Fixed points of nonexpansive mappings and Chebyshev centers in Banach spaces with norms of type (KK)

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1. The norm of a Banach space is said to be a *Kadec-Klee (KK)* norm provided that on the unit sphere sequences converge in norm whenever they converge weakly. In [4] Huff reformulates the (KK) property and introduces two successively stronger notions, namely *uniformly Kadec-Klee (UKK)* and *nearly uniformly convex (NUC)*. We recall his definitions.

For every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$\text{(UKK)}: \quad \left.\begin{array}{l} \|x_n\| \le 1 \quad (n = 1, 2, \dots) \\ x_n \xrightarrow{w} x \\ \operatorname{sep}(x_n) \ge \varepsilon \end{array}\right\} \Rightarrow \|x\| \le 1 - \delta$$

($\operatorname{sep}(x_n)$ is defined as $\inf\{\|x_n - x_m\| : m \neq n\}$.)

For every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$\text{(NUC)}: \quad \left.\begin{array}{l} \|x_n\| \le 1 \quad (n = 1, 2, \dots) \\ \operatorname{sep}(x_n) \ge \varepsilon \end{array}\right\} \Rightarrow \operatorname{co}(x_n) \cap B_{1-\delta}(0) \neq \phi$$

($\operatorname{co}(x_n)$ denotes the convex hull of $\{x_n : n \in N\}$, and $B_r(x)$ the closed ball with center x and radius r.)

It is shown in [4] that X is (NUC) if and only if X is (UKK) and reflexive. Clearly (NUC) is implied by uniform convexity (UC), so that we have

$$\text{(UC)} \Rightarrow \text{(NUC)} \Rightarrow \text{(UKK)} \Rightarrow \text{(KK)}.$$

It turns out that all of these notions are different: for each pair of properties there exists a space having the weaker of the two but failing to be isomorphic to any space with the stronger one (cf [4].) In this paper we shall be concerned with the property (UKK) and with a weakening of it which we now define.

Definition. (The norm of) a Banach space is called *weakly uniformly Kadec-Klee (WUKK)* if there exist an $\varepsilon < 1$ and a $\delta > 0$ such that

$$\left.\begin{array}{ll} \|x_n\| \le 1 & (n = 1, 2, \ldots) \\ x_n \xrightarrow{w} x & \\ \mathrm{sep}(x_n) \ge \varepsilon & \end{array}\right\} \Rightarrow \|x\| \le 1 - \delta$$

For dual Banach spaces we shall also consider the corresponding dual properties denoted by (KK*), (UKK*) and (WUKK*), respectively.
(KK*) (for general dual spaces) and (UKK*) and (WUKK*) (for duals with w*-sequentially compact unit ball) are obtained by replacing w-convergence with w*convergence in the above definitions. In section 3 we shall extend the definitions of (UKK*) and (WUKK*) to general dual spaces.

We now recall some notions from fixed point theory. A mapping $T: C \to X$ defined on a subset C of a Banach space X is said to be *non-expansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. We say that a [dual] Banach space X has the [*dual*] *fixed point property (FPP)* [*(FPP*)*] if for every w-compact [w*compact] convex subset $C \subset X$ and for every nonexpansive $T: C \to C$, T has a fixed point $\in C$. It is known (cf [1]) that $L^1[0,1]$ does not have (FPP). On the other hand a classical result of Kirk ([5]) states that if weakly compact convex sets in X have normal structure, then X has (FPP). (A convex set $K \subset X$ is said to have *normal structure* if for each bounded convex subset C of K which is not a singleton, there exists at least one point $x \in C$ with $\sup\{\|x - y\| : y \in C\} < \mathrm{diam}\ C$. Such a point x is called *nondiametral.*) A corresponding dual result is true for dual Banach spaces (cf. [6]). We now recall the concept of Chebyshev center. Let B and C be subsets of a Banach space and let B be bounded. For each $x \in C$ define

$$r(x) := \sup\{\|x - y\| : y \in B\}$$

and put

$$r_0 := \inf\{r(x) : x \in C\}.$$

Then the (possibly empty) set $\{x \in C: r(x) = r_0\}$ is called the *Chebyshev center* of B with respect to C and r_0 the *radius* of B w.r.t. C. It is well known that if C is w-compact and convex then Chebyshev centers w.r.t. C are non-empty, w-compact and convex; because the function r is continuous and convex and therefore w-*l*.s.c. If C is w-compact and convex and has normal structure, then the (non-empty) Chebyshev center A of C w.r.t. itself is strictly contained in C. Furthermore, A is invariant under T for any nonexpansive $T: C \to C$ if C is minimal w.r.t. being w-compact, convex and T-invariant. These facts (which contradict each other) form the proof of Kirk's theorem. It also follows from the second fact and the Schauder-Tychonoff theorem that if Chebyshev centers w.r.t. w-compact convex sets are compact, then X has (FPP). Similarly, if Chebyshev centers w.r.t. w*-compact convex sets in a dual Banach space can be shown to be compact and non-empty, then X has (FPP*).

