

Banach Lattices and the Weak Fixed Point Property

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ABSTRACT

Banach space properties that imply the weak fixed point property are investigated in a Banach lattice setting.

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1. INTRODUCTION

A Banach space is said to have the weak fixed point property (w-FPP) if every nonexpansive mapping on every nonempty weak compact convex set has a fixed point. The weak fixed point property and Banach lattices has not been the subject of many papers in the last twenty or so years; see Sine [23], Soardi [24], Maurey [15], Elton *et al.* [7], Borwein and Sims [3], Lin [12], Sims [19] and [20], and Khamsi and Turpin [11]. This is despite the fact that many examples have an order theoretic nature, see for example Borwein and Sims [3].

Hoping to generate renewed interest in the w-FPP and Banach lattices, this paper revisits the property of weak orthogonality from Borwein and Sims [3] and Sims [19]. We then consider Banach lattices with uniformly monotone norm, a property that was exploited in Elton *et al.* [7]. Along the way, other properties known to be associated with the w-FPP in Banach spaces are studied in the context of Banach lattices.

2. PRELIMINARIES

The usual approach to proving that a particular Banach space has the w-FPP is to assume that it does not have this property and obtain a contradiction. Thus there is a nonempty weak compact convex set C with a fixed point free nonexpansive mapping T where $T : C \rightarrow C$. Using the weak compactness of C and the nonexpansiveness of T it can be shown that there exists, in C , a weak null sequence with certain properties involving the norm. So most approaches to the w-FPP problem have involved weak null sequences and their relationship to the norm. In Banach lattices, the lattice structure can be added to this mix. The following definitions reflect this situation.

Opial's condition, from Opial [17], states

$$\text{if } x_n \rightharpoonup 0 \text{ and } x \neq 0 \text{ then } \limsup_n \|x_n\| < \limsup_n \|x_n - x\|.$$

Nonstrict Opial condition has the strict inequality replaced by ' \leq '. Uniform Opial's condition in Prus [18] is a strengthening of Opial's condition:

for every $\epsilon > 0$ there is an $r > 0$ such that

$$1 + r \leq \liminf_n \|x_n + x\|$$

for each $x \in X$ with $\|x\| \geq 1$ and each sequence (x_n) , $x_n \rightharpoonup 0$, with $\liminf_n \|x_n\| \geq 1$.

There is an Opial's modulus, introduced in Lin *et al.* [13], defined as

$$r(c) := \inf\{\liminf_n \|x_n + x\| - 1 : \|x\| \geq c, x_n \rightarrow 0 \text{ and } \liminf_n \|x_n\| \geq 1\}$$

for $c > 0$.

X has the uniform Opial's condition if and only if $r(c) > 0$ for all $c > 0$ and the nonstrict Opial condition if and only if $r(c) \geq 0$ for all $c > 0$, see Dalby [5].

A slightly different but related approach produces the following property, due to Sims [22]. A Banach space has property(K) if there exists $K \in [0, 1)$ such that whenever $x_n \rightarrow 0$, $\|x_n\| \rightarrow 1$ and $\liminf_n \|x_n - x\| \leq 1$ we have $\|x\| \leq K$.

If K is not the same across X but depends on the sequence (x_n) , then the condition is called property (k). Property(K) with $K = 0$ is equivalent to Opial's condition and Dalby [5] showed that a Banach space has property(K) if and only if $r(1) > 0$. Sims [22] proved that property(K) implies weak normal structure.

Next some definitions for Banach lattices. A Banach lattice is said to be weakly orthogonal if whenever $x_n \rightarrow 0$ then

$$\lim_n \||x_n| \wedge |x|\| = 0 \text{ for all } x \in X.$$

Sims [20] showed that weakly orthogonal Banach lattices have the w-FPP and a Banach space X has the w-FPP if there exists a weakly orthogonal Banach lattice Y with $d(X, Y) < \sqrt{5} - 1$ where $d(X, Y)$ is the Banach-Mazur distance between X and Y . In [4], Dalby extended the distance to $\frac{\sqrt{33}-3}{2}$.

