E,\mathcal{A}) \equiv$ sp. of all bded fns of E into $\mathcal{A}$

with $\|f\|_\infty = \sup_{a\in E} \|f(a)\|$ & operations $+, \cdot, \times$ point-wise,

a Banach algebra if $\mathcal{A}$ is.

If E is a compact Hausdorff top. sp. a special subalg of $\ell^\infty(E,\mathcal{A})$ is $\mathcal{C}(E,\mathcal{A})$ the sp. of all cont. fns $E \to \mathcal{A}$.

Special case: function algs. $\mathcal{C}(E,\mathbb{C}) \equiv \mathcal{C}(E)$. (or $\mathbb{R}$)

In particular The disk algebra.

$$\mathcal{A}(\Delta) = \{ f \in \mathcal{C}(\Delta) : f \text{ analytic on int } \Delta \}$$

$\Delta = \{ z \in \mathbb{C} : |z| \le 1 \}$

3) Wiener algebra, $\mathcal{W}$ — set of all $a : [0, 2\pi] \to \mathbb{C}$

$: t \longmapsto \sum_{k\in\mathbb{Z}} \alpha_k e^{ikt}$

with

$\|a\| = \sum_{k\in\mathbb{Z}} |\alpha_k| < \infty$ (absolutely conv Fourier series)

with point-wise operations,

so $(ab)(t) = a(t)\,b(t) = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \alpha_k \beta_{n-k} e^{int}$

4) Discrete group algebra:

$\ell_1(G)$, G any grp, all fns $f : G \to \mathbb{C}$ with

$\|f\| = \sum_{s\in G}' |f(s)| < \infty$,

with $+, \cdot$ point wise & $\times$ convolution

ie $f \times g(s) = f * g(s) = \sum' f(t)\, g(t^{-1}s)$

N.B. In particular $l_1(\mathbb{Z}) \underset{\text{isometric}}{\overset{\text{isomorph}}{\simeq}} \mathcal{W}$.

5) General Group algebras

Eg $\mathcal{L}_1(\mathbb{R})$ – elts f "a.e." Lebesgue equivalence classes of integrable fns from $\mathbb{R}$ to $\mathbb{C}$, with

$$\|f\| = \int_{\mathbb{R}} |f(t)| \, d\mu(t)$$

point wise + & · and × convolution

ie $f \times g(s) = f * g(s) = \int_{\mathbb{R}} f(t)\, g(s-t)\, d\mu(t)$

[Can extend to any "locally compact group with μ a left invariant Haar measure.]

Completion of a n.l. alg. $\mathcal{A}$

Let $\hat{\mathcal{A}}$ denote the completion of $(\mathcal{A}, +, \cdot, \|\cdot\|)$ and for $a, b \in \hat{\mathcal{A}}$ define

$$ab = \lim_{n\to\infty} a_n b_n \quad \text{where } \underset{\in\mathcal{A}}{a_n} \to a,\ \underset{\in\mathcal{A}}{b_n} \to a$$

EXERCISE : i) Show ab is well defined (ie independent of particular choice for (a_n) & (b_n)

ii) Show that with this product $\hat{\mathcal{A}}$ is a Banach alg. with $\mathcal{A}$ a dense sub-algebra.

Adjoining an identity, Let $\mathcal{A}$ be a n (Banach) alg.
Form $\mathcal{A} \times \mathbb{C}$ (or $\mathbb{R}$ if $\mathcal{A}$ is over $\mathbb{R}$) and define

$(a, \lambda) + (b, \mu) = (a+b, \lambda+\mu)$, $\mu(a, \lambda) = (\mu a, \mu\lambda)$
& $(a, \lambda)(b, \mu) = (ab + \mu a + \lambda b, \lambda\mu)$;
$\|(a, \lambda)\| = \|a\| + |\lambda|$

n. (Banach) alg. with unit $(0,1)$ and that $\mathcal{A} \to \mathcal{A}\times\mathbb{C} : a \mapsto (a,0)$ is an isometric isomorphism of $\mathcal{A}$ into $\mathcal{A}\times\mathbb{C}$.

Complexification

$\mathcal{A}$ a real alg.

Form $\mathcal{A}\times\mathcal{A}$ and define

$$(a,b)+(c,d) = (a+c,\ b+d)$$
$$(\alpha+i\beta)(a,b) = (\alpha a - \beta b,\ \beta a + \alpha b)$$
$$(a,b)\times(c,d) = (ac-bd,\ ad+bc)$$

EXERCISE Show that with these operations $\mathcal{A}\times\mathcal{A}$ is a complex algebra. Also that $\mathcal{A}\to\mathcal{A}\times\mathcal{A} : a\mapsto(a,0)$ is an isomorphism of $\mathcal{A}$ into $\mathcal{A}\times\mathcal{A}$.

Also, if $\mathcal{A}$ is a real normed alg., define

$$\|(a,b)\| = \sup_{0\le\theta<2\pi}\left(\|\cos\theta\, a - \sin\theta\, b\| + \|\sin\theta\, a + \cos\theta\, b\|\right)$$

(Kaplansky)

Show that this defines a norm on $\mathcal{A}$ and that $\|(a,0)\| = \sqrt{2}\|a\|$ so that $\mathcal{A}\to\mathcal{A}\times\mathcal{A} : a\mapsto(a,0)$ is a homeomorphism of $\mathcal{A}$ into $\mathcal{A}\times\mathcal{A}$.

REMARK: It is possible to equip $\mathcal{A}\times\mathcal{A}$ with a norm so that the above embedding is an isometry — see Rickart Thms 1·3·1 & 1·3·2 for example.

The regular representations of $\mathcal{A}$

Let $\mathcal{B}(\mathcal{A})$ denote the set of bounded lin. operators of $(\mathcal{A},+,\cdot,\|\cdot\|)$ into itself.

For each $a\in\mathcal{A}$ define $L_a : \mathcal{A}\to\mathcal{A} : b\mapsto ab$

clearly L_a is linear &

$$\|L_a\| = \sup_{\|b\|=1}\|L_a(b)\| = \sup_{\|b\|=1}\|ab\| \le \sup_{\|b\|=1}\|a\|\|b\| = \|a\|$$

Further it is easily verified that

$$L_{x+y} = L_x + L_y,\quad L_{\lambda x} = \lambda L_x \quad \& \quad L_{xy} = L_x \circ L_y$$

So the mapping

$\mathcal{A} \to B(\mathcal{A}) : a \mapsto L_a$ is an isomorphism of $\mathcal{A}$ into $B(\mathcal{A})$ — known as the left regular representation of $\mathcal{A}$ in $B(\mathcal{A})$.

[Similarly, we may define the rt. regular representation of $\mathcal{A}$ in $B(\mathcal{A})$ by $a \mapsto R_a$ where $R_a : \mathcal{A} \to \mathcal{A} : b \mapsto ba$.
— Since $R_{xy}(b) = bxy = R_y \circ R_x(b)$ we see that $a \mapsto R_a$ is an isomorphism of $\mathcal{A}^r$ into $B(\mathcal{A})$ where $\mathcal{A}^r$ denotes the reverse alg. of $\mathcal{A}$
ie $\mathcal{A}^r = \mathcal{A}(+, \circ, \otimes)$ where $a \otimes b = b \times a = ba$.]

Also, if $a_n \to a$ then we have

$$\|L_{a_n} - L_a\| = \|L_{a_n - a}\| \leq \|a_n - a\| \to 0 \text{ so}$$

$a \mapsto L_a$ is continuous

When $\mathcal{A}$ has an identity we have

$$\|a\| \geq \|L_a\| \geq \left\|L_a\left(\frac{e}{\|e\|}\right)\right\| = \frac{1}{\|e\|}\|ae\| = \frac{1}{\|e\|}\|a\|$$

so $a \mapsto L_a$ is a homeomorphism.

In particular, if $\mathcal{A}$ is unital

$\|a\| = \|L_a\|$ & $a \mapsto L_a$ is an isometric isomorphism of $\mathcal{A}$ into $B(\mathcal{A})$.

Since any normed alg $\mathcal{A} \xrightarrow[\text{isomph.}]{\text{isometrically}} \mathcal{A} \times \mathbb{C}$ (unital) $\xrightarrow[\text{isomph}]{\text{isometrically}} B(\mathcal{A} \times \mathbb{C})$

we have

<u>1st Representation Thm 1.1</u> <u>Every normed algebra is isometrically isomorphic to some subalgebra of $B(X)$ for some n. l. sp. X.</u>

(<u>N.B.</u> When $\mathcal{A}$ is unital we may take $X = (\mathcal{A}, +, \cdot, \|\cdot\|)$.)

EXERCISE: Let $(\mathcal{A}, +, \cdot, \times)$ be an alg. with idenity for which $\exists\ \|\cdot\|$ s.t. $(\mathcal{A}, +, \cdot, \|\cdot\|)$ is a Banach space & w.r.t. $\|\cdot\|$ $\times$ is separately continuous
(ie if $a_n \to a$ then $a_n b \to ab$ & $ba_n \to ba$, each $b \in \mathcal{A}$)
Show that $\exists$ an equivalent norm, $\|\cdot\|'$, w.r.t. which $\mathcal{A}$ is a Banach algebra. As a consequence note that $\times$ was infact jointly continuous to begin with.

§2 Let $\mathcal{A}$ denote an algebra with indenity e.*

$a \in \mathcal{A}$ is regular (invertible) if $\exists$ a (nec. !) $a^{-1} \in \mathcal{A}$ s.t. $aa^{-1} = a^{-1}a = e$.

The set of regular elements, denoted by $\mathcal{G}(\mathcal{A})$, clearly forms a group under $\times$. $[\, a, b \in \mathcal{G}(\mathcal{A}) \Rightarrow (ab^{-1})^{-1} = ba^{-1} \in \mathcal{G}(\mathcal{A})$

$a \in \mathcal{A} \setminus \mathcal{G}(\mathcal{A})$ is termed singular (or non-invertible)

The spectrum of $a \in \mathcal{A}$ is the set of scalars

$$\sigma(a) = \{\lambda \in \mathbb{C}\ (\text{or } \mathbb{R}) : a - \lambda e \notin \mathcal{G}(\mathcal{A})\}$$

EXERCISE: determine $\sigma(f)$ for any $f \in \mathcal{C}(E)$.

The complement $R(a) = \mathbb{C} \setminus \sigma(a) = \{\lambda : a - \lambda e \text{ is regular}\}$ is termed the resolvant of a.

