SUPPLEMENTARY NOTES

ON

METRIC SPACES

FOR

MATH221— Analytic Methods II

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C O N T E N T S

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TABLE OF NOTATIONS

N	- the set of natural numbers, $\{1, 2, 3, \ldots, n, \ldots\}$.
Q	- the field of rational numbers.
R	- the ordered field of real numbers.
C	- the field of complex numbers.
$X \times Y$	- the Cartesian product of the two sets X and Y i.e. the set $\{(x, y): x \in X, y \in Y\}$ of all ordered pairs with first element a member of X and second element a member of Y.
R^n (C^n)	- The Cartesian product of R (C) with itself n times.
V^n	- R^n regarded as a vector space, with addition and scalar multiplication defined component wise.
$\underset{\sim}{x}$	- an element of R^n (C^n) or V^n $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)$.
$X \backslash A$	- the complement of A in X i.e. the set $\{x \in X: x \notin A\}$.
[a, b]	- the closed interval $\{x \in R: a \leqslant x \leqslant b\}$ (Note: it is implicit in the notation that $a \leqslant b$).
(a, b)	- the open interval $\{x \in R: a < x < b\}$ (Note: it is implicit in the notation that $a < b$).
[a, b) and (a, b]	- Half open intervals in R.
C(I)	- the set of continuous functions with domain the interval I.
C[a, b] and C(a, b)	- special cases of C(I).
$f: X \to Y$	- a mapping from the set X into the set Y.
$f: x \mapsto f(x)$	- a notation for the function f which maps x to f(x) ($x \in$ some implicit domain X).
[x]	- the greatest integer function (the largest integer not exceeding x in value).
$\|x\|$	- the absolute value function $x \mapsto \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0. \end{cases}$
$p \Rightarrow q$	- p implies q, or q if p.
$p \Leftrightarrow q$	- p if and only if q, or $(p \Rightarrow q)$ and $(q \Rightarrow p)$.
iff	- if and only if.

d_1	- a metric on $\begin{cases} R^n \text{ given by } d(\underset{\sim}{x}, \underset{\sim}{y}) = \sum_{i=1}^{n} \lvert x_i - y_i \rvert \\ \text{(including the case } n = 1) \\ C(I) \text{ given by } d(f, g) = \int_I \lvert f - g \rvert. \end{cases}$
d_2	- a metric on R^n (C^n) given by $d(\underset{\sim}{x}, \underset{\sim}{y}) = \left(\sum_{i=1}^{n} \lvert x_i - y_i \rvert^2 \right)^{\frac{1}{2}}$.
d_∞	- a metric on $\begin{cases} R^n \text{ given by } d(\underset{\sim}{x}, \underset{\sim}{y}) = \underset{i=1,2,..,n}{\text{Max}} \lvert x_i - y_i \rvert \\ C(I) \text{ given by } d(f, g) = \underset{x \in I}{\text{Max}} \lvert f(x) - g(x) \rvert. \end{cases}$
ℓ_p^n	- the metric spaces (V^n, d_p) where $p = 1, 2, \infty$.
E^n	- ℓ_2^n, Euclidean n-space.
$B_r(x)$	- the open ball of radius r centre x.
Int(A)	- the interior of the set A.
d(A)	- the diameter of the set A, $\underset{x,y \in A}{\text{Sup}} d(x, y)$.
A'	- the derived set of A.
$\overline{A}$	- the closure of A, $A \cup A'$.
bdry A	- the boundary of A, $\overline{A} \cap \overline{(X \setminus A)}$
$\{A_\lambda : \lambda \in \Lambda\}$	- an indexed family of sets.
f(A)	- for $f: X \to Y$, $A \subseteq X$, the set $\{f(x): x \in A\} \subseteq Y$.
$f^{-1}(A)$	- for $f: X \to Y$, $A \subseteq Y$, the set $\{x \in X: f(x) \in A\}$.

Common symbols not used in the notes, which may be useful in your working -

$\exists$	- there exists.
$\forall$	- for all.

PURE MATHEMATICS 211-23

External Notes

ANALYSIS IN METRIC SPACES

PRELIMINARIES

Lecture 1 *Definition and Examples*

In both mathematics and common language the notion of *distance* is often used figuratively, for example:

> 'Orange is a colour nearer to red than violet.'
>
> 'When a massive particle moves in a gravitational field it follows the path of shortest "distance" (geodesic) in space-time.'
>
> 'The larger the value of $n \in N$, the closer the polynomial
>
> $$p_n(x) = \sum_{m=0}^{n} x^m/m!$$
>
> is to the function exp.'

We develop an extended "theory" of distance (encompassing many cases like those above) which has proved to be fundamental for much modern mathematics. Our primitive concept will be of a set, for which each ordered pair of elements (x, y) has associated with it a real number, $d(x, y)$ - the "distance" from x to y - which satisfies certain conditions set out below.

Except in specific examples, we will not be concerned with the nature of the objects comprising our set, or with how the distances between them are "calculated".

DEFINITION: A *metric space* is a set X, equipped with a *metric* d, i.e. a function $d: X \times X \to R$ which satisfies:

(M1) $d(x, y) \geqslant 0$ all $x, y \in X$

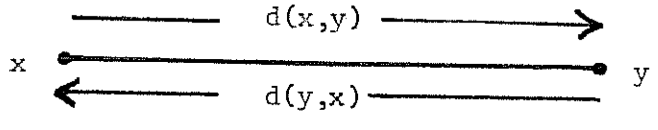
(M2) $d(x, y) = 0 \Leftrightarrow x = y$

(M3) $d(x, y) = d(y, x)$ all $x, y \in X$ (symmetry)

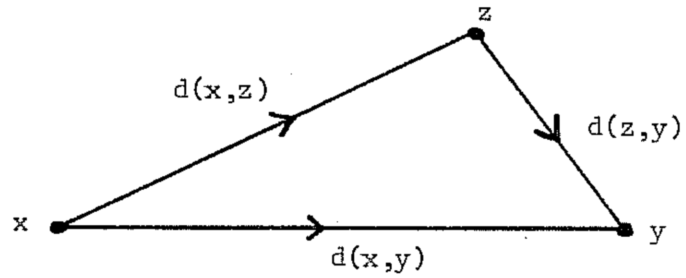
(M4) $d(x, y) \leqslant d(x, z) + d(z, y)$ all $x, y, z \in X$ (triangle inequality).

H. Minkowski described these four requiremnts on the function d as "properties which any notion of distance ought to possess".

Thus (M3) stipulates that the "distance" from x to y is the same as that from y to x



i.e. d is symmetric in its two arguments, while (M4) expresses the minimality of "distance": the distance from x to y via any intermediate point z cannot be "shorter" than the direct distance from x to y.



NOTATION: The metric space consisting of set X and metric d is denoted by (X, d).

EXAMPLES: (1) The set R with the *usual* metric

$$d_1(x, y) = |x - y| \text{ all } x, y \in R$$

is a metric space.

EXERCISE: Check that this is indeed so, by proving that the axioms (M1) to (M4) are satisfied by d_1.

NOTE: The metric d_1 agrees with 'ordinary' distance between points on a line.

(2) (a) R^2, whose elements, $\underset{\sim}{x} = (x_1, x_2)$, are ordered pairs of real numbers, together with the Euclidean metric

$$d_2(\underset{\sim}{x}, \underset{\sim}{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \text{ all } \underset{\sim}{x}, \underset{\sim}{y} \in R^2$$

forms a metric space.

(b) Similarly, C (the field of complex numbers) equipped with the (unitary) metric

$$d_2(z_1, z_2) = |z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \text{ all } z_1, z_2 \in C$$

where $z_j = x_j + iy_j$ $(j = 1, 2)$

is also a metric space.

NOTE: The elements of R^2 (or C) can be identified with points in the plane (via a set of rectangular Cartesian axes) and then d_2 gives the distance between points in the sense of Euclidean geometry.

(3) Any set X can be rendered a metric space by using the discrete metric defined by

$$d(x, y) = \begin{cases} 0 \text{ if } x = y \\ 1 \text{ if } x \neq y \end{cases}$$

Proof (that the discrete metric is a metric). Clearly (M1) and (M2) are satisfied, while (M3) holds since = and ≠ are symmetric relations.

To establish (M4) for $x, y, z \in X$ we must consider the following cases $x = y \neq z$, $x = y = z$, $x \neq y = z$, $z = x \neq y$, $x \neq y \neq z \neq x$.
In case 1, $d(x, y) = 0$ while $d(x, z) = d(z, y) = 1$ so

$$(0 =)\ d(x, y) \leqslant d(x, z) + d(z, y) (= 2).$$

The other cases may be handled similarly. ∎

NOTE: While the discrete metric is certainly pathological it is of considerable importance in the construction of counter-examples.

OBSERVATION: From these examples we see that the underlying set does not determine a metric uniquely; many different metrics can be defined on the same set and so several distinct metric spaces can share the same set.

EXAMPLES: (1) R can be equipped with the usual metric $d_1(x, y) = |x - y|$ and the discrete metric, d. Clearly these two metrics are not the same

$$d_1(2, 3.8) = 1.8$$

while

$$d\,(2, 3.8) = 1.$$

Frequently the appropriate metric to use is determined by the type of problem under consideration, as the following example illustrates.

(2) In elementary plane geometry the most used metric on R^2 is

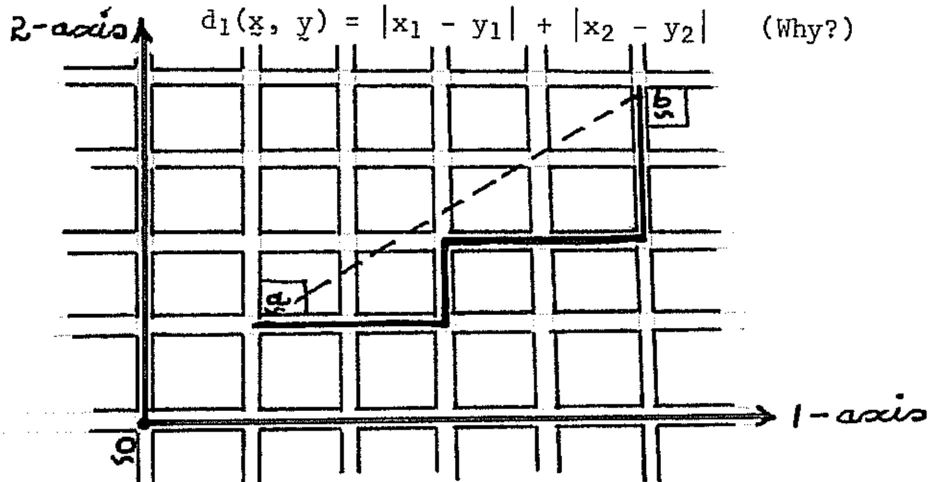
$$d_2(\underset{\sim}{x}, \underset{\sim}{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

However, for a law abiding motorist, the distance between site $\underset{\sim}{a}$ and $\underset{\sim}{b}$ on the map below is 7, not

$$d_2(\underset{\sim}{a}, \underset{\sim}{b}) = 5$$

The appropriate metric for our motorist to use would be

$$d_1(\underset{\sim}{x}, \underset{\sim}{y}) = |x_1 - y_1| + |x_2 - y_2| \quad \text{(Why?)}$$



Collateral Reading.

Giles "Analysis of Metric Spaces",
University of Newcastle, Lecture Notes in Mathematics, No. 1,
Ch. 0 and p.19.

A good informal introduction to the notion of a metric space may be found in W.W. Sawyer "A Path to Modern Mathematics", Pelican, Ch. 10, pp.187-221.

PROBLEMS.

1. Prove $d_1(\underset{\sim}{x}, \underset{\sim}{y}) = |x_1 - y_1| + |x_2 - y_2|$ is a metric on R^2.

2. Prove $d(x, y) = \min\{1, |x - y|\}$ is a metric on R.

3. Let (X, d) be a metric space. Show that d^* defined by

$$d^*(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

is also a metric on X.

*4. Let $(X^{(1)}, d^{(1)})$, $(X^{(2)}, d^{(2)})$ be two metric spaces.
Show that each of the following define metrics on the Cartesian product $X^{(1)} \times X^{(2)} = \{\underset{\sim}{x} = (x_1, x_2): \; x_1 \in X^{(1)}, x_2 \in X^{(2)}\}$

(i) $d(\underset{\sim}{x}, \underset{\sim}{y}) = \underset{i=1,2}{\operatorname{Max}} d^{(i)} (x_i, y_i)$

(ii) $d(\underset{\sim}{x}, \underset{\sim}{y}) = \sum_{i=1}^{2} d^{(i)} (x_i, y_i)$ – (Note: the metric in Prob. 1 is a special case of this.)

5. (very useful and <u>important</u>) In any metric space (X, d) prove the following inequality

$$|d(x, z) - d(z, y)| \leqslant d(x, y) \quad \text{all } x, y, z \in X$$

**6. For any set $X \neq 0$, show that $d: X \times X \to R$ satisfying

(M1') $d(x, y) = 0 \Leftrightarrow x = y$

and (M2') $d(x, y) \leqslant d(x, z) + d(y, z)$ for all $x, y, z \in X$,

is a metric on X.

<u>i.e.</u> The four axioms M1 - M4 of a metric space could be replaced by these two slightly more intricate ones.

<u>NOTES</u> on lecture 1.

1. Because the study of metric spaces is an abstract one, it is essential that you become completely familiar with the definitions and notations as well as some of the more basic examples.

2. Drawing diagrams which interpret the various definitions, constructions and results in the familiar space (R^2, d_2) is a valuable aid to understanding, and a practice which you should actively adopt.

3. (HISTORICAL) The definition of a metric space was first given in 1906 by M. Fréchet (Paris) who for many years pioneered the study of such spaces and their application to other areas of mathematics. In the same year (1906), and perhaps independently, Herman Minkowski (1864-1909) gave a similar definition in the course of his investigations into "the geometry of numbers". It was toward the end of last century that mathematicians (due to the work of Klein, Hilbert and many others) began to appreciate the power of generalized methods (such as those represented by the study of metric spaces) and so initiated the study of abstract systems - vector spaces, metric spaces, normed spaces, topological spaces, groups, rings, categories etc. - which have proved central to much twentieth century mathematics. Because a prototype for many of these structures is 'ordinary' 1, 2 or 3 dimensional space, they are often referred to as spaces and their elements as points; hence metric space (metric, from the latin *metor* - measure). The study of such a structure has proved valuable for several reasons, three of which are

(a) By retaining only essential features of a situation their consequences can be studied more simply in a less cluttered environment; and

(b) any conclusions of such a study are immediately applicable to any particular realisation of the structure. Thus a result can be simultaneously established for a number of apparently distinct situations and so hitherto unsuspected connections revealed.

(c) Once we recognize our object of study as a metric space we may in part transfer to it our "intuition" concerning familiar metric spaces such as (R^2, d_2).

Lecture 2. *Further Examples - Normed Linear Spaces.*

For many of the more important metric spaces the metric is constructed from additional structure carried by the underlying set.

EXAMPLE. Because R is a totally ordered field we are able to construct the absolute value function $x \mapsto |x|$, form $x - y$ (for all $x, y \in R$), and so define the uniform metric by $d_1(x, y) = |x - y|$. This example is generalized by

PROPOSITION: *Let X be an abelian (commutative) group, with respect to a binary operation denoted by +, on which an absolute sub-additive norm (a.s.n.) is defined,* i.e. a function

$$x \mapsto \|x\| \in R \text{ (all } x \in X)$$

satisfying:

(n1) $\|x\| \geqslant 0$ all $x \in X$;

(n2) $\|x\| = 0 \Leftrightarrow x = 0$;

(asn3) $\|-x\| = \|x\|$ all $x \in X$;

(n4) $\|x + y\| \leqslant \|x\| + \|y\|$ all $x, y \in X$,

then

$$d(x, y) = \|x - y\| \text{ (all } x, y \in X)$$

is a metric on X *(the metric induced by the a.s.n.* $\|\cdot\|$*).*

[EXAMPLE. $\|x\| = |x|$ (all $x \in R$ or C) defines an a.s.n. on R (or C) which induces the usual (uniform) metric.

REMARK. Clearly an a.s.n., $\|\cdot\|$, generalizes the absolute value (modulus) function $|\cdot|$ (hence the notation) and so $\|x\|$ may be thought of as the "distance" of x from the origin (group identity) 0.]

Proof. By (n1) $d(x, y) = \|x - y\| \geqslant 0$ (all $x, y \in X$) so (M1).

$$\begin{aligned} d(x, y) = 0 &\Leftrightarrow \|x - y\| = 0 \\ &\Leftrightarrow x - y = 0, \text{ by (n2)} \\ &\Leftrightarrow x = y \text{ so (M2).} \end{aligned}$$

$$\begin{aligned} d(x, y) = \|x - y\| &= \|-(y - x)\| \text{ (group properties)} \\ &= \|y - x\|, \text{ by (a.s.n.3)} \\ &= d(y, x), \text{ so (M3).} \end{aligned}$$

$$\begin{aligned} d(x, y) = \|x - y\| &= \|(x - z) + (z - y)\| \\ &\leqslant \|x - z\| + \|z - y\| \text{ by (n4)} \\ &= d(x, z) + d(z, y), \text{ so (M4).} \end{aligned}$$

∎

This result is useful, since it is frequently less tedious to establish (n1) to (n4) than to prove directly that d is a metric.

