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OPERATOR THEORY

Convergence of Picard Iterates of Nonexpansive Mappings

by

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Summary. Let X be a Banach space, C a closed subset of X, and $T : C \to C$ a nonexpansive mapping. Conditions are given which assure that if the fixed point set $F(T)$ of T has nonempty interior then the Picard iterates of the mapping T always converge to a point of $F(T)$. If T is asymptotically regular, it suffices to assume that the closed subsets of X are densely proximinal and that nested spheres in X have compact interfaces. Such spaces include, among others, those which have Rolewicz's property (β). If X has strictly convex norm the asymptotic regularity assumption can be dropped and the nested sphere property holds trivially. Consequently the result holds for all reflexive locally uniformly convex spaces.

1. Preliminaries. In 1979 J. Moreau [13] proved that if C is a closed subset of a Hilbert space H, if $T : C \to C$ is nonexpansive, and if the fixed point set $F(T)$ of T has nonempty interior, then for every $x \in C$ the Picard iterates of T at x converge strongly to a point of $F(T)$. Subsequently B. Beauzamy observed that this result also holds in uniformly convex spaces; see e.g., [7, pp. 219–222], for a proof. In this note we show that this fact extends to reflexive locally uniformly convex spaces. We also show that if either C is convex or T is asymptotically regular, similar results hold for spaces satisfying Rolewicz's property (β).

We begin with some definitions and basic facts. Throughout X will denote a Banach space. We use B_X and S_X to denote the respective closed unit ball and unit sphere of X. For brevity B_r will denote the closed ball centered at the origin with radius $r \geqslant 0$, and S_r the surface of B_r. Thus $S_X = \{x \in X : \|x\| = 1\}$ and, for $r \neq 1$, $S_r = \{x \in X : \|x\| = r\}$. In

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general we use $B_r[x]$ and $S_r[x]$ to denote respectively the closed ball and sphere centered at $x \in X$ of radius r.

DEFINITION 1.1. A mapping $T : C \to C$ is said to be *asymptotically regular* if

$$\lim_{n\to\infty} \|T^n(x) - T^{n+1}(x)\| = 0 \quad \text{for all } x \in C.$$

A mapping $T : C \to C$ is said to be *nonexpansive* if $\|T(x) - T(y)\| \leqslant \|x - y\|$ for every $x, y \in C$. Asymptotic regularity arises naturally in the study of nonexpansive mappings. If $T : C \to C$ is nonexpansive and if C is bounded and convex, then a fundamental result of Ishikawa [6] assures that for any $t \in (0,1)$ the mapping $T_t : C \to C$ defined by

$$T_t(x) = (1-t)x + tT(x), \quad x \in C,$$

is always asymptotically regular. Morever $F(T_t) = F(T)$.

Another concept plays a key role in what follows.

DEFINITION 1.2. A set C is said to be *densely proximinal* in X if the set

$$D(C) := \{x \in X : \exists\, p(x) \in C \text{ with } \|x - p(x)\| = \operatorname{dist}(x, C)\}$$

is dense in X.

Our results depend on two well-known elementary facts. Suppose C is a closed subset of X and $T : C \to C$.

Remark 1.1. If T is nonexpansive and $\lim_{k\to\infty} T^{n_k}(x) = z \in F(T)$, then

$$\lim_{n\to\infty} T^n(x) = z.$$

Remark 1.2. If T is continuous and asymptotically regular, and if

$$\lim_{k\to\infty} T^{n_k}(x) = z,$$

then $z \in F(T)$.

2. Main results. This section contains our most general results. Specific conditions under which they apply are given in the next section.

We say that X has the *nested sphere property* if nested spheres in X have compact interfaces. By this we mean that for each $x, y \in X$, with $x \neq y$, and $r > 0$,

$$S_r[x] \cap S_d[y]$$

is compact whenever $d = r + \|x - y\|$.

THEOREM 2.1. *Let X be a Banach space whose nonempty closed subsets are densely proximinal and suppose X has the nested sphere property. Let*

C be a closed subset of X and suppose $T : C \to C$ is both nonexpansive and asymptotically regular with $\operatorname{int}(F(T)) \neq \emptyset$. *Then for each $x \in C$ the Picard sequence $\{T^n(x)\}$ converges to a point of $F(T)$.*

Spaces having strictly convex norm trivially satisfy the nested sphere property since in this case the prescribed intersection reduces to a single point. (This is discussed in more detail in Section 4.) In fact, as the next theorem shows, in spaces with strictly convex norm the asymptotic regularity assumption can be dropped as well.