$\rho(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}$ is the spectral radius of $a \in \mathcal{A}$.

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* In case $\mathcal{A}$ has no idenity: define $a \in \mathcal{A}$ to be quasi-regular (singular) if $(a, 1)$ is regular (singular) in $\mathcal{A} \times \mathbb{C}$

Note 1) If $(a,1)(b,\lambda) = e_{\mathcal{A}\times\mathbb{C}} = (0,1)$ we must have $\lambda = 1$ and $ab + a + b = 0$ thus a is quasi-regular iff $\exists\ a^\circ$ s.t. $a \circ a^\circ = a^\circ \circ a =$ [illegible]
where "$\circ$" is defined by $a \circ b = ab + a + b$ $(\forall a, b \in \mathcal{A})$ quasi-prod[illegible]

2) To define a spectrum in the absence of an idenity, let
[illegible]

Unless otherwise stated we will henceforth take $\mathcal{A}$ to be a unital Banach algebra.

By analogy with $\frac{1}{1-x} = 1 + \sum_{n=1}^{\infty} x^n$, consider the "formal" series

$$e + \sum_{n=1}^{\infty} a^n$$

The partial sums S_N will form a Cauchy sequence & hence converge to some $s \in \mathcal{A}$ provided the series $1 + \sum_{n=1}^{\infty} \|a^n\|$ converges, which, by the root test, happens provided $r(a) = \limsup_n \|a^n\|^{1/n} < 1$.

Further, when this happens we have

$$(e-a)s = s(e-a) = \underset{N\to\infty}{\text{limit}} \left[(e-a)S_N = S_N - aS_N = e - \underset{\|\cdot\|\to 0}{a^{N+1}}\right] = e$$

and we have established

Theorem 2.1: If $a \in \mathcal{A}$ is s.t. $r(a) = \limsup_n \|a^n\|^{\frac{1}{n}} < 1$, then $e-a \in \mathcal{G}(\mathcal{A})$

Corollary 2.2: $\|a\| < 1 \Rightarrow e-a \in \mathcal{G}(\mathcal{A})$

$[\, r(a) \leq \|a\| \,]$

Corollary 2.3: $\rho(a) \leq r(a)$, in particular $\rho(a) \leq \|a\|$

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* cont. from previous page. $\quad a \circ b = (a+e)(b+e) - e$,

so $0 \neq \lambda \notin \sigma(a) \Leftrightarrow 0 = (a-\lambda e)(a-\lambda e)^{-1} - e$

$\Leftrightarrow 0 = \left((-\lambda)^{-1}a + e\right)\left((-\lambda)^{-1}a + e\right)^{-1})$

$= \left((-\lambda)^{-1}a + e\right)\left([((-\lambda)^{-1}a+e)^{-1} - e] + e\right)$

$= \left((-\lambda)^{-1}a\right) \circ [((-\lambda)^{-1}a + e)^{-1} - e]$

$\Leftrightarrow \quad (-\lambda)^{-1}a$ is quasi ~~singular~~ reg

So $\sigma(a) \cup \{0\} = \underbrace{\{\lambda \in \mathbb{C} : (-\lambda)^{-1} a \text{ is quasi-singular}\}}\cup\{0\}$.

Pf. Let λ be s.t. $|\lambda| > r(a)$ then $r(\frac{a}{\lambda}) = \limsup_n \|(\frac{a}{\lambda})^n\|^{\frac{1}{n}}$
$$= \frac{r(a)}{|\lambda|} < 1$$
thus $e - \frac{a}{\lambda} \in \mathcal{G}(\mathcal{A})$ so $\lambda e - a \in \mathcal{G}(\mathcal{A})$ or $\lambda \notin \sigma(a)$.

Corollary 2.4: Let $a \in \mathcal{G}(\mathcal{A})$ and $b \in \mathcal{A}$ have $\|b\| < \|a^{-1}\|^{-1}$, then $a - b \in \mathcal{G}(\mathcal{A})$. Inparticular $\mathcal{G}(\mathcal{A})$ is open.

Pf. $$a - b = a(e - a^{-1}b)$$
and $\|a^{-1}b\| \le \|a^{-1}\|\,\|b\| < 1$
so $e - a^{-1}b$ & hence $a - b \in \mathcal{G}(\mathcal{A})$.

Corollary 2.5: $a \mapsto a^{-1}$ is continuous on $\mathcal{G}(\mathcal{A})$.

Pf. For $\|h\|$ suff small $a \in \mathcal{G}(\mathcal{A}) \Rightarrow a - h \in \mathcal{G}(\mathcal{A})$ and
$$\|a^{-1} - (a-h)^{-1}\| \le \|a^{-1}\|\,\|e - (e - \underbrace{a^{-1}h}_{\to 0 \text{ as } h \to 0 \text{ so may assume has norm } < 1 \text{ in which case}})^{-1}\|$$
$$= \|a^{-1}\|\,\|a^{-1}h + (a^{-1}h)^2 + \cdots\|$$
$$\le \frac{\|a^{-1}\|^3\|h\|^2}{1 - \|a^{-1}h\|} \longrightarrow 0 \text{ as } \|h\| \to 0.$$

Corollary 2.6: For $a \in \mathcal{A}$, the resolvent $R(a)$ is open.

Pf. $R(a) = \{\lambda : \lambda e - a \in \mathcal{G}(\mathcal{A})\}$
$= \{\lambda : \psi_a(\lambda) \in \mathcal{G}(\mathcal{A})\}$ where $\psi_a : \mathbb{C} \to \mathcal{A} : \lambda \mapsto \lambda e - a$
$= \psi_a^{-1}(\mathcal{G}(\mathcal{A}))$
$=$ open, as ψ_a continuous & $\mathcal{G}(\mathcal{A})$ open.

Corollary 2.7: For $a \in \mathcal{A}$, $\sigma(a)$ is a compact subset of $\mathbb{C}$.

Pf. $\sigma(a) = \mathbb{C} \setminus R(a)$ is closed & by 2.3 bdd (by $\|a\|$).

1) If $0 \notin \sigma(a)$, show that $\sigma(a^{-1}) = \{\lambda^{-1} : \lambda \in \sigma(a)\}$.

EXERCISES 2) Let $\mathcal{A}$ be a normed alg with identity, show that $\sigma(ab)\setminus\{0\} = \sigma(ba)\setminus\{0\}$ for $\forall a,b \in \mathcal{A}$.
In case either a or $b \in \mathcal{G}(\mathcal{A})$, show that
$\sigma(ab) = \sigma(ba)$.

3) Let G be an open subset of $\mathbb{C}$ containing $\sigma(a)$ for a an element of the unital Banach algebra $\mathcal{A}$.
Show that $\exists\, \delta > 0$ s.t. $\sigma(b) \subset G$ whenever $\|a-b\| < \delta$;
That is, the set valued mapping $\mathcal{A} \to 2^{\mathbb{C}} : a \mapsto \sigma(a)$ is "upper semi-continuous".
What can you say about the mapping $a \mapsto \rho(a)$?

We continue to investigate topological properties of $\mathcal{G}(\mathcal{A})$ etc. by considering $\operatorname{bdry} \mathcal{G}(\mathcal{A})$.

Lemma 2.8: Let $a \in \operatorname{bdry}\mathcal{G}(\mathcal{A})$ and let $(a_n) \subset \mathcal{G}(\mathcal{A})$ be s.t. $a_n \to a$, then $\{a_n^{-1}\}$ is unbounded.

Proof. Assume $\|a_n^{-1}\| \le M$ $\forall n$, some $M > 0$, then
$$\|a_m^{-1} - a_n^{-1}\| = \|a_m^{-1}(a_n - a_m)a_n^{-1}\|$$
$$\le M^2 \|a_n - a_m\| \to 0$$
so (a_n^{-1}) is Cauchy & hence convgt to some $b \in \mathcal{A}$

Now $e = a_n^{-1}a_n \to ba$
& $e = a_n a_n^{-1} \to ab$
thus $a \in \mathcal{G}(\mathcal{A}) \cap \operatorname{bdry}\mathcal{G}(\mathcal{A})$
$\divideontimes$ $\mathcal{G}(\mathcal{A})$ open.

Corollary 2.9: If $a \in \operatorname{bdry}\mathcal{G}(\mathcal{A})$ $\exists$ seq $(a_n) \subset \mathcal{G}(\mathcal{A})$ with $a_n \to a$ & $\|a_n^{-1}\| \nearrow \infty$

[Extract an appropriate subsequence ...

Definition: $a \in \mathcal{A}$ is a *topological divisor of zero* if $\exists\ (x_n) \in \mathcal{A}$, $\|x_n\| = 1\ \forall n$, with $x_n a,\ a x_n \to 0$.

Note: a a top. div. of zero $\Rightarrow a \notin \mathcal{G}(\mathcal{A})$. [Exercise]

Theorem 2.10: $a \in \text{bdry}\, \mathcal{G}(\mathcal{A}) \Rightarrow a$ is a top. div. of zero.

Pf. take $(a_n) \subset \mathcal{G}(\mathcal{A})$ as in 2.9 & let $x_n = \dfrac{a_n^{-1}}{\|a_n^{-1}\|}$

then $$x_n a = \frac{a_n^{-1}}{\|a_n^{-1}\|} a = \underbrace{\frac{a_n^{-1}(a - a_n)}{\|a_n^{-1}\|}}_{\xrightarrow{\|\cdot\|} 0 \text{ as } a_n \to a} + \underbrace{\frac{e}{\|a_n^{-1}\|}}_{\to 0 \text{ as } \|a_n^{-1}\| \nearrow \infty}$$

Similarly, $a x_n \to 0$.

Notation: We will write $\mathcal{B} \leq \mathcal{A}$ to indicate $\mathcal{B}$ is a subalgebra of $\mathcal{A}$ (with same unit as $\mathcal{A}$)

Observation: if $a \in \mathcal{B} \leq \mathcal{A}$ is a top. div. of zero in $\mathcal{B}$ then a is a top. div. of zero in $\mathcal{A}$.

