Fixed point and common fixed point theorems of contractive multivalued mappings on complete metric spaces

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Abstract

In this paper we give a common fixed point type generalization for some multi-valued contractive mappings on complete metric spaces. Our results extend some recent results of Y. Feng, S. Liu[Y. Feng and S. Liu, Fixed point theorem of multi-valued contractive mappings, J. Math. Anal. Appl. 317(2006)103-112], N. Mizoguchi, W. Takahashi[N. Mizoguchi, W. Takahashi. Fixed point theorem for multi-valued mappings on complete metric spaces, J. Math. Anal. Appl. 141(1989)177-188], D. Klim, D. Wardowski[D. Klim, D. Wardowski, Fixed point theorems for set valued contractions in complete metric spaces, J. Math. Anal. Appl. 334(2007)132-139]. We show that some common fixed point contraction theorems for multi-valued mappings are straightforward consequence of our results.

Keywords. common fixed point, multi-valued map .

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1 Introduction

Throughout this paper we denote by N the set of positive integers and by R the set of real numbers.

Let(X, d) be a metric space, we denote by $CB(X)$, $Cl(X)$and $N(X)$ the class of all nonempty closed and bounded, all nonempty and closed and all nonempty subsets of X respectively.
Let H be the Hausdorff metric with respect to d, that is

$$H(A, B) = max\{\sup_{x\in B} d(x, A), \sup_{y\in A} d(y, B)\},$$

for all A, B in $CB(X)$. where $d(x, A) = \inf_{y\in A} d(x, y)$.

Nadler [1] Extended the Banach contraction theorem for multi-valued mappings and Nadler's fixed point theorem has been extended in many directions. The following generalization was given by Mizoguchi and Takahashi [2].
Theorem 1.(Mizoguchi,Takahashi [2]) Let (X, d) be a complete metric space and let $T : X \longrightarrow CB(X)$. If there exist a function $\alpha : [0, \infty) \longrightarrow [0, 1)$ such that $\limsup_{r\longrightarrow t+} \alpha(r) < 1$ for all $t \in [0, \infty)$ and

$$H(Tx, Ty) \leq \alpha(d(x, y))d(x, y),$$

for all $x, y \in X$, then T has a fixed point.

Another proof of the above theorem was given by Dafer and Kaneko[3] and recently another proof for this theorem was given by T. Suzuki [4].
Y. Feng and S. Liu [5] proved the following theorem.
Theorem 2. (Feng, Liu [5]) Let (X, d) be a complete metric space and let $T : X \longrightarrow CB(X)$. If there exist $b, c \in (0, 1)$such that $c < b$ and for any $x \in X$ there is $y \in Tx$ satisfying the following conditions:

- $bd(x, y) \leq d(x, Tx)$,
- $d(y, Ty) \leq cd(x, y)$.

Then T has a fixed point in X provided the function $D(x) = d(x, Tx)$ is lower semi-continuous.
Recently D. Klim and D. Wardowski [6] extended Feng and Liu's theorem in the following sense.
Theorem 3.(Klim, Wardowski [6]) Let (X, d) be a complete metric space and let $T : X \longrightarrow Cl(X)$. Assume the following conditions hold.
For each $x \in X$ there is $y \in Tx$ such that :

- $bd(x, y) \leq d(x, Tx)$,

- $d(y, Ty) \leq \alpha(d(x, y))d(x, y)$,

where $\alpha : [0, \infty) \longrightarrow [0, b)$ is such that $\limsup_{r \longrightarrow t+} \alpha(r) < b$ for all $t \in [0, \infty)$. Then T has a fixed point in X provided $D(x) = d(x, Tx)$ is lower semi-continuous.
M. Berinde and V. Brinde [7] extended Mizoguchi and Takahashi's theorem as follows.
Theorem 4. Let (X, d) be a complete metric space, $L \geq 0$ and let $T : X \longrightarrow CB(X)$ be a generalized (α, L) contraction, i.e., a mapping for which there exists a function $\alpha : [0, \infty) \longrightarrow [0, 1)$ satisfying $\limsup_{r \longrightarrow t+} \alpha(r) < 1$, for every $t \in [0, \infty)$ such that

$$H(Tx, Ty) \leq \alpha(d(x, y))d(x, y) + Ld(y, Tx),$$

for all $x, y \in X$. Then T has at least one fixed point.