In this paper we investigate (UKK), (WUKK) and (UKK*), (WUKK*) in connection with (FPP), respectively (FPP*). It turns out (section 2) that (WUKK) implies that w-compact convex sets have normal structure. Hence (WUKK) implies (FPP), by Kirk's theorem. In case (UKK) holds, more can be said. Namely, Chebyshev centers w.r.t. w-compact convex sets are compact (and convex). In section 3 we extend the definition of (WUKK*) and (UKK*) to general dual spaces.
It is then shown that the corresponding dual results are true: (WUKK*) implies (FPP*), while (UKK*) implies that Chebyshev centers w.r.t. w*-compact convex sets are non-empty and compact. Our results here include the case of ℓ^1 (for which a slightly stronger result was proved by Lim in [6]) and many others. Section 4 contains examples. They demonstrate the usefulness of the property (WUKK): almost always it is much easier to check than normal structure. Among other things we show that in Theorem 3 the requirement $\varepsilon < 1$ in (WUKK*) cannot be relaxed.
An easy example shows that (WUKK) does not imply compactness of Chebyshev centers. It is also proved that neither (KK) ⇒ (WUKK) nor (WUKK) ⇒ (KK) are true.

2. *THEOREM 1. A Banach space* X *satisfying* (WUKK) *has* (FPP).
Proof. It suffices to show that every w-compact convex subset C of X consisting of more than one point contains a non-diametral point. Suppose not. Then, by a method of Brodskii-Milman [2], there exists a sequence $(x_n) \subset C$ satisfying

$$\lim_{n\to\infty} d(x_{n+1}, \operatorname{co}\{x_1,\dots,x_n\}) = \operatorname{diam} C \tag{1}$$

Any subsequence of (x_n) again satisfies (1), so we may, by weak compactness, assume that $x_n \xrightarrow{w} x$. By applying first a translation and then a multiplication, we may further simplify the situation and assume that $x_n \xrightarrow{w} 0$ and $\operatorname{diam} C = 1$. Since the weak and the norm closure of $\operatorname{co}(x_n)$ coincide, (1) implies in particular that $\lim_{n\to\infty} \|x_n\| = 1$. Now let $\varepsilon < 1$ and $\delta > 0$ be as in the definition of (WUKK). Choose $n_0 \in \mathbb{N}$ such that $\|x_{n_0}\| > 1 - \delta$ and such that $d(x_{n+1}, \operatorname{co}\{x_1,\dots,x_n\}) > \varepsilon$ whenever $n \ge n_0$. Consider now the sequence $(x_{n_0} - x_n)_{n=n_0+1}^{\infty}$. Clearly $\|x_{n_0} - x_n\| \le 1$ $(n = n_0+1, n_0+2, \dots)$, $\operatorname{sep}(x_{n_0} - x_n) \ge \varepsilon$ and $x_{n_0} - x_n \xrightarrow{w} x_{n_0}$. This contradicts (WUKK) since $\|x_{n_0}\| > 1 - \delta$.

THEOREM 2. If a Banach space X *has* (UKK), *then Chebyshev centers w.r.t. w-compact convex sets are compact (and non-empty and convex).*

Proof. Let $C \subset X$ be w-compact and convex and let $B \subset X$ be bounded. Let A be the Chebyshev center of B w.r.t. C and r_0 its radius. We have observed earlier that A is w-compact, convex and non-empty.
If A is not compact, then A contains a sequence (x_n) with $\operatorname{sep}(x_n) \geq \varepsilon$, for some $\varepsilon > 0$. By passing to a subsequence we may assume that $x_n \xrightarrow{w} x$. Choose $\delta = \delta(\frac{\varepsilon}{r_0}) > 0$ as in the definition of (UKK) and fix $y \in B$. By definition we have $\|r_0^{-1}(x_n - y)\| \leq 1$ $(n = 1,2,\ldots)$, $\operatorname{sep}(r_0^{-1}(x_n - y)) \geq r_0^{-1}\varepsilon$ and $r_0^{-1}(x_n - y) \xrightarrow{w} r_0^{-1}(x - y)$. Thus (UKK) implies $\|x - y\| \leq (1-\delta)r_0$. Since $y \in B$ was arbitrary this contradicts the definition of r_0 as the radius of B w.r.t. C.

