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A characterization of Banach-star-algebras by numerical range

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It is known that in a B^*-algebra every self-adjoint element is hermitian. We give an elementary proof that this condition characterizes B^*-algebras among Banach*-algebras.

By A we mean a complex Banach*-algebra with a one, e, where $\|e\| = 1$. Following F.F. Bonsall [2] we define the *algebra numerical range* of an element $a \in A$ by $V(a) = \{f(a) : f \in D(e)\}$, where $D(e)$ is the set of normalised states of A, that is

$$D(e) = \{f \in A^* : f(e) = \|f\| = 1\} .$$

We say that an element $h \in A$ is *self-adjoint* if $h = h^*$, and following G. Lumer [3] we say that h is *hermitian* if $V(h) \subset R$. Furthermore we call h *positive hermitian* if $V(h) \subset [0, \infty)$.

G. Lumer [3] has proved that in a B^*-algebra every self-adjoint element is hermitian. By improving a result of I. Vidav [8], T.W. Palmer [4] has shown that this property characterizes B^*-algebras among Banach*-algebras.

The aim of this paper is to furnish a simpler proof of Palmer's result. More precisely we establish the following theorem.

THEOREM A. *A is a B^*-algebra if and only if every self-adjoint*

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element of A *is hermitian.*

Palmer actually shows that a Banach algebra in which every element a has a decomposition $a = u + iv$, where u and v are hermitian, is a B^*-algebra. However in this case $a \mapsto a^* = u - iv$ defines an involution [8, Hilfssatz 2c] for which every self-adjoint element is hermitian, as in Theorem A.

A.M. Sinclair [7] has proved the remarkable equality

(a) $$\nu(h) = \|h\| \text{ , for all hermitian } h \in A \text{ .}$$

Using this we show that every self-adjoint element is hermitian if and only if the square of every self-adjoint element is positive hermitian. This equivalence and Sinclair's result provide the essential techniques for our proof.

It is well known [2] that the spectrum, $\sigma(a)$ of any element a , is contained in the numerical range of that element. Defining the spectral radius of a by $\nu(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}$, and similarly the numerical radius of a by $w(a) = \sup\{|\lambda| : \lambda \in V(a)\}$, we therefore obtain the inequality

(b) $$\nu(a) \le w(a) \text{ , for all } a \in A \text{ .}$$

H. Bohnenblust and S. Karlin [1, p. 129] have proved the following inequality between the norm and the numerical radius.

(c) $$\frac{1}{e}\|a\| \le w(a) \le \|a\| \text{ , for all } a \in A \text{ .}$$

We now investigate properties of A when every self-adjoint element is hermitian.

LEMMA 1. *If every self-adjoint element of* A *is hermitian, then:*

(i) every hermitian element is self-adjoint;

(ii) $V(a^*) = \overline{V(a)}$ *, for all* $a \in A$ *;*

(iii) the involution $*$ *is continuous.*

Proof. Take any $f \in D(e)$ and $a \in A$, let $a = u + iv$, $a^* = u - iv$ (u, v self-adjoint) then

(i) if a is hermitian, $f(a) = f(u) + if(v) \in R$, therefore $f(v) = 0$, all $f \in D(e)$, so $w(v) = 0$ and hence by (c),

$v = 0$ and $a = u$, which is self-adjoint;

(ii) $f(a^*) = f(u) - if(v) = \overline{f(u) + if(v)} = \overline{f}(a) \in \overline{V(a)}$, therefore $V(a^*) \subseteq \overline{V(a)}$ and by symmetry $V(a^*) = \overline{V(a)}$;

(iii) from (ii) $w(a) = w(a^*)$ and consequently, by (c),

$$\frac{1}{e}\|a\| \leq \|a^*\| \leq e\|a\| . \qquad //$$

LEMMA 2. *The self-adjoint elements of* A *are hermitian if and only if the square of every self-adjoint element of* A *is positive hermitian.*

Proof. Let the square of every self-adjoint element of A be positive hermitian; then for any self-adjoint $h \in A$ and $f \in D(e)$, $f(h) = f\left(\frac{1}{2}(h+e)^2-\frac{1}{2}h^2-\frac{1}{2}e\right)$. Therefore

$$f(h) = \tfrac{1}{2}f\left((h+e)^2\right) - \tfrac{1}{2}f(h^2) - \tfrac{1}{2} \in R \text{ and so } V(h) \subset R .$$