SPECIAL CASE: If X is a vector (or linear) space (i.e. admits scalar multiplication) over R or C and (asn3) is replaced by the stronger requirement

(n3) $\|\lambda x\| = |\lambda| \|x\|$ all x, y $\in$ X and scalars λ,

then $\|\cdot\|$ is a *norm* on X and X a normed linear space.
Remember, every example of a normed linear space
(e.g. R^2 with $\|\underset{\sim}{x}\| = \sqrt{x_1^2 + x_2^2}$, all $\underset{\sim}{x} = (x_1, x_2) \in R^2$)
is also an example of a metric space with $d(x, y) = \|x - y\|$.
However, *not every metric on a vector space is induced by a norm.*
EXAMPLE. The discrete metric d on any non-trivial vector space X is not induced by a norm.
Proof. Assume the contrary i.e.

$d(x, y) = \|x - y\|$ for all x, y $\in$ X and some norm, $\|\cdot\|$, on X.

Then for $x \neq y$ we have $\frac{1}{2}x \neq \frac{1}{2}y$ and so

$$\begin{aligned} 1 &= d(\tfrac{1}{2}x, \tfrac{1}{2}y) \\ &= \|\tfrac{1}{2}(x - y)\| \\ &= \tfrac{1}{2}\|x - y\| \\ &= \tfrac{1}{2}d(x, y) = \tfrac{1}{2} \end{aligned}$$

a contradiction, so no such norm can exist. ■

FUNCTION SPACES. (Perhaps the most important examples of metric spaces, and the ones to which our ensuing theory has the most immediate and important applications.)

Much mathematical analysis concerns the approximation of functions (e.g. exp, ℓn, y such that $x^2y'' + xy' + (x^2 - \nu^2)y = 0$)
by other (usually more "tractable") functions.
E.g. Approximation by polynomials - Taylor's Theorem etc.
Approximation by trigonometric polynomials (or their equivalents) - Fourier Series and Harmonic analysis.
To measure the "goodness" of these approximations, a notion of 'the distance between functions', is necessary. Consequently, we address ourselves to the problem of defining suitable metrics on *the set* C[a, b] *of all real valued functions defined and continuous on the closed interval* $[a, b] \equiv \{x \in R: a \leqslant x \leqslant b\}$.*

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* Recall, $f:[a,b] \to R$ is *continuous*, if for each $\varepsilon > 0$ and $x \in [a,b]$, there exists a $\delta > 0$ ($\delta \equiv \delta(\varepsilon,x)$, although in fact δ can be chosen independently of x - Heine's Theorem) such that

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon \text{ (where } y \in [a,b]).$$

OBSERVATION: C[a, b] *is a vector space over* R *with 'addition' and 'scalar multiplication' defined point wise,* i.e. given any f, g $\in$ C[a, b]

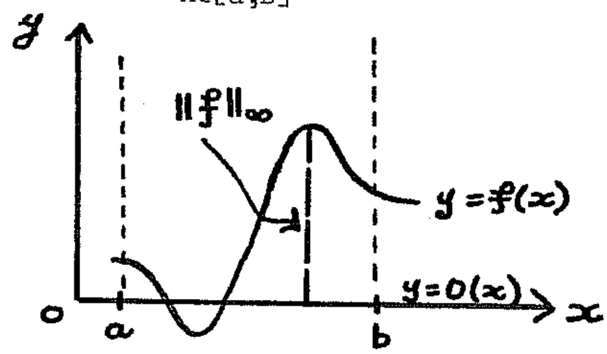
f + g is defined by $(f + g)(x) = f(x) + g(x)$, all $x \in [a, b]$

($f + g \in C[a, b]$ since sums of continuous functions are continuous) and for $\lambda \in R$, λf is defined by $(\lambda f)(x) = \lambda f(x)$ all $x \in [a, b]$. (The proof is an exercise in linear algebra.) Hence, to construct a metric for C[a, b] it suffices to define a norm on C[a, b] measuring the proximity of any f to the zero function $0(0(x) = 0$ all $x \in [a, b])$. This may be done in many distinct ways.

1. At any $x \in [a, b]$, the function $f \in C[a, b]$ differs in value from the zero function, 0, by $|f(x)|$. It is reasonable to take the largest such difference in value to be the "distance" between f and 0.

Accordingly, define the uniform norm on C[a, b] by

$$\|f\|_\infty = \max_{x\in[a,b]} |f(x)|$$



Clearly: $\|f\|_\infty \geq 0$; $\|f\|_\infty = 0 \Leftrightarrow |f(x)| = 0$ all x

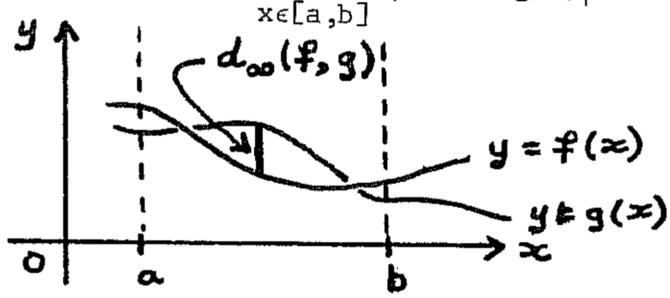
$\Leftrightarrow f = 0$;

$$\|rf\|_\infty = \max_{x\in[a,b]} |rf(x)| = \max_{x\in[a,b]} |r||f(x)| = |r|\|f\|_\infty;$$

$$\begin{aligned}\|f + g\|_\infty &= \max_{x\in[a,b]} |f(x) + g(x)| \\ &\leq \max_{x\in[a,b]} (|f(x)| + |g(x)|) \\ &\leq \max_{x\in[a,b]} |f(x)| + \max_{x\in[a,b]} |g(x)| \\ &= \|f\|_\infty + \|g\|_\infty\end{aligned}$$

and so $\|\cdot\|_\infty$ is indeed a norm on C[a, b], inducing the *uniform metric*

$$d_\infty(f, g) = \|f - g\|_\infty = \max_{x\in[a,b]} |f(x) - g(x)|.$$



E.g. If $f(x) = x^3 + x + 1$

and $g(x) = x^3 + x^2 + \frac{1}{2}x + 1$ all $x \in [0, 1]$

then in $C[0, 1]$, $d(f, g) = \max_{0 \leqslant x \leqslant 1} |\frac{1}{2}x - x^2|$

$$= \frac{1}{2} \qquad \text{(check)}$$

2. The "area" between $f \in C[a, b]$ and the zero function, 0, on $[a, b]$ defines an alternative norm to $\|\cdot\|_\infty$, viz

$$\|f\|_1 = \int_a^b |f(x)|dx \left(= \int_a^b |f|\right)$$

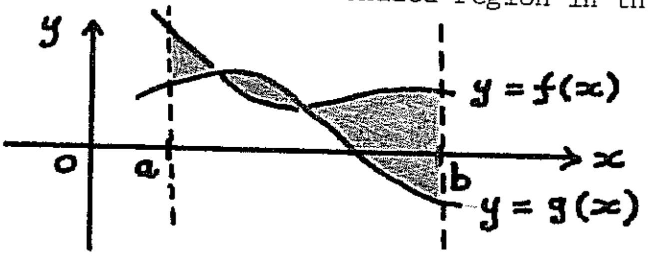
which exists for all $f \in C[a, b]$ since $|f|$ is continuous, hence integrable.

EXERCISE. Prove that $\|\cdot\|_1$ is a norm for $C[a, b]$.

The metric induced by this norm,

$$d_1(f, g) = \int_a^b |f - g|$$

is represented by the area of the shaded region in the following sketch



E.g. For f, g as above

$$d_1(f, g) = \int_0^1 |\tfrac{1}{2}x - x^2|dx = \int_0^{\frac{1}{2}} (\tfrac{1}{2}x - x^2)dx + \int_{\frac{1}{2}}^1 (x^2 - \tfrac{1}{2}x)dx$$

$$= \frac{1}{8} .$$

HIGHER-DIMENSIONAL EUCLIDEAN SPACES

Let $X = R^n$ - the set of ordered n-tuples of real numbers. Then X is a vector space (V^n) with addition and scalar multiplication defined component wise.

PROPOSITION. *For* $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n) \in X$,

$$\|\underset{\sim}{x}\|_1 = |x_1| + |x_2| + \ldots + |x_n|,$$

$$\|\underset{\sim}{x}\|_2 = \sqrt{x_1{}^2 + x_2{}^2 + \ldots + x_n{}^2}$$

and $$\|\underset{\sim}{x}\|_\infty = \text{Max}\{|x_1|, |x_2|, \ldots, |x_n|\}$$

define norms on X.

The proofs for $\|\cdot\|_1$ and $\|\cdot\|_\infty$ generalise problem (1.4) and may be proved by induction from it. (Alternatively the norm axioms (N1) to (N4) can be verified directly.)

EXERCISE. Prove that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ do define norms on X.

Clearly $\|\cdot\|_2$ satisfies (N1), (N2) and (N3), so it only remains to prove (N4) i.e.

$$\|\underset{\sim}{x} + \underset{\sim}{y}\|_2 \leqslant \|\underset{\sim}{x}\|_2 + \|\underset{\sim}{y}\|_2$$

or

$$\left(\sum_{j=1}^{n} (x_j + y_j)^2\right)^{\frac{1}{2}} \leqslant \left(\sum_{j=1}^{n} x_j{}^2\right)^{\frac{1}{2}} + \left(\sum_{j=1}^{n} y_j{}^2\right)^{\frac{1}{2}}$$

(MINKOWSKI'S Inequality).

Squaring both sides preserves the inequality and so it suffices to prove

$$\sum_{j=1}^{n} (x_j{}^2 + 2x_jy_j + y_j{}^2) \leqslant \sum_{j=1}^{n} x_j{}^2 + 2\left(\sum_{j=1}^{n} x_j{}^2 \sum_{j=1}^{n} y_j{}^2\right)^{\frac{1}{2}} + \sum_{j=1}^{n} y_j{}^2$$

or, cancelling terms,

$$\sum_{j=1}^{n} x_jy_j \leqslant \left(\sum_{j=1}^{n} x_j{}^2 \sum_{j=1}^{n} y_j{}^2\right)^{\frac{1}{2}}$$

(which implies

$$\left(\sum_{j=1}^{n} x_jy_j\right)^2 \leqslant \sum_{j=1}^{n} x_j{}^2 \sum_{j=1}^{n} y_j{}^2$$

- the inequality of Cauchy-Schwarz - Buniakovski.)

Now, since $a^2 - 2ab + b^2$ $(= (a - b)^2) \geqslant 0$ for all $a, b \in R$ we have

$$2\,\frac{x_j}{\left(\sum_{j=1}^{n} x_j{}^2\right)^{\frac{1}{2}}}\;\frac{y_j}{\left(\sum y_j{}^2\right)^{\frac{1}{2}}} \leqslant \frac{x_j{}^2}{\sum_{j=1}^{n} x_j{}^2} + \frac{y_j{}^2}{\sum_{j=1}^{n} y_j{}^2}$$

for $j = 1, 2, \ldots, n$

Summing these inequalities over j, gives

$$\frac{2\sum_{j=1}^{n} x_j y_j}{\left(\sum_{j=1}^{n} x_j^2 \sum_{j=1}^{n} y_j^2\right)^{\frac{1}{2}}} \leqslant 2$$

or
$$\sum_{j=1}^{n} x_j y_j \leqslant \left(\sum_{j=1}^{n} x_j^2 \sum_{j=1}^{n} y_j^2\right)^{\frac{1}{2}}$$

as required.

NOTE. The metric induced on X by $\|\cdot\|_\infty$,

$$d_\infty(\underset{\sim}{x}, \underset{\sim}{y}) = \underset{j=1,2,..,n}{\mathrm{Max}} |x_j - y_j|$$

is known as the *uniform* (or *Supremum*) *metric* for R^n.

$\|\cdot\|_2$ induces the *Euclidean Metric*

$$d_2(\underset{\sim}{x}, \underset{\sim}{y}) = \left(\sum_{j=1}^{n} (x_j - y_j)^2\right)^{\frac{1}{2}},$$

which is a direct generalization of the 'ordinary' distance between points in 1, 2 or 3 dimensional space.
(V^n, d_2) is frequently referred to as Euclidean n-space, denoted by E^n or ℓ_2^n.

Collateral Reading.

Giles, op cit Ch 1, sec 1, pp.19-25.
Simmons, G.F. "Introduction to Topology and Modern Analysis" McGraw-Hill, Ch 2, sec 14 and 15, pp.80-90 - see also sec 9, pp.51-58.

PROBLEMS.

1. Let X be a vector space and $\|\cdot\|$ a norm on it. Show that

$$d(x, y) = \mathrm{Min}\{1, \|x - y\|\}$$

defines a metric on X (see problem 1.2) which is not induced by any norm.

2. Let X be a normed linear space, norm $\|\cdot\|$. Prove the inequality

$$\big|\|x\| - \|y\|\big| \leqslant \|x - y\|, \text{ all } x, y \in X.$$

(see problem 1.5).

3. Using Taylor's Theorem with remainder, obtain estimates for

$$d_\infty(f, p_n) \text{ and } d_1(f, p_n) \text{ in } C[0, 1]$$

when

(i) $f = \exp$ and $p_n(x) = \sum_{m=0}^{n} x^m/m!$

and (ii) $f = \sin$ and $p_n(x) = \sum_{m=0}^{n} (-1)^m\, x^{2m+1}/(2m+1)!$

4. For $p = 1, 2$ and ∞ graph the following set of points in R^2

$$\{\underset{\sim}{x} = (x, y): \ \|\underset{\sim}{x}\|_p = 1\}.$$

*5. Show that the set ℓ_2, whose elements

$$\underset{\sim}{x} = (x_1, x_2, \ldots, x_n, \ldots)$$

are infinite sequences of real numbers such that $\sum_{j=1}^{\infty} x_j^2 < \infty$, is a vector space over R with component wise definitions of addition and scalar multiplication.

Further show that it is a normed linear space with norm given by

$$\|\underset{\sim}{x}\|_2 = \left(\sum_{j=1}^{\infty} x_j^2\right)^{\frac{1}{2}}$$

The space ℓ^2 with norm $\|\cdot\|_2$, known as <u>Hilbert('s) space</u>, is of considerable importance and represents an <u>infinite</u> dimensional version of 'ordinary Euclidean space'.

<u>NOTES</u> on lecture 2.

For each real number $p \geqslant 1$

$$\|\underset{\sim}{x}\|_p = \left(\sum_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}}$$

defines a norm on R^n (or C^n), of which $\|\cdot\|_1$ and $\|\cdot\|_2$ are special cases.

Further for each $\underset{\sim}{x} \in R^n$

$$\underset{p\to\infty}{\text{Limit}} \|\underset{\sim}{x}\|_p = \underset{j=1,2,\ldots,n}{\text{Max}} |x_j| = \|x\|_\infty$$

(hence the notation).

Similarly

$$\|f\|_p = \left(\int_a^b |f|^p\right)^{\frac{1}{p}}$$

defines an infinite family of distinct norms on C[a, b].

The truth of these remarks follows from a generalized Cauchy-Schwartz inequality, known as Hölder's inequality

$$\left|\int_a^b f\bar{g}\right| \leqslant \left(\int_a^b |f|^p\right)^{\frac{1}{p}} \left(\int_a^b |g|^q\right)^{\frac{1}{q}} \qquad \text{(where } q = \frac{p}{p-1} \text{ if } p > 1\text{)}$$

from which an appropriate Minkowski inequality follows.

These norms play a dominant role in much modern mathematics and theoretical physics (particularly in Quantum Mechanics).

Lecture 3 *Convergence and Cauchy Sequences.*

RECALL. A *sequence* (of elements) of the set X is a function $a : \mathbb{N} \to X$, and we usually write a_n for $a(n)$ - the image of $n \in \mathbb{N}$ under the function a.

NOTATION. The sequence $a : \mathbb{N} \to X$ of X is denoted by

$$\{a_n\}_{n=1}^{\infty} \equiv a_1, a_2, a_3, \ldots, a_n, \ldots$$

(when the context makes it clear that we are dealing with a sequence and not a set, we will sometimes write $\{a_n\}$ instead of $\{a_n\}_{n=1}^{\infty}$).