2 Main Results

First we prove two common fixed point theorem of multi-valued contractive mappings on complete metric spaces.

Theorem 2.1. Let (X, d) be a complete metric space and let $f, g : X \longrightarrow Cl(X)$. If there exist constant $b \in (0, 1)$, non negative real numbers β, γ such that $b + 2\beta + 2\gamma < 1$ and a function $\alpha : X \longrightarrow [0, b)$ such that f, g satisfy the following conditions :

$$d(y, gy) \leq \alpha(d(x, y))d(x, y) + \beta\{d(x, fx) + d(y, gy)\} + \gamma d(x, gy), \tag{1}$$

for all $y \in I_b^x$ and,

$$d(x, fx) \leq \alpha(d(x, y))d(x, y) + \beta\{d(x, fx) + d(y, gy)\} + \gamma d(y, fx), \tag{2}$$

for all $x \in gy$, where

$$I_b^x = \{y \in fx : bd(x, y) \leq d(x, fx)\}.$$

Then f, g have a common fixed point provided $I(x) = d(x, fx)$ is lower semi-continuous.
Proof. First we show that every fixed point of f is a fixed point of g and conversely. Let $z \in fz$, by putting $x = y = z$ in (1) we get

$$d(z, gz) \leq \alpha(d(z, z))d(z, z) + \beta\{d(z, fz) + d(z, gz)\} + \gamma d(z, gz) = (\beta + \gamma)d(z, gz).$$

Since $\beta+\gamma<1$ hence $d(z,gz)=0$. A similar argument shows that every fixed point of g is also a fixed point of f, define :

$$A(t)=\frac{\alpha(t)+\beta+\gamma}{1-(\beta+\gamma)}, \tag{3}$$

$$A=\frac{b+\beta+\gamma}{1-(\beta+\gamma)}. \tag{4}$$

By assumption $0<A<1$ so we can choose $b<\delta<1$ such that $\frac{A}{\delta}<1$, let $y\in fx$ from (1) we have:

$$d(y,gy)\leq \alpha(d(x,y))d(x,y)+\beta\{d(x,fx)+d(y,gy)\}+\gamma d(x,gy)$$

$$\leq \alpha(d(x,y))d(x,y)+\beta\{d(x,fx)+d(y,gy)\}+\gamma d(x,y)+\gamma d(y,gy).$$

Which in turn yields

$$d(y,gy)\leq A(d(x,y))d(x,y), \tag{5}$$

for all $y\in I_x^b$. Similarly we get

$$d(x,fx)\leq A(d(x,y))d(x,y), \tag{6}$$

for all $x\in gy$. Let $x_0\in X$, since $\delta<1$ there exist $x_1\in fx_0$ such that $\delta d(x_0,x_1)<d(x_0,fx_0)$. Since $b<\delta$ we have $x_1\in I_{x_0}^b$ and from (5) we get $d(x_1,gx_1)\leq A(d(x_1,x_0))d(x_1,x_0)$, hence there is $x_2\in gx_1$ such that $d(x_1,x_2)<Ad(x_1,x_0)$, and from (6) we have

$$d(x_2,fx_2)\leq A(d(x_2,x_1))d(x_2,x_1)<A^2d(x_1,x_0)<\frac{A}{\delta}Ad(x_0,fx_0)<Ad(x_0,fx_0).$$