THEOREM 2.2. *Let X be a Banach space whose norm is strictly convex and whose nonempty closed subsets are densely proximinal. Let C be a closed subset of X and suppose $T : C \to C$ is nonexpansive with* $\operatorname{int}(F(T)) \neq \emptyset$. *Then, for each $x \in C$ the Picard sequence $\{T^n(x)\}$ converges to a point of $F(T)$.*

Proof of Theorem 2.1. Let $x \in C$ and assume $\{T^n(x)\}$ does not converge to a point of $F(T)$. Then by Remark 1.2 the set $O(x) := \{x, T(x), T^2(x), \ldots\}$ is discrete, hence closed and so densely proximinal. Thus there exist $p \in \operatorname{int}(F(T))$ and $n_0 \in \mathbb{N}$ such that

$$d := \operatorname{dist}(p, O(x)) = \|p - T^{n_0}(x)\| \geqslant 0.$$

If $d = 0$ there is nothing to prove so we assume $d > 0$. Since T is nonexpansive,

$$\|p - T^n(x)\| = d \quad \text{for all } n \geqslant n_0.$$

Also, since $p \in \operatorname{int}(F(T))$, there exists $q \in F(T)$ with $q \neq p$ such that

$$\|p - q\| + \|q - T^{n_0}(x)\| = \|p - T^{n_0}(x)\|.$$

Let $\|q - T^{n_0}(x)\| = r$ and suppose there exists $n \geqslant n_0$ such that $\|q - T^n(x)\| < r$. Then

$$\|p - T^n(x)\| \leqslant \|p - q\| + \|q - T^n(x)\| < \|p - q\| + r = d.$$

Since this contradicts the definition of d it must be the case that

$$\|q - T^n(x)\| = r \quad \text{for all } n \geqslant n_0.$$

Hence $T^n(x) \in S_d[p] \cap S_r[q]$, $n \geqslant n_0$, and by assumption this intersection is compact. Therefore $\{T^n(x)\}$ has a subsequence which converges to a point $z \in C$. By Remarks 1.1 and 1.2, $z \in F(T)$ and $\lim_{n\to\infty} T^n(x) = z$.

Proof of Theorem 2.2. Let $x \in C$, and choose $p \in \operatorname{int}(F(T))$ and $z \in \overline{O(x)}$ so that

$$\|p - z\| = \operatorname{dist}(p, \overline{O(x)}) := d.$$

If $d = 0$ the result follows from Remark 1.1. Otherwise choose $q \in F(T)$, $q \neq p$, so that

$$\|p - q\| + \|q - z\| = \|p - z\|.$$

If $\|q - T(z)\| < \|q - z\|$ then $\|p - T(z)\| < \|p - z\|$ with $T(z) \in \overline{O(x)}$, and this contradicts the definition of d. Thus it must be the case that $\|q - T(z)\| = \|q - z\|$, from which

$$\|p - q\| + \|q - z\| = \|p - z\| = \|p - q\| + \|q - T(z)\|.$$

This implies that both z and $T(z)$ lie in $S_d[p] \cap S_r[q]$ where $r = \|q - z\|$. Since the norm of X is strictly convex this can only happen if $T(z) = z$. □

3. Some consequences. It is known that nonempty closed subsets of reflexive locally uniformly convex Banach spaces are densely proximinal (Lau [12], Theorem 5). Thus the following is a corollary to Theorem 2.2.

COROLLARY 3.1. *Let X be a reflexive locally uniformly convex Banach and let C be a closed subset of X. Suppose $T : C \to C$ is nonexpansive with* $\operatorname{int}(F(T)) \neq \emptyset$. *Then for each $x \in C$ the Picard sequence $\{T^n(x)\}$ converges to a point of $F(T)$.*

There are consequences of Theorem 2.1 as well. Let X be a Banach space and for $x \in X \setminus B_X$ set

$$D(x, B_X) = \operatorname{conv}(\{x\} \cup B_X);$$
$$R(x, B_X) = D(x, B_X) \setminus B_X.$$

The set $D(x, B_X)$ is called the *drop* of x. In [14] S. Rolewicz introduced the following as an extension of the so-called 'drop property'. Here α denotes the usual Kuratowski measure of noncompactness.