[Let $\{n_k\}_{k=1}^{\infty}$ be a strictly increasing sequence of natural numbers i.e. $n : \mathbb{N} \to \mathbb{N} : k \mapsto n(k) = n_k$. Then for any sequence $\{a_n\}$ the composite $a \circ n$ is a *subsequence of* $\{a_n\}$,

$$\{a_{n_k}\} \text{ or } \{a_{n_k}\}_{k=1}^{\infty} \equiv a_{n_1}, a_{n_2}, \ldots, a_{n_k}, \ldots]$$

DEFINITION. A sequence $\{x_n\}$ of points of the metric space (X, d) is *convergent* if there is a point $x \in X$ for which, given any $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$n \geqslant N \Rightarrow d(x_n, x) < \varepsilon.$$

In which case we say the sequence $\{x_n\}$ converges to (has limit) x and write $d(x_n, x) \to 0$. Provided the metric space within which we are working is clearly understood we may write $\underset{n\to\infty}{\text{Limit}}\, x_n = x$ or simply $x_n \to x$ to mean the sequence $\{x_n\}$ converges to x. (Sometimes, to emphasize the metric w.r.t. which convergence is taking place we may write $x_n \overset{d}{\to} x$.)

NOTE. This definition of convergence corresponds to the definition of convergence in $\mathbb{R}$ with our general concept of distance replacing the usual one in $\mathbb{R}$, i.e. $|x_n - x|$ becomes $d(x_n, x)$.

THEOREM 3.1: *A convergent sequence* $\{x_n\}$ *of the metric space* (X, d) *has a unique limit.*

Proof. Assume $x_n \to x$ and $x_n \to y$, then for any $\varepsilon > 0$ there exist $N_1, N_2 \in \mathbb{N}$ such that

$$n \geqslant N_1 \Rightarrow d(x_n, x) < \frac{\varepsilon}{2}$$

$$n \geqslant N_2 \Rightarrow d(x_n, y) < \frac{\varepsilon}{2}$$

But then, for all $n \geqslant \text{Max}\{N_1, N_2\}$

$$d(x, y) \leqslant d(x, x_n) + d(x_n, y), \text{ by the triangle inequality}$$
$$< \varepsilon$$

and, since ε is arbitrary, this implies $d(x, y) = 0$ or $x = y$. ∎

DEFINITION. In the metric space (X, d) the *diameter* of $A \subseteq X$ is

$$d(A) = \sup_{x,y \in A} d(x, y)$$

i.e. the supremum or least upper bound* of the distances between pairs of points in A (which may be finite or infinite. By convention, take $d(\phi) = -\infty$).

A is *bounded* if $|d(A)| < \infty$.

THEOREM 3.2: *In a metric space the points of a convergent sequence form a bounded set.*

Proof. Let $x_n \to x$ in (X, d), then there exists $N \in \mathbb{N}$ such that $d(x_n, x) < 1$ all $n > N$.

So, let $r = \text{Max } \{d(x_1, x), d(x_2, x), \ldots, d(x_N, x), 1\}$ ($<\infty$, why?)

then $d(x_n, x) \leqslant r$ for all $n \in \mathbb{N}$

whence $d(x_n, x_m) \leqslant d(x_n, x) + d(x, x_m)$

$\leqslant 2r$ for all $n, m \in \mathbb{N}$

and so $d(\{x_n : n = 1, 2, \ldots\}) \leqslant 2r$

i.e. the points of the sequence form a bounded set of diameter less than or equal to 2r.

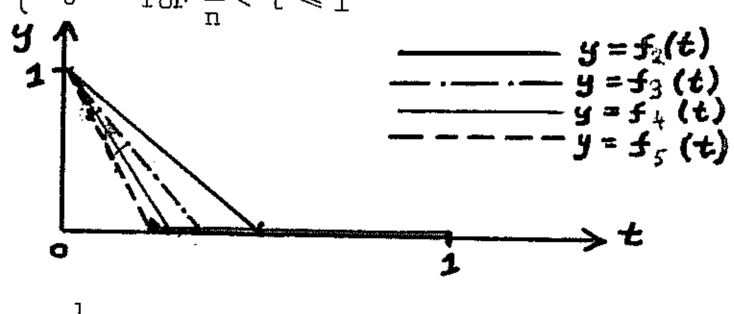
■

REMARK. The property of convergence is not inherent in a sequence but depends on both X and d.

EXAMPLES. (1) Dependence on d.

Take X = C[0, 1] and for each $n \in \mathbb{N}$ define $f_n \in X$ by

$$f_n(t) = \begin{cases} 1 - nt & \text{for } 0 \leqslant t \leqslant \frac{1}{n} \\ 0 & \text{for } \frac{1}{n} < t \leqslant 1 \end{cases}$$



Then $\|f_n - 0\|_1 = \int_0^1 f_n(t)dt = \frac{1}{2n} \to 0$

so $f_n \to 0$ in $(C[0, 1], d_1)$.

---

* We cannot use maximum as the diameter may not be attained. For example the open interval (0, 1) has diameter 1 w.r.t. the usual metric on R but there is no pair of points $x, y \in (0, 1)$ with $|x - y| = 1$.

However $\|f_n - 0\|_\infty = \max_{t\in[0,1]} |f_n(t)| = 1$

so $f_n \not\to 0$ in $(C[0, 1], d_\infty)$.

(2) Dependence on X.

For X = R and d_1 the usual metric on R,

$x_n = \frac{1}{n} \to 0 \in X$.

However if we take X = (0, 1], $x_n = \frac{1}{n}$ does not converge, as the point toward which the sequence is tending (0) is not a member of X. In a more general situation it may be difficult to identify the "missing" limit point and so this can represent a real problem.

We investigate one property of convergent sequences which is independent of X (but not of d).

DEFINITION. A sequence $\{x_n\}$ of the metric space (X, d) is a *Cauchy Sequence* if given $\varepsilon > 0$ there exists an $N \in N$ such that

$$m, n \geqslant N \Rightarrow d(x_m, x_n) < \varepsilon,$$

i.e. points in the tail of the sequence become arbitrarily 'close' together.

EXAMPLE. For X = (0, 1] with the usual metric $d_1(x, y) = |x - y|$ the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots$ is a Cauchy sequence (although from above it is not a convergent sequence. Thus not every Cauchy sequence need be a convergent sequence. To see that $\{\frac{1}{n}\}_{n=1}^{\infty}$ is a Cauchy sequence, note that $|\frac{1}{n} - \frac{1}{m}| < \varepsilon$ whenever $m, n > [\frac{1}{\varepsilon}]$.

THEOREM 3.3: *Every convergent sequence of a metric space is a Cauchy sequence.*

Proof. Let $x_n \to x$ in (X, d), then given $\varepsilon > 0$ there exists $N \in N$ such that

$$d(x_n, x) < \frac{\varepsilon}{2}, \text{ all } n \geqslant N$$

whence, for $m, n \geqslant N$ we have

$$d(x_n, x_m) \leqslant d(x_n, x) + d(x, x_m) \quad \text{(triangle inequality)}$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

as required. ■

DEFINITION. The metric space (X, d) is *complete* if every Cauchy sequence is a convergent sequence i.e. $(d(x_n, x_m) \to 0$ as $n, m \to \infty) \Rightarrow$ (there exists $x \in X$ such that $x_n \to x$).

We will single out the class of complete metric spaces for a more intensive study in later lectures. We now investigate some of our example spaces for completeness.

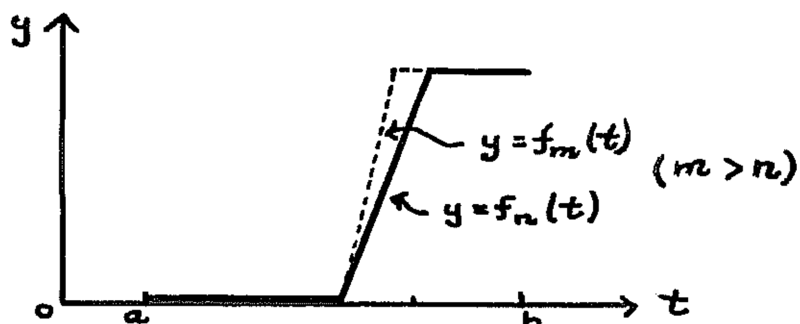
EXAMPLES.

(1) By assumption (or construction) (R, d_1) and (R, d_2) are complete (as is (C, d_2)).

(2) $(C[a, b], d_1)$ is not complete

Proof. The sequence defined by

$$f_n(t) = \begin{cases} 0 \text{ for } a \leqslant t \leqslant \frac{b + a}{2} \\ n(t - \frac{a + b}{2}) \text{ for } \frac{b + a}{2} < t < \frac{b + a}{2} + \frac{1}{n} \\ 1 \text{ for } \frac{b + a}{2} + \frac{1}{n} \leqslant t \leqslant b \end{cases}$$



is a Cauchy sequence, since for $m > n$

$$d_1(f_n, f_m) = \|f_n - f_m\|_1$$

$$= \int_{\frac{a+b}{2}}^{\frac{a+b}{2}+\frac{1}{n}} m(t - \frac{b + a}{2}) - n(t - \frac{b + a}{2})dt$$

$$\leqslant \frac{1}{2n} \to 0 \text{ as } m, n \to \infty.$$

However $f_n \to f$ where $f(x) = \begin{cases} 0 & a \leqslant t \leqslant \frac{b + a}{2} \\ 1 & \frac{b + a}{2} < t \leqslant b \end{cases}$

and $f \notin C[a, b]$ (f is discontinuous at $\frac{b + a}{2}$).

[Note also, it is necessary to show $f_n \xrightarrow{d_1} f$ where $f_n, f \epsilon C[a,b] \Rightarrow f_n(x) \to f(x) \forall x \epsilon [a,b]$, before the proof given here is complete.

(3) $(C[a, b], d_\infty)$ is complete, hence the importance of the uniform metric

$$d_\infty(f, g) = \|f - g\|_\infty = \underset{x\epsilon[a,b]}{\text{Max}} |f(x) - f(x)|$$

To see this it suffices to know that a uniform limit of continuous functions is continuous (see appendix to this lecture).

(4) From the completeness of R follows the completeness of the spaces ℓ_p^n ($p = 1, 2, \infty$ and $n \in N$).

The completeness of ℓ_2^n is a consequence of the following result. A similar analysis establishes the other two cases.

PROPOSITION. *The sequence of points*

$$\underset{\sim}{x}_k = (x_{k1}, x_{k2}, \ldots, x_{kn}) \in \ell_2^n \quad (k = 1, 2, 3, \ldots)$$

is a Cauchy sequence if and only if for each $i \in \{1, 2, \ldots, n\}$ *the sequence of real numbers* $\{x_{ki}\}_{k=1}^{\infty}$ *is convergent to some* $x_i \in R$ *under the usual metric, and then* $\underset{\sim}{x}_k \to \underset{\sim}{x} = (x_1, x_2, \ldots, x_n)$ in ℓ_2^n.

(<u>i.e.</u> convergence in ℓ_2^n is equivalent to component-wise convergence in (R, d_1).)

Proof. ($\Leftarrow$) We in fact show that $x_{ki} \to x_i$ as $k \to \infty$ ($i = 1, 2, \ldots, n$) implies $\underset{\sim}{x}_k \to \underset{\sim}{x}$, and so by Theorem 3.3 $\{\underset{\sim}{x}_k\}$ is certainly a Cauchy sequence.

Now, given $\varepsilon > 0$, for each i there exists N_i such that

$$k \geqslant N_i \Rightarrow |x_{ki} - x_i| < \varepsilon/\sqrt{n}.$$

So taking $N = \text{Max}\,\{N_1, N_2, \ldots, N_n\}$ (Note: for N to exist we require n to be finite) we have

$$|x_{ki} - x_i| < \varepsilon/\sqrt{n} \quad \text{for all } i \text{ whenever } k \geqslant N$$

and so

$$d_2(\underset{\sim}{x}_k, \underset{\sim}{x}) = \|\underset{\sim}{x}_k - \underset{\sim}{x}\|_2 = \left(\sum_{i=1}^{n} (x_{ki} - x_i)^2\right)^{\frac{1}{2}}$$

$$\leqslant \left(\sum_{i=1}^{n} (\varepsilon/\sqrt{n})^2\right)^{\frac{1}{2}} \quad \text{for } k \geqslant N$$

$$= \left(n\frac{\varepsilon^2}{n}\right)^{\frac{1}{2}} = \varepsilon \quad \text{for } k \geqslant N.$$

and so $\underset{\sim}{x}_k \to \underset{\sim}{x}$ as $k \to \infty$.

($\Rightarrow$) If $\{\underset{\sim}{x}_k\}$ is a Cauchy sequence, then

$$d_2(\underset{\sim}{x}_k, \underset{\sim}{x}_j) = \|\underset{\sim}{x}_k - \underset{\sim}{x}_j\|_2 = \left(\sum_{i=1}^{n} (x_{ki} - x_{ji})^2\right)^{\frac{1}{2}} \to 0$$

as $j, k \to \infty$. Hence for each $i \in \{1, 2, \ldots, n\}$ $|x_{ki} - x_{ji}| \to 0$ as $k, j \to \infty$ <u>i.e.</u> for each i $\{x_{ki}\}_{k=1}^{\infty}$ is a Cauchy sequence of real numbers and so, by the completeness of R converges to some $x_i \in R$.

But, then, by the first part of the proof

$$\underset{\sim}{x}_k \to \underset{\sim}{x} = (x_1, x_2, \ldots, x_n) \text{ (and so } \ell_2^n \text{ is complete).} \quad \blacksquare$$

Collateral Reading.

Giles *op cit* Ch. 2 pp. 45 to 50 excluding Theorems 2 and 3.

Simmons *op cit* Ch. 2, sec. 12, pp. 70-72.

PROBLEMS.

1. Show that in $(C[0, 1], d_\infty)$ the sequence of Taylor polynomials $p_n(x) = \sum_{m=0}^{n} x^m/m!$ is convergent to exp. Is the same true in $(C[0, 1], d_1)$? (See problem 2.3.)

2. Let $x_n \to x$ and $y_n \to y$ in the metric space (X, d). Show that $d(x_n, y_n) \to d(x, y)$ in (R, d_1).

*3. Show that in a metric space (X, d), where d is the discrete metric, a convergent sequence $\{x_n\}$ has only a finite number of points in its range.

4. Show that (X, d), where d is the discrete metric, is a complete metric space.

5. Using the discrete metric, give a further example to show that convergence depends on the choice of metric.

*6. Show that a Cauchy sequence in any metric space (X, d) is convergent if and only if it has a convergent subsequence.

7. In some metric space (X, d) give a <u>counter example</u> to the proposition:
   If the sequences $\{x_n\}$ and $\{y_n\}$ are such that $\{d(x_n, y_n)\}$ is convergent then $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences.

*8. Let $(X^{(1)}, d^{(1)})$ and $(X^{(2)}, d^{(2)})$ be two complete metric spaces, show that $X^{(1)} \times X^{(2)}$ is a complete metric space under the two metrics of problem 1.4.

*9. Prove that Hilbert space, ℓ_2, (see problem 2.5) is complete.

<u>NOTES</u> on lecture 3.

1. The spaces ℓ_p^n $(p \geqslant 1, n \in N)$ can all be shown to be complete.
2. None of the spaces $(C[a, b], d_p)$ $(1 \leqslant p < \infty)$ are complete. The problem of "adding" in additional functions (and extending the definition of the metric to cover these new functions) so as to "complete" these spaces is a major motivation for Lebesgue's theory of integration and measure.
3. For any metric space (X, d) it is possible to find a minimal complete super-space $(\tilde{X}, \tilde{d})$ known as the <u>completion</u> of (X, d). (<u>i.e.</u> $X \subseteq \tilde{X}$ and $\tilde{d}(x, y) = d(x, y)$ for all $x, y \in X$.)
   One construction of $(\tilde{X}, \tilde{d})$ from (X, d), due to Cauchy, allows, as a special case, the real numbers R to be axiomatically derived from the rational numbers Q.

4. For a detailed discussion of the many different types of convergence possible in function spaces the interested reader is referred to Korevaar, "Mathematical Methods", Acad. Press, Part 2, section 1.1, p. 162.

APPENDIX to lecture 3.

UNIFORM CONVERGENCE in C[a, b].

Let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $(C[a, b], d_\infty)$.

i.e. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d_\infty(f_n, f_m) = \|f_n - f_m\|_\infty = \max_{x\in[a,b]} |f_n(x) - f_m(x)| < \varepsilon$$

whenever $n, m \geqslant N$.

Consequently, for each $x \in [a, b]$

$$|f_n(x) - f_m(x)| < \varepsilon \text{ for } n, m \geqslant N,$$

i.e. $\{f_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence in the complete metric space (R, d_1) and so is convergent to some unique limit which we choose to denote by $f(x)$.

Define a function f on $[a, b]$ by $x \mapsto f(x)$. Then, for every $x \in [a, b]$ $f_n(x) \overset{d_1}{\to} f(x)$ and we say f is the point wise limit of the sequence $\{f_n\}$. (This type of convergence - point wise convergence - is important in real analysis but peripheral to metric analysis, since it does not represent convergence with respect to any of the metrics on C[a, b].)

[In general point wise convergent is 'weaker' than uniform convergence (convergence with respect to the uniform metric d_∞) i.e.

$$(\text{uniform convergence}) \Rightarrow (\text{point wise convergence})$$

however, it may happen that $f_n \to f$ point wise but $f_n \not\to f$ uniformly. Give an example illustrating this (Hint: see example 2, p. 17 of lecture 3).]