COROLLARY. If X *has* (NUC), *then* X *has normal structure.*

Proof. (NUC) implies reflexivity ([4]), so every closed bounded convex set C is w-compact. If C is compact, then it is well-known to have a non-diametral point. If not, then the Chebyshev center A of C w.r.t. itself is compact by Theorem 2. Thus $A \subsetneq C$, hence r_0(= the radius) < diam C. Any point of A is therefore non-diametral.

Remark. It was pointed out in [4] that there exist (NUC) spaces which fail to be superreflexive: every ℓ^2-sum of finite-dimensional spaces has (NUC).

3. We now turn to conjugate Banach spaces and begin by noting that in duals of separable spaces (or more generally, in spaces for which the dual unit ball is w*-sequentially compact) (WUKK*) and (UKK*) may be reformulated as follows.

If (*) denotes the property:

$$\left.\begin{array}{l} A \text{ a subset of the closed unit ball containing} \\ \text{a sequence } (x_n) \text{ with } \operatorname{sep}(x_n) > \varepsilon \end{array}\right\} \Rightarrow w^*\text{-cl}A \cap B_{1-\delta}(0) \neq \emptyset;$$

then the dual space has (WUKK*) if (*) holds for some $\varepsilon \in (0,1)$ and $\delta > 0$ and has (UKK*) if for every $\varepsilon \in (0,1)$ (*) holds for some $\delta = \delta(\varepsilon) > 0$. We take these as the definitions for (WUKK*) and (UKK*) in general dual spaces.

LEMMA. *Let* X *be a dual space in which* (*) *holds for a given* $\varepsilon \in (0,1)$ *and* $\delta > 0$. *If* C, *a* w*-*closed convex subset of* X, *and* $x_1, x_2, \ldots, x_n \in X$ *are such that:*

$$C \text{ contains a sequence } (y_n) \text{ with } \operatorname{sep}(y_n) > \varepsilon$$

and

$$C \subseteq B_1(x_i) \text{ for } i = 1,2,\ldots,n,$$

$$C \cap \left(\bigcap_{i=1}^{n} B_{1-\delta}(x_i) \right) \neq \emptyset.$$

Proof. First note that by assumption the lemma is true when $n = 1$. Now, assume the lemma were false. Then there is a largest $n (\geq 1)$ for which the conclusion remains valid. Denote this largest value of n by n_0. Then there exists a w^*-closed convex $C \subseteq X$ containing a sequence with separation constant greater than ε and $x_1, x_2, \ldots, x_{n_0}, x_{n_0+1} \in X$ with $C \subset B_1(x_i)$ $(i = 1, 2, \ldots, n_0+1)$ for which

$$C \cap \left(\bigcap_{i=1}^{n_0+1} B_{1-\delta}(x_i) \right) = \emptyset.$$

Let $E = C \cap B_{1-\delta}(x_1) \cap \ldots \cap B_{1-\delta}(x_{n_0})$. Then by the definition of n_0, $E \neq \emptyset$, and E is a w^*-closed convex subset of X. Further $E \cap B_{1-\delta}(x_{n_0+1}) = \emptyset$, so there exists a w^*-continuous linear functional f and k with $\sup f(E) < k < \inf f(B_{1-\delta}(x_{n_0+1}))$. Let

$$C_1 = \{x \in C\colon f(x) \geq k\} \quad \text{and} \quad C_2 = \{x \in C\colon f(x) \leq k\}.$$

Then

$$C_2 \subset C \subset B_1(x_{n_0+1})$$

while

$$C_2 \cap B_{1-\delta}(x_{n_0+1}) = \emptyset,$$

so by assumption, C_2 cannot contain any sequence with separation constant greater than ε, and so, since $C = C_1 \cup C_2$, we conclude that C_1 does contain such a sequence. Thus, C_1 is a w^*-closed set containing a sequence with separation constant greater than ε and

$$C_1 \subset C \subseteq B_1(x_i) \quad (i = 1, 2, \ldots, n_0),$$

but

$$C_1 \cap \left(\bigcap_{i=1}^{n_0} B_{1-\delta}(x_i) \right) = C_1 \cap \left[C \cap \left(\bigcap_{i=1}^{n_0} B_{1-\delta}(x_i) \right) \right] = C_1 \cap E = \emptyset,$$

contradicting the choice of n_0 and establishing the lemma.

