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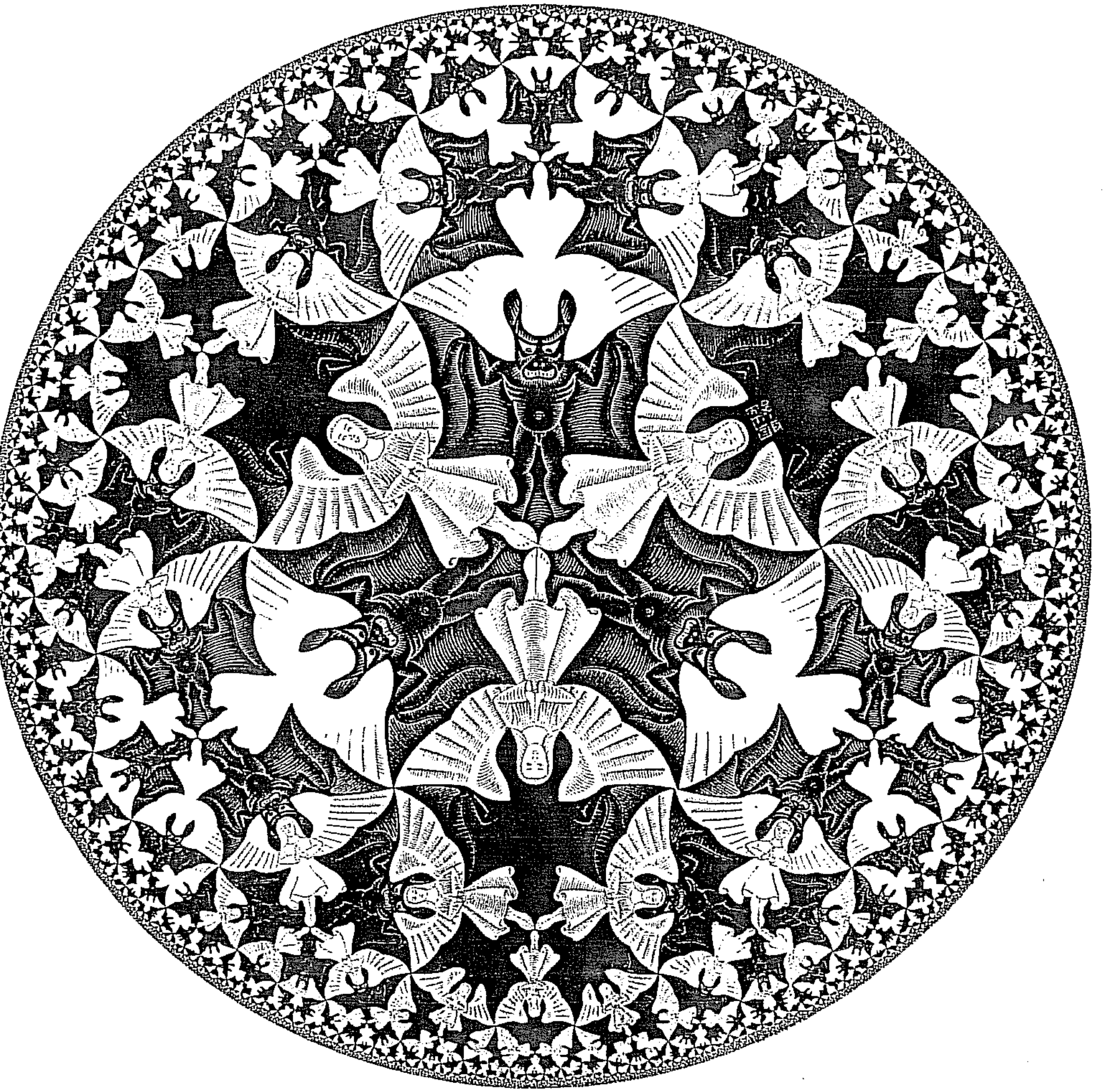


An image of the plane where radial distances might have been distorted according to the metric

$$d(x,y) = \frac{|x - y|}{1 + |x - y|}$$

Cirkellimiet IV - M.C. Escher.
(Reproduced for the private use of students in PM211.)