So $a \in \text{bdry}\, \mathcal{G}(\mathcal{B}) \Rightarrow a$ a top. div of zero in $\mathcal{A}$
$\Rightarrow a \notin \mathcal{G}(\mathcal{A})$

so $\text{bdry}\, \mathcal{G}(\mathcal{B}) \subseteq \mathcal{A} \setminus \mathcal{G}(\mathcal{A})$ ------- ①

trivially, of course, $\mathcal{G}(\mathcal{B}) \subseteq \mathcal{G}(\mathcal{A})$
so $\text{bdry}\, \mathcal{G}(\mathcal{B}) \subseteq \overline{\mathcal{G}(\mathcal{A})}$ ------ ②

combining ① and ② we therefore have

Theorem 2.11: If $\mathcal{B}$ is a Banach (ie closed) subalg. of $\mathcal{A}$, then $\text{bdry}\, \mathcal{G}(\mathcal{B}) \subseteq \text{bdry}\, \mathcal{G}(\mathcal{A})$

Application to Spectra

however, if $\lambda \in \text{bdry}\, \sigma_{\mathcal{B}}(a) \subseteq \sigma_{\mathcal{B}}(a)$, as closed, then
$\exists\, \lambda_n \to \lambda$ s.t. $\lambda_n e - a \in \mathcal{G}(\mathcal{B})$
i.e. $\lambda e - a \in \text{bdry}\, \mathcal{G}(\mathcal{B}) \subseteq \text{bdry}\, \mathcal{G}(\mathcal{A})$
so $\lambda \in \sigma_{\mathcal{A}}(a)$

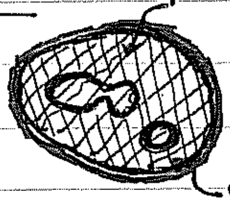
Hence $\text{bdry}\, \sigma_{\mathcal{B}}(a) \subseteq \sigma_{\mathcal{A}}(a)$ — ④

Combining ③ & ④ we obtain

Theorem 2.12: For $a \in \mathcal{B} \leq \mathcal{A}$ we have
$$\sigma_{\mathcal{A}}(a) \subseteq \sigma_{\mathcal{B}}(\mathcal{A})$$

$$\text{but, } \text{bdry}\, \sigma_{\mathcal{B}}(\mathcal{A}) \subseteq \text{bdry}\, \sigma_{\mathcal{A}}(a)$$

filled
components of
the complement
can be "filled"
in



[a bdry pt of a larger set which happens to be in some subset must also be a bdry pt of that subset.]

Corollary 2.13: For $a \in \mathcal{B} \leq \mathcal{A}$ $\quad \rho_{\mathcal{B}}(a) = \rho_{\mathcal{A}}(a)$.

* EXERCISE: (very important)

Let $\mathcal{A}$ be a Banach alg. with unit, e. For $a \in \mathcal{A}$, show $\exists$ a maximal commutative subalg. containing a and e, denote it by $\mathcal{C}_m(a)$.

Show that $\mathcal{C}_m(a)$ is closed (hence a Banach subalg) and.

Show that $\sigma_{\mathcal{C}_m(a)}(a) = \sigma(a)$.

Lemma 2.14: For $\mathcal{A}$ a unital Banach algebra and fixed $a \in \mathcal{A}$ and $f \in \mathcal{A}^*$ (the dual space of $(\mathcal{A}, +, \cdot, \|\cdot\|)$) the function defined by
$$\phi : R(a) \longrightarrow \mathbb{C} : \lambda \longmapsto f([\lambda e - a]^{-1})$$
is analytic on $R(a)$.

Pf. $\dfrac{\phi(\lambda + h) - \phi(\lambda)}{h} = f\left(\dfrac{[\lambda e + he - a]^{-1} - (\lambda e - a)^{-1}}{h}\right)$

So by Corollary 2·5, letting $h \to 0$ we have

$$\frac{d\phi}{d\lambda} = -f\left(\left([\lambda e - a]^{-1}\right)^2\right) \text{ exists for all } \lambda \in R(a)$$

As corollaries we have

Theorem 2·15:- For $a \in \mathcal{A}$, a unital Banach algebra, $\sigma(a) \neq \phi$.

Pf. If $\sigma(a) = \phi$ for any $a \in \mathcal{A}$, then ϕ of lemma 2·14 is entire.

Further

$$|\phi(\lambda)| \leq \|f\| \, \|[\lambda e - a]^{-1}\|$$
$$= \underbrace{\frac{\|f\|}{|\lambda|}}_{\to 0} \, \|\underbrace{\left(e - \frac{a}{\lambda}\right)^{-1}}_{\to e}\| \quad \text{as } \lambda \to \infty \text{ (by 2·5)}$$
$$\to 0 \quad (*)$$

Thus ϕ is bounded on $\mathbb{C}$ and hence, by Liouville's Thm, $\phi(\lambda) = \text{const} = 0$ by $(*)$.
Thus, by the Hahn-Banach Theorem
$[\lambda e - a]^{-1} = 0$ (for all λ) which is impossible.

Theorem 2·16 (Gelfand-Mazur) If $\mathcal{A}$ is a complete, unital normed division algebra (ie every non-zero elt of $\mathcal{A}$ is regular), then $\mathcal{A}$ is isometrically isomorphic to $\mathbb{C}$.

Pf. By 2·15, for each $a \in \mathcal{A}$ $\exists \lambda_a \in \sigma(a)$ ie $\lambda_a e - a$ is singular, so by assumption $\lambda_a e - a = 0$
ie $a = \lambda_a e$ and $\|a\| = |\lambda_a|$.
The mapping $a \mapsto \lambda_a$ is then the required isometric isomorphism.

Theorem 2.17:- (Spectral Radius Formula)
In a unital Banach algebra we have
$$\rho(a) = r(a)\left(= \limsup_n \|a^n\|^{\frac{1}{n}}\right).$$

Pf. For λ with $|\lambda| > \rho(a)$ we have by 2.15 that $\phi(\lambda)$ is analytic and so may be expanded as a Laurent series.

Also,
$$\phi(\lambda) = \lambda f\left([e - \tfrac{a}{\lambda}]^{-1}\right)$$
$$= \lambda f\left(e + \sum_{n=1}^{\infty} \left(\tfrac{a}{\lambda}\right)^n\right) \quad \text{for } |\lambda| > r(a) \text{ by } 2.1$$
$$= f(e)\lambda + \sum_{n=1}^{\infty} f(a^n)\,\lambda^{1-n}$$

By the uniqueness of power series, this is the Laurent expansion and therefore converges for $|\lambda| > \rho(a)$.

Consequently we must have that
$$f(a^n)\,\lambda^{1-n} \to 0 \quad \text{for } |\lambda| > \rho(a)$$
and so, in particular
$$f(a^n)\,\lambda^{1-n} \text{ is bded for } \forall n \text{ \& each } f.$$
Thus, by the uniform boundedness principle, we have
$$\|a^n\|\,|\lambda^{1-n}| \text{ is bded.}$$
ie $\exists K > 0$ with $\|a^n\| \leq K|\lambda|^{n-1} \ \forall n.$
or $\|a^n\|^{\frac{1}{n}} \leq K^{\frac{1}{n}}\,|\lambda|\,|\lambda|^{-\frac{1}{n}}$

and so
$$r(a) = \limsup_n \|a^n\|^{\frac{1}{n}} \leq |\lambda| \limsup_n \underbrace{K^{\frac{1}{n}}|\lambda|^{-\frac{1}{n}}}_{\to 1}$$

ie $r(a) \leq |\lambda|$ for all λ with $|\lambda| > \rho(a)$
and so we conclude
$$r(a) \leq \rho(a).$$
This, together with 2.3, yields the desired result.

§3. Unital Commutative Banach algebras $\mathcal{A}$.

Defn: We will denote by $\underline{H}$ the set of all multiplicative linear functionals (complex homomorphisms) of $\mathcal{A}$; that is, $f \in H$ iff $f : \mathcal{A} \xrightarrow{\text{linear}} \mathbb{C}$ & $f(ab) = f(a)f(b)$.

$H_0 = H \setminus \{0\}$ is the set of non trivial mult. lin. fnls.

Proposition 3·1:- For $f \in H_0$ we have

i) $f(e) = 1$

ii) $a \in \mathcal{G}(\mathcal{A}) \Rightarrow f(a^{-1}) = \frac{1}{f(a)}$, in particular $f(a) \neq 0$.

iii) For $a \in \mathcal{A}$ $f(a) \in \sigma(a)$

iv) $\|f\| = 1$, in particular $f \in \mathcal{A}^*$.

Note: i) & iv) together show $f \in \mathcal{D}(e)$ the set of support functionals to $B[\mathcal{A}]$ at e.

Pf. i) $f(e) = f(e^2) = f(e)^2$ so either $f(e) = 1$ or $f(e) = 0$, but $f(e) = 0 \Rightarrow f(a) = f(ae) = f(a)f(e) = 0 \ \forall a \in \mathcal{A}$ $\divideontimes$ $f \in H_0$.

ii) $f(a)f(a^{-1}) = f(aa^{-1}) = f(e) = 1$.

iii) $f(f(a)e - a) = 0$ so by ii) $f(a)e - a \notin \mathcal{G}(\mathcal{A})$ ie $f(a) \in \sigma(a)$.

iv) $|f(a)| \leq \rho(a)$ by iii)

$\leq \|a\|$ by 2·3

Thus $\|f\| \leq 1$ but by i) $\|f\| \geq |f(e)| = 1$.

Proposition 3·2:- In the relative w^*-topology H_0 is a compact Hausdorff space.

Proof. For each $a, b \in \mathcal{A}$ define

$\varphi_{ab} : \mathcal{A}^* \longrightarrow \mathbb{C} : f \longmapsto f(ab) - f(a)f(b)$,

then $\varphi_{ab} = \widehat{ab} - \hat{a} \cdot \hat{b}$, where $\hat{a} = J(a)$ the natural

pt. wise product

embedding of $\mathcal{A}$ into $(\mathcal{A}, +, \cdot, \|\cdot\|)^{**}$, is w^*-continuous and so we have

$\{f : \|f\| = f(e) = 1\}$ w^* compact. $f(e)=1$ $\|f\| \in 1$

$$H_0 = \underbrace{\underbrace{\underbrace{\bigcap_{a,b\in\mathcal{A}} \phi_{ab}^{-1}(\overset{\text{closed}}{\downarrow}\!\!0)}_{w^*\text{-closed}}}_{w^*\text{ closed}} \cap \underbrace{\mathcal{D}(e)}_{w^*\text{-compact}}}_{w^*\text{-compact}}$$

To further our investigation we introduce the following notions.