DEFINITION 3.1. A Banach space X is said to have *property* (β) if for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in X \setminus B_X$ satisfies $1 < \|x\| < 1+\delta$, then $\alpha[R(x, B_X)] < \varepsilon$. If δ depends on x then X is said to be *locally* (β).

Rolewicz observed in [14] that all uniformly convex spaces have property (β) and that spaces with property (β) are always nearly uniformly convex (NUC) in the sense of Huff [5]. Spaces with property (β) have been the subject of a number of more recent papers. It is known, for example, that if a Banach space is nearly uniformly smooth and NUC then it has property (β) [9]. Also, a uniformly smooth space which has Kadec-Klee norm has property (β) [8]. (For related results see Kutzarova, et al. [11].)

Closed subsets of spaces which have property (β) are densely proximinal. This is a consequence of Theorem 4 of [12] and the fact that a space is NUC if and only if it is reflexive and has uniformly Kadec-Klee norm [5].

Also property (β) implies the nested sphere property. This follows from Proposition 3.1 of [8] and is also a consequence of the results of Section 4 of this paper. Therefore the following is a corollary of Theorem 2.1.

COROLLARY 3.2. *Let X be a Banach space which has property (β) and let C be a closed subset of X. Suppose $T : C \to C$ is both nonexpansive and asymptotically regular with* $\operatorname{int}(F(T)) \neq \emptyset$. *Then for each $x \in C$ the Picard sequence $\{T^n(x)\}$ converges to a point of $F(T)$.*

Because of Ishikawa's result this leads to the follow fact for convex C.

COROLLARY 3.3. *Let X be a Banach space which has property (β), let C be a bounded closed and convex subset of X, and suppose $T : C \to C$ is nonexpansive with* $\operatorname{int}(F(T)) \neq \emptyset$. *Let $T_t = tI + (1-t)T$ for $t \in (0,1)$. Then for each $x \in C$ the Picard sequence $\{T_t^n(x)\}$ converges to a point of $F(T)$.*

Finally we remark that if a Banach space has a Kadec-Klee norm then it is locally uniformly convex if and only if it is locally (β) and has strictly convex norm [10]. Moreover, a Banach space is reflexive with a smooth Kadec-Klee norm if and only if its nonempty closed subsets are densely proximinal and the corresponding distance functions possess minimal subdifferential mappings; see [1, Corollary 6.3].

4. Spaces with NSP. In light of Theorem 2.1 it seems natural to focus more attention on the nested sphere property (NSP). While this property is weaker than strict convexity of the norm, we observe here that convex subsets of spheres in such spaces must be compact. Hence if a Banach space X has the NSP, then it has the fixed point property for spheres, a fact useful in the study of the structure of the fixed point sets of nonexpansive mappings [4].

By a *face* of the ball $B_r[x]$ we understand a maximal convex subset of $S_r[x]$. For $y \in S_r[x]$ let

$$\mathfrak{F}_y = \{F : F \text{ is a face of } B_r[x] \text{ with } y \in F\}.$$

LEMMA 4.1. *If $r > 0$ and $x \neq y$, then for $d = r + \|x - y\|$,*

$$S := S_r[x] \cap S_d[y] = x + r \cup \{F : F \in \mathfrak{F}_{(x-y)/\|x-y\|}\}.$$

Proof. First we note that $S' := \frac{1}{d}(S - y) = B_t[(1-t)z] \cap S_X$, where $t = r/d$ and $z = (x-y)/\|x-y\| \in S_X$. Let $w \in S'$. Then $\|w\| = 1$ and $w = (1-t)z + tv$ for some $v \in S_X$. Choose $f \in S_{X^*}$ so that $f(w) = 1$, whence $f(z) = f(v) = 1$. Thus $v \in F = f^{-1}(1) \cap S_X \in \mathfrak{F}_z$; so

$$w \in (1-t)z + t \cup \{F : F \in \mathfrak{F}_z\}.$$

Conversely, if $w \in (1-t)z + t \cup \{F : F \in \mathfrak{F}_z\}$, then $w \in (1-t)z + tF$ for some $F \in \mathfrak{F}_z$ and courtesy of the Hahn Banach theorem $F = f^{-1}(1) \cap S_X$ for some $f \in S_{X^*}$ with $f(z) = 1$. Thus $w = (1-t)z + tv$ for some $v \in S_X$ with $f(v) = 1$. It follows that $f(w) = 1$, thus $w \in S_X$ and $w \in B_t[(1-t)z]$. That is, $w \in S'$. Therefore we conclude that

$$S' = (1-t)z + t \cup \{F : F \in \mathfrak{F}_z\},$$

and thus

$$S = y + d(1-t)z + dt \cup \{F : F \in \mathfrak{F}_z\} = x + r \cup \{F : F \in \mathfrak{F}_{(x-y)/\|x-y\|}\}$$

as required. □

Noting that A is compact if and only if $x + rA$ is compact for some $r > 0$, the following is an immediate consequence of the above lemma.