Note that Borwein and Sims [3] used a slightly weaker definition of weak orthogonality, namely when $x_n \rightarrow 0$ then

$$\lim_m \lim_n \||x_m| \wedge |x_n|\| = 0.$$

It has become the practice to use the stronger definition when referring to weak orthogonality, see for example Sims [19] and Garcia-Falset [8].

The norm of a Banach lattice is said to be uniformly monotone if given $\epsilon > 0$ there is a $\delta > 0$ such that if $x, y \geq 0$ with $\|y\| = 1$ and $\|x + y\| \leq 1 + \delta$ then $\|x\| \leq \epsilon$.

An equivalent definition is:

There exists a strictly increasing continuous function δ on $[0,1]$ with $\delta(0) = 0$ so that if $x, y \geq 0$ with $1 = \|y\| \geq \|x\|$ then $\|x + y\| \geq 1 + \delta(\|x\|)$.

Birkhoff [2] was responsible for the first version and the second version appeared in Katznelson and Tzafriri [10]. Akcoglu and Sucheston [1] considered both these and several other formulations. They showed that the two definitions are equivalent and in Orlicz function spaces they are equivalent to the Δ_2 condition.

Note that a Banach lattice, X , that has a uniformly monotone norm is weak sequentially complete so it cannot contain a subspace isomorphic to c_0 . In particular, X has order continuous norm. Recall that the norm is said to be order continuous if $\inf\{\|x\| : x \in A\} = 0$ for every downward directed set $A \subset X$ such that $\inf(A) = 0$.

The norm of a Banach lattice is said to be strictly monotone if $x > y \geq 0$ implies $\|x\| > \|y\|$.

X having a uniformly monotone norm is equivalent to $\tilde{X} := l_\infty(X)/c_0(X)$ having a strictly monotone norm. See for example Elton *et al.* [7].

Finally, a Banach lattice has a p -superadditive norm if

$$(\|x\|^p + \|y\|^p)^{1/p} \leq \|x + y\| \quad \text{for all disjoint } x, y.$$

A p -superadditive norm is a uniformly monotone norm.

The w-FPP is separably determined, see for example Goebel and Kirk [9]. So throughout this paper X will be assumed to be an infinite dimensional separable Banach lattice. So if X is σ -Dedekind complete then the norm is order continuous, see Lindenstrauss and Tzafriri [14] or Meyer-Nieberg [16] for details.

3. RESULTS

First a result concerning the nonstrict Opial condition.

Proposition 1. *If X is a Banach lattice with order continuous norm then X satisfies the nonstrict Opial condition for positive weak null sequences.*

Proof: Let $x_n \rightarrow 0$ where $x_n \geq 0$ for all n , then by proposition 2.3.4 of Meyer-Nieberg [16], there exists a disjoint sequence, (x_n^*) , of positive elements in B_{X^*} such that

$$\limsup_n x_n^*(x_n) = \limsup_n \|x_n\|.$$

Order continuity of the norm implies that $x_n^* \xrightarrow{*} 0$. So for any $x \in X$

$$\begin{aligned} \limsup_n \|x_n + x\| &\geq \limsup_n x_n^*(x_n + x) \\ &= \limsup_n x_n^*(x_n) \\ &= \limsup_n \|x_n\|. \end{aligned}$$

■

The situation involving Opial's condition and uniform Opial's condition is left for the moment until weak orthogonality has been dealt with.

Proposition 2. *If X is a Banach lattice with order continuous norm then the lattice operations are weak sequentially continuous if and only if X is weakly orthogonal, and hence has the w-FPP.*

Proof: (\Rightarrow) Let $x_n \rightarrow 0$ then $|x_n| \rightarrow 0$. For $x \in X$ let

$$y_n := |x_n| \wedge |x|.$$

Then $y_n \rightarrow 0$ and $0 \leq y_n \leq |x|$. If $\lim_n \|y_n\| \neq 0$ then by taking subsequences we have $\inf_n \|y_n\| \geq \alpha$ for some $\alpha > 0$.

Let $z_n := y_n/\alpha$ then $z_n \rightarrow 0$, $0 \leq z_n \leq |x|/\alpha$ and $\|z_n\| \geq 1$ for all n .