So there is $x_3\in fx_2$ such that $d(x_2,x_3)<Ad(x_2,x_1)$.
Continuing this process we can iteratively choose a sequence $\{x_n\}_{n=0}^{\infty}$ such that $x_{2n}\in gx_{2n-1}$ and $x_{2n+1}\in fx_{2n}$ for which the followings hold for all nonnegative integers $n\in N$. Either $x_{n+1}=x_n$ for some nonnegative integer n, in which case the proof is completed, or $x_{n+1}\neq x_n$ for all nonnegative integers n, then

$$\delta d(x_{2n},x_{2n+1})<d(x_{2n},fx_{2n}), \tag{7}$$

$$d(x_{2n+1},x_{2n+2})<Ad(x_{2n},x_{2n+1}), \tag{8}$$

$$d(x_{2n+2},fx_{2n+2})<Ad(x_{2n},fx_{2n}). \tag{9}$$

From (9) we get

$$d(x_{2n},fx_{2n})\leq A^n d(x_0,fx_0), \tag{10}$$

for all n=1, 2,3,.. and from (10) we get

$$\sum_{n=0}^{\infty}d(x_n,x_{n+1})=\sum_{n=0}^{\infty}d(x_{2n},x_{2n+1})+\sum_{n=0}^{\infty}d(x_{2n+1},x_{2n+2})$$

$$\leq \sum_{n=0}^{\infty} \frac{1}{\delta} d(x_{2n}, fx_{2n}) + \sum_{n=0}^{\infty} \frac{A}{\delta} d(x_{2n}, fx_{2n}) < \infty.$$

Hence the sequence $\{x_n\}_{n=0}^{\infty}$ is Cauchy and since X is complete, there is some $z \in X$ such that $\lim_{n \longrightarrow \infty} x_n = z$. Now $D(x) = d(x, fx)$ is lower semi-continuous and from (10) we get

$$d(z, fz) \leq \liminf_{n \longrightarrow \infty} d(x_{2n}, fx_{2n}) = 0.$$

Since fz is closed $z \in fz$, so by first part of the proof z is also a fixed point of g and the proof is complete. ■

By putting $\beta = \gamma = 0$ in above theorem we have the following corollary.

Corollary 2.2. Let (X, d) be a complete metric space and let $f, g : X \longrightarrow Cl(X)$. If there exist constant $b \in (0, 1)$ and $\alpha : X \longrightarrow [0, b)$ and f, g satisfy the following conditions :

$$d(y, gy) \leq \alpha(d(x, y))d(x, y), \tag{11}$$

for all $y \in I_b^x$ where $I_b^x = \{y \in fx : bd(x, y) \leq d(x, fx)\}$, and

$$d(x, fx) \leq \alpha(d(x, y))d(x, y), \tag{12}$$

for all $x \in gy$.

Then f, g have a common fixed point provided $D(x) = d(x, fx)$ is lower semi-continuous.

The following theorem generalize theorem 2.1 of D. Klim and D. Wardowski [6].

Theorem 2.3. Let (X, d) be a complete metric space and let $f, g : X \longrightarrow Cl(X)$. If there exists a constant $b \in (0, 1)$ such that f, g satisfy the following conditions:

for all $x \in X$ there is $y \in gx$ such that

- $bd(x, y) \leq d(x, fx)$,
- $d(y, fy) \leq \alpha(d(x, y))d(x, y)$,

where $\alpha : X \longrightarrow [0, b)$ is such that $\limsup_{t \longrightarrow r+} \alpha(t) < b$ for every $r \in [0, \infty)$.

Then f, g have a common fixed point provided $D(x) = d(x, fx)$ is lower semi-continuous.

Proof. Let $x_0 \in X$, since gx_0 is nonempty there is $x_1 \in gx_0$ and by assumption we have:

- $bd(x_0, x_1) \leq d(x_0, fx_0)$,

- $d(x_1, fx_1) \leq \alpha(d(x_0, x_1))d(x_0, x_1)$.