TABLE OF NOTATIONS

N, $\mathbb{N}$	- the set of natural numbers, $\{1,2,3,\ldots,n,\ldots\}$.
Q	- the field of rational numbers.
R, $\mathbb{R}$	- the ordered field of real numbers
C	- the field of complex numbers.
$X \times Y$	- the Cartesian product of the two sets X and Y i.e. the set $\{(x,y): x \in X, y \in Y\}$ of all ordered pairs with first element a member of X and second element a member of Y.
R^n	- The Cartesian product of R with itself n times - the set of ordered n-tuples of real numbers, usually regarded as a vector space over R with respect to component wise definitions of vector addition and scalar multiplication.
$\underset{\sim}{x}$	- an element (vector) of R^n $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)$
$[a,b]$	- the closed interval $\{x \in R: a \leqslant x \leqslant b\}$ (Note: it is implicit in the notation that $a \leqslant b$).
(a,b)	- the open interval $\{x \in R: a < x < b\}$ (Note: it is implicit in the notation that $a < b$).
$[a,b)$ and $(a,b]$	- Half open intervals in R.
St[a,b]	- the vector space of step functions on [a,b].
Reg[a,b]	- the space of regulated functions on [a,b].
B or B[a,b]	- the vector space of bounded functions on [a,b].
C(I)	- the set of continuous real valued functions with domain the interval I, usually regarded as a vector space over R with respect to point-wise defined operations of vector addition and scalar multiplication.
C[a,b]	- a special case of C(I).

$|x|$ – the absolute value function $x \mapsto \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$

d – a metric function.

(X,d) – the metric space consisting of the set X equipped with the metric function d.

$\|.\|$ – a norm function

$(X,\|.\|)$ – the normed linear space consisting of the vector space X equipped with the norm function $\|.\|$.

$\langle .,. \rangle$ – an inner-product.

$\|.\|_p$ – the norm function on $\mathcal{R}^n$ defined by

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} .$$

Note: $\|\underset{\sim}{x}\|_\infty = \lim_{p\to\infty} \|\underset{\sim}{x}\|_p = \text{Max}\ \{|x_i| : i=1,2,\ldots,n\}$

<u>or</u> the norm function on $\mathcal{C}[a,b]$ defined by

$$\|f\|_p = \left(\int_a^b |f|^p\right)^{\frac{1}{p}}$$

Note: $\|f\|_\infty = \sup_{x\in[a,b]} |f(x)|$ may also be defined on $\mathcal{B}[a,b]$.

d_p – the metric induced by $\|.\|_p$, $d_p(x,y) = \|x-y\|_p$.

ℓ_p^n – the finite dimensional normed linear space $(\mathcal{R}^n, \|.\|_p)$.

ℓ_p – the infinite dimensional analogue of ℓ_p^n; the space of all p-summable sequences with

$$\|\underset{\sim}{x}\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}}$$

where $\underset{\sim}{x} = x_1, x_2, \ldots, x_n, \ldots$.

c_0 – the subspace of ℓ_∞ consisting of all sequence which converge to 0.

$S_r(x)$	- the sphere centre x and radius r, $S_r(x) = \{y \in X: d(y,x) = r\}$
$B_r(x)$	- the open ball centre x and radius r, $B_r(x) = \{y \in X: d(y,x) < r\}$
$B_r[x]$	- the closed ball centre x and radius r, $B_r[x] = \{y \in X: d(y,x) \leq r\}$
$B[X]$	- the (closed) unit ball of the normed linear space $(X, \|\cdot\|)$; $B[X] = B_1[0] = \{y \in X: \|y\| \leq 1\}$
Int(A)	- the interior of the set A.
diam(A)	- the diameter of the set A, $\sup_{x,y \in A} d(x,y)$.
$\bar{A}$	- the closure of A, the set of all limit points of A.
bdry A	- the boundary of A, $\bar{A} \cap \overline{(X \setminus A)}$
(x_n) or $(x_n)_{n=1}^{\infty}$	- the sequence $x_1, x_2, \ldots, x_n, \ldots$.
$\{A_\lambda: \lambda \in \Lambda\}$	- an indexed family of sets.
$\bigcap_{\lambda \in \Lambda} A_\lambda$	- the intersection of the indexed family of sets $\{A_\lambda: \lambda \in \Lambda\}$; $\bigcap_{\lambda \in \Lambda} A_\lambda = \{x: x \in A_\lambda \text{ for all } \lambda\}$. Sometimes written $\bigcap_{n=1}^{\infty} A_n$ when the indexed family $\{A_1, A_2, \ldots, A_n, \ldots\}$ is countable.
$\bigcup_{\lambda \in \Lambda} A_\lambda$	- the union of the indexed family of sets $\{A_\lambda: \lambda \in \Lambda\}$, consisting of all x which belong to at least one A_λ.
$X \setminus A$	- The complement of A in X <u>i.e.</u> the set $\{x \in X: x \notin A\}$.
$A \Delta B$	- the symmetric difference of A and B, $A \Delta B = (A \cup B) \setminus (A \cap B)$

$f : X \to Y$	- a mapping from the set X into the set Y.
$f : x \mapsto f(x)$	- a notation for the function f which maps x to f(x) ($x \in$ some implicit domain X). Sometimes combined with the previous notation to give $f:X \to Y:x \mapsto f(x)$, specifying both the domain and range of f as well as the "rule".
$f(A)$	- for $f : X \to Y$, $A \subseteq X$, the set $\{f(x) : x \in A\} \subseteq Y$.
$f^{-1}(A)$	- for $f : X \to Y$, $A \subseteq Y$, the set $\{x \in X : f(x) \in A\}$.
$p \Rightarrow q$	- p implies q, or q if p.
$p \Leftrightarrow q$	- p if and only if q, or $(p \Rightarrow q)$ and $(q \Leftarrow p)$.
Sup A	- for $A \subset \mathbb{R}$ the supremum (least upper bound) of A.
Inf A	- for $A \subset \mathbb{R}$ the infemum (greatest lower bound) of A.