By corollary 2.3.5 of Meyer-Nieberg [16], for $0 < \beta < 1$ there exists a subsequence (z_{n_k}) and disjoint (w_k) such that $0 \leq w_k \leq z_{n_k}$ and $\|w_k\| \geq \beta > 0$ for all k . So $0 \leq w_k \leq |x|/\alpha$ for all k . Then the order continuous norm and theorem 2.4.2 of Meyer-Nieberg [16] means $\|w_k\| \rightarrow 0$, a contradiction.

(\Leftarrow) This follows the ideas contained in the proof of proposition 2.3.23 in Meyer-Nieberg [16]. That is, let $x_n \rightarrow 0$ and by theorem 2.5.9 of [16], it suffices to show that $|x_{n_k}| \rightarrow 0$ for every subsequence such that $(|x_{n_k}|)$ is weak Cauchy.

From lemma 2.5.11 of [16], there exists an increasing, positive sequence (y_k) such that $|x_{n_k}| - y_k \rightarrow 0$.

Weak orthogonality implies that

$$\lim_k \| |x_{n_k}| \wedge |x| \| = 0 \text{ for all } x \in X$$

and

$$\lim_k \| |x_{n_k}| - y_k \wedge |x| \| = 0 \text{ for all } x \in X.$$

Therefore

$$|x_{n_k}| \wedge |x| + |x_{n_k}| - y_k \wedge |x| \rightarrow 0.$$

But

$$\begin{aligned}
|x_{n_k}| \wedge |x| + ||x_{n_k}| - y_k| \wedge |x| &\geq (|x_{n_k}| + ||x_{n_k}| - y_k|) \wedge |x| \\
&\geq |y_k| \wedge |x| \\
&= y_k \wedge |x| \\
&\geq 0.
\end{aligned}$$

Therefore $\lim_k \| y_k \wedge |x| \| = 0$.

But since (y_k) is increasing so is $(y_k \wedge |x|)$ which means that $y_k = 0$ for all k and so $|x_{n_k}| \rightarrow 0$. ■

So Banach lattices with lattice operations weak sequentially continuous and order continuous norm have the w-FPP, as do Banach spaces whose Banach-Mazur distance from such lattices is less than $\frac{\sqrt{33}-3}{2}$

The lattice operations of any abstract M space are weak sequentially continuous. See for example Meyer-Nieberg [16], proposition 2.1.11. But lemma 1.b.10 of Lindenstrauss and Tzafriri [14] states that an abstract M space has order continuous norm if and only if it is order isometric to $c_0(\Gamma)$, for some index set Γ . So proposition 2 includes c_0 but excludes any M space with an order unit, for example $C(K)$ where K is an infinite compact Hausdorff space. Also see Borwein and Sims [3] for further consequences of proposition 2.

If the Banach lattice is atomic then by proposition 2.5.23 of Meyer-Nieberg [16] the lattice operations are weak sequentially continuous and so we have the following corollary.

Corollary 3. *Let X be an atomic Banach lattice with order continuous norm then X is weakly orthogonal, and hence has the w-FPP.*

It is well known that if X is a Banach lattice and $c_0 \not\hookrightarrow X$ (by Lindenstrauss and Tzafriri [14] this is equivalent to X not containing a sublattice order isomorphic to c_0) then X has order continuous norm which leads to the following.

Corollary 4. *Let X be an atomic Banach lattice where $c_0 \not\hookrightarrow X$ then X is weakly orthogonal, and hence has the w-FPP.*

Khamsi and Turpin [11] considered Banach spaces with a vector lattice structure satisfying:

$$(\alpha) \quad (x^+ \leq y^+ \text{ and } x^- \leq y^-) \Rightarrow \|x\| \leq \|y\|, \quad x, y \in X;$$

$$(\beta) \text{ for some real constant } k < 2, |x| \leq |y| \Rightarrow \|x\| \leq k\|y\|, \quad x, y \in X.$$

Instead of the weak topology, the topology, τ , studied was the coarsest topology on X for which the map $x \rightarrow \| |x| \wedge u \|$ is continuous at 0 for every $u \in X, u \geq 0$. Khamsi and Turpin showed that every nonexpansive map on every nonempty τ -compact convex subset has a fixed point. For weakly orthogonal Banach lattices this is the w-FPP result of Sims [19].