Continuing this process we can choose an iterative sequence $\{x_n\}_{n=0}^{\infty}$ such that the following conditions hold for $n \in N$.

- $x_{n+1} \in gx_n$,
- $bd(x_n, x_{n+1}) \leq d(x_n, fx_n)$,
- $d(x_{n+1}, fx_{n+1}) \leq \alpha(d(x_n, x_{n+1}))d(x_n, x_{n+1})$.

With the same argument as used in the proof of theorem 2.1 of [5] we deduce that $\lim_{n\longrightarrow\infty} d(x_n, fx_n) = 0$ and $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence and since X is complete so it converges to some point $z \in X$. Now

$$0 \leq d(z, fz) \leq liminf_{n\longrightarrow\infty} d(x_n, fx_n) = 0,$$

and fz is closed, so $z \in fz$. It is easy to see that every fixed point of f is a fixed point of g and the proof is complete . ■

Theorem 2.4. Let (X, d) be a complete metric space and let $f, g : X \longrightarrow Cl(X)$ and $\alpha : [0, \infty) \longrightarrow [0, 1)$ be a function such that $\limsup_{t\longrightarrow r+} \alpha(t) < 1$ for all $r \in [0, \infty)$ and f, g satisfy the following conditions :

$$d(y, gy) \leq \alpha(d(x, y))d(x, y) \tag{13}$$

for all $y \in fx$ and

$$d(x, fx) \leq \alpha(d(x, y))d(x, y) \tag{14}$$

for all $x \in gy$.

Then f and g have a common fixed point provided $D(x) = d(x, fx)$ is lower semi-continuous.

Proof. (This proof is inspired by T. Suzuki [4]). Define $\beta : [0, \infty) \longrightarrow [0, 1)$ by

$$\beta(t) = \frac{1 + \alpha(t)}{2}.$$

Then $\limsup_{r\longrightarrow t+} \beta(r) < 1$, $\alpha(t) < \beta(t) < 1$ for all $t \in [0, \infty)$.

Let $x_0 \in X$ and $x_1 \in fx_0$ from (13) we have,

$$d(x_1, gx_1) \leq \alpha(d(x_0, x_1))d(x_0, x_1)$$

So there exists $x_2 \in gx_1$ such that:

$$d(x_1, x_2) \leq \beta(d(x_0, x_1))d(x_0, x_1),$$

since $x_2 \in gx_1$ hence from (14) we have

$$d(x_2, fx_2) \leq \alpha(d(x_0, x_1))d(x_0, x_1).$$

Continuing this process we can choose an iterative sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_{2n} \in gx_{2n-1}, x_{2n+1} \in fx_{2n}$ and,

$$d(x_{n+1}, x_{n+2}) \leq \beta(d(x_n.x_{n+1}))d(x_n, x_{n+1}). \tag{15}$$

Since $\beta(t) < 1$ for all $t \in [0, \infty)$ so $\{d(x_n, x_{n+1})\}_{n=0}^{\infty}$ is a decreasing sequence in R and must converge to some nonnegative real number $d \in R$.

Since $\limsup_{t \longrightarrow d+} \beta(t) < 1$, we can choose $r \in [0, 1)$ and $\upsilon \in N$ such that

$\beta(d(x_n, x_{n+1})) < r$ for every $n \geq \upsilon$. Hence

$$\Sigma_{n=1}^{\infty}(d(x_n, x_{n+1})) \leq \Sigma_{n=1}^{\upsilon}(d(x_n, x_{n+1})) + \Sigma_{n=\upsilon+1}^{\infty} r^n(d(x_1, x_0)) < \infty.$$

So $\{x_n\}$ is a Cauchy sequence and since X is complete, it converges to some $z \in X$ and since $x_{2n+1} \in fx_{2n}$,

$$d(z, fz) \leq liminf_{n \longrightarrow \infty} d(x_{2n}, fx_{2n}) \leq$$

$$liminf_{n \longrightarrow \infty}(d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, fx_{2n})) = 0,$$

hence $z \in fz$ and by putting $x = y = z$ in (13) we get

$$d(z, gz) \leq \alpha(d(z, z))d(z, z),$$

so $z \in gz$ and this completes the proof. ■

Theorem 4 of M. Berinde and V. Berinde [7] is a straightforward consequence of the following corollary which is obtained by simply taking $f = g$ in Theorem 2.4.