Because of the particular construction of f above,

($f(x) = \lim\limits_{m\to\infty} f_m(x)$ where $\{f_m\}$ is a Cauchy sequence in $(C[a, b], d_\infty)$),

we have, for $n, m \geqslant N$

$$|f_n(x) - f_m(x)| < \varepsilon \text{ for all } x \in [a, b]$$

and so

$$\lim_{m\to\infty}|f_n(x) - f_m(x)| = |f_n(x) - \lim_{m\to\infty} f_m(x)| = |f_n(x) - f(x)| < \varepsilon$$

for all $x \in [a, b]$ and $n \geqslant N$.

Whence $\max\limits_{x\in[a, b]}|f_n(x) - f(x)| < \varepsilon$ whenever $n \geqslant N$, or f_n converges uniformly to f

i.e. $d_\infty(f_n - f) \to 0$.

We now show $f \in C[a, b]$ and so establish the completeness of $(C[a, b], d_\infty)$. To do this we must show f is continuous, i.e. given $\varepsilon > 0$ and any $x_0 \in [a, b]$ we must find $\delta > 0$ such that

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon \quad (x \in [a, b]).$$

Now

$$\begin{aligned}|f(x) - f(x_0)| &= |f(x) - f_n(x) + f_n(x) - f_n(x_0) + f_n(x_0) - f(x_0)| \\ &\leqslant |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|\end{aligned}$$

and since $f_n \overset{d_\infty}{\rightarrow} f$ there exists $N \in \mathbb{N}$ such that

$$n \geqslant N \Rightarrow |f(x) - f_n(x)|,\ |f_n(x_0) - f(x_0)| \leqslant d_\infty(f_n, f) < \frac{\varepsilon}{3}.$$

So for any fixed $n > N$

$$|f(x) - f(x_0)| \leqslant \frac{\varepsilon}{3} + |f_n(x) - f_n(x_0)| + \frac{\varepsilon}{3}$$

but $f_n \in C[a, b]$ so there exists $\delta > 0$ such that

$$|x - x_0| < \delta \Rightarrow |f_n(x) - f_n(x_0)| < \frac{\varepsilon}{3}$$

and so for this δ we have $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| \leqslant \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$, as required, to show f is continuous.

TOPOLOGY OF METRIC SPACES

Lecture 4 *Open Sets*

DEFINITION. For (X, d) a metric space, $x_0 \in X$ and $0 < r \in R$, the set

$$\underline{B_r(x_0)} \equiv \{x \in X: d(x_0, x) < r\}$$

is the *open ball with centre* x_0 *and radius* r.

NOTE. In some books (e.g. Simmons) open balls are termed open spheres, while others (e.g. Giles) denote them by $B(x_0, r)$.

EXAMPLES. (1) In $\ell_2^3 = (R^3, d_2)$,

$$B_r(\underset{\sim}{x}_0) \equiv \{\underset{\sim}{x} \in R^3: d_2(\underset{\sim}{x}_0, \underset{\sim}{x}) < r\}$$

$$= \{(x_1, x_2, x_3): (x_1 - x_{01})^2 + (x_2 - x_{02})^2 + (x_3 - x_{03})^2 < r^2\}$$

accords precisely with our intuitive idea of such an object, being the set of points enclosed within the spherical shell of radius r centred at $\underset{\sim}{x}_0 = (x_{01}, x_{02}, x_{03})$.

(2) In $\ell_2^2 = (R^2, d_2)$ [or (C, d_2)],

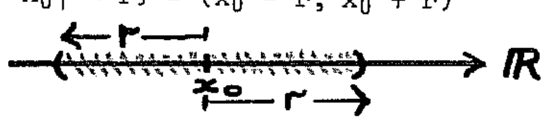
$$B_r(\underset{\sim}{x}_0) = \{(x_1, x_2): (x_1 - x_{01})^2 + (x_2 - x_{02})^2 < r^2\}$$

is the circular disc (excluding the rim) of radius r and centre (x_{01}, x_{02}).

[This provides a convenient and serviceable pictorial representation of a general open ball.]

(3) In (R, d_1) open balls correspond to open intervals,

$$B_r(x_0) = \{x \in R: |x - x_0| < r\} = (x_0 - r, x_0 + r)$$



(4) For any set X with the discrete metric d

$$B_r(x_0) = \begin{cases} \{x_0\} & \text{if } r \le 1 \\ X & \text{if } r > 1 \end{cases} \qquad \text{(prove this)}$$

NOTE. In any metric space (X, d) an open ball, $B_r(x_0)$, is never empty, since $x_0 \in B_r(x_0)$. (Why?)

DEFINITION. For a metric space (X, d), any subset, $A \subseteq X$ is an <u>*open set*</u> if it is a union of open balls,

<u>i.e.</u> $A = \bigcup_{\lambda \in \Lambda} B_{r_\lambda}(x_\lambda)$, for some sets of points $\{x_\lambda: \lambda \in \Lambda\} \subseteq X$ and real numbers $\{r_\lambda: \lambda \in \Lambda\}$ (Λ is a suitable index set).

OBSERVATION. An open ball is an open set, since

$$B_r(x_0) = \cup\{B_r(x_0)\}, \text{ where } \{B_r(x_0)\}$$

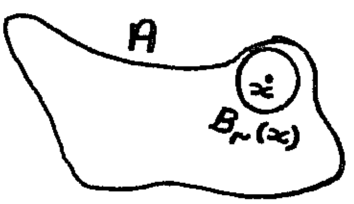
is the singleton set whose only element is $B_r(x_0)$.

EXAMPLE. For any set X, with the discrete metric d, every subset $A \subseteq X$ is open, since

$$A = \bigcup_{a \in A} \{a\} \text{ and } \{a\} = \{x \in X: d(a, x) < \tfrac{1}{2}\} = B_{\frac{1}{2}}(a).$$

We develop some alternative characterizations of open sets, which will prove to be of subsequent use.

DEFINITION. In the metric space (X, d) the point $x \in A \subseteq X$ is an *interior point* of the set A if there exists an $r_x > 0$ such that $B_{r_x}(x) \subseteq A$, i.e. if there is an open ball centred on x which is contained entirely within A.



The set of all interior points of A is the *interior of* A, which we shall denote by Int A.

A useful concept in metric analysis is given by -

DEFINITION. In the metric space (X, d) the set $A \subseteq X$ is a *neighbourhood* (n'hood) of the point $x \in X$ if $x \in \text{Int } A$.

Further, if A is an open set we say A is an *open neighbourhood* of x.

THEOREM 4.1: *In any metric space* (X, d) *the following are equivalent statements about the set* $A \subseteq X$;

(i) *for every* $x \in A$ *there exists* $r_x > 0$ *such that* $B_{r_x}(x) \subseteq A$,

(ii) *every point of* A *is an interior point of* A,

(iii) A *is a neighbourhood of each of its points*

(iv) Int A = A

(v) *A is open.*

Proof. It is immediate from the definitions that (i) ⇔ (ii) ⇔ (iii) ⇔ (iv), it is therefore sufficient to show (i) ⇔ (v).

(i) ⇒ (v): Form (i) for each $x \in A$ we have $x \in B_{r_x}(x) \subseteq A$, thus $A \subseteq \bigcup_{x \in A} B_{r_x}(x) \subseteq A$, so A is a union of open balls and therefore open.

To prove (v) ⇒ (i) we first establish,

LEMMA. *If* $x \in B_r(x_0)$ *- an open ball in the metric space* (X, d) *- , then* $x \in \text{Int } B_r(x_0)$ i.e. *there exists* $r_x > 0$ *such that* $B_{r_x} \subseteq B_r(x_0)$, which establishes the result for open balls.

Proof (of lemma). Since $x \in B_r(x_0)$, $d(x, x_0) < r$ set $r_x = r - d(x, x_0)$ (> 0), then for any $y \in B_{r_x}(x)$ $d(y, x_0) \leqslant d(y, x) + d(x, x_0)$

$$< r - d(x, x_0) + d(x, x_0)$$

$$= r \;, \quad \text{so } y \in B_r(x_0), \text{ all } y \in B_{r_x}(x),$$

whence $$B_{r_x}(x) \subseteq B_r(x_0) \text{ as required.}$$

□

Returning to the proof of (v) ⇒ (i), since A is open $A = \bigcup_{\lambda\in\Lambda} B_\lambda$ for some family of open balls $\{B_\lambda : \lambda \in \Lambda\}$ and so if $x \in A$, $x \in B_\lambda$ for some $\lambda \in \Lambda$. Whence by the lemma, there exists $r_x > 0$ such that $B_{r_x}(x) \subseteq B_\lambda \subseteq A$, as required to complete the proof of the theorem.

■

FUNDAMENTAL PROPERTIES OF OPEN SETS.

THEOREM 4.2 (The 'algebra' of open sets):

Let (X, d) *be a metric space, then*

(i) ϕ *and* X *are open sets*

(ii) *A union of open sets is an open set*, and

(iii) *the intersection of a finite number of open sets is an open set.*

Proof.

(i) Since ϕ contains no points, it is clear that every point of ϕ is the centre of an open ball contained in ϕ.
While, for any $x \in X$, $B_r(x) \subseteq X$ for every $r > 0$, thus $X = \bigcup_{x\in X} B_1(x)$ and so X is open.

(ii) Let $\{G_\lambda : \lambda \in \Lambda\}$ be a family of open sets of (X, d).
Then, since each G_λ is open we have

$$G_\lambda = \bigcup_{\gamma\in\Gamma_\lambda} B_\gamma \text{ where } B_\gamma \text{ is an open ball (all } \gamma \in \Gamma_\lambda).$$

Thus, the union of open sets,

$$G = \bigcup_{\lambda\in\Lambda} G_\lambda = \bigcup_{\lambda\in\Lambda} \Big(\bigcup_{\gamma\in\Gamma_\lambda} B_\lambda\Big)$$

is a union of open balls and consequently is itself open.

(iii) Let $\{G_k : k = 1,2, \ldots, n\}$ be a finite family of open sets,
$x \in G = \bigcap_{k=1,2,\ldots,n} G_k \Rightarrow x \in G_k$ (all $k = 1, 2, \ldots n$) then, since each G_k is open, there exists $r_k > 0$ such that $B_{r_k}(x) \subseteq G_k$ $(k = 1,2,\ldots,n)$.
Let $r = \text{Min}\{r_1, r_2, \ldots, r_n\}$ (which exists and is strictly positive, since there are only a finite number of the r_k), then clearly $x \in B_r(x) \subseteq B_{r_k}(x)$ (all k) and so $B_r(x) \subseteq G = \bigcap_k G_k$.

(iii) (continued)

But x was any point of G, so G is open. ■

The finiteness condition in (iii) above cannot be dropped.

EXAMPLE.

$\{B_{\frac{1}{n}}(0): n \in N\}$ is an infinite family of open sets in (R, d_1) such that their intersection

$$\bigcap_{n \in N} B_{\frac{1}{n}}(0) = \{0\} \qquad \text{(prove)}$$

which is not an open set (prove).

The next theorem characterizes the interior of any set.

THEOREM 4.3: *For* $A \subseteq X$, (X, d) *a metric space, the interior of* A, Int A *is the 'largest' open set contained in* A, i.e. if G is any open set contained in A, then $G \subseteq \text{Int } A$.

Proof. We must first show that Int A is open for any set $A \subseteq X$. Accordingly, $x \in \text{Int } A \Rightarrow$ there exists $r_x > 0$ such that $B_r(x) \subseteq G \subseteq A$. Now $B_{r_x}(x)$ is an open set so for $y \in B_{r_x}(x)$ there exists $r_y > 0$ such that $B_{r_y}(y) \subseteq B_{r_x}(x)$ which is contained in A. Thus each $y \in B_{r_x}(x)$ is an interior point of A or $B_{r_x}(x) \subseteq \text{Int } A$. *But* x was an arbitrary point of Int A so every point of Int A is the centre of an open ball $(B_{r_x}(x))$ contained in Int A which is therefore open. Now, let $G \subseteq A$ be an open subset of A, then $x \in G \Rightarrow$ there exists $r_x > 0$ such that $B_{r_x}(x) \subseteq B \subseteq A \Rightarrow x \in \text{Int } A$ so $G \subseteq \text{Int } A$, establishing the maximality of Int A.

Collateral Reading.

Giles *op cit* Ch. 1, sec. 1.3, pp. 32-39 (exclude Theorems 5 and 8).

(Sec. 1.2 may also prove interesting reading.)

Simmons *op cit* Ch 2, sec. 10, pp. 59-64.

PROBLEMS.

1. Sketch the following open balls
   (i) $B_1(\underset{\sim}{0})$ in (R^2, d_1)
   (ii) $B_1(\underset{\sim}{0})$ in (R^2, d_∞).

2. (a) Show that the following is a metric on R^2.
   $d: R^2 \times R^2 \to R$ defined by

$$d(\underset{\sim}{x}, \underset{\sim}{y}) = \begin{cases} \|\underset{\sim}{x}\|_2 + \|\underset{\sim}{y}\|_2 & \text{if } \underset{\sim}{x} \neq \underset{\sim}{y} \\ 0 & \text{if } \underset{\sim}{x} = \underset{\sim}{y} \end{cases}$$

2. (a) (continued)

(Recall, for $x = (x_1, x_2) \in R^2$, $\|\underset{\sim}{x}\|_2 = \sqrt{x_1^2 + x_2^2}$.)
This metric is known as the POST OFFICE METRIC (can you see why?).

(b) For d as in part (a), sketch the open balls $B_1(\underset{\sim}{0})$ and $B_{\frac{1}{2}}((1,1))$ in (R^2, d).

3. Let A be a subset of the metric space (X, d) such that the diameter of A, d (A), is less than r. If $A \cap B_r(x) \neq \phi$ show $A \subseteq B_{2r}(x)$.

4. <u>The 'algebra' of interiors</u>
In a metric space (X, d) with $A, B \subseteq X$ prove
(i) if $A \subseteq B$ then Int $A \subseteq$ Int B;
(ii) Int $(A \cap B)$ = (Int A) $\cap$ (Int B);
(iii) Int $(A \cup B) \supseteq$ (Int A) $\cup$ (Int B);
(iv) Construct a counter-example to show that the reverse inclusion to that of part (iii) need not hold in general.

5. Prove that in any metric space the complement of any singleton set is open, and hence or otherwise show that the complement of any finite set is open.

6. In a given metric space (X, d) prove that every subset of X is open if and only if every singleton set is open.

*7. (a) Show that the singleton subsets of any non-trivial normed linear space cannot be open sets with respect to the metric induced by the norm.

(b) Show that except for $\{\underset{\sim}{0}\}$ all other singleton sets of R^2 are open with respect to the Post Office Metric of problem 2(a).
Hence, conclude that there is no norm on R^2 which induces the Post Office Metric.

*8. Two metrics d, d' on the set X are said to be <u>equivalent</u> metrics if they give rise to the same family of open sets.
Prove that for any metric space (X, d) d and d^* are equivalent metrics, where $d^*(x, y) = \frac{d(x, y)}{1 + d(x, y)}$ (see problem 1.3).

<u>NOTES</u> on lecture 4

1. It varies from author to author whether our definition or one of the equivalent statements (i) - (iv) of theorem 4.1 is used as the definition of open set. Thus, Giles takes (i) to be his definition.

2. Any family of subsets T of the set X which have the properties (i), (ii) and (iii) of theorem 4.2,

   (i.e. (i) $\phi, X \in T$;

   (ii) $\bigcup_{\lambda \in \Lambda} T_\lambda \in T$ whenever $T_\lambda \in T$, all $\lambda \in \Lambda$;

   (iii) $\bigcap_{\lambda \in F} T_\lambda \in T$ whenever $T_\lambda \in T$, all $\lambda \in F$ - a finite index set)

   is termed a TOPOLOGY for X, and X equipped with T is called a *Topological space*. Theorem 4.2 shows that the family of open subsets of a metric space is a Topology for the space X.

   In general, however, there are topologies which do not coincide with the family of open sets generated by any possible metric on X. Thus topological spaces are more general than metric spaces and their study forms an important branch of modern mathematics. The question of characterizing those topologies which do arise from a metric is known as the *metrization problem*.

SUPPLEMENT to lecture 4.

A CHARACTERIZATION OF OPEN SETS in (R, d_1)

THEOREM 4.4: *In* (R, d_1) *every open set is the union of a countable family of disjoint open intervals.*

[Note. Since open intervals correspond to open balls, it is true by definition that every open set is a union of open intervals.]

Proof. Take G an open subset of (R, d_1) and x any point in G. Let I_x equal the union of all open intervals (open balls) which contain x and are contained in G.

Then

(i) $I_x \neq \phi$, since G is open and so there exists an $r_x > 0$ such that $x \in B_{r_x}(x) \subseteq G$, i.e. $(x - r_x, x + r_x) \subseteq I_x$.