THEOREM 3. If X *is a dual space with* (WUKK*), *then* X *has* (FPP*).

Proof. Let C be a non-empty w^*-compact convex subset and $T:C \to C$ non-expansive. By a standard application of Zorn's lemma we may replace C by a minimal (in the sense of inclusion) non-empty w^*-compact convex subset $C_1 \subset C$ such that $T(C_1) \subset C_1$. Since the function

$$z \to r(z) := \operatorname{Sup}\{\|z-y\| : y \in C_1\}$$

is a supremum of w^*-ℓ.s.c. functions and therefore itself w^*-ℓ.s.c., it follows that the Chebyshev center A of C_1 w.r.t. itself is a non-empty w^*-compact convex subset of C_1. Further, by a standard argument A is invariant under T and so by minimality $A = C_1$. Now suppose that C_1 contains more than one point, then every point of C_1 is diametral and by an argument of Brodskii and Milman we may extract a sequence of points $(x_n) \subset C_1$ with $\|x_n - x_m\| > \varepsilon$ where $\varepsilon \in (0,1)$ is that in the definition of (WUKK*) and by a multiplication we have assumed without loss of generality that $r(C_1) = \min\{r(z) : z \in C_1\} = 1$.

For each $x \in C_1$ we have that $C_1 \subset B_1(x)$ and so by (WUKK*) for some $\delta > 0$, that

$$E_x = C_1 \cap B_{1-\delta}(x)$$

is a non-empty w^*-compact convex subset of C_1. Further, by the lemma the family $\{E_x : x \in C_1\}$ has the finite intersection property and so by the w^*-compactness of C_1 there exists an $x_0 \in C_1$ with $x_0 \in \cap\{E_x : x \in C_1\}$. For this x_0 we therefore have that

$$\|x_0 - x\| \le (1-\delta) \quad \text{for all} \quad x \in C_1,$$

so x_0 is a non-diametral point of C_1. This contradiction establishes that C_1 must consist of a single point, which is necessarily a fixed point of T.

THEOREM 4. If X *is a dual space with* (UKK*), *then Chebyshev centers w.r.t.* w^**-compact convex sets are non-empty compact and convex.*

Proof. Let $C \subset X$ be w^*-compact and convex and let $B \subset X$ be bounded. Assume the Chebyshev center A of B w.r.t. to C is not compact. Then A contains a sequence (x_n) with $\operatorname{sep}(x_n) > \varepsilon$ for some $\varepsilon > 0$ and so, since $A \subseteq C$, we have that C is a w^*-closed convex subset containing a sequence with positive separation constant ε. Using (UKK*) with this ε and assuming the Chebyshev radius of B w.r.t. C is 1 we have for each $x \in B$ that $A \subset B_1(x)$ and so for some $\delta > 0$

$$E_x = A \cap B_{1-\delta}(x)$$

is a non-empty w^*-compact convex subset of C.

The argument now proceeds as that of the last part of Theorem 3.

4. *Examples.*

a The most obvious example for Theorems 3 and 4 is ℓ^1. It is easily checked that ℓ^1 has the property (WUKK*) with any δ, ε satisfying $0 < \varepsilon < 2$, $\delta < \frac{1}{2}\varepsilon$. More generally, any ℓ^1-sum of finite-dimensional Banach spaces has (FPP*).

b Every Orlicz sequence space ℓ_M, with the Orlicz function M satisfying the Δ_2-condition, also satisfies the conditions of Theorem 3. In particular, therefore, any nonexpansive map T from the unit ball of such a space into itself has a fixed point.

c It is easily seen that a slight change in the norm of a uniformly convex space (depending on the modulus of convexity) preserves (WUKK). Therefore, by Theorem 1, any such space has (FPP).