Definition:- $\mathcal{I} \subseteq \mathcal{A}$ is an <u>ideal</u> in $\mathcal{A}$ if $\mathcal{I}$ is a subspace of $(\mathcal{A}, +, \cdot)$ and $\mathcal{A}\mathcal{I} \subseteq \mathcal{I}$.
"$\{ai : a\in\mathcal{A}, i\in\mathcal{I}\}$"

$\mathcal{I}$ is <u>proper</u> if $\mathcal{I} \neq \{0\}$ or $\mathcal{A}$, and a (proper) <u>maximal</u> ideal if it is not contained in any other proper ideal of $\mathcal{A}$.

EXERCISE:-

(a) Using Zorn's lemma, show that any proper ideal of $\mathcal{A}$ is contained in a maximal ideal.
In particular, if $\mathcal{A}$ is not a field deduce that $\mathcal{A}$ contains a maximal ideal.

(b) If $\mathcal{I}$ is a maximal ideal of $\mathcal{A}$ show that
i) $\mathcal{I}$ is closed [Hint: For any ideal 1st show that $\bar{\mathcal{I}}$ is also an ideal, then note that if $\mathcal{I}$ is proper $\mathcal{I} \subset \mathcal{A}\setminus\mathcal{G}(\mathcal{A})$ and so $\bar{\mathcal{I}} \subseteq \mathcal{A}\setminus\mathcal{G}(\mathcal{A})$ — a proper subset as it does not contain e.]
ii) $\mathcal{A}/\mathcal{I}$ is a field wrt $+, \times$.

(c) If $\mathcal{I}$ is a closed ideal in $\mathcal{A}$, prove that the quotient $\mathcal{A}/\mathcal{I}$ is an algebra which is a complete normed algebra (assuming $\mathcal{A}$ is) when equipped with the quotient norm
$\|a+\mathcal{I}\| = \inf_{i\in\mathcal{I}} \|a+i\|$. [You may assume $\mathcal{A}/\mathcal{I}$ is a Banach space wrt the quotient norm.]

Proposition 3·3:- $f \in H_0 \implies \operatorname{Ker} f$ is a maximal ideal in $\mathcal{A}$.

Pf. For $a \in \mathcal{A}$ and $i \in \operatorname{Ker} f$ we have
$f(ai) = f(a)f(i) = f(a)\cdot 0 = 0$ so $\mathcal{A}\operatorname{Ker} f \subseteq \operatorname{Ker} f$
a closed subspace of co-dimension 1 in $\mathcal{A}$, and so a maximal ideal.

We now establish the converse.

Proposition 3·4:- If $\mathcal{J}$ is a maximal ideal of $\mathcal{A}$ then $\mathcal{J} = \operatorname{Ker} f$ for some $f \in H_0$.

Pf. Since $\mathcal{J}$ is maximal, by Exercise b) ii) and c) $\mathcal{A}/\mathcal{J}$ is a complete normed field & so by 2·16 $\exists$ an isometric isomorphism $\eta : \mathcal{A}/\mathcal{J} \to \mathbb{C}$.
Since the quotient mapping $q : \mathcal{A} \to \mathcal{A}/\mathcal{J}$ is a complex homomorphism we have that
$f : \mathcal{A} \to \mathbb{C} : a \mapsto \eta q(a)$ is an element of H_0
Clearly $\mathcal{J} \subseteq \operatorname{Ker} f$, so $\mathcal{J} = \operatorname{Ker} f$ as $\mathcal{J}$ is maximal & by 3·3 $\operatorname{Ker} f$ is proper.

Corollary 3·5:- If $\mathcal{A} \not\cong \mathbb{C}$ then $H_0 \neq \emptyset$.

Pf $\mathcal{A} \not\cong \mathbb{C} \implies \mathcal{A}$ is not a field (2·16) so by Exercise (a) $\mathcal{A}$ contains a maximal ideal $\mathcal{J}$ & so by 3·4 $\exists f \in H_0$ with $\operatorname{Ker} f = \mathcal{J}$.

Corollary 3·6:- For $a \in \mathcal{A}$, $\sigma(a) = \{f(a) : f \in H_0\}$

Pf $\lambda \in \sigma(a) \implies \lambda e - a$ is singular, so $\mathcal{J}_1 = \mathcal{A}(\lambda e - a)$ is a proper ideal of $\mathcal{A}$ containing $\lambda e - a$. By Exercise (a) it may be extended to a maximal ideal $\mathcal{J}$. Let $\mathcal{J} = \operatorname{Ker} f$, $f \in H_0$ (3·4), then $f(\lambda e - a) = 0$ so $\lambda = f(a)$. So $\sigma(a) \subseteq \{f(a) : f \in H_0\}$

Comment :- We have established : $\exists$ a 1-1 correspondence between the set $\mathcal{M}$ of maximal ideals of $\mathcal{A}$ and the set of (kernels of) multiplicative linear functionals H_0. For this reason H_0 is often identified with $\mathcal{M}_0$ & referred to as the maximal ideal space of $\mathcal{A}$. It is otherwise known as the "Carrier Space" of $\mathcal{A}$.

Application to General (unital) Banach Algebras

Theorem 3.7 (Spectral Mapping Theorem): *Let $a \in \mathcal{A}$ and $p(a)$ a polynomial in a (ie, $p(a) = \alpha_0 e + \sum_{k=1}^{n} \alpha_k a^k$, $\alpha_k \in \mathbb{C}$) then*

$$\sigma(p(a)) = p(\sigma(a)) = \{ p(\lambda) : \lambda \in \sigma(a) \;\&\; p(\lambda) = \alpha_0 + \sum_{k=1}^{n} \alpha_k \lambda^k \}$$

Pf. Let $\mathcal{C}_m(a)$ be a maximal commutative subalg. containing a and e. Clearly $p(a) \in \mathcal{C}_m(a)$, whence

$$\lambda \in \sigma(p(a)) \iff \lambda = f(p(a)) \quad \text{some } f \in H_0(\mathcal{C}_m(a))$$
$$\iff \lambda = p(f(a))$$
$$\iff \lambda \in p(\sigma(a)).$$

Remark: The same argument applies to show $\sigma(F(a)) = F(\sigma(a))$ where F is any "function" of a expressible as a convergent power series in e, a (and $(\lambda e - a)^{-1}$ when they exist) — and so includes all functions "analytic" on an open set containing $\sigma(a)$. By a density argument (Weierstrass' Thm) it also includes all continuous functions. The leads to a general "Symbolic Functional Calculus" for elts of $\mathcal{A}$.

Theorem 3·8: (*Spectral Radius Formulae*)

For $a \in \mathcal{A}$

$$\rho(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}$$
$$= r(a) \left(= \limsup_n \|a^n\|^{1/n}\right) \quad —— ①$$
$$= \lim_n \|a^n\|^{1/n}. \quad —— ②$$
$$= \inf_n \|a^n\|^{1/n} \quad —— ③$$

Pf. ① has already been proved (2·17).
Clearly ② and ③ will follow provided we show $\rho(a) \le \inf_n \|a^n\|^{1/n}$.

Now, $\lambda \in \sigma(a^n)$ iff $\lambda = \mu^n$ for some $\mu \in \sigma(a)$ (3·7)

So $\|a^n\| \ge \rho(a^n) = \sup_{\lambda \in \sigma(a^n)} |\lambda| = \sup_{\mu \in \sigma(a)} |\mu|^n = \rho(a)^n$

Thus $\rho(a) \le \|a^n\|^{1/n}$ for all n and so
$\rho(a) \le \inf_n \|a^n\|^{1/n}$. ∎

The Gelfand Mapping (Transform)

For each $a \in \mathcal{A}$ define the "evaluation" function $\hat{a}$ by $\hat{a}(f) = f(\hat{a})$, all $f \in H_0$.
The notation "$\hat{a}$" is appropriate since
$\hat{a} = J(a)\big|_{H_0}$, this also shows that $\hat{a}$ is

natural embedding into $\mathcal{A}^{**}$

an element of $\mathcal{C}(H_0)$ where we regard H_0 as a compact Hausdorff space in the realative w^*-topology, and so we are led to define the
Gelfand Mapping $G : \mathcal{A} \longrightarrow \mathcal{C}(H_0) : a \longmapsto \hat{a}$.

We begin by noting that G is a continuous homomorphism of the unital commutative Banach algebra $\mathcal{A}$ into $(\mathcal{C}(H_0), \|\cdot\|_\infty)$. Continuous, as
$\|\hat{a}\|_\infty = \sup |f(a)| = \rho(a) \le \ldots$

EXERCISE: Show that G is an isometry iff $\rho(a) = \|a\|$ for all $a \in \mathcal{A}$ and that this happens iff $\|a^2\| = \|a\|^2$ for all $a \in \mathcal{A}$.

PROPOSITION 3.9: $a \in \mathcal{G}(\mathcal{A}) \Longleftrightarrow \hat{a} \in \mathcal{G}(\mathcal{C}(H_0))$

Pf ($\Rightarrow$) $\hat{a}(f) \neq 0 \; \forall f \in H_0$ (3.1(ii)) so $\hat{a}^{-1} = \frac{1}{\hat{a}}$ exists

($\Leftarrow$) if $\hat{a}^{-1}$ exists then $0 \notin \hat{a}(H_0) = \sigma(a)$ by 3.6.
so $a \in \mathcal{G}(\mathcal{A})$.

Definition: The (Jacobson) Radical of $\mathcal{A}$

$$R(\mathcal{A}) = \bigcap \{ \mathcal{I} : \mathcal{I} \text{ is a maximal ideal in } \mathcal{A} \}$$

Thus, $a \in R(\mathcal{A}) \Longleftrightarrow a \in$ every maximal ideal

$\Longleftrightarrow a \in \operatorname{Ker} f$ for every $f \in H_0$ (3.3/4)

$\Longleftrightarrow \sigma(a) = \{0\}$ (3.6)

$\Longleftrightarrow \rho(a) = 0$ (2.15)

We therefore have the characterization

$$R(\mathcal{A}) = \{ a \in \mathcal{A} : \rho(a) = 0 \}$$

We say $\mathcal{A}$ is semi-simple if $R(\mathcal{A}) = \{0\}$

Since G is linear and $\|\hat{a}\|_\infty = \rho(a)$ we therefore have

PROPOSITION 3.10: $G : \mathcal{A} \to \mathcal{C}(H_0)$ is a continuous isomorphism of $\mathcal{A}$ onto a subalgebra (not necessarily closed*) of $\mathcal{C}(H_0)$ iff $\mathcal{A}$ is semi-simple.