(ii) Clearly I_x is an open set (why?). In fact I_x is an open interval. To show this, it suffices to prove $(a, b) \subseteq I_x$ whenever $a < b$ and $a, b \in I_x$. Now, if $a, b \in I_x$ then, by the definition of I_x, there exists open intervals (c, d) and (c', d') in I_x such that $x \in (c, d) \cap (c', d')$ with $a \in (c, d)$ and $b \in (c', d')$ hence $a, b \in (c, d) \cup (c', d') \subseteq I_x$ but $(c, d) \cup (c', d')$ is an open interval and so $(a, b) \subseteq I_x$.

(iii) <u>If $y \in I_x$ then $I_y = I_x$</u>, for I_x is an open interval (by (ii)) containing y and contained in G, so by definition of I_y, $I_x \subseteq I_y$ and similarly $I_y \subseteq I_x$.

(iv) <u>For x, y $\in$ G, either $I_x = I_y$ or $I_x \cap I_y = \phi$.</u>

Assume $z \in I_x \cap I_y$ then $z \in I_x$ so by (iii) $I_x = I_z$ similarly $z \in I_y$ so $I_y = I_z$, whence $I_x = I_y$. We have therefore proved that the family of sets $\{I_x : x \in G\}$ is a family of disjoint open intervals and clearly

$$G = \bigcup_{x \in G} I_x \text{ (as } x \in I_x \subseteq G \text{ for all } x \in G)$$

it therefore only remains to prove that there are only a countable number of distinct I_x's.

Let $Q_G = Q \cap G$ (the <u>countable</u> set of rational numbers in G), define $f: Q_G \to \{I_x : x \in G\}$ by $f(q) = I_q$ (which is unique by (iv)), then clearly f is onto, since each I_x, being an interval, contains a rational point q_x and so $I_x = I_{q_x} = f(q_x)$ whence $\{I_x : x \in G\}$ is countable. ∎

See Giles, p. 37 for an alternative $f: Q \overset{\text{onto}}{\to} \{I_x : x \in G\}$, and Ch. 0, pp. 12-14 for work on countable (denumerable) sets.

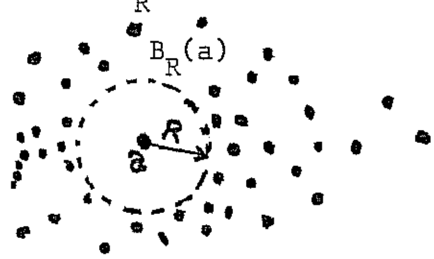
Lecture 5. *Cluster points and Closed Sets.*

DEFINITION. In a metric space (X, d) with $A \subseteq X$, $x \in X$ is a ***cluster point*** of A if each open ball centred on x contains a point of A distinct from x, i.e. if $(B_r(x)\backslash\{x\}) \cap A \neq \phi$ for all $r > 0$.

NOTES. (1). It is not necessary that $x \in A$, *for example,* in (R, d_1), 0 is a cluster point of $\{\frac{1}{n}: n \in N\}$ (prove), alternatively every real number is a cluster point of Q.

(2). Cluster points are sometimes termed limit points (e.g. Simmons) or points of accumulation.

DEFINITION. For $A \subseteq X$, (X, d) a metric space, the *derived set* of A, denoted by A' is the set of all cluster points of A.
$a \in A\backslash A'$ is termed an *isolated* point of A.
Clearly $a \in A$ is an isolated point of A (i.e. not a cluster point) iff there exists some $R > 0$ such that $B_R(a) \cap A = \{a\}$



(dots represent points of A)

THEOREM 5.1: *If $x \in A'$ for some set $A \subseteq X$, (X, d) a metric space, then for any $r > 0$, $B_r(x)$ contains an infinite number of (distinct) points of A.*

Proof. Assume there is some $R > 0$ such that $B_R(x) \cap A$ contains only a finite number of points $a_1, a_2, \ldots, a_n$ and possibly x, take

$$r = \min_{i=1,2,\ldots,n} \{d(a_i, x)\} \quad \text{then}$$

$$a_1, a_2, \ldots, a_n \notin B_r(x) \cap A \subseteq B_R(x) \cap A \subseteq \{a_1, a_2, \ldots, a_n, x\}$$

so $(B_r(x)\backslash\{x\}) \cap A = \phi$ and x is not a cluster point of A, contradicting our hypothesis, so no such R can exist. ■

COROLLARY. *Any finite subset of a metric space has no cluster points and so consists solely of isolated points.*

There is an intimate connection between the notion of a cluster point and the idea of convergent sequence. Roughly $(a \in A') \Rightarrow$ (there exists a sequence $\{a_n\}$ of A with $a_n \to a$). The unqualified converse of this is not true, *for example*, let A be the singleton set $\{1\}$ then in (R, d_1) the sequence $1, 1, 1, \ldots, 1, \ldots$ converges to 1 but 1 is not a cluster point of A (why?).

The situation is made precise by

THEOREM 5.2: *Let* $A \subseteq X$ *and* (X, d) *be a metric space, then* $x \in A'$ *if and only if there exists a sequence* $\{a_n\}$*, with* $a_n \in A\backslash\{x\}$ for all $n \in N$, *such that* $a_n \overset{d}{\to} x$.

Proof. ($\Leftarrow$) Given any $r > 0$ there exists $N \in N$ such that $d(a_n, x) < r$ for all $n \geqslant N$ (definition of convergence) so $a_n \in B_r(x)\backslash\{x\}$ for all $n \geqslant N$

<u>i.e.</u> $(B_r(x)\backslash\{x\}) \cap A \neq \phi$ and so $x \in A'$.

($\Rightarrow$) We construct the sequence $\{a_n\}$ inductively as follows.

Since $x \in A'$, $(B_1(x)\backslash\{x\}) \cap A \neq \phi$ so we may choose an $a_1 \in (B\ (x)\backslash\{x\}) \cap A$. Now, assume there exist $a_1, a_2, \ldots, a_n \in A$ such that

$$d(a_j, a) < \text{Min}\{d(a_{j-1}, a), \tfrac{1}{j}\} \qquad j = 1, 2, \ldots, n$$

then, setting $r = \text{Min}\{d(a_n, x), \frac{1}{n+1}\}$ we have $a_1, a_2, \ldots, a_n \notin B_r(x)$ so, as $x \in A'$, there exists an $a_{n+1} \in (B_r(x)\backslash\{x\}) \cap A$

<u>i.e.</u> $$d(a_{n+1}, a) < r < \text{Min}\{d(a_n, a), \tfrac{1}{n+1}\}$$

by choice, and so we have inductively constructed a sequence $\{a_n\}$ with $a_n \in A$ all n, $d(a_n, a) < \frac{1}{n}$, <u>i.e.</u> $a_n \to a$, and further $d(a_{n+1}, a) < d(a_n, a)$ so $a_{n+1} \neq a_n$ for any n, <u>i.e.</u> all the a_n are distinct and so at most one of them could equal x. Should this be the case that one term of the sequence could be deleted from it without effecting the convergence. Hence, in either case, the desired sequence has been constructed.

■

<u>CLOSED SETS</u>

DEFINITION. For the metric space (X, d) and $A \subseteq X$, A is a <u>*closed*</u> *set* if $A' \subseteq A$ <u>i.e.</u> A is a closed set if it contains all its cluster points.

The next theorem gives a powerful characterization of closed sets, as well as indicating the close connection between the open and closed sets in a metric space.

THEOREM 5.3: *In the metric space* (X, d) *the subset* $A \subseteq X$ *is closed if and only if its complement* $X\backslash A$ *is open.*

Proof. ($\Rightarrow$) $x \in X\backslash A \Rightarrow x \notin A'$ (as A is closed, so $A' \subseteq A$) $\Rightarrow$ there exists an $r_x > 0$ s.t.

$B_{r_x}(x) \cap A = \phi$, whence $B_{r_x}(x) \subseteq X\backslash A$ and so, since x is an arbitrary point of $X\backslash A$ the set is open (by Theorem 4.1 (i)).

(⇐) if $x \in X\backslash A$ which is open, then there exists an $r_x > 0$ such that $B_{r_x}(x) \subseteq X\backslash A$ so $B_{r_x}(x) \cap A = \phi$ whence $x \notin A'$.

Thus $A' \cap (X\backslash A) = \phi$ or $A' \subseteq A$, and so A is closed. ∎

REMARKS. 1. For practical purposes the style of argument used in the above proof is almost as important as the result itself.

2. In the more general context of topological spaces (see notes to lecture 4) and even in some works on metric spaces the result of the above theorem is taken as a definition for closed sets, i.e. a set is closed, by definition, if its complement is open, in which case the above theorem becomes a structure theorem for closed sets in terms of cluster points and would read,"a set is closed iff it contains all its cluster points".

OBSERVATION (very important). While the above theorem establishes an intimate relationship between open and closed sets the two concepts are not mutually exclusive.

i.e. Any of the following can happen.

(i) A is open but not closed (e.g. (a, b) in (R, d_1));

(ii) A is closed but not open (e.g. $[a, b]$ in (R, d_1));

(iii) A is neither open or closed (e.g. $[a, b)$ in (R, d_1));

(iv) A is both open and closed (e.g. every subset of any set equipped with the discrete metric is both open and closed - prove).

Thus, in general, from a knowledge that A is open (closed) nothing can be inferred as to whether or not it is closed (open).

THEOREM 5.4: (The 'algebra' of closed sets).

Let (X, d) be a metric space, then

(i) ϕ and X are closed;

(ii) An intersection of closed sets is a closed set;

(iii) The union of a finite number of closed sets is a closed set.

Proof. Throughout we use the characterization of closed sets given by Theorem 5.3 in conjunction with Theorem 4.2, of which this theorem is the analogue for closed sets.

(i) $X \backslash \phi = X$ is open, so ϕ is closed

similarly $X\backslash X = \phi$ is open, so X is closed.

(ii) Let $\{F_\alpha : \alpha\in\Lambda\}$ be a family of closed sets, then

$$F = \bigcap_{\alpha\in\Lambda} F_\alpha = X \backslash \bigcup_{\alpha\in\Lambda}(X/F_\alpha) \quad \text{(deMorgan's Theorem)}$$

(ii) *but*, $X\backslash F_\alpha$ is open, and so $\bigcup_{\alpha\in\Lambda} (X \backslash F_\alpha)$ is open, whence F is the complement of an open set and so F is closed.

(iii) Let $\{F_1, F_2, \ldots, F_n\}$ be a finite family of closed sets, then

$$F = \bigcup_{m=1}^{n} F_m = X\backslash \bigcap_{m=1}^{n} (X\backslash F_m) \qquad \text{(de Morgan)}$$

and so, since finite intersections of open sets are open $\bigcap_{m=1}^{n} (X\backslash F_m)$ is open, whence F is closed.

NOTE. As with finite intersections of open sets, the finiteness condition in (iii) cannot be dropped.

EXAMPLE. In (R, d_1)

$$\bigcup_{n=2}^{\infty}[\tfrac{1}{n}, 1 - \tfrac{1}{n}] = (0, 1) \qquad \text{(prove)}$$

which is not closed (prove).

THE CLOSURE OPERATION.

NOTATION. For any $A \subseteq X$, (X, d) a metric space, let $\underline{\overline{A} = A \cup A'}$ $\underline{\text{i.e.}}$ the points of $\overline{A}$ are the points of A together with the cluster points of A. Intuitively $\overline{A}$ consists of the points of A together with all the points arbitrarily close to A.

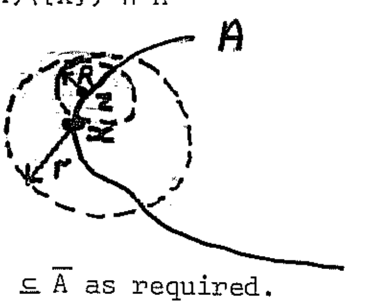
THEOREM 5.5: *For any metric space* (X, d) *and* $A \subseteq X$, $\overline{A}$ *is the smallest closed set containing* A.

Proof. We first show $\overline{A}$ is closed. Thus, suppose $x \in (\overline{A})'$ (we need to show $x \in \overline{A}$), then for any $r > 0$ $\exists$ $y \in (B_r(x)\backslash\{x\}) \cap \overline{A}$ so, as $y \in \overline{A}$, either $y \in A$ or $y \in A'$.

If $y \in A$, then $(B_r(x)\backslash\{x\}) \cap A \neq \phi$.

If $y \in A'$, there exists $z \in (B_R(y)\backslash\{y\}) \cap A \subseteq (B_r(x)\backslash\{x\}) \cap A$

where $R = \text{Min}\{r - d(x, y), d(x, y)\}$.



So in either case $(B_r(x)\backslash\{x\}) \cap A \neq \phi$ whence $x \in A' \subseteq \overline{A}$ as required.

Now assume F is a closed set containing A, consider $x \in X\backslash F$ an open set, so there exists $r > 0$ such that $B_r(x) \subseteq X\backslash F$, but $X\backslash F \subseteq X\backslash A$, $(A \subseteq F)$, so $B_r(x) \subseteq X\backslash A$ or $B_r(x) \cap A = \phi$ whence $x \notin \overline{A}$. Thus

$(X\backslash F) \cap \overline{A} = \phi$ (as x was any element of $X\backslash F$)

and so $\overline{A} \subseteq F$, establishing the minimality of $\overline{A}$.

Because $\overline{A}$ is the smallest closed set containing A we offer

DEFINITION. For any metric space (X, d) and $A \subseteq X$, the *closure of* A is $\overline{A} = A \cup A'$.

EXAMPLES. In (R, d_1) $\overline{(a, b)} = [a, b]$ (prove), and $\overline{Q} = R$.

The following concept, together with those of open set and interior, plays an important role in advanced calculus.

DEFINITION. For (X, d) a metric space and $A \subseteq X$, the *boundary of* A, denoted by bdry A is given by

$$\text{bdry } A = \overline{A} \cap \overline{(X \backslash A)}.$$

Thus bdry A consists of all those points which are arbitrarily close to both A and its complement $X\backslash A$.

EXAMPLES. In (R, d_1) the boundary of an open interval (a, b) is what we might expect, viz

$$\text{bdry } (a, b) = \{a, b\}$$

$$(= \overline{(a, b)} \cap \overline{(R\backslash(a, b))}$$

$$= [a, b] \cap \big((-\infty, a] \cup [b, \infty)\big)$$

as $R\backslash(a, b) = (-\infty, a] \cup [b, \infty)$

is the complement of an open set and so is closed.)

However this is not always the case, e.g.

$$\text{bdry } Q = \overline{Q} \cap \overline{(R\backslash Q)}$$

$$= R \cap R = R.$$

while, for $A \subseteq X$, X any set and d the discrete metric
bdry $A = \overline{A} \cap \overline{(X\backslash A)} = A \cap (X\backslash A) = \phi$.

Despite these observations the boundary of a set does behave in an intuitively pleasing way. For example in one of the problems you are asked to show that Int A $\cup$ bdry A = $\overline{A}$ and Int A $\cap$ bdry A = ϕ for any $A \subseteq X$, (X, d) any metric space.

Collateral Reading.

Giles *op cit* Ch. 2, sec. 2, p. 56 up to p. 64.

Simmons *op cit*

PROBLEMS.

1. Closed Balls

   In view of the definition of an open ball, $B_r(x)$, it seems natural to define a '*closed*' *ball* as the set

1. (continued)

$$B_r[x_0] = \{x \in X: d(x, x_0) \leqslant r\}$$

for any metric space (X, d).

(i) Show that $B_r[x_0]$ is indeed a closed subset of (X, d);

(ii) Give a counter-example to the "likely" proposition:

$$\overline{B_r(x_0)} = B_r[x_0].$$

2. ('Algebra' of closures)

Show that, for $A, B \subseteq X$ and (X, d) a metric space,

(i) $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$

(ii) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

(iii) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$, what can you say about the reverse inclusion?

3. For (X, d) a metric space and $A \subseteq X$, show

(i) $(\text{Int } A) \cap (\text{bdry } A) = \phi$, and

(ii) $\overline{A} = (\text{Int } A) \cup (\text{bdry } A)$.

4. DEFINITION. $A \subseteq X$ is a *dense* subset in the metric space (X, d) if $\overline{A} = X$. (E.g. Q is dense in R.)

Prove that the following statements are equivalent in (X, d)

(i) $A \subseteq X$ is dense;

(ii) the only closed superset of A is X;

(iii) the only open set disjoint from A is ϕ;

(iv) A has a non-trivial intersection with every non-empty open set of (X, d).

*5. Using the result of the supplement to lecture 4, characterize the closed subsets of (R, d_1).

**6. For a normed linear space, show that the only sets which are both open and closed, with respect to the metric induced by the norm, are the whole space and the empty set.

7. Let A be a closed subset in the complete metric space (X, d). Show that any Cauchy sequence $\{a_n\}$ with $a_n \in A$, for all $n \in N$, converges to a point of A.

Lecture 6. *Mappings between Metric Spaces, Continuity*

DEFINITION. A *mapping (function)* from the metric space (X, d) into the metric space (Y, d') associates with each point $x \in X$ a unique point $y \in Y$ which is often denoted by $f(x)$.

It will be convenient to use the following suggestive

NOTATIONS: (1) $f: X \to Y$, $f: x \mapsto f(x)$ or even $f: X \to Y : x \mapsto f(x)$, indicate the mapping f from X into Y 'carrying' x to $f(x)$.