Garcia-Falset [8] extended this set up to have $k \leq 2$ but required the additional condition of the alternate-signs Banach-Saks property. In this paper Garcia-Falset called a Banach space, X , weakly orthogonal if X satisfies (α) and (β) and if for each weakly null sequence (x_n) in X , $\lim_n \| |x_n| \wedge |x| \| = 0$ for every $x \in X$. To obtain the w-FPP the additional condition was the weak Banach-Saks property.

Related to the foregoing is the following.

Question: If X is a weakly orthogonal Banach lattice with norm $\| \cdot \|$, does X with the new norm $\|x\|_1 := \|x^+\| \vee \|x^-\|$ satisfy the w-FPP? It is straight forward to show that $\| \cdot \|_1$ is an equivalent Banach space norm that satisfies (α) and (β) . So $(X, \| \cdot \|_1)$ satisfies the w-FPP if X has the weak Banach-Saks

property.

To obtain Opial's condition, the condition that X must have an order continuous norm has to be strengthened to X having a uniformly monotone norm.

Note that a Banach lattice that is weakly orthogonal has the Banach space property, WORTH:

$$\text{if } x_n \rightarrow 0 \text{ then } \limsup_n \|x_n - x\| = \limsup_n \|x_n + x\| \text{ for all } x \in X.$$

This in turn implies the nonstrict Opial condition.

Proposition 5. *If X is a Banach lattice with uniformly monotone norm and whose lattice operations are weak sequentially continuous then X satisfies Opial's condition.*

Proof: Recall that a uniformly monotone norm implies that $c_0 \not\hookrightarrow X$ and so by proposition 2, X is weakly orthogonal. Thus X has WORTH. Assume that X does not satisfy Opial's condition then there exists $x_n \rightarrow 0$ and a nonzero $x \in X$ such that

$$\limsup_n \|x_n\| \not\leq \limsup_n \|x_n + x\|.$$

Since X satisfies the nonstrict Opial condition we have

$$\limsup_n \|x_n\| = \limsup_n \|x_n + x\|.$$

Without loss of generality we may assume $\lim_n \|x_n\| = \lim_n \|x_n + x\| = 1$ and $\inf_n \|x_n\| > 0$. So $x_n/\|x_n\| \rightarrow 0$ and $|x_n|/\|x_n\| \rightarrow 0$.

Nonstrict Opial condition implies

$$\begin{aligned}
1 &\leq \limsup_n \|x_n/\|x_n\| + x\| \\
&\leq \limsup_n \| |x_n|/\|x_n\| + |x| \| \\
&= \limsup_n \| |x_n|/\|x_n\| - |x| \| \quad \text{using WORTH} \\
&\leq \limsup_n \|x_n/\|x_n\| + x\|.
\end{aligned}$$

Therefore

$$1 \leq \limsup_n \| |x_n|/\|x_n\| + |x| \| = \limsup_n \|x_n/\|x_n\| + x\|.$$

Also

$$\begin{aligned}
\limsup_n \|x_n/\|x_n\| + x\| &\leq \limsup_n \|x_n/\|x_n\| - x_n\| + \lim_n \|x_n + x\| \\
&= \lim_n |1/\|x_n\| - 1| \|x_n\| + \lim_n \|x_n + x\| \\
&= 1.
\end{aligned}$$

Thus using the weak lower semi-continuity of the norm

$$1 = \limsup_n \| |x_n|/\|x_n\| + |x| \| \geq \| |x| \|.$$

This means that $\| |x| \| \leq \| |x_n|/\|x_n\| \|$ for all n . The uniformly monotone norm means there exists a strictly increasing continuous function δ on $[0, 1]$ where

$$\| |x_n|/\|x_n\| + |x| \| \geq 1 + \delta(\|x\|) \quad \text{for all } n.$$

Letting $n \rightarrow \infty$ we have

$$1 = \limsup_n \| |x_n|/\|x_n\| + |x| \| \geq 1 + \delta(\|x\|) > 1.$$

A contradiction. ■

It can be shown that a Banach lattice satisfying the conditions of proposition 5 has property(K) and so has weak normal structure.