Common symbols not used in the notes, which may be useful in your working -

$\exists$	- there exists.
$\forall$	- for all.
iff	- if and only if.

ABSTRACT AND FUNCTIONAL ANALYSIS

Chapter 1 : BASIC THEORY

§1.1 *Definitions and Examples* - *Metric spaces, Normed linear spaces and Inner-product spaces.*

METRIC SPACES

In both mathematics and common language the notion of *distance* is often used figuratively, for example:

> 'Orange is a colour nearer to red than violet.'
>
> 'When a massive particle moves in a gravitational field it follows the path of shortest "distance" (geodesic) in space-time.'
>
> 'For x small, $x - \frac{x^3}{6}$ is a closer approximation to sin x than is x.'

Such notions are made precise when it is possible to assign a numeric value to the *distance* between the objects under consideration. We develop an extended "theory" of distance (encompassing many cases like those above) which has proved to be fundamental for much modern mathematics. Except in specific examples, we will not be concerned with the nature of the objects or with how the distances between them are "calculated".

Our primitive concept will be that of a given set X together with a function d which assigns to each ordered pair of elements (x,y) of X a real number d(x,y) which we take to be the "distance" from x to y.

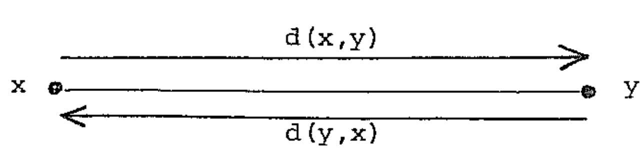
We will require the function d to satisfy certain conditions, which as the German Mathematician Herman Minkowski (1864-1909) remarked in 1906, "any notion of distance ought to possess". These four conditions are:

(M1) $d(x,y) \geq 0$ for all points $x,y \in X$;

(M2) $d(x,y) = 0$ if and only if $x = y$;

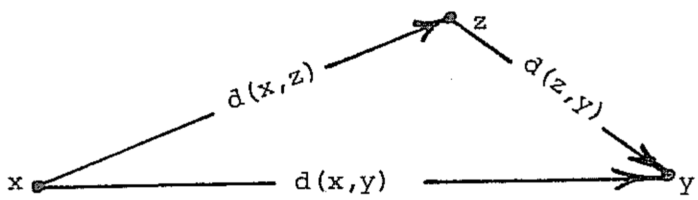
(M3) $d(x,y) = d(y,x)$ for all points $x,y \in X$

That is, d is a symmetric function of its two arguments - the distance from x to y is the same as the distance from y to x.



(M4) $d(x,y) \le d(x,z) + d(z,y)$ for all $x,y,z \in X$.

This inequality is usually referred to as the triangle inequality and may be interpreted as the statement that the length of one side of a triangle cannot exceed the sum of the lengths of the other two sides.



Equivalently, the distance from x to y via any intermediate point z cannot be "shorter" than the direct distance from x to y.

A function $d: X \times X \to R$ which satisfies the above four properties is termed a metric (from the latin *metor* - measure) on X. The set X together with a metric d is referred to as a metric space and denoted by (X,d). (M1) to (M4) are the axioms for a metric space. For any particular X and d, to prove (X,d) is a metric space it is necessary to show d is a metric on X; that is, to verify that d satisfies each of the axioms (M1) to (M4). When developing the general theory of metric spaces we build from the axioms. All results must ultimately be consequences of only the four axioms.

EXAMPLES (1) *The set of real numbers R with the usual metric*

$d_1(x,y) = |x - y|$, *for all* $x,y \in R$, *is a metric space.*

Proof. (M1), (M2) and (M3) are easily verified (do so). To establish (M4) we use the well known inequality $|a + b| \le |a| + |b|$ and observe that

$$\begin{aligned} d(x,y) &= |x - y| \\ &= |x - z + z - y| \quad \text{(The z's cancel)} \\ &= |(x - z) + (z - y)| \\ &\le |x - z| + |z - y| \\ &= d(x,z) + d(z,y). \end{aligned}$$

(2) *The set of ordered pairs of real numbers* R^2 - the set of points on the plane identified with their coordinates with respect to a set of Cartesian axes - *together with the Euclidean metric*

$$d_2(\underset{\sim}{x},\underset{\sim}{y}) = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$$

where $\underset{\sim}{x} = (x_1,x_2)$ *and* $\underset{\sim}{y} = (y_1,y_2) \in R^2$, *is a metric space.*

This is a special case of a wider class of metrics which arise from inner-products. Since these will be considered later we omit a direct proof for the above statement.