* EXERCISE: Let $\mathcal{A}$ be a semi-simple unital commutative Banach algebra. Show that $\hat{\mathcal{A}} = G(\mathcal{A})$ is a closed subalgebra of $\mathcal{C}(H_0) \Longleftrightarrow \rho$ is a norm on $\mathcal{A}$ equivalent to $\|\cdot\|$, and that this happens iff $\exists\, k > 0$ with $\|x\|^2 \leq k \|x^2\|$

NOTE: The Gelfand mapping provides representation theorems for commutative (unital) Banach algebras as subalgebras of algebras of the type $\mathcal{C}(E)$ of Example 2 in §1.

We conclude the present section by sketching an illustrative application of the Commutative Theory to "Classical" Harmonic analysis [For a "classical" proof of the same result see for example

R. R. Goldberg "Fourier Transforms"
Cambridge Tracts in Mathematics & Mathematical Physics 52.]

Recall: The Wiener alg. $\mathcal{W}$ is the unital commutative Banach alg of complex valued fns on $[0, 2\pi]$ with absolutely conv. Fourier Series; ie, fns of the form

$$\phi : [0, 2\pi] \to \mathbb{C} : \theta \mapsto \phi(\theta) = \sum_{n=-\infty}^{\infty} \phi_n e^{in\theta}$$

$$\text{where } \sum_{n=-\infty}^{\infty} |\phi_n| < \infty.$$

Operation are defined point-wise, in particular the alg. product is given by

$$\phi\psi(\theta) = \phi(\theta)\psi(\theta) = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \phi_k \psi_{n-k} e^{in\theta}$$

The norm of ϕ is $\|\phi\| = \sum_{n=-\infty}^{\infty} |\phi_n|$.

We begin by noting that for each fixed $\theta_0 \in [0, 2\pi]$ the "evaluation" mapping $f : \phi \mapsto \phi(\theta_0)$ defines a multiplicative linear fnl on $\mathcal{W}$.

We show that every element of $H_0(\mathcal{W})$ has this form.

Let $f \in H_0$ and let I denote the elt of $\mathcal{W}$ defined by $I(\theta) = e^{i\theta}$ so $\|I\| = 1$ and $I^n = e^{in\theta}$

Then, if $r > 0$ and $\theta_0 \in [0, 2\pi]$ are such that $f(I) = re^{i\theta_0}$ we have

$$r = |f(I)| \leq \|f\| \, \|I\| = 1$$

while

$$r = |f(I)| = \frac{1}{|f(I^{-1})|} \geq \frac{1}{\|f\| \|I^{-1}\|}$$

$$= 1 \quad \text{as } I^{-1}(\theta) = e^{-i\theta}.$$

So $f(I) = e^{i\theta_0}$ and $f(I^n) = f(I)^n = e^{in\theta_0}$

Now for any $\phi \in \mathcal{W}$ we have

$$\phi(\theta) = \sum_{n=-\infty}^{\infty} \phi_n e^{in\theta} = \sum_{n=-\infty}^{\infty} \phi_n I^n(\theta)$$

$$\text{or } \phi = \sum_{n=-\infty}^{\infty} \phi_n I^n,$$

$$\text{so } f(\phi) = \sum_{n=-\infty}^{\infty} \phi_n f(I^n) = \sum_{n=-\infty}^{\infty} \phi_n f(I)^n = \sum_{n=-\infty}^{\infty} \phi_n e^{in\theta_0}$$

$$= \phi(\theta_0) \quad \text{as required.}$$

From this 1-1 correspondence between H_0 and pts of $[0, 2\pi]$ we have for each $\phi \in \mathcal{W}$ that

$$\sigma(\phi) = \{ f(\phi) : f \in H_0(\mathcal{W}) \}$$
$$= \{ \phi(\theta_0) : \theta_0 \in [0, 2\pi] \}$$
$$= \phi([0, 2\pi]), \text{ the range of } \phi$$

In particular, if $0 \notin \phi([0, 2\pi])$ we have $\phi \in \mathcal{G}(\mathcal{W})$.

Now the "algebraic" inverse of ϕ, $1/\phi$, is unique, so $1/\phi \in \mathcal{W}$ and we have the classical result:

If ϕ has an absolutely convergent Fourier Series and $\phi(\theta) \neq 0$ for all $\theta \in [0, 2\pi]$, then $1/\phi$ also has an absolutely convergent Fourier series.

§4. *-algebras.

By an *involution* on the alg. $\mathcal{A}$ we mean a mapping
$\mathcal{A} \to \mathcal{A} : a \mapsto a^*$
satisfying

i) $(a^*)^* = a$

ii) $(\lambda a)^* = \bar{\lambda} a^*$

iii) $(a+b)^* = a^* + b^*$

iv) $(ab)^* = b^* a^*$

(ie $a \mapsto a^*$ is an anti isomorphism of period 2.)

By a *-subalg we will mean a subalg B which is closed under "*" ie $a \in B \Rightarrow a^* \in B$.

A Banach alg with an involution will be termed a Banach *-algebra.

Many of the common Banach algs are naturally *-algs.

EXAMPLES:

1) The prototype of all *-algs is $\mathbb{C}$ with $\lambda^* = \bar{\lambda}$ (conjugation).

2) $\mathcal{C}(K)$ with involution $f^* = \bar{f}$ (the conjugate f^n) is a Banach *-alg.
(K compact Hausdorff)

3) $\mathcal{B}(H)$, H an Hilbert sp. with involution $T \to T^*$ is a Banach *-alg., where T^* is the adjoint of T; ie, the unique operator s.t.
$(Tx, y) = (x, T^*y) \quad \forall x, y \in H.$

Henceforth we will assume $\mathcal{A}$ denotes a Unital Banach *-algebra, unless otherwise specified.

EXERCISE: 1) Prove $e^* = e$

2) If $a \in \mathcal{G}(\mathcal{A})$ show that $(a^*)^{-1} = (a^{-1})^*$ and so conclude that $a \in \mathcal{G}(\mathcal{A}) \Leftrightarrow a^* \in \mathcal{G}(\mathcal{A})$ ie $\mathcal{G}(\mathcal{A})$ is closed under *.

3) Show that $\sigma(a^*) = \overline{\sigma(a)} = \{\bar{\lambda} : \lambda \in \sigma(a)\}$, in particular then $\rho(a^*) = \rho(a)$.

Many of the special kinds of complex numbers have analogues in $\mathcal{A}$. Following the terminology for $\mathcal{B}(H)$ we define :-

$a \in \mathcal{A}$ is <u>self-adjoint</u> if $a = a^*$

<u>normal</u> if a & a^* commute

<u>unitary</u> if $a^* = a^{-1}$ (c.f. complex numbers on the unit ⊙, <u>ie</u> of the form $e^{i\theta}$)

EXERCISE: 1) Show that aa^*, $a+a^*$ and $i(a-a^*)$ are self-adjoint for any $a \in \mathcal{A}$.

2) If a, b and ab are self-adjoint deduce that a and b commute.

<u>Note</u>: <u>Every element $a \in \mathcal{A}$ has a unique decomposition into self-adjoint elements p & q, viz $a = p + iq$ such that $a^* = p - iq$.</u>

[take $p = \frac{1}{2}(a+a^*)$ and $q = \frac{1}{2i}(a-a^*)$]

p & q are sometimes referred to as the "real" and "imaginary" parts of a.

So far we have made no assumption concerning the continuity of the involution. In each of the 3 examples given we have that

$$\|a\|^2 = \|aa^*\| \quad \text{———} \quad (B)$$

EXERCISE: Verify this assertion.

An algebra for which (B) holds is termed a B^*-algebra. The remainder of the course will be devoted to the study of this important class of $*$-algebras. In particular we aim to characterise such algebras by developing an important representation Theorem due to Gelfand & Naimark.

EXERCISE: For a normal element a in a B^*-algebra prove that $\|a\| = \|a^*\|$.

Lemma 4.1. Let $\mathcal{A}$ be a unital B^*-alg, then if $a \in \mathcal{A}$ is self-adjoint we have $\rho(a) = \|a\|$.

Pf. $\|a\|^2 = \|aa^*\| = \|a^2\|$ as $a = a^*$
& in general $\|a\|^{2n} = \|a^{2n}\|$, the result now follows from 3.8, since $\lim_n \|a^n\|^{\frac{1}{n}} = \lim_n \|a^{2n}\|^{\frac{1}{2n}} = \|a\|$.

Corollary 4.2: In a B^*-alg the norm is given by $\|a\| = \sqrt{\rho(aa^*)}$

EXERCISE: i) Show that $\rho(a) = \|a\|$ for any normal element ~~a~~ of a B^*-alg.
ii) Deduce that a commutative B^*-alg. is semi-simple and that the Gelfand mapping is an isometric isomorphism in this case.

We now develop an area of general Banach alg Theory which is particularly useful for the further study of B^*-algs.

NUMERICAL RANGE

Recall: $\mathcal{D}(e) = \{ f \in \mathcal{A}^* : f(e) = \|f\| = 1 \}$ is the set of support fns to $B[\mathcal{A}]$ at e.

The numerical range of $a \in \mathcal{A}$ is $V(a) = \{ f(a) : f \in \mathcal{D}(e) \}$

The numerical radius of $a \in \mathcal{A}$ is
$$\nu(a) = \sup \{ |\lambda| : \lambda \in V(a) \}$$
$$= \sup \{ |f(a)| : f \in \mathcal{D}(e) \}$$

Proposition 4·3: For a unital Banach alg. $\mathcal{A}$

i) for $a \in \mathcal{A}$ $V(a)$ is a compact convex subset of $\mathbb{C}$ containing $\sigma(a)$.

ii) $\rho(a) \leq \nu(a) \leq \|a\|$

iii) $V(a+b) \subseteq V(a) + V(b)$ & $V(\lambda a) = \lambda V(a)$

iv) ν is a semi-norm on $(\mathcal{A}, +, \cdot)$ — indeed in can be shown that ν is an equivalent linear space norm to $\|\cdot\|$; $\frac{1}{e}\|a\| \leq \nu(a) \leq \|a\|$.