Uniform Opial's condition can be found by using the spaces l_p , $1 < p < \infty$, as guides.

Proposition 6. *If X is a Banach lattice with p -superadditive norm, $1 < p < \infty$, and whose lattices operations are weak sequentially continuous then X satisfies the uniform Opial's condition with $r(c) \geq (1 + c^p)^{1/p} - 1$.*

Proof: Recall that a norm is p -superadditive if

$$\|x\|^p + \|y\|^p \leq \|x + y\|^p \quad \text{for all disjoint } x, y.$$

It can be shown that this condition is equivalent to the same inequality where x and y are merely ≥ 0 . See for example proposition 2.8.2 of Meyer-Nieberg [16].

Let $x_n \rightarrow 0$, $\liminf_n \|x_n\| \geq 1$ and $\|x\| \geq c > 0$. Then

$$\| |x_n| \|^p + \| |x| \|^p \leq \| |x_n| + |x| \|^p \quad \text{for all } n.$$

So

$$\left(\liminf_n \|x_n\|^p + \|x\|^p \right)^{1/p} \leq \liminf_n \| |x_n| + |x| \|.$$

Using weak orthogonality and a similar argument to that in proposition 5 we have

$$\liminf_n \| |x_n| + |x| \| = \liminf_n \|x_n + x\|$$

and thus

$$\liminf_n \|x_n + x\| \geq (1 + c^p)^{1/p} = 1 + [(1 + c^p)^{1/p} - 1].$$

Which means that X satisfies the uniform Opial's condition with

$$r(c) \geq (1 + c^p)^{1/p} - 1.$$

■

This proposition covers the cases of l_p , $1 < p < \infty$. Note that if the norm is p -additive and X is atomic and separable then X is isometrically isomorphic to l_p .

It is a long standing conjecture in metric fixed point theory that reflexivity and the fixed point property are linked. This means that the presence or absence of c_0 and l_1 is of interest, which leads to the following proposition.

Proposition 7. *Let X be a Banach space. If $c_0 \hookrightarrow X$ then X does not have property(K).*

Proof: $c_0 \hookrightarrow X$ if and only if there exists a sequence (ϵ_n) in $(0, 1)$ where $\epsilon_n \rightarrow 0$ and a sequence (x_n) in X such that

$$(1 - \epsilon_n) \sup_{k \geq n} |t_k| \leq \left\| \sum_{k=n}^{\infty} t_k x_k \right\| \leq (1 + \epsilon_n) \sup_{k \geq n} |t_k| \quad \text{for all } (t_k) \in c_0, \text{ for all } n \in \mathbb{N}.$$

Without loss of generality $\epsilon_n \downarrow 0$. Note that $x_n \rightarrow 0$ and $\lim_n \|x_n\| = 1$.

Fix $n \in \mathbb{N}$ then

$$1 - \epsilon_n \leq \|x_n - x_k\| \leq 1 + \epsilon_n \text{ for all } k > n.$$

So

$$1 - \epsilon_n \leq \lim_k \|x_n - x_k\| \leq 1 + \epsilon_n.$$

Therefore $r(\|x_n\|) \leq \lim_k \|x_n - x_k\| - 1$ for all n and $r(\|x_n\|) \leq \epsilon_n$ for all n .

Taking $n \rightarrow \infty$ we have $r(1) \leq 0$ which implies $r(1) = 0$. But X has property(K) if and only if $r(1) > 0$.



Another way of viewing this is: if X has an equivalent norm which satisfies property(K) then X does not contain an isomorphic copy of c_0 .

Note: Thus if X is a Banach lattice where $l_1 \not\hookrightarrow X$ and X has property(K) then X is reflexive.

Dalby [6] showed that if X^* satisfies the condition that $R(X^*) < 2$ and has the nonstrict *Opial property then X satisfied property(K). So if X is a Banach lattice with X^* order continuous, $R(X^*) < 2$ and having the nonstrict *Opial property then X is reflexive.

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