THEOREM 4.2. *A Banach space X has the NSP if and only if for all $x \in S_X$, $\cup\{F : F \in \mathfrak{F}_x\}$ is compact.*

One corollary to this is the fact that spaces with strictly convex norm always have the NSP (for in a strictly convex space $\mathfrak{F}_x = \{\{x\}\}$). Other corollaries include:

COROLLARY 4.3. *If X has the NSP, then faces of the unit ball are compact.*

Proof. Given a face F_0 of B_X, F_0 is a closed subset of $\cup\{F : F \in \mathfrak{F}_x\}$ for any $x \in F_0$. □

COROLLARY 4.4. *If X has a smooth norm, then X has the NSP if and only if faces of the unit ball are compact.*

Proof. For a smooth space each $\mathfrak{F}_x$, $x \in S_X$, is a singleton set. □

Thus a smooth reflexive space with Kadec-Klee norm has the NSP. In [8] it is shown that such spaces are always locally (β). Therefore the fact that they have the NSP is also a consequence of the following characterization of the NSP via a weakening of the locally (β) property.

DEFINITION 4.1. A Banach space X is said to have property (β_∞) if for each $x \in S_X$

$$\cap_{\delta>0}\overline{R((1+\delta)x, B_X)}$$

is compact.

THEOREM 4.5. *For a Banach space X the following are equivalent.*
(i) *X has property (β_∞).*

(ii) *X has NSP.*

P r o o f. To see that (i) implies (ii), suppose $x \in S_X$, let $d > 0$, and let

$$S = S_d[x] \cap S_{d+1}.$$

It suffices to show that (i) implies S is compact. To this end let $y \in S$, let $\overline{y} = y/\|y\|$, and let $m_y = \alpha\overline{y} + (1-\alpha)x$ for $\alpha \in (0,1)$. Then

$$\begin{aligned}\|y - m_y\| &\leqslant \alpha\|y - \overline{y}\| + (1-\alpha)\|y - x\| = \alpha(\|y\| - 1) + (1-\alpha)\|y - x\| \\ &\leqslant \alpha(\|x\| + \|y - x\| - 1) + (1-\alpha)\|y - x\| = \|y - x\|.\end{aligned}$$

Hence $\|y - m_y\| \leqslant d$. Also $1 + d = \|y\| \leqslant d + \|m_y\|$. Therefore $\|m_y\| \geqslant 1$; in fact, $\|m_y\| = 1$.

Now let $\varepsilon > 0$ be arbitrary, let $k = (\frac{1+\varepsilon}{1+\alpha\varepsilon})$, and let $\widetilde{m}_y = km_y$. Thus $\|\widetilde{m}_y\| = k > 1$. On the other hand a straightforward calculation shows that

$$\widetilde{m}_y = \beta\overline{y} + (1-\beta)(1+\varepsilon)x,$$

where $\beta = \alpha k \in (0,1)$. Therefore $\widetilde{m}_y \in R((1+\varepsilon)x, B_X)$. Since $\widetilde{m}_y \to \overline{y}$ as $\alpha \to 1$, $\overline{y} \in \overline{R((1+\varepsilon)x, B_X)}$, and since this is true for each $\varepsilon > 0$

$$\overline{y} \in \cap_{\varepsilon>0}\overline{R((1+\varepsilon)x, B_X)}.$$

Thus by (i) the set $\{\overline{y} : y \in S\}$ is compact. Since $S = \{\|y\|\overline{y} : y \in S\}$, S is compact as well, completing the proof that (i) implies (ii).