[By identifying the point $\underset{\sim}{x}$ with the complex number $u = x_1 + ix_2$ and $\underset{\sim}{y}$ with $v = y_1 + iy_2$ we see that the modulus $|u-v| = d(\underset{\sim}{x},\underset{\sim}{y})$. This shows that $d(u,v) = |u-v|$ defines a metric on C the set of complex numbers.]

An alternative metric on R^2 is provided in the following example.

(3) *For* $\underset{\sim}{x}, \underset{\sim}{y} \in R^2$

$$d(\underset{\sim}{x},\underset{\sim}{y}) = \begin{cases} 0, \text{ if } \underset{\sim}{x} = \underset{\sim}{y} \\ \sqrt{x_1^2+x_2^2} + \sqrt{y_1^2+y_2^2}, \text{ otherwise} \end{cases}$$

defines a metric on R^2 *which is sometimes referred to as the "post-office" metric (can you see why.)*

A proof of this is more easily given after the notion of a normed linear space has been introduced and will be called for in a later exercise.

It is indeed possible to define a metric on any set, as the following example illustrates.

(4) *Any set X can be rendered a metric space by using the discrete metric defined by*

$$d(x,y) = \begin{cases} 0 \text{ if } x = y \\ 1 \text{ if } x \neq y \end{cases}$$

Proof (That the discrete metric is a metric). Clearly (M1) and (M2) are satisfied, while (M3) holds since = and $\neq$ are symmetric relations.

To establish (M4) for $x, y, z \in X$ we must consider the following cases $x = y \neq z$, $x = y = z$, $x \neq y = z$, $z = x \neq y$, $x \neq y \neq z \neq x$.

In case 1, $d(x,y) = 0$ while $d(x,z) = d(z,y) = 1$ so

$$(0 =)\ d(x,y) \leqslant d(x,z) + d(z,y)\ (=2).$$

The other cases may be handled similarly.

NOTE: While the discrete metric is certainly pathological it is of considerable importance in the construction of counter-examples.

<u>Observation</u> From the last three examples we see that the underlying set does not determine a metric uniquely; many different metrics can be defined on the same set and so several distinct metric spaces can share the same set. For example; - for the two points $\underset{\sim}{x} = (\frac{1}{2},0)$ and $\underset{\sim}{y} = (1,0)$ in R^2 we have:

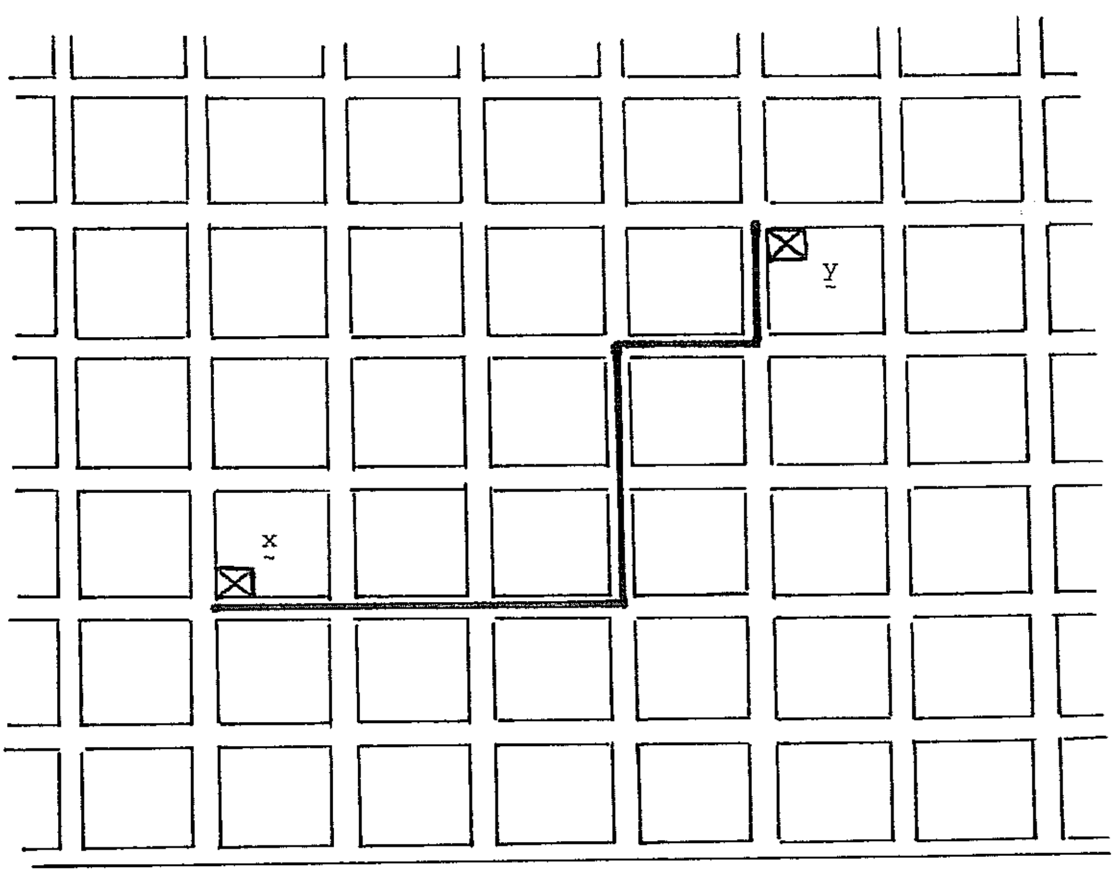
Using the Euclidean metric $d_2(\underset{\sim}{x},\underset{\sim}{y}) = \frac{1}{2}$;