Let every self-adjoint element be hermitian. Clearly we need only consider self-adjoint h with $\nu(h) \leq 1$; then, since $\nu(h^2) \leq 1$, we have $\sigma(h^2) \subseteq [0, 1]$. Hence $\sigma(e-h^2) \subseteq [0, 1]$ and therefore $\nu(e-h^2) \leq 1$. By (a) and (b), $\nu(k) = w(k)$ for any self-adjoint $k \in A$. Hence it follows that for any $f \in D(e)$,

$$1 = f(h^2) + f(e-h^2) \leq f(h^2) + |f(e-h^2)| \leq f(h^2) + \nu(e-h^2) \leq f(h^2) + 1$$

and therefore $f(h^2) \geq 0$. //

LEMMA 3. *If every self-adjoint element of* A *is hermitian, then* $\|x\|\|x^*\| \leq 4\|xx^*\|$ *for all* $x \in A$ *, (that is,* A *is an Arens*-algebra).*

Proof. Let $x = u + iv$, $x^* = u - iv$ (u, v self-adjoint); then $xx^* + x^*x = 2u^2 + 2v^2$. For any $f \in D(e)$, by Lemma 2, $f(u^2)$, $f(v^2) \geq 0$, so we have $2f(u^2)$, $2f(v^2) \leq 2\left(f(u^2)+f(v^2)\right) = f(xx^*+x^*x)$ and therefore

$$2\max\{w(u^2), w(v^2)\} \leq w(xx^*+x^*x) \leq w(xx^*) + w(x^*x) .$$

But

$$w(xx^*) = \nu(xx^*) = \nu(x^*x) = w(x^*x)$$

and

$$w(u^2) = \nu(u^2) = \nu(u)^2 \text{ , (similarly for } v) ,$$

by (a) and [5, Lemma 1.4.17].

Therefore,

$$\left(\max\{\nu(u),\ \nu(v)\}\right)^2 \le \nu(xx^*) = \|xx^*\| \ .$$

Further

$$\|x^*\|,\ \|x\| \le \|u\| + \|v\| = \nu(u) + \nu(v) \le 2\max\{\nu(u),\ \nu(v)\} \ ,$$

(Lemma 2 *(iii)*) therefore

$$\frac{1}{4}\|x\|\|x^*\| \le \left(\max\{\nu(u),\ \nu(v)\}\right)^2 \ .$$

Combining these inequalities we have,

$$\|x\|\|x^*\| \le 4\|xx^*\| \ . \qquad //$$

S. Shirali and W.M. Ford [6] have proved that A is symmetric, that is, $-1 \notin \sigma(xx^*)$ for any $x \in A$, provided $\sigma(h) \subset R$ for all self-adjoint $h \in A$. We show that when every self-adjoint element of A is hermitian, their proof may be shortened, as in the following lemma.

LEMMA 4. *If every self-adjoint element of* A *is hermitian then* A *is symmetric.*

Proof. For any $f \in D(e)$, by Lemma 2,

$$f(xx^*) + f(x^*x) = 2f(u^2) + 2f(v^2) \ge 0 \quad (u,\ v \text{ as in Lemma 3})$$

so $f(xx^*) \ge -f(x^*x)$. Therefore if $\lambda \le 0$, $\lambda \in \sigma(x^*x) = \sigma(xx^*)$ there exists $f \in D(e)$ such that $f(xx^*) \ge -\lambda \ge 0$. Hence

$$\sup\{f(xx^*)\} \ge -\inf\{\lambda : \lambda \in \sigma(xx^*)\} \ .$$

But

$$\sup\{f(xx^*)\} = \sup\{\lambda : \lambda \in \sigma(xx^*)\} \ ,$$

otherwise, for $\alpha > \|xx^*\|$, we would have

$$w(\alpha e+xx^*) = \sup\{f(\alpha e+xx^*)\} \ne \sup\{\lambda : \lambda \in \sigma(\alpha e+xx^*)\} = \nu(\alpha e+xx^*)$$

contradicting (a). Therefore $\sup\{\lambda : \lambda \in \sigma(xx^*)\} \ge -\inf\{\lambda : \lambda \in \sigma(xx^*)\}$ thus establishing the result of [6, Lemma 5]. The result now follows by the reasoning of [6, Section 3, p. 278]. //

LEMMA 5. *If every self-adjoint element of* A *is hermitian, then, for an equivalent renorming,* A *is a* B^**-algebra.*

Proof. From Lemmas 2 and 1 *(iii)*, we have by [5, Theorem 4.7.3] that

$f(xx^*) \geq 0$ $(f \in D(e))$ whenever $\sigma(xx^*) \subset [0, \infty)$, but if $-\delta^2 \in \sigma(xx^*)$, then $-1 \in \sigma\left[\delta^{-1}x(\delta^{-1}x)^*\right]$, contradicting Lemma 4. Therefore $f(xx^*) \geq 0$, for all $f \in D(e)$, in which case the Cauchy-Schwartz inequality, $|f(xy^*)|^2 \leq f(xx^*)f(yy^*)$, holds [5, 4.5 (2)]. Using this and (a) it is easily verified that $\|x\|_0^2 = \|xx^*\| = w(xx^*) = \sup\{f(xx^*) : f \in D(e)\}$ is a norm on A satisfying $\|x\|_0^2 = \|xx^*\|_0$. But