The reverse implication is quicker. Let $y \in \cap_{\delta>0}\overline{R((1+\delta)x, B_X)}$. Then for each $n \in \mathbb{N}$ there exists $y_n \in R((1+\frac{1}{n})x, B_X)$ such that $y_n \to y$. Let $z_n = \frac{1}{2}((1+\frac{1}{n})x + y_n)$. Then $1 \leqslant \|z_n\| \leqslant 1 + \frac{1}{n}$ and $z_n \to z := \frac{1}{2}(x+y)$. It follows that $\|z\| = 1$.

On the other hand,

$$\|\frac{1}{2}x - z_n\| = \frac{1}{2}\|y_n + \frac{1}{n}x\| \to \frac{1}{2}.$$

Thus $z \in S_X \cap S_{\frac{1}{2}}[x/2]$. Therefore for each $y \in \cap_{\delta>0}\overline{R((1+\delta)x, B_X)}$ the point $z = \frac{1}{2}(x+y)$ lies in the compact set $S_X \cap S_{\frac{1}{2}}[x/2]$; hence

$$\cap_{\delta>0}\overline{R((1+\delta)x, B_X)}$$

is compact. □

5. The uniformly convex case. In uniformly convex spaces Theorem 2.2 holds for an even wider class of mappings. The proof is an easy consequence of the following fact, due independently to Edelstein [2] and Steckin [15].

PROPOSITION 5.1. *Let X be a uniformly convex Banach space. Then for* *each $d > 0$ and each $c, c' \in X$ satisfying $0 < \|c - c'\| = r < d$,*

$$\lim_{\varepsilon \to 0} \operatorname{diam}(B_{d-r+\varepsilon}[c] \cap (X \setminus B_d[c'])) = 0$$

Moreover the convergence is uniform for all such c, c' lying in any bounded subset of X.

A mapping $T : C \to C$ $(C \subset X)$ is said to be *asymptotically nonexpansive* [3] if there exists a sequence $\{k_n\}$ of reals with $k_n \to 1$ such that $\|T^n(x) - T^n(y)\| \leqslant k_n \|x - y\|$ for all $x, y \in C$.

THEOREM 5.2. *Let C be a closed subset of a unformly convex Banach space and suppose $T : C \to C$ is asymptotically nonexpansive with $\operatorname{int}(F(T)) \neq \emptyset$. Then for each $x \in C$, the Picard sequence $\{T^n(x)\}$ converges to a point of $F(T)$.*

P r o o f. Let $x \in C$ and $p \in \operatorname{int}(F(T))$. Since

$$\|p - T^{n+m}(x)\| = \|T^m(p) - T^{n+m}(x)\| \leqslant k_m \|p - T^n(x)\|$$

with $k_m \to 1$, it follows that $\lim_{n\to\infty} \|p - T^n(x)\|$ always exists. Moreover,

$$d := \lim_{n \to \infty} \|p - T^n(x)\| = \inf_{n \in \mathbb{N}} \|p - T^n(x)\|.$$

If $d = 0$ there is nothing to prove. Otherwise there exists $r > 0$, with $r < d$, and $q \in F(T)$ such that $B_r[q] \subseteq F(T)$. For each $n \in \mathbb{N}$ choose $q_n \in F(T)$ so that $\|p - q_n\| = r$ and also so that

$$\|p - q_n\| + \|q_n - T^n(x)\| = \|p - T^n(x)\|.$$

It follows that $\lim_{n\to\infty} \|q_n - T^n(x)\| = d - r$. Let $\varepsilon > 0$. Then for n sufficiently large $T^n(x) \in B_{d-r+\varepsilon}[q_n]$. On the other hand $T^n(x) \in \overline{X \setminus B_d[p]}$ for all $n \in \mathbb{N}$. By Proposition 5.1

$$\lim_{\varepsilon \to 0} \operatorname{diam}[B_{d-r+\varepsilon}[q_n] \cap (\overline{X \setminus B_d[p]})] = 0.$$

This implies $\{T^n(x)\}$ is a Cauchy sequence. Since T is continuous, the conclusion follows. □

Clearly the assumptions on T can be weakened in the above theorem. In particular it need only be assumed that $\|T^n(x) - T^n(y)\| \leqslant k_n \|x - y\|$ hold for $y \in F(T)$ and for sufficiently large n. Indeed, any condition on T which assures

$$\lim_{n \to \infty} \|p - T^n(x)\| = \inf_{n \in \mathbb{N}} \|p - T^n(x)\|$$

for $p \in F(T)$ suffices if T or one of its iterates is continuous.

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