Using the post office metric $d(\underset{\sim}{x},\underset{\sim}{y}) = 1\frac{1}{2}$;

Using the discrete metric $d(\underset{\sim}{x},\underset{\sim}{y}) = 1$.

Frequently the appropriate metric to use is determined by the type of problem under consideration, as the following example illustrates.

(5) In elementary plane geometry the most used metric on R^2 is the Euclidean Metric of example (2). However, for a law abiding motorist, the distance between site $\underset{\sim}{x}$ and $\underset{\sim}{y}$ on the map below is 7, not $d_2(\underset{\sim}{x},\underset{\sim}{y}) = 5$



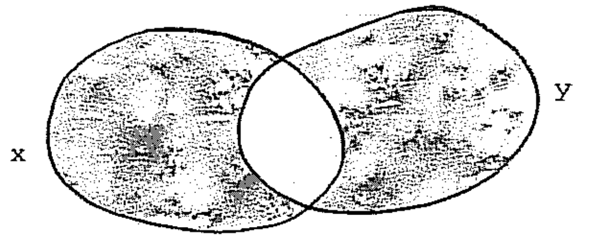
The appropriate metric for our motorist to use would be

$$d_1(\underset{\sim}{x},\underset{\sim}{y}) = |x_1 - y_1| + |x_2 - y_2| \qquad \text{(Why?)}$$

The proof that this is indeed a metric on R^2 will be deferred until we have introduced the concept of normed linear space.

(6) *For any given set* A *let* X *denote the family of all finite subsets of* A. That is subsets x of A with only a finite number of elements. Let $x \Delta y$ denote the symmetric difference of the two subsets x,y of A. That is;

$$x \Delta y = (x \cup y) \setminus (x \cap y)$$



Define $d: X \times X \to R$ *by* $d(x,y) = \#(x\Delta y)$, *the number of elements in* $x\Delta y$. *Then,* d *is a metric on* X. The proof of this is left as an EXERCISE. (Note, the "points" of X are themselves sets - finite subsets of A.)

<u>EXERCISES</u>

1. Prove $d(x,y) = \min\{1, |x-y|\}$ is a metric on R.

2. Let (X,d) be a metric space. Show that d* defined by

$$d^*(x,y) = \frac{d(x,y)}{1 + d(x,y)}$$

   is also a metric on X. (Note that $d^*(x,y) < 1$ for all $x,y \in X$.)

3. (Very useful and <u>important</u>) In any metric space (X,d) prove the following inequality

$$|d(x,z) - d(z,y)| \leq d(x,y) \quad \text{all } x, y, z \in X$$

[This may be interpreted as: the difference in lengths of two sides of a triangle cannot exceed the length of the third side.]

4. For any set $X \neq \emptyset$, show that $d: X \times X \to R$ satisfying

(M1') $d(x,y) = 0 \Leftrightarrow x = y$

and (M2') $d(x,y) \leqslant d(x,z) + d(y,z)$ for all $x, y, z \in X$,

is a metric on X.

i.e. The four axioms (M1) - (M4) of a metric space could be replaced by these two slightly more intricate ones due to S. Banach.