$$\|xx^*\| \geq \frac{1}{4} \|x\|\|x^*\| \geq \frac{1}{4} e^{-1}\|x\|^2 ,$$

by Lemmas 3 and 1 *(iii)*; also,

$$\|xx^*\| \leq \|x\|\|x^*\| \leq e\|x\|^2 ,$$

by Lemma 1 *(iii)*. So

$$\frac{1}{2} e^{-\frac{1}{2}}\|x\| \leq \|x\|_0 \leq e^{\frac{1}{2}}\|x\| ,$$

that is $\| \ \|_0$ and $\| \ \|$ are equivalent. //

COROLLARY 5.1. *The two norms of Lemma 5 agree on the self-adjoint elements.*

Lemma 5 also follows from a result of B. Yood [9, Theorem 2.7]. For if every self-adjoint element h of A is hermitian, then its spectrum is real and by (a) $\|h\| = \nu(h)$. However, because of the additive properties of the numerical range we have been able to give a more concise and revealing proof.

We now introduce the following Lemma, which is implicit in the work of Palmer [4].

LEMMA 6. *If A is a B^*-algebra in an equivalent norm $\| \ \|_0$, such that for all self-adjoint elements h of A , $\|h\| = \|h\|_0$, then A is a B^*-algebra in the given norm.*

Proof. Since A with $\| \ \|_0$ is a B^*-algebra, by [4, Lemma 1] its unit ball, $B_0 = \{x \in A : \|x\|_0 \leq 1\}$ is the closed convex hull of the set of elements of the form $\exp(ih)$, where h is hermitian. By [8, Hilfssatz 1], $\|\exp(ih)\| = 1$ so $B_0 \subset B$, that is $\|x\| \leq \|x\|_0$, for all

$x \in A$; therefore $\|xx^*\| \le \|x\|\|x^*\| \le \|x\|_0\|x^*\|_0 = \|xx^*\|_0 = \|xx^*\|$ and so A with $\| \ \|$ is a B^*-algebra. //

Combining Lemma 6 with Lemma 5 and Corollary 5.1 we obtain the sufficiency in Theorem A. Necessity follows from [3, Lemma 20].

As in [10, Corollary 1] Theorem A can be stated in the apparently stronger form:

THEOREM A[1]. *A is a B^*-algebra if and only if the set of hermitian self-adjoint elements of A is dense in the set of self-adjoint elements.*

Since the set of self-adjoint elements is closed in A , it is sufficient to establish the following lemma.

LEMMA 7. *The set of hermitian elements of A is closed.*

Proof. Let $\{h_n\}$ be any sequence converging to h , with $V(h_n) \subset R$, for all n . For any $\varepsilon > 0$ there exists N so that $\|h_n-h\| \le \varepsilon$ whenever $n \ge N$. If $\lambda \in V(h)$ then $\lambda = f(h)$ for some $f \in D(e)$. Let $\lambda_n = f(h_n)$ for all n , then $|\lambda_n-\lambda| = |f(h_n-h)| \le \|h_n-h\| \le \varepsilon$ for $n \ge N$. So λ is the limit of a sequence of real numbers and therefore λ is real. //

[Added 16 November 1970]. We give an example to show that

If every hermitian element is self-adjoint, then A is not necessarily a B^*-algebra even under equivalent renorming.

Let $X = l_\infty^2$ and take $A = L(X)$, all the 2×2 matrices with complex entries.

If $a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is such that $V(a) \subset R$, then it is well known [2] that $f_x(ax) \in R$ for all $x \in X$ with $\|x\| = 1$ and all $f_x \in X^*$ such that $f_x(x) = \|f\| = 1$. Let $x = (1, 0)$; then $f_x = (1, 0)$ and so $f_x(ax) \in R$ implies that $a_{11} \in R$. Similarly $a_{22} \in R$.

Now choose $x = (1, \lambda)$ for any complex λ where $0 < |\lambda| < 1$. Then $f_x = (1, 0)$, $f(ax) = a_{11} + a_{12}\lambda \in R$ and therefore $a_{12} = 0$. Similarly

$a_1 = 0$. It follows that $a \in A$ is hermitian if and only if

$a = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$ for $\alpha, \beta \in R$.

Define the involution $*$ on A by

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^* = \begin{bmatrix} \bar{a}_{11} & -\bar{a}_{21} \\ -\bar{a}_{12} & \bar{a}_{22} \end{bmatrix} ;$$

then every hermitian element is self-adjoint (but not conversely). However $*$ is not proper (that is $aa^* = 0$ does not imply $a = 0$) ; for example take $a = \begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix}$; and so A cannot be a B^*-algebra for any norm.

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