(2) For $A \subseteq X$
$\underline{f(A)} = \{y \in Y\colon \text{there exists } x \in A \text{ with } y = f(x)\}$.

(3) For $B \subseteq Y$
$f^{-1}(B) = \{x \in X\colon f(x) \in B\}$, this is not to be confused with the inverse of f which may or may not exist.

DEFINITION. For metric spaces (X, d) and (Y, d') $f: X \to Y$ is *continuous* at $x_0 \in X$ if, given $\varepsilon > 0$, there exists $\delta > 0$ such that
$d(x, x_0) < \delta \Rightarrow d'(f(x), f(x_0)) < \varepsilon$,
or equivalently
$x \in B_\delta(x_0) \Rightarrow f(x) \in B_\varepsilon(f(x_0))$
or $f(B_\delta(x_0)) \subseteq B_\varepsilon(f(x_0))$.

NOTE. This definition extends the familiar definition of local continuity for a $f: R \to R$ which is the special case $(X, d) = (Y, d') = (R, d_1)$.
Local continuity in metric spaces can be characterized in terms of sequences, as the next theorem shows.

THEOREM 6.1 (SEQUENTIAL CONTINUITY): *Let* (X, d), (Y, d') *be metric spaces, then* $f: X \to Y$ *is continuous at* $x_0 \in X$ *if and only if for every sequence* $\{x_n\}$ *with* $x_n \to x_0$, $f(x_n) \to f(x_0)$.

Proof. ($\Rightarrow$) Since f is continuous at x_0, for any $\varepsilon > 0$ there is a $\delta > 0$ with

$$f(B_\delta(x_0)) \subseteq B_\varepsilon(f(x_0))$$

Now if $x_n \to x_0$, there exists $N \in \mathbb{N}$ such that

$$d(x_n, x_0) < \delta \text{ for all } n \geq N \text{ and so}$$

$$f(x_n) \in B_\varepsilon(f(x_0)) \text{ for all } n \geq N$$

$$\text{or} \quad f(x_n) \to f(x_0).$$

(⇐) Assume $f(x_n) \to f(x_0)$ whenever $x_n \to x_0$, but f is not continuous at x_0 i.e. there exists an $\varepsilon > 0$ such that $f(B_r(x_0)) \not\subseteq B_\varepsilon(f(x_0))$ all $r > 0$. Thus in particular for each $n \in \mathbb{N}$, there exists an $x_n \in B_{\frac{1}{n}}(x_0)$ such that

$f(x_n) \notin B_\varepsilon(f(x_0))$.

The sequence $\{x_n\}$ so constructed is such that $d(x_n, x_0) < \frac{1}{n}$ i.e. $x_n \to x_0$, but $d(f(x_n), f(x_0)) > \varepsilon$ all n so $f(x_n) \not\to f(x_0)$ a contradiction to our assumption.

DEFINITION: For metric spaces (X, d), (Y, d'), $f: X \to Y$ is *continuous* if f is continuous at each $x \in X$.

(This is sometimes referred to as f being globally continuous.)

COROLLARY (to Theorem 5.1): If (X, d) and (Y, d') are metric spaces, then $f: X \to Y$ is continuous if and only if f preserves convergent sequences i.e. for any sequence $\{x_n\}$ convergent to x we have $f(x_n) \to f(x)$.

NOTE. (1) It is not true, that for continuous $f: X \to Y$ if $f(x_n) \to f(x)$ then $x_n \to x$. (E.g. in $(\mathbb{R}, d_1)$ for $f: x \mapsto x^2$ and $x_n = (-1)^n$, $f(x_n) = 1 \to f(1)$ but $-1, 1, -1, 1, -1, \ldots, \not\to 1$.)

(2) This corollary provides the simplest way of proving a mapping is discontinuous at x, viz,

by selecting a sequence $x_n \to x$ for which $f(x_n) \not\to f(x)$.

The next theorem provides a very general, and powerful characterization of continuous mappings which is often used as a definition of continuity, particularly in the setting of topological spaces (see Notes to Lecture 4).

THEOREM 6.2: *Let* (X, d) *and* (Y, d') *be metric spaces, then* $f: X \to Y$ *is continuous if and only if for any open set* $G \subseteq Y$, $f^{-1}(G) = \{x \in X: f(x) \in G\}$ *is an open set of* X.

i.e. the inverse image of open sets is open or the pull back of an open set under f is open.

(NOTE. This does not assert that f maps open sets to open sets - see Problem 5.)

Proof. (⇒) Let G be an open subset in Y, then for any $x \in f^{-1}(G)$, $f(x) \in G$ which is open and so there exists $r_x > 0$ with $B_{r_x}(f(x)) \subseteq G$.

Now, by the definition of continuity, there exists a $\delta_x > 0$ with

$f(B_{\delta_x}(x)) \subseteq B_{r_x}(f(x)) \subseteq G$

so $\quad B_{\delta_x}(x) \subseteq f^{-1}(G)$ and so

$$f^{-1}(G) = \bigcup_{x \in f^{-1}(G)} B_{\delta_x}(x) \text{ is open.}$$

($\Leftarrow$) Since, $f^{-1}(G)$ is open whenever G is, for any $x \in X$ and $\varepsilon > 0$, we have $f^{-1}(B_\varepsilon(f(x))$ is an open set containing x and so there exists $\delta > 0$ such that $B_\delta(x) \subseteq f^{-1}(B_\varepsilon(f(x))$ or $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$ and f is continuous. $\blacksquare$

EXAMPLES.

Let $f: X \to Y$, where (X, d) and (Y, d') are metric spaces, be such that $d'(f(x), f(y)) \leqslant Md(x, y)$ for all $x, y \in X$ and some $M > 0$,

(Such an f is said to satisfy a *Lipschitz condition*), then f is continuous, for clearly given $\varepsilon > 0$

$$f(B_{\frac{\varepsilon}{M}}(x)) \subseteq B_\varepsilon(f(x)) \text{ for all } x \in X.$$

[Remark: The mean value theorem for derivatives asserts that every differentiable functions on (a, b) satisfies a Lipschitz condition with $(X, d) = ((a, b), d_1)$ and $(Y, d') = (R, d_1)$.]

Of special importance later, will be the case when $M < 1$, $X = Y$ and $d = d'$, in which case f is called a *(strict) contraction* on (X, d).

Another particular case of special interest, occurs when $d'(f(x), f(y)) = d(x, y)$ for all $x, y \in X$. Such an f is called an *isometry* from (X, d) into (Y, d').

Note: An isometry is necessarily 1 to 1, i.e. $f(x) = f(y) \Rightarrow x = y$ or $f^{-1}(\{x\})$ is singleton (prove).

If f is an isometry of (X, d) into (Y, d') we can readily see that (f(X), d') is essentially the same as (X, d) and we speak of Y containing a (isomorphic) copy of X, viz f(X).

Note further, that if f is also onto, i.e. $f(X) = Y$, then the inverse map $f^{-1}: Y \to X$ exists and

$$d(f^{-1}(x), f^{-1}(y)) = d'(f(f^{-1}(x)), f(f^{-1}(y)) = d'(x, y)$$

so f^{-1} is an isometry (and so continuous) from Y onto X.

More generally, if $f: X \to Y$ is 1 to 1 (so the inverse $f^{-1}: f(X) \to X$ exists) and both f and f^{-1} are continuous, then f is termed a *homeomorphism* of X into Y and X, f(X) are homeomorphic or topologically equivalent. The latter is an appropriate name since the open sets of f(X) are precisely the images under f of open sets in X and vice versa.)

REMARK. A property P is a *topological invariant* for metric spaces if, whenever X, Y are homeomorphic under f and $A \subseteq X$ has P then f(A) also has P.

EXAMPLES. It is easily seen that the property of "being open" is a topological invariant (prove), as also is "being closed".

CONTINUITY IN NORMED LINEAR SPACES.

Let X be a normed linear space, with norm $\|\cdot\|$, and

Y be a normed linear space with norm $\|\cdot\|'$.

We aim to characterize continuity for the important class of linear mappings $T: X \to Y$.

RECALL. $T: X \to Y$ is *linear* if

$$T(x + \lambda y) = T(x) + \lambda T(y) \text{ for all } x, y \in X \text{ and scalars } \lambda.$$

(In this context the term mapping (or function) is sometimes replaced by transformation or operator.)

DEFINITION. A linear mapping $T: X \to Y$ between normed linear spaces is *bounded* if for all $x \in X$ $\|T(x)\|' \leq M\|x\|$ for some $M > 0$.

THEOREM 6.3: *A linear mapping between normed linear spaces is continuous if and only if it is bounded.*

(Here 'continuous' means continuous w.r.t. the metrics induced by the respective norms.)

Proof. ($\Leftarrow$) Given $\varepsilon > 0$ and any $x \in X$, if $y \in B_{\frac{\varepsilon}{M}}(x)$ i.e. $\|x - y\| < \frac{\varepsilon}{M}$, then

$$\begin{aligned}\|T(x) - T(y)\|' &= \|T(x - y)\|' && \text{(by linearity)}\\ &\leq M\|x - y\| && \text{(by boundedness)}\end{aligned}$$

and so $T(y) \in B_{\varepsilon}(T(x))$, whence T is continuous.

($\Rightarrow$) Since T is continuous, it is in particular continuous at 0 i.e. given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d(x, 0) < \delta \Rightarrow d'(T(x), T(0)) < \varepsilon$$

or $\|x\| < \delta \Rightarrow \|T(x)\|' < \varepsilon$ (as $T(0) = 0$, by linearity).

Now for any $x \in X$,

$$\left\|\frac{\delta}{2\|x\|}x\right\| = \frac{\delta}{2\|x\|}\|x\| < \delta \qquad \text{(by n (iii))}$$

so $$\left\|T\left(\frac{\delta}{2\|x\|}x\right)\right\|' = \frac{\delta}{2\|x\|}\|T(x)\|' \qquad \text{(by linearity of T and n (iii))}$$

$$< \varepsilon$$

or $$\|T(x)\|' < \frac{2\varepsilon}{\delta}\|x\|.$$

whence T is bounded with $M \geqslant \frac{2\varepsilon}{\delta}$.

COROLLARY. *A linear mapping between normed linear spaces is continuous iff it is continuous at* 0 (prove).

Collateral Reading:

Giles, *op cit*, Ch. 3, p. 76-93.

The work on pages 88-90 is relevant to the lecture supplement dealing with uniform continuity.

Simmons *op cit*.

PROBLEMS.

1. Let (X, d) be a metric space and x_0 a fixed element of X. Show that the mapping

$$f: X \to R: x \mapsto d(x, x_0)$$

is continuous.

2. Show that the evaluation functional

$$F: C[a, b] \to R: f \mapsto f(x_0) \qquad (x_0 \text{ a fixed point of } [a, b]),$$

is a continuous mapping from $(C[a, b], d_\infty)$ into (R, d_1). Is this still true if $C[a, b]$ is considered with the metric d_1.

3. Let (X, d), (Y, d'), (Z, d'') be metric spaces and $f: X \to Y$, $g: Y \to Z$ be continuous mappings.
Show the composite $g\circ f: X \to Z$ is continuous.

4. Theorem 6. asserts that convergent sequences are preserved under continuous mappings. Show that this is not necessarily true for Cauchy sequences, i.e. it may happen that $\{x_n\}$ is a cauchy sequence in (X, d), $f: X \to Y$ is continuous and $\{f(x_n)\}$ is not a Cauchy sequence in (Y, d').
(Hint: Consider $f: (0, \infty) \to (0, \infty): x \mapsto 1/x$.)

5. A mapping $f: X \to Y$ between the two metric spaces (X, d), (Y, d') is *open* if $f(A)$ is an open subset of Y whenever A is an open subset of X (i.e. f carries open sets to open sets).
Show that not every continuous mapping need be open.
(Hint: Consider $f: R \to R: x \mapsto c$, a constant.)

6. Prove that the metric spaces $\left(\left(-\frac{\pi}{2}, \frac{\pi}{2}\right), d_1\right)$ and (R, d_1) are homeomorphic,

*7. If f is a homeomorphism of (X, d) onto (Y, d') show that

$$d''(x, y) = d(f^{-1}(x), f^{-1}(y)) \quad (\text{for all } x, y \in Y)$$

is an equivalent metric (see Problem 4.8) to d' on Y.

8. If T is a linear transformation from V^n to V^m then

$$T(x_1, x_2, \ldots, x_n) = \left(\sum_{i=1}^{n} t_{1i}x_i, \sum_{i=1}^{n} t_{2i}x_i, \ldots, \sum_{i=1}^{n} t_{mi}x_i\right)$$

for some $m \times n$ matrix $[t_{ji}]$ (refer linear algebra).
Show that T defines a bounded (hence continuous) linear mapping from ℓ_1^n to ℓ_1^m.

*9. Prove that $\text{Ker}(T) = T^{-1}(\{0\})$ is a closed subset if T is a continuous linear mapping between normed linear spaces.

*10. Let X and Y be normed linear spaces with norms $\|\cdot\|$ and $\|\cdot\|'$ respectively. Show that the linear mapping $T: X \to Y$ is a homeomorphism if and only if there exists $m, M > 0$ such that

$$m\|x\| \leqslant \|T(x)\|' \leqslant M\|x\| \quad \text{for all } x \in X.$$

*11. Let M denote the family of all metric spaces. Show that "(X, d) is homeomorphic to (Y, d')" defines an equivalence relationship on M (and so metric spaces may be partitioned into classes of homeomorphic spaces).

**12. Prove that $T: C[0, 1] \to C[0, 1]$ defined by $T(f)(x) = \int_0^x F(t, f(t))dt$ is continuous with respect to the metric d_∞ if $F: R^2 \to R$ is continuous.

Notes on lecture 6

1. The algebraic structure of any vector space X corresponds to a number of 'natural' mappings. From the additive structure we derive the translations

$$t_y: X \to X: x \mapsto x + y \ (= y + x), \text{ for each } y \in X.$$

Similarly, scalar multiplication produces

$$d_\lambda: X \to X: x \mapsto \lambda x, \text{ for each scalar } \lambda,$$

and $f_x: R \to X: \lambda \mapsto \lambda x$, for each $x \in X$.

These last two mappings are readily seen to be linear.
Further, for R with the metric d_1 and $\|\cdot\|$ any norm on X, the continuity of t_y follows easily while that of d_λ and f_x is equivalent to n(iii), which asserts boundedness for these mappings. A metric (or topology) on X is said to be *compatible with the algebraic structure of* X if the above mappings are continuous with respect to it.
Thus any norm induced metric is compatible with the algebraic structure.
(EXERCISE. Show the mapping $x \to \|x\|$ is continuous from (X, d) to (R, d_1) where d is the metric induced by the norm $\|\cdot\|$.)

2. Let X be any vector space of finite dimension n over the field R, $\|\cdot\|$ a norm on X and $\{\underset{\sim}{b}_1, \underset{\sim}{b}_2, \ldots, \underset{\sim}{b}_n\}$ a basis for X.
It may be shown (see Giles, Theorem 1, p. 132) that the 'natural' isomorphism

$$\phi: X \to V^n: (\lambda_1\underset{\sim}{b}_1 + \lambda_2\underset{\sim}{b}_2 + \ldots + \lambda_n\underset{\sim}{b}_n) \mapsto (\lambda_1, \lambda_2, \ldots, \lambda_n)$$

is a homeomorphism between X and $\ell_2^n = (V^n, d_2)$.

Thus X and ℓ_2^n are topologically equivalent. In particular then the family of spaces ℓ_p^n ($1 \leqslant p \leqslant \infty$, n fixed) are topologically equivalent.
Further, since ϕ can be taken to be the identity map, each of the metrics d_p ($1 \leqslant p \leqslant \infty$) give rise to the same open sets (although of course the open balls differ from metric to metric).
Consequently there is only one norm topology possible for V^n. (Although there are many different norms they all induce equivalent metrics.)
In fact, it can be shown that there is only one topology for V^n compatible with the algebraic structure.
With respect to this unique topology any linear transformation (mapping) $T: V^n \to V^m$ is continuous (see problem 8).* This explains the relative unimportance of metric, or continuity, arguments in finite dimensional linear algebra.
(Precisely the same remarks apply to finite vector spaces over the complex field C.)

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*Note: By the earlier remarks, this implies the continuity of any linear transformation between finite dimensional vector spaces over R (or C).

SUPPLEMENT to Lecture 6.

UNIFORM CONTINUITY in METRIC SPACES

DEFINITION. For (X, d), (Y, d') metric spaces, $f: X \to Y$ is *uniformly continuous* on X if given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$f(B_\delta(x)) \subseteq B_\varepsilon(f(x)) \text{ for all } x \in X.$$

REMARK. The definition of global continuity given earlier may be stated as: "Given $\varepsilon > 0$, for each $x \in X$ there exists a $\delta > 0$ such that $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$", and we see that there the value of δ may depend on the particular x under investigation, $\delta \equiv \delta(x)$. In uniform continuity there exists a universal δ applicable for all $x \in X$. Equivalent to this is the requirement that for $\delta(x)$ as above,

$$(\delta =) \operatorname*{infimum}_{x \in X} \delta(x) > 0.$$

It is thus clear that uniform continuity is stronger than continuity

i.e. (f uniformly continuous) $\Rightarrow$ (f continuous).