REMARKS:

1) *A good informal introduction to the notion of a metric space may be found in W.W. Sawyer "A Path to Modern Mathematics", Pelican, Ch.10, pp.187-221. I strongly suggest you read this.*
2) *Because the study of metric spaces is an abstract one, it is essential that you become completely familiar with the definitions and notations as well as some of the more basic examples.*
3) *Drawing diagrams which interpret the various definitions, constructions and results in the familiar space* (R^2, d_2) *is a valuable aid to understanding, and a practice which you should actively adopt.*
4) *(HISTORICAL) The concept of metric space is essentially due to the French Mathematician Maurice Fréchet (1878-1973), though our definition is that given by the German Mathematician Felix Hausdorff (1868-1942) in 1914. Fréchet introduced the notion in his Doctoral thesis to the University of Paris in 1906 and for many years pioneered the study of such spaces and their applications to other areas of Mathematics. It was toward the end of last century that mathematicians (due to the work of Klein, Hilbert and many others) began to appreciate the power of generalized methods (such as those represented by the study of metric spaces) and so initiated the study of abstract systems - vector spaces, metric spaces, normed spaces, topological spaces, groups, rings, categories etc. - which have proved central to much twentieth century mathematics. Because a prototype for many of these structures is 'ordinary' 1, 2 or 3 dimensional space, they are often referred to as spaces and their elements as points. The study of such a structure has proved valuable for several reasons, some of which are:*
   (a) *By retaining only essential features of a situation their consequences can be studied more simply in a less cluttered environment.*

(b) *Any conclusions of such a study are immediately applicable to any particular realisation of the structure. Thus a result can be simultaneously established for a number of apparently distinct situations.*

(c) *By recognising the common structure of familiar examples such as 1, 2 or 3 dimensional space and other spaces it is possible to transfer some of our intuition about 'ordinary' space to less familiar situations.*

NORMED LINEAR SPACES

For many of the more important examples of metric spaces the metric is defined in terms of additional structure carried by the space X. Of particular importance is the case when X is a vector space on which a *norm* function is defined.

DEFINITION : A <u>norm</u> on the vector (or linear) space X is a function $X \to R: x \mapsto \|x\|$ which satisfies the following axioms.

(N1) $\|x\| \geq 0$ for all $x \in X$.

(N2) $\|x\| = 0$ if and only if $x = 0$.

(N3) $\|\lambda x\| = |\lambda| \, \|x\|$ for all $x \in X$ and scalars $\lambda \in R$.

(N4) $\|x + y\| \leq \|x\| + \|y\|$ for all $x,y \in X$.

[(N4) is sometimes referred to as the triangle inequality.]

REMARK : *It should be apparent that a norm function is a generalization of the absolute value function on R (or the modulus function on C). The norm of the vector x, $\|x\|$, may be thought of as the "length" of the vector, or the distance from x to the origin (the zero vector 0 in X).*

A vector space X together with a norm, $\|.\|$, on it will be referred to as a <u>normed linear space</u> and denoted by $(X,\|.\|)$.

The concept of a normed linear space is essentially due to the Polish mathematician Stefan Banach (1892-1945), who first considered the idea in his doctoral dissertation of 1920. Banach (together with collegues) continued to work on the theory and applications of such spaces for the rest of his life.

Before presenting examples of normed linear space we establish the link between them and metric spaces.

PROPOSITION 1 :- *If* $(X, \|.\|)$ *is a normed linear space, then*

$$d(x,y) = \|x - y\|$$

defines a metric on X which we refer to as the metric induced by the norm $\|.\|$.

PROOF : Using the fact that $\|.\|$ is a norm, ie. satisfies (N1) to (N4) we must show that $d(x,y)$ as defined above satisfies (M1) to (M4).

That $d(x,y) \geq 0$ for all $x,y \in X$ follows directly from (N1).

Next note that

$$\begin{aligned} d(x,y) = 0 &\Leftrightarrow \|x - y\| = 0 \\ &\Leftrightarrow x - y = 0 \quad \text{(by N2)} \\ &\Leftrightarrow x = y \end{aligned}$$

and that

$$\begin{aligned} d(y,x) &= \|y - x\| \\ &= \|-1(x - y)\| \\ &= |-1|\|x - y\| \quad \text{(by N3)} \\ &= \|x - y\| \\ &= d(x,y). \end{aligned}$$

To establish (M4) we argue as follows.

$$\begin{aligned} d(x,y) &= \|x - y\| \\ &= \|x - z + z - y\| \\ &= \|(x - z) + (z - y)\| \\ &\leq \|x - z\| + \|z - y\| \quad \text{(by N4)} \\ &= d(x,z) + d(z,y). \end{aligned}$$