Corollary 2.5. Let (X, d) be a complete metric space and let $f : X \longrightarrow Cl(X)$ and $\alpha : X \longrightarrow [0, 1)$ be a function such that $\limsup_{t \longrightarrow r+} \alpha(t) < 1$ for all $r \in [0, \infty)$ and f satisfy:

$$d(y, fy) \leq \alpha(d(x, y))d(x, y), \tag{16}$$

for all $y \in fx$.

Then f has a fixed point provided $I = d(x, fx)$ is lower semi-continuous.

Some common fixed point theorems can be easily proved by using the theorem 2.4. We state some of them by way of illustration.

Corollary 2.6. Let (X, d) be a complete metric space and let $f, g : X \longrightarrow Cl(X)$ and $\alpha : [0, \infty) \longrightarrow [0, 1)$ be a function such that $\limsup_{t \longrightarrow r+} \alpha(t) < 1$ for all $r \in [0, \infty)$ and f, g satisfy the following conditions:

- $d(y, fy) \leq \alpha(d(x, y))d(x, gx)$ for all $y \in gx$,

- $d(x, gx) \leq \alpha(d(x, y))d(y, fy)$ for all $x \in fy$.

Then fixed point sets of f and g coincide and will be nonempty if either $d(x, fx)$ or $d(x, gx)$ is lower semi-continuous.
Proof. If $y \in gx$ then,

$$d(y, fy) \leq \alpha(d(x, y))d(x, gx) \leq \alpha(d(x, y))(d(x, y)+d(y, gx)) = \alpha(d(x, y))d(x, y).$$

So $d(y, fy) \leq \alpha(d(x, y))d(x, y)$ for every $y \in gx$. Similarly $d(x, gx) \leq \alpha(d(x, y))d(x, y)$ for all $x \in fy$, now by theorem 2.4 the proof is complete. ■
Corollary 2.7. Let (X, d) be a complete metric space and let $f, g : X \longrightarrow Cl(X)$ and f, g satisfy the following condition:

$$H(fx, gy) \leq \alpha(d(x, y))m(x, y)$$

where $\alpha : X \longrightarrow [0, 1)$ is a function such that $\limsup_{t \longrightarrow r+} \alpha(t) < 1$ for all $r \in [0, \infty)$, $m(x, y) = max\{d(x, y), d(x, fx), d(y, gy), \frac{d(x,gy)+d(y,fx)}{2}\}$ for all $x, y \in X$. Then f, g have a common fixed point provided $I = d(x, fx)$ is lower semi-continuous.
Proof. Let $y \in fx$ then

$$d(y, gy) \leq H(fx, gy) \leq$$

$$\alpha(d(x, y))max\{d(x, y), d(x, y) + d(y, fx), d(y, gy), \frac{d(x, y) + d(y, gy)}{2}\}$$

$$= \alpha(d(x, y))max\{d(x, y), d(y, gy), \frac{d(x, y) + d(y, gy)}{2}\} =$$

$$\alpha(d(x, y))max\{d(x, y), d(y, gy)\},$$

if $d(x, y) < d(y, gy)$ then $d(y, gy) \leq \alpha(d(x, y))d(y, gy)$, a contradiction. So