The converse of this is false.

For example. In $((0, 1), d_1)$, $f: x \mapsto \frac{1}{x}$ is continuous (as it is differentiable at all $x \in (0, 1)$) however it is not uniformly continuous. To see this, assume it were uniformly continuous, then for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x - y| < \delta \Rightarrow \left|\frac{1}{x} - \frac{1}{y}\right| < \varepsilon \quad (\text{for all } x, y \in X)$$

but for

$$0 < x < \text{Min}\{2\delta, \tfrac{1}{\varepsilon}\},\ y = \frac{1}{2}x.$$

$$|x - y| < \delta \text{ while } \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{2}{x}\right| > 2\varepsilon,$$

a contradiction, so f is not uniformly continuous.

THEOREM 6. : *Let* (X, d), (Y, d') *be metric spaces* $f: X \to Y$ *a uniformly continuous mapping and* $\{x_n\}$ *a Cauchy sequence in* (X, d), *then* $\{f(x_n)\}_{n=1}^{\infty}$ *is a Cauchy sequence in* (Y, d').

i.e. Cauchy sequences are preserved under uniformly continuous mappings. (A Cauchy sequence is not necessarily preserved under continuous mappings - see Problem 4, although convergent sequences are.)

Proof. Given $\varepsilon > 0$ there exists $\delta > 0$ such that $d'(f(x), f(y)) < \varepsilon$ whenever $d(x, y) < \delta$ (uniform continuity of f), further there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \delta$ for all $n, m \geqslant N$ (definition of Cauchy sequence) hence, for $n, m \geqslant N$ we have $d'(f(x_n), f(x_m)) < \varepsilon$ and so $\{f(x_n)\}$ is Cauchy. ■

THEOREM 6. (EXTENSION THEOREM): *Let* (X, d) *be a metric space,* (Y, d') *be a* *complete metric space,* A *a dense subset of* X, *and* $f: X \to Y$ *a uniformly continuous mapping on* (A, d). *Then there exists a unique uniformly continuous mapping* $\tilde{f}: X \to Y$ *with* $\tilde{f}(x) = f(x)$ *for all* $x \in A$.

i.e. f has a unique uniformly continuous extension to X.

RECALL. A is a *dense* subset of X if $\overline{A} = X$.

Proof. If $A = X$ there is nothing to prove, so assume $X \backslash A \neq \phi$.
Now if $x \in X\backslash A$, x is a cluster point of A ($\overline{A} = A \cup A' = X$) and so, there exists a sequence $\{x_n\}$ with $x_n \in A$ (all n) such that $x_n \to x$.
By theorem 6. $\{f(x_n)\}$ is a Cauchy sequence in (Y, d') which is complete and so $\lim_{n\to\infty} f(x_n)$ exists in Y.
Hence define $\tilde{f}$ by

$$\tilde{f}: x \mapsto \begin{cases} x & \text{if } x \in A \\ \lim_{n\to\infty} f(x_n) & \text{if } x \in X\backslash A, \text{ where } x_n \to x \text{ and } x_n \in A \end{cases}$$

We must show:

$\tilde{f}$ is well defined; - thus let $\{x_n\}$ and $\{y_n\}$ be two sequences converging to $x \in X\backslash A$, then

$$d'(f(x_n), f(y_n)) \leqslant d'(f(x_n), f(x)) + d'(f(x), f(y_n)) \to 0$$

So $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n)$ and $\tilde{f}$ is well defined at x.

$\tilde{f}$ is uniformly continuous; - since $f (= \tilde{f}|_A)$ is uniformly continuous on A we have: Given $\varepsilon > 0$ there exists $\delta > 0$ such that $f(B_\delta(x) \cap A) \subseteq B_{\frac{\varepsilon}{3}}(f(x))$ for all $x \in A$.

Now, let $d(x, y) < \frac{\delta}{2}$ $(x, y \in X)$ then there exists $a_1, a_2 \in A$ such that $d(a_1, x)$ and $d(a_2, y)$ are less than $\frac{\delta}{4}$ (density of A) and $d'(f(a_1), \tilde{f}(x))$ and $d'(f(a_2), \tilde{f}(y))$ are less than $\frac{\varepsilon}{3}$ (definition of $\tilde{f}$).
Whence, for any $x, y \in X$ with $d(x, y) < \frac{\delta}{2}$ we have

$$d'(\tilde{f}(x), \tilde{f}(y)) \leqslant d'(\tilde{f}(x), \tilde{f}(a_1)) + d'(\tilde{f}(a_1), \tilde{f}(a_2)) + d'(\tilde{f}(a_2), \tilde{f}(y)) \leqslant \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

and so $\tilde{f}$ is uniformly continuous.

$\tilde{f}$ is unique; - assume $g: X \to Y$ is uniformly continuous and $g(x) = f(x)$ all $x \in A$. Then $g(x) = \tilde{f}(x)$ for all $x \in A$. Now for $x \in X\backslash A$, let $x_n \to x$ where $x_n \in A$ (all n), then

$$d'(\tilde{f}(x), g(x)) \leqslant d'(\tilde{f}(x), f(x_n)) + d'(f(x_n), g(x))$$
$$= d'(\tilde{f}(x), f(x_n)) + d'(g(x_n), g(x))$$
$$\to 0 \text{ (by the definition of } \tilde{f} \text{ and the uniform continuity of } g)$$

Hence $\tilde{f}(x) = g(x)$ all $x \in X$ and so $\tilde{f}$ is unique.

REMARK. It is often natural to construct a uniformly continuous function on (Q, d_1); this theorem allows us to extend such a function to R.

Example. Using the laws of indices we can define the function $x \mapsto a^x$ (for fixed $a \in (0, \infty)$) and all $x \in Q$, an application of the theorem allows us to extend the domain of definition to R. In this way the function exp may be obtained $(a = e)$.

EXERCISE. (a) Show that any bounded linear transformation from a normed linear space into a complete normed linear space (Banach space) is uniformly continuous.

(*b) In (R, d_1) show that there is a unique continuous function $f: R \to R$ satisfying the functional equation $f(x + y) = f(x) + f(y)$, with $f(1) = 1$.

COMPLETE METRIC SPACES

Lecture 7. *Fixed Point Theorems.*

DEFINITION. $x_0 \in X$ is a *fixed point* of the mapping $f: X \to X$ if $f(x_0) = x_0$.

REMARK. Many problems of mathematics and its applications can be formulated as 'does f have a fixed point' for some appropriate f?
Also a great number of the procedures in 'numerical' analysis and approximation theory amount to obtaining successive approximations to the fixed points of appropriate mappings (e.g. Newton's method of locating zeros).

The sole purpose of this lecture is to establish a powerful fixed point theorem for contraction mappings on a complete metric space. Some applications of the result are examined in the appendices.

RECALL. For any metric space (X, d), $f: X \to X$ is a (strict) *contraction* if there exists $k \in (0, 1)$ such that

$$d(f(x), f(y)) \leq kd(x, y) \text{ for all } x, y \in X.$$

THEOREM 7.1 (Banach's Fixed Point Theorem): *Let* $f: X \to X$ *be a contraction on the complete metric space* (X, d), *then* f *has a unique fixed point.*

Proof. Take any point $x_1 \in X$ and inductively construct the sequence of points

$$\begin{aligned}
x_2 &= f(x_1)\\
x_3 &= f(x_2) = f^2(x_1)\\
x_4 &= f(x_3) = f^3(x_1)\\
&\dots\\
x_{n+1} &= f(x_n) = f^n(x_1)\\
&\dots
\end{aligned}$$

We first show that $\{x_n\}_{n=\phi}^{\infty}$ is a Cauchy sequence in (X, d).
Thus, without loss of generality take $m < n$ ($m, n \in N$), then

$$\begin{aligned}
d(x_m, x_n) &= d(f^{m-1}(x_1), f^{n-1}(x_1))\\
&\leq kd(f^{m-2}(x_1), f^{n-2}(x_1))\\
&\leq k^2 d(f^{m-3}(x_1), f^{n-3}(x_1))\\
&\leq \dots.\\
&\leq k^{m-1} d(x_1, f^{n-m}(x_1))
\end{aligned}$$

(as f is a contraction)

$$\leq k^{m-1}\{d(x_1, f(x_1)) + d(f(x_1), f^2(x_1)) + \dots + d(f^{n-m-1}(x_1), f^{n-m}(x_1))\}$$

(by extended application of the triangle inequality)

$$\leq k^{m-1}\{d(x_1, f(x_1) + kd(x_1, f(x_1)) + k^2 d(x_1, f(x_1)) + \dots + k^{n-m-1} d(x_1, f(x_1))\}$$

(again, since f is a contraction)

$$\leqslant k^{m-1} d(x_1, f(x_1))\,(1 + k + k^2 + \ldots + k^{n-m-1})$$

Now, as $0 < k < 1$,

$$1 + k + k^2 + \ldots + k^{n-m-1} < \sum_{j=0}^{\infty} k^j = \frac{1}{1 - k}$$

(sum of an infinite geometric progression), whence

$$d(x_m, x_n) \leqslant \frac{k^{m-1}}{1 - k} d(x_1, f(x_1))$$

$$\to 0 \text{ as } m \text{ (and hence } n) \to \infty.$$

Thus $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence, and so, by the completeness of (X, d), there exists $x_0 \in X$ with $x_n \to x_0$.
We now show x_0 is a fixed point of f.
Now $d(x_0, f(x_0)) \leqslant d(x_0, x_n) + d(x_n, f(x_0))$

$$\leqslant d(x_0, x_n) + kd(x_{n-1}, x_0) \quad (\text{as } x_n = f(x_n - 1).)$$

$$\to 0 \text{ as } x_n, x_{n-1} \to x_0$$

whence $d(x_0, f(x_0)) = 0$ and so

$$f(x_0) = x_0.$$

That x_0 is the unique fixed point of f follows, for if $f(y) = y$, then

$$d(x_0,y) = d(f(x_0),f(y)) \leqslant kd(x_0,y)$$

which is impossible unless $y = x_0$ (as $0 < k < 1$). ∎

NOTE. Almost as important as the result itself is the 'constructive' nature of the proof, which shows that the image of any $x_1 \in X$ under successive iterates of f, i.e. $x_1, f(x_1), f^2(x_1), \ldots, f^n(x_1), \ldots$, gives rise to increasingly accurate approximations for x_0, the fixed point of f.
Further in many applications, the error in the n'th such approximation can be estimated. For instance

$$d(x_0, x_n) = d(f^{n-1}(x_0), f^{n-1}(x_1)) \quad (\text{as } f^m(x_0) = x_0)$$

$$\leqslant k^{n-1} d(x_0, x_1),$$

so provided $d(x_0, x_1)$ can be estimated (frequently $d(X) < \infty$, in which case $d(x_0, x_1) \leqslant d(X)$ provides such an estimate) we have

$$(\text{error in n'th approximation}) \leqslant k^{n-1}(d(x_0, x_1))$$

and so at each iteration the error is decreased by a factor of at least $\frac{1}{k}$.

Collateral Reading.

A good introductory discussion of fixed point theorems and their importance is to be found in Courant and Robbins "What is Mathematics?", Ch. VIII.

Marvin Shinbrot "Fixed Point Theorems", *Scientific American,* January 1966, reprinted in: Readings from Scientific American "Mathematics in the Modern World".

Giles *op cit* p.52.

Simmons *op cit* Appendix One, pp. 337-340.

For an interesting discussion of fixed point theorems and their applications, the student should consult:

Rosenlicht "Introduction to Analysis", Scott, Foresmand, 1968.

Hille "Methods in Classical and Functional Analysis", Addison-Wesley, 1972.

PROBLEMS:

1. Let $f: [0, 1] \to [0, 1]$ be continuous with respect to the metric d_1, show that f has a fixed point.
   (Hint: Apply the intermediate value theorem to $x \mapsto f(x) - x$.)
2. Assume that the temperature varies continuously as we move around the equator of the earth. Prove that at any instant of time there is at least one pair of antipodal points on the equator which are at the same temperature.
3. Let $f: X \to X$ be such that $d(f(x), f(y)) < d(x, y)$ all $x, y \in X$, $x \neq y$, d a metric on X. Show that f can have at most one fixed point.
4. Let $f: [a, b] \to R$ be such that $f(a) < 0 < f(b)$, f' exists and is continuous on $[a, b]$ and further there exists constants m, M with
   $0 < m \leqslant \underset{x \in [a, b]}{\text{Max}} f'(x) \leqslant M$.
   Show that for a suitable choice of constant k the mapping g defined by $g(x) = x - k f(x)$ is a contraction on $([a, b], d_1)$. (Hint: Apply the mean-value theorem to show $g(x) - g(y) = (x - y)(1 - kf'(z))$, for some $z \in (x, y)$.
   Hence, conclude that f has a unique zero in (a, b).
5. Let $a \in R$ be such that $|a + 1| < 1$. Show that the mapping f defined by
   $$f(x) = \tfrac{1}{2}\big((1 + x)^2 - (1 + a)\big)$$
   is a contraction on $X = \{x: \ |1 + x| \leqslant |1 + a|\}$ with respect to the metric d_1. Hence conclude that a has a unique square root in X.
   [REMARK. This result remains true in some more general spaces, where it plays an important role by establishing the existence of square roots for certain elements.]

6. Let $A = [a_{ij}]$ be an $n \times n$ matrix, such that $\sum_{j=1}^{n} |a_{ij}| < 1$ for all $i \in \{1, 2, \ldots, n\}$. Prove that the matrix equation $(I - A)\underset{\sim}{X} = \underset{\sim}{B}$, where $\underset{\sim}{X}$ and $\underset{\sim}{B}$ are $n \times 1$ column vectors, has a unique solution $\underset{\sim}{X}$ for each choice of $\underset{\sim}{B}$.
(Hint: Consider the affine mapping on $\underline{\ell_\infty^n}$ defined by

$$T: \underset{\sim}{X} \mapsto A\underset{\sim}{X} + \underset{\sim}{B}.)$$

7. In any complete metric space (X, d) find an estimate for the error in the n'th approximation $x_n = f^{n-1}(x_1)$ to the fixed point of the contraction $f: X \to X$ which is applicable even if $d(X) = \infty$.
[Hint: Refer to the proof of the Banach fixed point theorem.]

APPENDIX I to lecture 7.

IMPLICIT FUNCTIONS

Our aim is to illustrate the power of fixed point Theorems by proving, via the Banach Fixed point Theorem, the following simple

IMPLICIT FUNCTION THEOREM.

Let $x, y \in R$ *be related by*

(1) $y = ax + R(y)$ *where* $R(0) = 0$ *and for* $|y| < r$

R *satisfies the Lipschitz Condition*

$$|R(y_1) - R(y_2)| \leqslant k|y_1 - y_2|$$

where k *is a fixed constant with* $0 < k < 1$.

Then there exists a unique continuous function f *with* $f(0) = 0$ *and domain* $D = \{x: |x| \leqslant \rho < \frac{1-k}{|a|} r\}$ *such that* $y = f(x)$, *all* $x \in D$.

i.e. the relation implies y is functionally related to x at least in a neighbourhood of 0.

Proof. If a solution exists it will belong to the subspace X of $C\,[-\rho, \rho]$ consisting of those functions g with $g(0) = 0$ and $\|g\|_\infty = \underset{|x|\leqslant\rho}{\text{Max}} |g(x)| \leqslant r$.

with the induced metric, $d_\infty(g, h) = \underset{|x|\leqslant\rho}{\text{Max}} |g(x) - h(x)|$ for all $g, h \in X$.

It is easily verified that (X, d_∞) is a *complete* metric space.

Further observe that f is a solution if and only if

$T(f) = f$ where T is the operator on X defined by

$$T(g)(x) = ax + R(g(x)) \text{ for all } |x| < \rho,\ g \in X.$$

Thus, provided we can show T is a *strict contraction* mapping into X, the desired result will follow upon invoking the Banach fixed point Theorem.

But $\|T(g)\|_\infty \leqslant |a|\,|x| + k\|g\|_\infty \leqslant |a|\,|\rho| + kr \leqslant r$ by the choice of $\rho\left(< \frac{1-k}{|a|} r\right)$ so $T(g) \in X$.

Further $d_\infty(T(g), T(h)) = \underset{|x|\leqslant\rho}{\text{Max}} |R(g(x)) - R(h(x))|$

$$\leqslant k \underset{|x|\leqslant\rho}{\text{Max}} |g(x) - h(x)| = kd_\infty(g, h),$$

so T is a strict contraction and the result follows. ∎

Application: INVERSE FUNCTION THEOREM

Take $f \in C^1\ (x_0 - r_1, x_0 + r_1)$ *with* $f'(x_0) \neq 0$. *If* $f(x_0) = y_0$ *we aim to show there exists a unique function* g, *domain* $D \equiv (y_0 - r_2, y_0 + r_2)$ *for some* $r_2 > 0$, *such that if* $y = f(x)$ *then* $x = g(y)$ *all* $y \in D$.