Pf. i) $\lambda \in \sigma(a) \Rightarrow \lambda = f(a)$ some $f \in H_0(\mathcal{C}_m(a))$ — $\mathcal{C}_m(a)$ a maximal comm. subalg containing a & e — by Hahn-Banach Thm. f may be extended to $\mathcal{A}$ yielding an elt $\tilde{f} \in \mathcal{D}(e)$. Whence

$\lambda = \tilde{f}(a) \in V(a)$ and so $\sigma(a) \subseteq V(a)$.

Convexity & compactness are immediate since $V(a)$ is the image of the w^*-compact convex set $\mathcal{D}(e)$ under the w^*-cont. linear fnl $\hat{a} \in \mathcal{A}^{**}$.

ii) is imediate from i) & the defn of $V(a)$ since $\lambda \in V(a) \Rightarrow \lambda = f(a)$ so $|\lambda| \leq \|f\|\|a\| = \|a\|$.

iii) is an immediate consequence of the defn of $V(a)$ and iv) follows directly from iii).

Defn: $a \in \mathcal{A}$ is Hermitian if $V(a) \subset \mathbb{R}$.

Lemma 4·4: $a \in \mathcal{A}$ is hermitian if

$$\nu(a+\lambda e)^2 \leq \|[a+\lambda e][a+\bar{\lambda} e]\| \text{ for all } \lambda \in \mathbb{C}.$$

Pf. For $f \in \mathcal{D}(e)$, let $f(a) = \alpha + i\beta$ $(\alpha, \beta \in \mathbb{R})$, then for $\gamma \in \mathbb{R}$

$$\begin{aligned}\gamma^2 + 2\gamma\beta + \beta^2 &= (\gamma+\beta)^2 \\ &= |i\gamma + i\beta|^2 \\ &= |f(a - \alpha e + i\gamma e)|^2 \\ &\leq \nu(a + (-\alpha + i\gamma)e)^2\end{aligned}$$

$\leq \| [a+(-\alpha+i\gamma)e][a+(-\alpha-i\gamma)e] \|$
$= \| [a-\alpha e]^2 + \gamma^2 e \|$
$\leq \| [a-\alpha e]^2 \| + \gamma^2$

ie $2\gamma\beta + \beta^2 \leq \| [a-\alpha e]^2 \|$ — a fixed constant not depending on γ, which is impossible unless $\beta = 0$ ie $f(a) = \alpha \in \mathbb{R}$.

Corollary 4.5: In a B^*-alg, $\mathcal{A}$ an element a is self-adjoint iff it is hermitian.

Pf. ($\Rightarrow$) $\nu(a+\lambda e)^2 \leq \|a+\lambda e\|^2$
$= \| [a+\lambda e][a+\lambda e]^* \|$
$= \| [a+\lambda e][a+\bar{\lambda} e] \|$ as $a = a^*$

so a is hermitian.

($\Leftarrow$) Let $a = p+iq$, p, q self-adj & so hermitian by ($\Rightarrow$), then for $f \in \mathcal{D}(e)$, since a is also hermitian, we have

$$f(a) = \underset{\in \mathbb{R}}{f(p)} + i\underset{\in \mathbb{R}}{f(q)} \in \mathbb{R}$$

so $f(q) = 0 \quad \forall f \in \mathcal{D}(e)$, therefore $\nu(q) = 0$ and so by 4.1 & 4.3 (ii) we have $\|q\| = 0$ or $q = 0$ & so $a = p$ is self-adjoint.

Corollary 4.6: $V(a^*) = \overline{V(a)} = \{\bar{\lambda} : \lambda \in V(a)\}$, in particular $\nu(a^*) = \nu(a)$.

Remark: The property that every self-adj elt is hermitian has been shown to characterize B^*-algs among all Banach*-algs (Berkson & Glickfeld — independently in 1966; although the result was substantially anticipated by Vidav in 1955). Indeed the weaker property; every elt of $\mathcal{A}$ has a hermitian decomposition ie, can be written as $p+iq$

where p, q are hermitian, characterises B^*-algs among all Banach algs (Palmer 1968).

EXERCISE: Let $\mathcal{A}$ be a unital B^*-alg. and $a \in \mathcal{A}$ a normal elt., show that $\varrho(a) = \nu(a) = \|a\|$ and deduce that $V(a) = \overline{co}\,\sigma(a)$.
Note that in particular this is true for self-adj elts.

The Gelfand Theory for Unital commutative B^*-algs.

By ii) of the Ex after Corol. 4.3 we had: the Gelfand mapping $G : \mathcal{A} \to \mathcal{C}(H_0) : a \mapsto \hat{a}$ is an isometric isomorphism, and so we have

Proposition 4.7: $\hat{\mathcal{A}}$ is a closed subalg of $\mathcal{C}(H_0)$

Proposition 4.8: G carries the given involution "$*$" in $\mathcal{A}$ to the natural involution of $\mathcal{C}(H_0)$ - see Ex 2.
ie G is a $*$ isomorphism

Pf. First note: If $a \in \mathcal{A}$ is self-adj, for each $f \in H_0$ we have $\hat{a}(f) = f(a) \in V(a)$ since $H_0 \in \mathcal{D}(e)$
$\subseteq \mathbb{R}$

ie $\hat{a}$ is real valued on H_0 and so $\hat{a}^* = \overline{\hat{a}} = \hat{a}$
ie $\hat{a}$ is self-adj w.r.t. natural involution.

Now for any $a \in \mathcal{A}$ we have $a = p + iq$, $a^* = p - iq$ (p, q self-adj) so $\widehat{a^*} = \hat{p} - i\hat{q} = (\hat{a})^*$
(since $\hat{p}^* = \hat{p}$ & $\hat{q}^* = \hat{q}$).

Corollary 4.9: $\hat{\mathcal{A}}$ is closed under conjugation
(ie $\hat{\mathcal{A}}$ is a $*$-subalg of $\mathcal{C}(H_0)$.)

Pf if $\hat{a} \in \hat{A}$ then $\bar{\hat{a}} = \widehat{a^*}$, by 4·8
$\in \hat{A}$

Corollary 4·10: $G : A \to \mathcal{C}(H_0)$ is onto iff $\hat{A} = \mathcal{C}(H_0)$

Pf. $\hat{A}$ is closed under conjugation, contains the constant functions, as $\widehat{\lambda e}(f) = f(\lambda e) = \lambda$ $\forall f \in H_0$ and separates the pts of H_0 (if f_1 & f_2 are two mult. lin fnls on A which agree on all pts of $\hat{A}$ then they are equal). So by the Stone-Weierstrass Thm, $\hat{A}$ is dense in $\mathcal{C}(H_0)$. The result now follows from 4·7.

We therefore have:

Theorem 4·11 (Commutative Gelfand-Naimark Representation Thm): G is an isometric $*$-isomorphism of A onto $\mathcal{C}(H_0)$.

Application to general B^*-algs:
(Square Root) lemma 4·12: Let A be a unital B^*-alg and $a \in A$ a self-adjoint element with $\sigma(a) \subseteq [0, \infty)$, then there exists $u \in A$ with $u = u^*$ and $u^2 = a$ (also $\sigma(u) \subseteq [0, \infty)$)

Proof. Let $\mathcal{C}_m(a)$ be a maximal commutative $*$subalg of A containing a & e & let $H_0 = H_0(\mathcal{C}_m(a))$, then $\hat{a}(H_0) = \sigma(a)$ is a +ve real valued function on H_0 so we can define the continuous real valued function
$\phi : H_0 \to \mathbb{R}^+$ by $\phi(f) = \sqrt{\hat{a}(f)}$
Since $\mathcal{C}(H_0) = \widehat{\mathcal{C}_m(a)}$ (4·11)
$\exists$ $u \in \mathcal{C}_m(a) \subseteq A$, with $\hat{u} = \phi$

But then $\hat{u}^* = \hat{u}$ (as $\hat{u}$ is real valued)
& $\hat{u}^2 = \hat{a}$

so again by 4·11
$$u^* = u \ \& \ u^2 = a \quad \text{as required.}$$

SYMMETRY

Defn:- A Banach $*$-alg is *symmetric* if $\sigma(aa^*) \subset [0,\infty)$ every $a \in \mathcal{A}$.

EXERCISE: Show that any commutative B^*-alg is symmetric

THEOREM 4·13 (Kaplansky - Fukamiya et.al):
Every B^*-algebra is symmetric.

Proof: Let $x = aa^*$ and let $\mathcal{C}_m(x)$ be a maximl commutative $*$-subalg of $\mathcal{A}$ containing x & e and set $H_0 = H_0(\mathcal{C}_m(x))$. Since $|\hat{x}| - \hat{x}$ is a real +ve valued fn in $\mathcal{C}(H_0)$, by 4·11, $\exists\ z \in \mathcal{C}_m(x) \subseteq \mathcal{A}$ with $z = z^*$ and $\hat{z} = \sqrt{|\hat{x}| - \hat{x}}$ so
$$\sigma(z^2 x) = \hat{z}^2 \hat{x}\,(H_0) = (|\hat{x}| - \hat{x})\,\hat{x}\,(H_0) \subseteq [-\infty, 0]$$

$\left[(|\hat{x}| - \hat{x})\,\hat{x}\,(\varphi) = \begin{cases} 0 & \text{if } \hat{x}(\varphi) \geq 0 \\ -\text{ve} & \text{if } \hat{x}(\varphi) < 0 \end{cases} \right]$

Now,
$$\begin{aligned} z^2 x &= z x z && (\text{as } z \in \mathcal{C}_m(x)) \\ &= z a a^* z && \text{defn of } x \\ &= (za)(za)^* && \text{as } z = z^* \end{aligned}$$

so
$$\sigma((za)(za)^*) \subset (-\infty, 0] \quad \text{——— ①}$$
But $\sigma((za)(za)^*) \subseteq \sigma((za)^*(za)) \cup \{0\}$ ——— ②
(by Exercise 2 after Corol 2·7)
and $(za)(za)^* + (za)^*(za) = 2p^2 + 2q^2$

where $za = p + iq$ & $(za)^* = p - iq$ (p, q self-adj)

So $(za)^*(za) = 2p^2 + 2q^2 - (za)(za)^*$

so $\sigma((za)^*(za)) \subseteq V((za)^*(za))$

$$\subseteq 2\cdot V(p^2) + 2V(q^2) - V((za)(za^*))$$

$$= \underbrace{\underbrace{2\,\mathrm{co}(\sigma(p^2))}_{\subseteq[0,\infty)} + \underbrace{2\,\mathrm{co}(\sigma(q^2))}_{\subseteq[0,\infty)} - \underbrace{\mathrm{co}(\sigma((za)(za)^*))}_{\subseteq(-\infty,0]}}_{\subseteq[0,\infty)}$$

So by ② $\sigma((za)(za)^*) \subseteq [0, \infty)$ & so by ①

$\sigma((za)(za)^*) = \{0\}$

$\therefore$ $\|(za)(za)^*\| = \rho((za)(za)^*)$ as in B^*-alg

$= 0$

so $za(za)^* = 0$

$\therefore$ $z^2 x = 0$

or $\hat{x}(|\hat{x}| - \hat{x}) = \hat{z}^2\hat{x} = 0$

so $\hat{x} = |\hat{x}|$

That is, $\hat{x}$ assumes +ve values on H_0 & hence

$\sigma(aa^*) = \sigma(x) = \hat{x}(H_0) \subseteq [0, \infty)$ as required.