$$d(y, gy) \leq \alpha(d(x, y))d(x, y) \tag{17}$$

for every $y \in fx$, similarly we have

$$d(x, fx) \leq \alpha(d(x, y))d(x, y) \tag{18}$$

for every $x \in gy$.
So result follows by Theorem 2.4. ■
Theorem 2.8. Let (X, d) be a complete metric space and let $f, g : X \longrightarrow Cl(X)$ be functions such that

$$H(fx, gy) \leq \alpha(d(x, y))d(x, y) + \beta(d(x, y))\{d(x, fx) + d(y, gy)\}$$

$$+\gamma(d(x, y))\{d(x, gy) + d(y, fx)\}$$

for every $x, y \in X$, where $\alpha, \beta, \gamma : [0, \infty) \longrightarrow [0, 1)$ are functions such that $0 \leq \alpha(t) + 2\beta(t) + 2\gamma(t) < 1$ for every $t \in [0, \infty)$ and

$$\limsup_{t \longrightarrow r+} (\alpha(t) + \beta(t) + \gamma(t)) < \limsup_{t \longrightarrow r+} (1 - (\beta(t) + \gamma(t)),$$

for every $r \in [0, \infty)$,then f, g have a common fixed point provided $D(x) = d(x, fx)$ is lower semi-continuous.

Proof. Define $\theta : [0, \infty) \longrightarrow [0, 1)$ by $\theta(t) = \frac{(\alpha+\beta+\gamma)(t)}{(1-(\beta+\gamma))(t)}$. Let $y \in fx$ then like above theorem we have

$$\begin{aligned} d(y, gy) &\leq H(fx, gy) \leq \alpha(d(x, y))d(x, y) + \beta(d(x, y)) \\ &\times \{d(x, y) + d(y, gy)\} + \gamma(d(x, y))\{d(x, y) + d(y, gy), \end{aligned}$$

so we have:

$$(1 - (\beta + \gamma)(d(x, y)))d(y, gy) \leq (\alpha + \beta + \gamma)(d(x, y))d(x, y),$$

and this in turn yields:

$$d(y, gy) \leq \theta(d(x, y))d(x, y) \tag{19}$$

for every $y \in fx$, similarly

$$d(x, fx) \leq \theta(d(x, y))d(x, y) \tag{20}$$

for every $x \in gy$. Now from (19), (20) and theorem 2.4 the proof is follows. ■

For completeness we extend theorem 4.2 and 4.3 of [3] as follows.

Theorem 2.9. Let (X, d) be a complete metric space and $f, g : X \longrightarrow N(X)$ be multi-valued mappings, let $\varphi : X \longrightarrow R$ be bounded from below and lower semi-continuous and let $\eta : [0, \infty) \longrightarrow [0, \infty)$ be nondecreasing, continuous and subadditive which $\eta^{-1}(\{0\}) = \{0\}$, and satisfying the following conditions:

- For any $x \in X$ there is $y \in fx$ such that $\eta(d(x, y)) \leq \varphi(x) - \varphi(y)$,
- For any $x \in X$ there is $y \in gx$ such that $\eta(d(x, y)) \leq \varphi(x) - \varphi(y)$,

Then f and g have a common fixed point.

Proof. Like lemma 4.1 of [3] we know that η define a partial order on X such that X has a maximal element with respect to it. Let x_0 be the maximal element of X then by the above conditions there exists $z \in fx_0$ and $w \in gx_0$ such that $x_0 \leq z$ and $x_0 \leq w$, since x_0 is maximal we must have $x_0 = w = z$ and the proof is complete.

Corollary 2.10. For (X, d), f, g, φ and η as above if η satisfies the stronger conditions:

- For any $x \in X$ and $y \in fx$, $\eta(d(x, y)) \leq \varphi(x) - \varphi(y)$,
- For any $x \in X$ and $y \in gx$, $\eta(d(x, y)) \leq \varphi(x) - \varphi(y)$,

then there is $x_0 \in X$ such that $fx_0 = gx_0 = \{x_0\}$.

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