<u>i.e.</u> f is invertible on a neighbourhood of x_0.

It suffices to show that such a g must satisfy a relation of the form (1). The following reasoning is due to Édouard Goursat (1858 - 1936) in 1903.

We rewrite $y = f(x)$ as

$$x - x_0 = f'(x_0)^{-1}(y - y_0) - R(x)$$

where
$$R(x) = f'(x_0)^{-1}[f(x) - f(x_0)] - (x - x_0).$$

This is of precisely the right form, and further, from the continuity of f' there exists $r_3 \in (0, r_1]$ such that

$$|f'(x) - f'(x_0)| < \tfrac{1}{2}|f'(x_0)| \text{ all } x \text{ with } |x - x_0| < r_3$$

whence, for $x_1, x_2 \in (x_0 - r_3, x_0 + r_3)$ we have

$$|R(x_1) - R(x_2)| = |f(x_1) - f(x_2) - (x_1 - x_2) f'(x_0)||f'(x_0)|^{-1}$$

$$= \left|\int_{x_2}^{x_1} [f'(x) - f'(x_0)]dx\right| \left|f'(x_0)\right|^{-1}$$

$$\leqslant \int_{x_2}^{x} |f'(x) - f'(x_0)|dx|f'(x_0)|^{-1}$$

$$\leqslant \tfrac{1}{2}|f'(x_0)||x_1 - x_2||f'(x_0)|^{-1}$$

by choice of r_3

and so R satisfies the required Lipschitz condition

$$|R(x_1) - R(x_2)| < \tfrac{1}{2}|x_1 - x_2|.$$

Thus an application of our simple implicit function Theorem gives a unique h such that

$x - x_0 = h(y - y_0)$ for all x with $|x - x_0| \leqslant r_2 = \max\{r_3, \tfrac{1}{2}|f'(x_0)|r_3\}$

whence $x = g(y) \equiv x_0 + h(y - y_0)$ as required.

REMARKS: (1) The proof of our simple implicit function theorem may be trivially extended to cover the case where $R \equiv R(x, y)$ provided the Lipschitz condition

$$|R(x_1, y_1) - R(x_2, y_2) \leqslant k[|x_1 - x_2| + |y_1 - y_2|], \quad 0 < k < 1,$$

is satisfied for $|x_1|, |x_2| < r_1$ and $|y_1|, |y_2| < r_2$ some $r_1, r_2 > 0$.

(2) Under appropriate assumptions on F, Goursat's arguments can be combined with this extended implicit function Theorem to obtain a version of the usual implicit function Theorem:

$$F(x, y) \equiv 0 \Rightarrow y = f(x) \text{ some function } f.$$

The calculations are however considerably more involved.

(3) Both versions of the Implicit function Theorem considered remain valid if x and y are allowed to be elements of a Banach algebra*, the only difference in the proof being the replacing of $|\cdot|$ by the norm in the appropriate places, while the Inverse function Theorem extends to cover complex valued functions of a complex variable.

Collateral Reading.

Essentially, the above considerations were extracted from the two books of Einar Hille, *"Analytic Function Theory"* Vol. I Gin, Boston, 1959 and *"Methods in Classical and Functional Analysis"*, Addison-Wesley, Massachusetts, 1972.

EXERCISE.

Let x and y be related implicitly by

$$x^3 + y^3 + x - y = 0.$$

Establish the existence of an $r > 0$ and function $f \in C(-r, r)$ with $f(0) = 0$, such that

$$y = f(x) \text{ for all } x \in (-r, r).$$

[Hint: Consider $T: X \to X$ where

$$T(g)(t) = t + t^3 + g^3(t) \text{ for all } g \in X,$$

a suitable subspace of $C(-r, r)$.]

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* i.e. a normed linear space, complete with respect to the induced metric in which a product is defined, which is distributive over + and satisfies $\|xy\| \leq \|x\|\|y\|$ and $(\lambda x)y = x(\lambda y) = \lambda(xy)$ for all x, y in it and $\lambda \in R(C)$.

APPENDIX II to lecture 7.

AN EXISTENCE UNIQUENESS RESULT FOR DIFFERENTIAL EQUATIONS

We develop conditions on $F: R^2 \to R$ under which the first order intial value problem

$$u'(t) = F(t, u(t)), \; u(t_0) = u_0$$

has a unique solution on a neighbourhood of the initial point t_0.
OBSERVE that, under the transformations

$$x = t - t_0, \quad y = u - u_0, \quad f(x, y) = F(x + t_0, y + u_0)$$

the problem has the more convenient formulation

$$y' = f(x, y), \; y(0) = 0,$$

with which we will work from now on.

CORRESPONDING INTEGRAL EQUATION.
Henceforth assume that f defines a continuous mapping from (A, d_2) into (R, d_1) where A is the rectangle

$$\{(x, y) \in R^2 : |x| \leqslant a, \; |y| \leqslant b\},$$

and let

$$X_h = \{\psi \in C[-h, h] : \|\psi\|_\infty = \max_{x \in [-h,h]} |\psi(x)| \leqslant b\}$$

then we have

LEMMA. *If* $\phi \in X_h$ *is such that* $\phi(x) = \int_0^x f(t, \phi(t))dt$, *then* $y = \phi(x)$ *is a solution of the initial value problem* $y' = f(x, y), \; y(0) = 0$.

Proof. From the assumption on f and the nature of ϕ follows the continuity of the composite $f(t, \phi(t))$ and so the fundamental theorem of calculus applies to give

$$\phi'(x) = \frac{d}{dx}\int_0^x f(t, \phi(t))dt = f(x, \phi).$$

It is also clear that $\phi(0) = \int_0^0 f(t, \phi(t))dt = 0.$

OPERATOR REFORMULATION.

For any $h > 0$ define the operator (mapping)

$$T: X_h \to C[-h, h] \text{ by}$$

$$T(\psi)(x) = \int_0^x f(t, \psi(t))dt \qquad \text{for all } \psi \in X_h.$$

Then the above lemma may be restated as
"If $\phi \in X_h$ is such that $T(\phi) = \phi$, then $y = \phi(x)$ is a solution of $y' = f(x, y), \; y(0) = 0$".

Thus to establish the existence of a solution to the initial value problem it suffices to find an $h > 0$ such that T has a <u>fixed point</u> in X_h. To achieve this we seek a value of $h > 0$ for which T is a contraction on (X_h, d_∞). The result then follows from Banach's fixed point theorem, since it is readily shown (do so) that (X_h, d_∞) is a complete metric space.

Now for $\psi \in X_h$

$$|T(\psi)(x)| = \left|\int_0^x f(t, \psi(t))dt\right|$$

$$\leqslant \begin{cases} \int_0^x |f(t, \psi(t))|dt & \text{for } x \geqslant 0 \\ \int_x^0 |f(t, \psi(t))|dt & \text{for } x < 0 \end{cases}$$

$$\leqslant M|x|$$

where $M = \max_{(x,y)\in A} |f(x, y)|$ ($< \infty$, by the continuity of f)

So $\|T(\psi)\|_\infty = \max_{x\in[-h,h]} |T(\psi)(x)| \leqslant Mh$

whence if $h \leqslant \frac{b}{M}$ we have $\|T(\psi)\|_\infty \leqslant b$ and so $T(\psi) \in X_h$ or $T: X_h \to X_h$.

The final step is to show that we can choose h so that in (X_h, d_∞)

$d_\infty(T(\psi), T(\phi)) \leqslant k\, d_\infty(\psi, \phi)$ for some $k \in (0, 1)$.

Regrettably this is not, in general, possible without further restricting f. We will <u>assume</u> that f satisfies a Lipschitz's condition in the second variable, <u>i.e.</u>

$$|f(x, y_1) - f(x, y_2)| \leqslant K|y_1 - y_2|$$

for some $K > 0$ and all $(x, y_i) \in A$ ($i = 1,2$). Then

$$d_\infty(T(\psi), T(\phi)) = \max_{x\in[-h,h]} |T(\psi)(x) - T(\phi)(x)|$$

$$= \max_{x\in[-h,h]} \left|\int_0^x f(t, \psi(t)) - f(t, \phi(t))dt\right|$$

$$\leqslant \max_{x\in[-h,h]} \int_0^x |f(t, \psi(t)) - f(t, \phi(t))|dt \quad (x \geqslant 0)$$

$$\leqslant \max_{x\in[-h,h]} K\int_0^x |\psi(t) - \phi(t)|dt$$

$$\leqslant \max_{x\in[-h,h]} \left(K \max_{t\in[-h,h]} |\psi(t) - \phi(t)| x \right)$$

$$= Kh \max_{t\in[-h,h]} |\psi(t) - \phi(t)|$$

$$= Kh\, d_\infty(\psi, \phi).$$ (The same result holding if $x < 0$.)

Thus $d_\infty(T(\psi), T(\phi)) \leqslant k d_\infty(\psi, \phi)$

where $k = Kh < 1$ provided $h < \frac{1}{K}$.

We have therefore proved:

For $h < \text{Min}\{\frac{b}{M}, \frac{1}{K}\}$, T *has a unique fixed point* $\phi_h \in X_h$. *Thus*

$$\phi_h(x) = \int_0^x f(t, \phi_h(t))dt \textit{ for } |x| < h$$

and so $y = \phi_h(x)$ *is a solution of* $y' = f(x, y)$, $y(0) = 0$ *on the neighbourhood* $(-h, h)$.

We now show that this solution is unique, at least for h sufficiently small.

Assume $y = \phi(x)$ is any solution of $y' = f(x, y)$, $y(0) = 0$, then a priori ϕ is differentiable and so continuous, thus there exists $h_1 \in (0, \text{Min}\{\frac{b}{M}, \frac{1}{K}\})$ such that

$$|x| < h_1 \Rightarrow |\phi(x)| < b$$

Hence, on the open interval $(-h_1, h_1)$ the composite $f(x, \phi(x))$

$$= \phi'(x) \quad \text{(by assumption)}$$

is continuous and therefore integrable. Whence the fundamental theorem of calculus applies to give

$$\phi(x) = \int_0^x f(t, \phi(t))dt = T(\phi)(x).$$

Thus, ϕ is a fixed point of T in X_h, which is unique by the above result.

In all, we have therefore proved

THEOREM: *If* $f: R^2 \to R$ *is continuous on some rectangle*

$$A = \{(x, y): |x| \leqslant a, |y| \leqslant b\}$$

and satisfies a Lipschitz condition

$$|f(x, y_1) - f(x, y_2)| \leqslant K|y_1 - y_2|$$

for all pairs of points $(x, y_1), (x, y_2) \in A$, *then there exists* $h > 0$ *such that the initial value problem* $y' = f(x, y)$, $y(0) = 0$ *has a unique solution on* $(-h, h)$.

REMARKS. (1) A sufficient (and often used) condition for f to satisfy a Lipschitz condition of the type referred to above is that $\frac{\partial f}{\partial y}$ be continuous in A, as may easily be seen upon an application of the Mean Value Theorem.

(2) From the proof of Banach's fixed point theorem and the remarks following it, for T defined by $T(\phi)(x) = \int_0^x f(t, \phi(t))dt$, the sequence of iterates

$$\phi_0,\ T(\phi_0),\ T^2(\phi_0),\ \ldots,\ T^n(\phi_0),\ \ldots,$$

for any continuous starting function ϕ_0 with $\phi_0(0) = 0$, form successive approximations (known as Picard's approximations) to the solution of $y' = f(x, y)$, $y(0) = 0$ (at least for sufficiently small values of x).

EXERCISE.

(i) Transform the initial value problem $u' = u$, $u(0) = 1$ into the form $y' = f(y)$, $y(0) = 0$ and repeat the general arguments of this appendix for this specific case.

(ii) Obtain the first five successive Picard approximations to the solution of $y' = f(y)$, $y(0) = 0$ (f as in (i)) starting with initial approximation $\phi_0(x) = 0$.

MISCELLANEOUS PROBLEMS

1. Let (X, d) be a metric space, for $x \in X$ and $A \subseteq X$ define the *distance from x to* A to be

$$d(x, A) = \inf_{a \in A} d(x, A).$$

   (i) Show $A \neq \phi$ implies $0 \leqslant d(x, A) < \infty$ (i.e. $d(x, A) = \infty$ iff $A = \phi$).

   (ii) Does $d(x, A) = 0$ imply $x \in A$?

   (iii) Prove $x \in \text{Int}(X \backslash A)$ iff $d(x, A) > 0$.

   (iv) Prove $x \in \overline{A}$ iff $d(x, A) = 0$, (hence characterise those $x \in$ bdry A in terms of distances from sets).

2. (SEPARATION THEOREMS)

   (a) If x, y are distinct points of the metric space (X, d), show that there exists a pair of disjoint open balls each of which is centred on one of the points. Because of this property metric spaces are said to satisfy the Hausdorff separation axiom.

   (b) In the metric space (X, d) let $x \notin A = \overline{A}$. Show there exists disjoint open sets G_1, G_2 with $x \in G_1$ and $A \subseteq G_2$.

   *(c) Let A_1, A_2 be any pair of disjoint closed subsets in the metric space (X, d). Show that there exists disjoint open sets G_1, G_2 with $A_i \subseteq G_i$ $(i = 1, 2)$. Because of this property we say metric spaces are normal spaces.

3. Let $f: X \to Y$ and $g: X \to Y$ be two continuous mappings between the metric spaces (X, d) and (Y, d') such that $f(a) = g(a)$ for all $a \in A$, a dense subset of X. Show that f and g are in fact identical.

4. The ruler function $r: [0, 1] \to Q \cap [0, 1]$ defined by

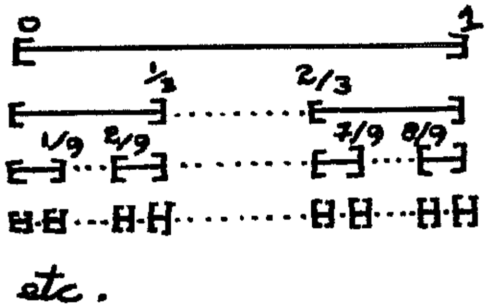
$$r(x) = \begin{cases} 0 \text{ if } x \text{ is irrational} \\ \frac{1}{q} \text{ if } x = \frac{p}{q} \end{cases}$$

where p and q are mutually prime integers (i.e. the greatest common divisor of p and q is 1),
has the property of being continuous at each irrational point, but discontinuous at every rational point. Prove this for, at least, the two special cases of $x = \frac{1}{2}$ and $x = \sqrt{2}/2$. (This function is considered in many of the standard

4. <u>(continued)</u>

books, including Spivak "Calculus".)

5. <u>Cantor's Ternary Set</u>, may be constructed inductively by deleting the open interval $(\frac{1}{3}, \frac{2}{3})$ from $[0,1]$, then deleting $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$, followed by the deletion of $(\frac{1}{27}, \frac{2}{27})$, $(\frac{7}{27}, \frac{8}{27})$, $(\frac{19}{27}, \frac{20}{27})$, $(\frac{25}{27}, \frac{26}{27})$ followed by ....



(see diagram)

Clearly, the points $\frac{1}{3}, \frac{2}{3}, \frac{1}{9}, \frac{2}{9}, \frac{7}{9}, \frac{8}{9}, \ldots$ are never deleted by this process and so $K \neq \phi$, in fact K contains infinitely many points, despite the fact that the total "length" of non-overlapping intervals deleted is readily seen to equal

$$\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \frac{8}{81} + \ldots = \frac{1}{3}\left[1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \ldots + \frac{2^n}{3^n} + \ldots\right]$$

a geometric progression whose sum is 1!

(i) Deduce that K is a closed subset of $([0, 1], d_1)$.

*(ii) (For latter parts you may assume the results of this part, if you feel unable to prove them and feel they would help.) Show the following are equivalent definitions of K

(a) $$K = [0, 1] \setminus \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\frac{3^n-1}{2}} \left(\frac{2m-1}{3^n}, \frac{2m}{3^n}\right)$$

(b) K consists of precisely those points in [0, 1] having <u>a</u> ternary representation (<u>i.e.</u> representation to base 3) of the form $\sum_{n=1}^{\infty} \frac{a_n}{3^n}$ with $a_n = 0$ or 2 for all $n \in N$.

(Note: $\frac{1}{9} = 0.01 = 0.0022\dot{2}$; (base 2) etc.)

(iii) Show Int $K = \phi$

(iv) Show bdry $K = K$

(v) Show $[0, 1]\setminus K$ is a dense subset of $[0, 1]$

*(vi) Show $K' = K$

*(vii) Show K is an uncountable set with cardinality that of the continuum

5. *(vii) (continued)

i.e. there exists a 1-1 and onto mapping from K to [0, 1]. (Thus although K has zero "length" it contains as "many" points as the original interval [0, 1].)

You will find many more problems both elementary and more advanced in the various reference books cited at the end of each lecture.

The interested student is encouraged to read further on the theory of metric spaces.

Topics which such reading might include are:

1. The Baire Category Theory
2. Compactness
3. Connected Sets.