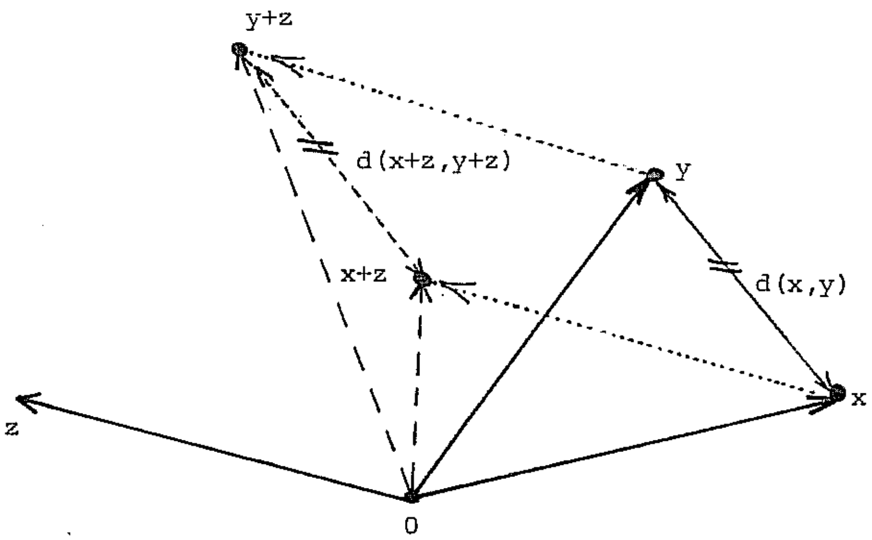
■

A metric induced by a norm has several special properties. For example

1) *translation invariant* ; that is,

$$d(x+z,y+z) = d(x,y) \quad \text{for all } x,y \text{ and } z \in X$$

[Proof. $d(x+z,y+z) = \|x + z - (y + z)\| = \|x - y\| = d(x,y)$].



2) *Provided* X *is a non-trivial vector space* (ie. , $X \neq \{0\}$, or X contains at least one non-zero point), *the value of the induced metric ranges over all positive numbers.*

[Proof. Choose $x \in X$, $x \neq 0$, then $\|x\| \neq 0$. Given any positive number c let $\lambda = c/\|x\|$ and observe that

$$\begin{aligned} d(\lambda x, 0) = \|\lambda x - 0\| &= \|\lambda x\| \\ &= |\lambda|\,\|x\| \\ &= c/\|x\| \cdot \|x\| \\ &= c. \end{aligned}$$]

Not all metrics have these properties. For example; the Post-office metric of Example 3 is not translation invariant $d((\frac{1}{2},0),(1,0)) = 1\frac{1}{2}$ while $d((1\frac{1}{2},0),(2,0)) = 3\frac{1}{2}$. The discrete metric (values 0 or 1) or the metric of Exercise 2 (values between 0 and 1) do not assume all possible positive values. These metrics cannot therefore be induced by a norm. Thus the concept of a metric space is a more general one than that of a normed linear space. Every norm induces a metric, however not all metrics arrise in this way.

EXERCISES.

1) Let $(X, \|.\|)$ be a normed linear space, establish the important inequality

$$\big|\|x\| - \|y\|\big| \leq \|x - y\| \quad \text{for all } x, y \in X.$$

2) If x is a non-zero element of the normed linear space $(X, \|.\|)$ show that $y = \frac{1}{\|x\|}\, x$ is an element of norm 1.

3) Show that the metrics of Example 6 and Exercise 1 are not induced by any norm.

EXAMPLES: (Note, each example of a normed linear space also provides a further example of a metric space.)

1) *Finite dimensional spaces.*

In order to fix notation we mention a large family of norms for $\mathbb{R}^n$, we will however only be interested in three particular cases.

Let $X = \mathbb{R}^n$ the vector space of all ordered n-tuples of real numbers (with addition and scalar multiplication defined component-wise).

For any real number p with $1 \le p < \infty$

$$\|\underset{\sim}{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$$

$$\text{where } \underset{\sim}{x} = (x_1, x_2, \ldots, x_n) \in X$$

defines a norm on X.

Further, $\lim_{p\to\infty} \|\underset{\sim}{x}\|_p = \text{Max}\{|x_1|, |x_2|, \ldots, |x_n|\}$ (can you prove this) and this last expression also defines a norm on X which, for consistency, we denote by $\|\underset{\sim}{x}\|_\infty$ $(= \underset{i=1,\ldots,n}{\text{Max}} |x_i|)$.

Throughout this course we will only consider $\mathbb{R}^n$ with one of the three norms

$$\|\underset{\sim}{x}\|_1 = |x_1| + |x_2| + \ldots + |x_n|$$

$$\|\underset{\sim}{x}\|_2 = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$$ (known as the <u>Euclidean norm</u> on $\mathbb{R}^n$.)

and $$\|\underset{\sim}{x}\|_\infty = \underset{i=1,\ldots,n}{\text{Max}} |x_i|$$ (Sometimes referred to as the <u>uniform or Tchebyscheff norm</u> on $\mathbb{R}^n$.)

We now prove that $\|.\|_1$ and $\|.\|_\infty$ are indeed norms on $\mathbb{R}^n$, the proof for $\|.\|_2$ will be given after the notion of an inner-product has been considered.

That (N1), (N2) and (N3) hold for both $\|.\|_1$ and $\|.\|_\infty$ is easily verified (do so). To establish (N4) in each case we proceed as follows.