Application:

Theorem 4.14 (Polar decomposition): Let a be an invertible ~~[illegible]~~ element of the unitary B^*-alg $\mathcal{A}$, then a has a (unique) decomposition as

$$a = up$$

where u is a unitary elt & p is +ve self-adjoint element [ie $\sigma(p) \subset [0, \infty)$] (cf $re^{i\theta}$)

proof: Since a is invertible so are a^* and a^*a^*

thus by 4.13 & 4.12 $\exists$ $p \in \mathcal{A}$ with

$p^* = p$, $p^2 = a^*a^*$ and $\sigma(p) \subseteq [0, \infty)$, further

$0 \notin \sigma(p)$ or by the fnl calculus we would have $0 \in \sigma(p^2)$

$\not\Leftarrow$ a^*a^* is invertible.

Let $u = ap^{-1}$ then

$a = up$

and $u^*u = (p^{-1})^* a^* a p$

$= p^{-1} a^* a p$

$= p^{-1} p^2 p = e$

so $u^* = u^{-1}$ (as u is invertible)

or u is unitary as required.

Positive Functionals ($\mathcal{A}$ unital B^*-alg.)

<u>Defn</u> $f \in \mathcal{A}^*$ is called a <u>positive functional</u> if $f(aa^*) \geq 0$ for all $a \in \mathcal{A}$.

<u>Examples</u>: 1) $f \in \mathcal{D}(e)$ is +ve, by symmetry and the exercise following corollary 4.6.

2) For $f \in \mathcal{D}(e)$ and $a \in \mathcal{A}$ define $f_a(b) = f(aba^*)$, then f_a is +ve, since

$f_a(xx^*) = f(axx^*a^*)$

$= f((ax)(ax)^*)$

≥ 0 by 1).

<u>Lemma 4.15: For any +ve fnl f and self-adj $a \in \mathcal{A}$ we have $f(a) \in \mathbb{R}$.</u>

<u>Pf</u> Let $f(a) = \alpha + i\beta$ $(\alpha, \beta \in \mathbb{R})$

Since f is +ve

$r = f((a+e)(a+e)^*) \geq 0$ ie $r \in \mathbb{R}$

but $r = f(aa^* + a + a^* + ee^*)$

$= f(aa^*) + f(a) + f(a^*) + f(ee^*)$

$\therefore f(a) + f(a^*) = r - f(aa^*) - f(ee^*) \in \mathbb{R}$.

$\therefore \operatorname{Im} f(a) + \operatorname{Im} f(a^*) = 0 \rightsquigarrow \operatorname{Im} f(a^*) = -\operatorname{Im}(f(a))$

Since $a = a^*$ this gives $\beta = \operatorname{Im} f(a) = 0$ as needed.

For any +ve fn^{l} f define
$$(a,b)_f = f(ab^*) \qquad a,b \in \mathcal{A}.$$

Lemma 4·16:
i) $(a,a)_f \geq 0 \quad \forall a \in \mathcal{A}$
ii) $(a+b,c)_f = (a,c)_f + (b,c)_f$
iii) $(\lambda a, c)_f = \lambda (a,c)_f$
iv) $(b,a)_f = \overline{(a,b)_f}$

Pf i), ii) & iii) are immediate

For iv), let $ab^* = p + iq \quad p, q$ self-adj
then
$$\begin{aligned}(b,a)_f = f(ba^*) &= f((ab^*)^*) \\ &= f((p+iq)^*) \\ &= f(p-iq) \\ &= f(p) - i f(q) \\ &= \overline{f(p) + i f(q)} \quad \text{as } f(p), f(q) \in \mathbb{R} \text{ by } 4{\cdot}15 \\ &= \overline{f(p+iq)} \\ &= \overline{(a,b)_f} \qquad //\end{aligned}$$

As a Corollary we have the Cauchy-Schwarz inequality.

lemma 4·17 $\quad |(a,b)_f|^2 \leq (a,a)_f\,(b,b)_f$

Proof.
First note that for any $\lambda \in \mathbb{C}$
$$\begin{aligned} 0 &\leq f((a+\lambda b)(a+\lambda b)^*) \\ &= f(aa^*) + \bar{\lambda} f(ab^*) + \lambda f(ba^*) + |\lambda|^2 f(bb^*) \\ &= (a,a)_f + 2\,\mathrm{Re}\,\overline{\lambda}(a,b)_f + |\lambda|^2 (b,b)_f \quad (4{\cdot}16)\end{aligned}$$

$$= (a,a)_f + \bar{\lambda}(a,b)_f + \lambda (b,a)_f + \lambda\bar{\lambda}(b,b)_f \quad \text{—— (1)}$$

So if both $(a,a)_f$ & $(b,b)_f = 0$ then
$$\mathrm{Re}\,(\bar{\lambda}(a,b)_f) \geq 0 \text{ for all } \lambda \in \mathbb{C}$$
which is only possible if $(a,b)_f = 0$ and so in this case the inequality holds.

Thus assume one of $(a,a)_f$ & $(b,b)_f$ is not zero, wlofg take it to be $(b,b)_f$.

Then taking $\lambda = -\frac{(b,a)_f}{(b,b)_f}$ in ① we

obtain

$$0 \leq (a,a)_f - \frac{(b,a)_f}{(b,b)_f}(a,b)_f$$

or $\underbrace{(a,b)_f (b,a)_f}_{\| \atop |(a,b)_f|^2} \leq (a,a)_f (b,b)_f$

$|(a,b)_f|^2$ (by 4.16 iv)

as required.

Observation: For any +ve fnl f and $a \in \mathcal{A}$ we have

$$\begin{aligned} |f(a)|^2 &= |f(ae^*)|^2 \\ &= |(a,e)_f|^2 \\ &\leq (a,a)_f (e,e)_f \quad \text{by } 4.17 \\ &= f(e)\, f(aa^*) \quad \text{as } ee^* = e^2 = e. \end{aligned}$$

ie, $|f(a)|^2 \leq f(e) f(aa^*)$ ——— ⊛

lemma 4.18 For any +ve fnl f & $a \in \mathcal{A}$

$$|f(a)|^2 \leq f(e)^2 \|aa^*\|$$

[In particular, if f is a +ve fnl with $f(e)=1$ then $f \in \mathcal{D}(e)$]

Proof. Noting that $\sigma(\rho(aa^*)e - aa^*) \subseteq [0,\infty)$, by 4.12 ($\sqrt{\ }$ - lemma) $\exists\ u \in \mathcal{A}$ with $u = u^*$ & $u^2 = \rho(aa^*)e - aa^*$

Hence

$$\begin{aligned} f(\rho(aa^*)e - aa^*) &= f(u^2) \\ &= f(uu^*) \\ &\geq 0 \end{aligned}$$

or $f(aa^*) \leq f(e)\,\rho(aa^*)$

Combining this with ⊛ above, gives

$$|f(a)|^2 \le f(e)^2 \rho(aa^*)$$
$$= f(e)^2 \, \|aa^*\|, \text{ as } \rho(aa^*) = \|aa^*\| \text{ (} aa^* \text{ is self-adj)}$$

We now detour from our main theme to obtain an "interesting" characterization of the set H_0 as a subset of $\mathcal{D}(e)$ in the case of a unital commutative B^*-alg.

Theorem 4.19 Let $\mathcal{A}$ be a unital commutative B^*-alg, then tfae

i) $f \in H_0(\mathcal{A})$

ii) $f(aa^*) = f(a)f(a^*)$ for every $a \in \mathcal{A}$ & $f \not\equiv 0$

iii) f is an extreme point of $\mathcal{D}(e)$.

Pf

i) $\Rightarrow$ ii) ✓

ii) $\Rightarrow$ iii): taking $a = e$ in ii) $\rightarrow f(e) = f(e)^2$ so either $f(e) = 0$ (impossible as then by 4.18 $f(a) = 0$ for all $a \in \mathcal{A}$ ✱ $f \not\equiv 0$) or $f(e) = 1$,

so ii) $\Rightarrow f \in \mathcal{D}(e)$

assume $f = \frac{1}{2}(f_1 + f_2)$ with $f_i \in \mathcal{D}(e)$ $(i = 1, 2)$

If $x \in \operatorname{Ker} f$ then

$$|f_1(x)|^2 \le f_1(xx^*)$$ (by 4.18 and the Ex that $f_1 \in \mathcal{D}(e) \Rightarrow f_1$ is +ve)

$$\le 2f(xx^*)$$ (by the assumption, as similarly $f_2(xx^*) \ge 0$)

$$= 2f(x)f(x^*) \quad \text{by ii)}$$

$$= 0$$

so $\operatorname{Ker} f_1 \supseteq \operatorname{Ker} f_2$ & f_1 agrees with f at e

thus $f_1 = f$ and so also $f_2 = f = f_1$.

iii) $\Rightarrow$ i): Assume f is an extreme point of $\mathcal{D}(e)$

we first prove a special case of i), viz:

i') $f(xx^* y) = f(xx^*) f(y) \quad \forall x, y \in \mathcal{A}$.

w.l.o.g. we may assume $\|xx^*\| < 1$ and then by 4.12 $\exists\ u \in \mathcal{A}$, $u = u^*$ & $u^2 = e - xx^*$.

