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A BANACH SPACE WHICH IS FULLY 2-ROTUND BUT NOT LOCALLY UNIFORMLY ROTUND

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ABSTRACT. A Banach space is fully 2-rotund if (x_n) converges whenever $\|x_n + x_m\|$ converges as $m, n \to \infty$ and locally uniformly rotund if $x_n \to x$ whenever $\|x_n\|$ and $\|(x_n + x)/2\| \to \|x\|$.

We show that l_2 with the equivalent norm

$$\|x\| = ((|x_1| + \|(x_2, \ldots, x_n, \ldots)\|_2)^2 + \|(x_2/2, \ldots, x_n/n, \ldots)\|_2^2)^{1/2}$$

is fully 2-rotund but not locally uniformly rotund, thus answering in the negative a question first raised by Fan and Glicksberg in 1958.

The Banach space $(X, \|\cdot\|)$ is *fully 2-rotund* $(2R)$ if (x_n) is a convergent sequence whenever $\|x_n + x_m\|$ converges as $m, n \to \infty$, and [3] *locally uniformly rotund* (*lur*) if $x_n \to x$ whenever $\|x_n\|$ and $\|(x_n + x)/2\| \to \|x\|$.

The property 2R was first considered by Šmul'yan [8]. It and several generalizations were the subject of an extensive investigation by Fan and Glicksberg [1 and 2]. In [2, p. 563] they raise the question of whether *lur* is a consequence of 2R or its generalizations. They show that a number of weaker properties than 2R imply analogous weakenings of *lur*. A converse question was posed by V. D. Mil'man [4, p. 97] and answered by Mark A. Smith [5], who gave an example of a reflexive *lur* Banach space which is not 2R.

We give an example of a 2R space which is not *lur*. More particularly we show that l_2 with the equivalent norm

$$\|\mathbf{x}\| = ((|x_1| + \|p\mathbf{x}\|_2)^2 + \|T\mathbf{x}\|_2^2)^{1/2},$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n, \ldots)$, $p\mathbf{x} = (0, x_2, x_3, \ldots, x_n, \ldots)$ and $T\mathbf{x} = (0, x_2/2, x_3/3, \ldots, x_n/n, \ldots)$, is 2R but not *lur*, since $\|(\mathbf{e}_1 + \mathbf{e}_n)/2\| \to 1$ while $\|\mathbf{e}_1 - \mathbf{e}_n\| \to 2$.

This space was used by Mark A. Smith [6 and 7] as an example of a *rotund*, indeed uniformly rotund in every direction, Banach space which has the Kadec property *H*, but is not *w-lur* or URWC. As noted by the referee, an l_2-sum of this space with the space of [5] provides an example of a reflexive rotund space with *H*, but which is neither *lur* nor 2R.

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Define

$$\alpha(\mathbf{x}) = |x_1| + \|p\mathbf{x}\|_2$$

and

$$\beta(\mathbf{x}) = \|T\mathbf{x}\|_2$$

then

$$\|\mathbf{x}\| = \|(\alpha(\mathbf{x}), \beta(\mathbf{x}))\|_2.$$

Now consider a sequence $(\mathbf{x}_n) \subset (l_2, \|\cdot\|)$ with $\|\mathbf{x}_n + \mathbf{x}_m\|$ converging, without loss of generality, to 2. In particular then $\|\mathbf{x}_n\| \to 1$ and we have

$$\begin{aligned}\|\mathbf{x}_n + \mathbf{x}_m\| &= \|(\alpha(\mathbf{x}_n + \mathbf{x}_m), \beta(\mathbf{x}_n + \mathbf{x}_m))\|_2 \\ &\leq \|((\alpha(\mathbf{x}_n), \beta(\mathbf{x}_n)) + (\alpha(\mathbf{x}_m), \beta(\mathbf{x}_m))\|_2 \\ &\leq \|\mathbf{x}_n\| + \|\mathbf{x}_m\| \to 2.\end{aligned}$$

Since l_2^2 is 2R, $\|(\alpha(\mathbf{x}_n), \beta(\mathbf{x}_n)) - (\alpha_0, \beta_0)\|_2 \to 0$. Also the above inequalities hold "component-wise" so it follows that

$$\alpha(\mathbf{x}_n + \mathbf{x}_m) \to 2\alpha_0 \quad \text{and} \quad \beta(\mathbf{x}_n + \mathbf{x}_m) \to 2\beta_0.$$

Hence,

$$|x_1^{(n)} + x_1^{(m)}| + \|p\mathbf{x}_n + p\mathbf{x}_m\|_2 \to 2\alpha_0.$$

Now consider the two possibilities:

CASE 1. $x_1^{(n)} \to x_1^{(\infty)}$, then

$$\|p\mathbf{x}_n + p\mathbf{x}_m\|_2 \to 2(\alpha_0 - |x_1^{(\infty)}|).$$

Thus since l_2 is 2R and complete we have

$$\|p\mathbf{x}_n - \mathbf{x}_\infty\|_2 \to 0.$$

Also, since T is continuous and $T\mathbf{x} = Tp\mathbf{x}$, we have

$$\|T\mathbf{x}_n - T\mathbf{x}_\infty\|_2 \to 0$$

and so

$$\|\mathbf{x}_n - (x_1^{(\infty)}, \mathbf{x}_\infty)\| \to 0.$$

CASE 2. $x_1^{(n)}$ does not converge.

In this case, extract a subsequence $\{\mathbf{x}_{n_k}\}$ with the following property:

$$x_1^{(n_{2k})} \to \liminf_n x_1^{(n)} \quad \text{and} \quad x_1^{(n_{2k+1})} \to \limsup_n x_1^{(n)}.$$

By Case 1 above, the subsequence $\{\mathbf{x}_{n_{2k}}\}$ converges to some $\mathbf{x}_E$ and the subsequence $\{\mathbf{x}_{n_{2k+1}}\}$ converges to some $\mathbf{x}_0$ with $\mathbf{x}_0 \neq \mathbf{x}_E$. However, as $k \to \infty$,

$$\left\| \frac{\mathbf{x}_{n_{2k}} + \mathbf{x}_{n_{2k+1}}}{2} \right\| \to 1 \quad \text{and so} \quad \left\| \frac{\mathbf{x}_0 + \mathbf{x}_E}{2} \right\| = 1$$

contradicting the rotundity of $(l_2, \|\cdot\|)$. Thus Case 2 cannot occur, and we conclude that $(l_2, \|\cdot\|)$ is 2R.

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