For $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)$ and $\underset{\sim}{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$

$$\begin{aligned}\|\underset{\sim}{x} + \underset{\sim}{y}\|_1 &= \|(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)\|_1 \\ &= |x_1 + y_1| + |x_2 + y_2| + \ldots + |x_n + y_n| \\ &\leqslant |x_1| + |y_1| + |x_2| + |y_2| + \ldots + |x_n| + |y_n|\end{aligned}$$

(the triangle inequality for the absolute value function)

$$\begin{aligned}&= |x_1| + |x_2| + \ldots + |x_n| + |y_1| + |y_2| + \ldots + |y_n| \\ &= \|\underset{\sim}{x}\|_1 + \|\underset{\sim}{y}\|_1.\end{aligned}$$

Also,

$$\begin{aligned}\|\underset{\sim}{x} + \underset{\sim}{y}\|_\infty &= \underset{i=1,\ldots,n}{\text{Max}} |x_i + y_i| \\ &\leqslant \underset{i=1,\ldots,n}{\text{Max}} (|x_i| + |y_i|) \\ &\leqslant \underset{i=1,\ldots,n}{\text{Max}} |x_i| + \underset{i=1,\ldots,n}{\text{Max}} |y_i| \\ &= \|\underset{\sim}{x}\|_\infty + \|\underset{\sim}{y}\|_\infty\end{aligned}$$

The normed linear space $(\mathbb{R}^n, \|.\|_p)$, where $p=1,2$ or ∞ is sometimes denoted by ℓ_p^n.

The metrics induced on $X = \mathbb{R}^n$ by these three norms are :-

$$\begin{aligned}d_1(\underset{\sim}{x}, \underset{\sim}{y}) &= \|\underset{\sim}{x} - \underset{\sim}{y}\|_1 \\ &= |x_1 - y_1| + |x_2 - y_2| + \ldots + |x_n - y_n|\end{aligned}$$

[Note; the "taxi-cab" metric of Example 5 of metric spaces is the particular case of this corresponding to n = 2.]

$$d_2(\underset{\sim}{x}, \underset{\sim}{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_n - y_n)^2}$$

[This is the Euclidean metric for $\mathbb{R}^n$, of which the metric in Example 2 is the special case with n = 2. $\mathbb{R}^n$ equipped with this metric is known as <u>n-dimensional Euclidean space</u>.]

$$d_\infty(\underset{\sim}{x},\underset{\sim}{y}) = \text{Max}\{|x_1 - y_1|,|x_2 - y_2|,\dots,|x_n - y_n|\}.$$

[The uniform or Tchebyscheff metric on R^n.]

2) Sequence spaces

The set of all (infinite) sequences of real numbers is a vector space under "term-wise" definitions of vector addition and scalar multiplication:

$$x_1,x_2,\dots,x_n,\dots + y_1,y_2,\dots,y_n,\dots$$
$$= x_1 + y_1,x_2 + y_2,\dots,x_n + y_n,\dots$$

and

$$\lambda(x_1,x_2,\dots,x_n,\dots) = \lambda x_1,\lambda x_2,\dots,\lambda x_n,\dots$$

Formally, the sequence $\underset{\sim}{x} = x_1,x_2,\dots,x_n,\dots$ is the function $\underset{\sim}{x} : N \to R : n \to x_n$, it is also a generalization of an ordered n-tuple (to an "ordered ∞-tuple"), accordingly it is possible to define norms analogous to those of the last example on various subspaces of R^∞.

(a) Let X be the subspace(in the vector space sense) of all bounded sequences; that is, sequences $\underset{\sim}{x} = x_1,x_2,\dots,x_n,\dots$ for which $\text{Sup}\{|x_1|,|x_2|,\dots,|x_n|,\dots\}$ is finite, then

$$\|\underset{\sim}{x}\|_\infty = \sup_n |x_n|$$

is a norm on X. X together with this norm is usually denoted by ℓ_∞ (or m).

Since all convergent sequences are necessarily bounded (see first year or later) an important subspace of ℓ_∞ is c_0; the set of all sequences which converge to 0 together with the "sup-norm", $\|.\|_\infty$, defined above.

(b) For p = 1 or 2 (indeed, for any real number p with $1 \leqslant p < \infty$), ℓ_p is the normed linear space of all "p-summable" sequences (that is, sequences $\underset{\sim}{x} = x_1,x_2,\dots,x_n,\dots$ for which $\sum_{n=1}^{\infty}|x_n|^p < \infty$) together with the norm

$$\|\underset{\sim}{x}\|_p = (|x_1|^p + |x_2|^p + \dots + |x_n|^p + \dots)^{1/p}$$

[That the spaces ℓ_p ($p = 1,2$ and ∞) are indeed normed linear spaces follows by "passage to limit" from the finite cases and the proofs are left as an (optional) EXERCISE.]