Define $\phi(y) = f(xx^* y) \qquad (y \in \mathcal{A})$
then
$$\phi(yy^*) = f(xx^* yy^*) = f((xy)(xy)^*) \geq 0 \quad \text{—①}$$
$$(f \in \mathcal{D}(e) \Rightarrow f \text{ +ve})$$

and
$$(f - \phi)(yy^*) = f((e - xx^*) yy^*)$$
$$= f(u^2 yy^*)$$
$$= f((uy)(uy)^*)$$
$$\geq 0 \qquad \text{—②}$$

Thus both ϕ & $f - \phi$ are +ve fnls,
also
$$0 \leq \phi(ee^*) = \phi(e) = f(xx^*) \leq \|xx^*\| < 1 \quad \text{—③}$$

so $f(e) - \phi(e) = 1 - \phi(e) \gneq 0$ & hence by 4.18 and ② $\dfrac{f - \phi}{f(e) - \phi(e)} \in \mathcal{D}(e)$

Further, if $\phi(e) = 0$ then $f(xx^*) = 0$ & by 4.18 $\phi \equiv 0$ so i') holds.
On the other hand if $\phi(e) \neq 0$ then
$$\frac{\phi}{\phi(e)} \in \mathcal{D}(e) \quad (③, ① \text{ \& } 4.18)$$

and $f = \phi(e) \dfrac{\phi}{\phi(e)} + (f - \phi)(e) \dfrac{f - \phi}{(f - \phi)(e)}$

is a convex combination of elts of $\mathcal{D}(e)$, since f is extreme we therefore have
$$\frac{\phi}{\phi(e)} = f \quad \text{which is i')}$$

To show $i') \Rightarrow i)$ it suffices to note that any $a \in \mathcal{A}$ may be written as

$$a = \frac{1}{3}\sum_{p=1}^{3} \omega^p z_p z_p^* \quad \text{where } \omega = e^{2\pi i/3}$$
$$\text{and } z_p = e + \omega^{-p} a \qquad (4)$$

As then,

$$f(ay) = \frac{1}{3}\sum_{p=1}^{3} \omega^p f(z_p z_p^* y)$$
$$= \frac{1}{3}\sum_{p=1}^{3} \omega^p f(z_p z_p^*) f(y) \quad \text{by } i')$$
$$= f\left(\frac{1}{3}\sum_{p=1}^{3} \omega^p z_p z_p^*\right) f(y)$$
$$= f(a) f(y), \text{ so } f \in H_0 .$$

[~~proof of (4)~~ — EXERCISE

$$\frac{1}{3}\sum_{p=1}^{3} \omega^p (e + \omega^{-p} x)(e + \omega^{-p} x)^*$$
$$= \frac{1}{3}\sum_{p=1}^{3} \omega^p (e + \omega^{-p} x)(e + \overline{\omega}^{-p} x^*)$$
$$= \frac{1}{3}\sum_{p=1}^{3} \omega^p \left(e + \omega^{-p} x + \overline{\omega}^{-p} x^* + (\omega\overline{\omega})^{-p} x x^*\right)$$
$$= \frac{1}{3}\underbrace{\sum_{p=1}^{3} \omega^p}_{=0} e + x + \frac{1}{3}\sum_{p=1}^{3} \omega^p \overline{\omega}^{-p} x^*$$
$$+ \frac{1}{3}\underbrace{\sum_{p=1}^{3} \omega^p}_{=0} x x^*$$

$$\sum_{p=1}^{3} \omega^{2p}$$
$$= \omega^2 + \omega^4 + \omega^6$$
$$= \omega^2 + \omega^1 + \underbrace{\omega^0}_{=\omega^3} = 0 \quad]$$

The cullminating result of this section is

THEOREM 4·20 (Gelfand - Naimark Representation)

Every (unital) B^*-algebra is isometrically $*$-isomorphic to a closed $*$-subalgebra of operators on some Hilbert space.

Note Historically, those Banach $*$-algebras which are isometrically $*$-isomorphic to closed $*$-subalgs of operators on some Hilbert space, were refered to as C^*-algebras. Thus 4·20 is often restated: Every B^*-alg. is a C^*-alg.; and today because of 4·20 the terms B^* & C^*-alg. have become somewhat interchangeable in the literature.

Proof. We break the proof into a number of steps:-

I) Let $f \in \mathcal{D}(e)$, so f is a +ve fnl, and define
$$N_f = \{ a \in \mathcal{A} : (a,a)_f = 0 \},$$
then N_f is a closed (right) ideal in $\mathcal{A}$.
To see this note:-
i) N_f is a subspace; Let $(a,a)_f = (b,b)_f = 0$ then by 4·18 $(a,b)_f = 0$ so
$$(a+\lambda b, a+\lambda b)_f = (a,a)_f + 2\operatorname{Re}\lambda(b,a)_f + |\lambda|^2(b,b)_f = 0$$
i.e. $a, b \in N_f \Rightarrow a+\lambda b \in N_f$.
ii) $N_f \mathcal{A} \subseteq N_f$. For $a \in N_f$ & $b \in \mathcal{A}$ we have
$$|(ab, ab)_f|^2 = |f((ab)(ab)^*)|^2 = |f(a(abb^*)^*)|^2 \leq (a,a)_f\,(abb^*, abb^*)_f = 0.$$

iii) N_f closed follows from the continuity of f, $a \mapsto a^*$ -x-.

II) Let $H_f = \mathcal{A}/N_f$

$$= \{ [a]_f : [a]_f = a + N_f,\ a \in \mathcal{A} \}$$

Then H_f is a linear space on which $([a]_f, [b]_f)_f \equiv (a,b)_f$ gives a well defined inner-product wrt which H_f is a Hilbert space, as

$(a+n,\ b+n)_f$ any $n \in N_f$

$= f(ab^*) + \cancel{f(nb^*) + f(an^*)} + f(nn^*)$

$= f(ab^*) \cancel{+}\ f(na^*)$ by 4.16(iv)

$= f(ab^*)$.

$= (a,b)_f$.

III) For any $b \in \mathcal{A}$ define

$$T_b^f : H_f \to H_f : [a]_f \mapsto [ab]_f$$

Since N_f is a right ideal, T_b^f is a well-defined linear mapping s.t.

$$T_{bc}^f = T_c^f T_b^f$$

Thus

$b \mapsto T_b^f$ is a $*$-homomorphism of $\mathcal{A}$ into $\mathcal{B}(H_f)$. [To see that $T_{b^*}^f = (T_b^f)^*$, note that;

$$\begin{aligned}(T_b^f [a]_f, [c]_f)_f &= (ab, c)_f \\ &= f(abc^*) \\ &= (a, cb^*)_f \\ &= ([a]_f, [cb^*]_f)_f \\ &= ([a]_f, T_{b^*}^f [c]_f)_f .\end{aligned}$$

We also have that $\|T_b^f\|_f \leqslant \|b\|$ so $T_b^f \in \mathcal{B}(H_f)$.

$$\Big[\quad \|T_b^f\|_f^2 = \sup\{\|T_b^f[a]_f\|_f^2 : \|[a]_f\|_f^2 = 1\}$$

$$= \sup\{\|[ab]_f\|_f^2 : \|[a]_f\|_f^2 = 1\}$$

$$= \sup\{(ab, ab)_f : (a,a)_f = 1\}$$

$$= \sup\{f((ab)(ab)^*) : f(aa^*) = 1\}$$

$$= \sup\{f(a(bb^*)a^*) : f(aa^*) = 1\}$$

$$= \sup\{f_a(bb^*) : f_a(e) = 1\}$$

where, from Ex 2 of +ve lin fnl, f_a is a +ve lin fnl, so

$$\leqslant \sup\{f_a(e)^2\|bb^*\| : f_a(e) = 1\} \quad (4.18)$$

$$= \|bb^*\| = \|b\|^2.$$

IV) Now, Let H be the "l_2-product of the spaces H_f s.t. $f \in \mathcal{D}(e)$" (— a substitution space)

ie

$$H = \bigoplus_{f \in \mathcal{D}(e)} {}_2\, H_f$$

$$= \Big\{ \eta : \mathcal{D}(e) \to \bigcup_{f \in \mathcal{D}(e)} H_f \;\Big|\; \eta(f) \in H_f \text{ all } f \;\&\; \sum_{f\in\mathcal{D}(e)} \|\eta(f)\|_f^2 < \infty \Big\}$$

Then H is a Hilbert space with inner-product

$$(\eta_1, \eta_2) = \sum_{f \in \mathcal{D}(e)} (\eta_1(f), \eta_2(f))_f \quad \left(\text{finite by C-Sch.}\right)$$

and norm

$$\|\eta\| = \sqrt{\sum_{f\in\mathcal{D}(e)} \|\eta(f)\|_f^2}.$$

For $b \in \mathcal{A}$, define

$T_b : H \to H$ by

$$T_b(\eta)(f) = T_b^f(\eta(f))$$

Then

and $b \longmapsto T'_b$ is a $*$-isomorphism of $\mathcal{A}$ into $\mathcal{L}(H)$.

We complete the proof by showing

V) $b \longmapsto T'_b$ is an isometry.

Now, $\|T_b(\eta)(f)\|_f \equiv \|T^f_b(\eta(f))\|_f$

$$\leq \|T^f_b\|_f \, \|\eta(f)\|_f$$
$$\leq \|b\| \, \|\eta(f)\|_f \quad (\text{by III})$$

Squaring and summing over $f \in \mathcal{D}(e)$ we have

$\|T'_b(\eta)\|^2 \leq \|b\|^2 \, \|\eta\|^2$.

i.e. $T'_b \in \mathcal{B}(H)$ with $\|T_b\| \leq \|b\|$.

Also, for any $f_0 \in \mathcal{D}(e)$, taking $\eta_0(f) = \begin{cases} [e]_{f_0} & f = f_0 \\ 0 & f \neq f_0 \end{cases}$

and noting that $\|[e]_{f_0}\|^2_{f_0} = f_0(ee^*) = f_0(e) = 1$ we have that $\|\eta_0\| = 1$ and so

$\|T_b\|^2 \geq \|T_b(\eta_0)\|^2 = \|T^{f_0}_b(\eta_0(f_0))\|^2_{f_0}$

$$= \|T^{f_0}_b([e]_{f_0})\|^2_{f_0}$$

$$= f_0(bb^*)$$

Thus $\|T_b\|^2 \geq \sup\limits_{f_0 \in \mathcal{D}(e)} f_0(bb^*)$

$= \nu(bb^*)$

$= \|bb^*\| \quad$ (Ex after 4.6)

$= \|b\|^2$