In many examples (FPP) or (FPP*) can be most easily verified by checking that the w - $[w^*]$ Opial condition holds. Recall that a [dual] space is said to satisfy the $w[w^*]$ Opial condition if $x_n \in X$ $(n = 1,2,\ldots)$, $x_n \xrightarrow{w} x_0$ $[x_n \xrightarrow{w^*} x_0]$ implies

$$\liminf \|x_0 - x_n\| < \liminf \|x - x_n\| \quad \text{for all} \quad x \neq x_0,$$

and that a [dual] space satisfying the $w[w^*]$ Opial condition has (FPP) [(FPP*)] (cf. [3]). The following example shows that sometimes we can get results even in cases where the Opial condition is not satisfied.

d Define an equivalent norm on ℓ^1 by

$$|||x||| = \max(\|x\|_1, (1+\alpha)\|x\|_\infty),$$

where $\|x\|_1 = \sum_{n=1}^{\infty} |x_n|$, $\|x\|_\infty = \sup_{n\in N} |x_n|$, and $\alpha > 0$. It is easily checked that for sufficiently small $\alpha > 0$ the property (WUKK*) holds for $|||.|||$. So $(\ell^1, |||.|||)$ has (FPP*) by Theorem 3. The w^*-Opial condition fails in this case, however, since $e_n \xrightarrow{w^*} 0$, $|||e_n||| = 1+\alpha$, but also $|||e_n - \alpha e_1||| = 1 + \alpha$ $(n = 2,3,\ldots)$.

e In [6] Lim considers the space ℓ^1 with the (dual) norm

$$|||x||| = \max(\|x^+\|_1, \|x^-\|_1),$$

where x^+ and x^- are the positive and negative part of x, respectively. He shows that $(\ell^1, |||.|||)$ does not have (FPP*). Indeed, $|||.|||$ does not have (WUKK*). For every ε *larger* than 1, however, there exists a $\delta > 0$ such that $|||x_n||| \le 1$ $(n = 1,2,\ldots)$, $x_n \xrightarrow{w^*} x$ and $\operatorname{sep}(x_n) \ge \varepsilon$ imply $|||x||| \le 1-\delta$. This shows that in Theorem 3 the requirement $\varepsilon < 1$ in (WUKK*) cannot be further relaxed. Note that $|||.|||$ also fails to be (KK*).

(f) Although (KK) and (WUKK) are both weakenings of (UKK), neither one of these properties implies the other, as the following examples show.

(i) $\left(\sum_{n=1}^{\infty} \oplus \ell^n\right)_2$ has (KK), but not (WUKK). The proof that (KK) holds is easy, and known (cf. [4].) To show that (WUKK) fails, let $\varepsilon < 1$ and $\delta > 0$ be arbitrary. Choose $n_0 \in \mathbb{N}$ so large that $2^{-1/n_0} > 1-\delta$ and consider the sequence (x_n) with

$$x_n := \underbrace{0 \oplus 0 \oplus \ldots \oplus 0}_{n_0-1} \oplus \underbrace{(2^{-1/n_0},0,\ldots,0,2^{-1/n_0}}_{n},0,\ldots) \oplus 0 \oplus \ldots .$$

Then $\|x_n\| = 1$ $(n = 1,2,\ldots)$, $\|x_n - x_m\| = 1$ $(n \ne m)$,

$x_n \xrightarrow{w} x = 0 \oplus \ldots \oplus 0 \oplus (2^{-1/n_0},0,\ldots) \oplus 0 \oplus \ldots$ and $\|x\| = 2^{-1/n_0} > 1-\delta$.

(ii) ℓ^2 with norm $|||x||| = \max(\|x\|_2, (1+\alpha)\|x\|_\infty)$ has (WUKK) for suitably small $\alpha > 0$, but not (KK). The first statement is obvious (see (c)). To see that $(\ell^2, |||.|||)$ fails (KK), observe that

$$e_1 + \alpha e_n \xrightarrow{w} e_1, \quad |||e_1 + \alpha e_n||| = |||e_1||| = 1+\alpha \qquad (n = 2,3,\ldots)$$

but $e_1 + \alpha e_n$ is not norm convergent to e_1.

We do not know whether Theorems 1 and 3 hold with (KK) [(KK*)] in place of (WUKK) [(WUKK*)].

(g) Finally let us note that the space $(\ell^2, |||.|||)$ above, although it has (WUKK) for small α, does not satisfy the conclusion of Theorem 2. Indeed, let $B := \{e_1, -e_1\}$ and let C be the unit ball. Then the Chebyshev center of B w.r.t. C contains the sequence $(\alpha e_n)_{n=2}^{\infty}$ and therefore fails to be compact.

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