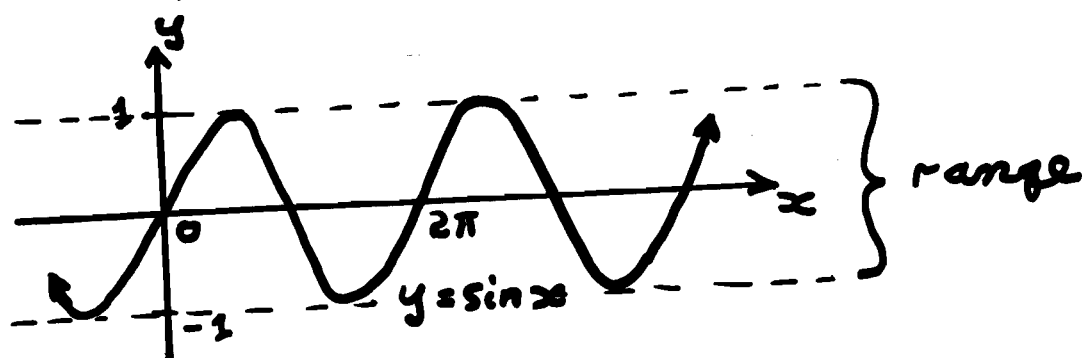


INVERSE FNS

To understand inverse functions we must first consolidate the notion of a function.

The set of values a function assumes is its range.

Eg. The range of $y = \sin x$ is $-1 \leq y \leq 1$



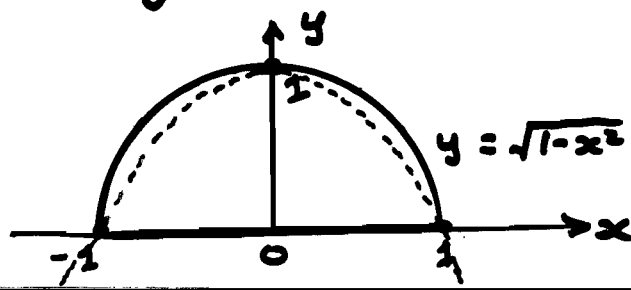
While it need not generally be the case, our functions will act on real numbers (i.e. the domain will be a set of numbers) and the values assigned will also be numeric (i.e. the range will be a set of numbers).

When the domain is not explicitly specified we take it to be all those numbers to which the rule f can meaningfully apply.

Eg. 1) The function $f(x) = \sqrt{1-x^2}$

has domain $-1 \leq x \leq 1$,

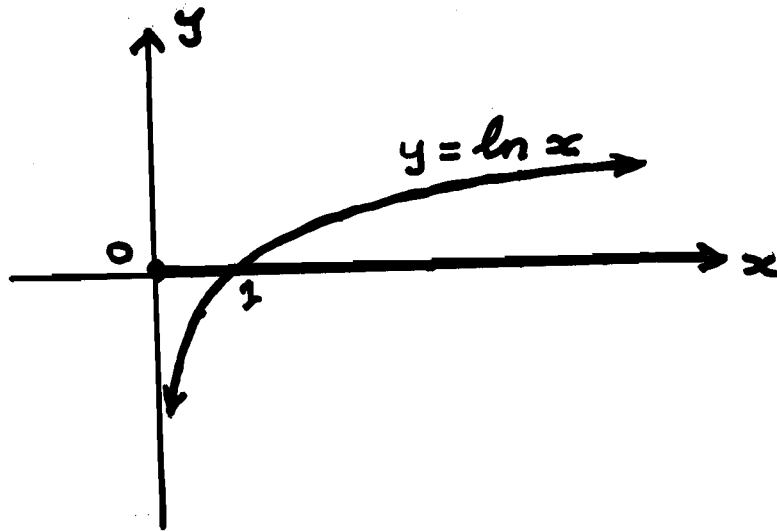
and range the interval from 0 to 1.



2) ln assigns to x the power, $\ln x$,⁴
to which e must be raised to
obtain x .

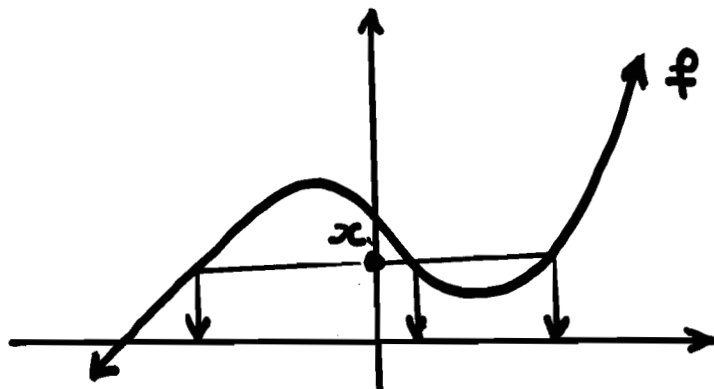
$$\ln x \text{ is the number such that}$$
$$e^{\ln x} = x$$

Since negative numbers do not
arise as any power of e
we see that the domain of
 $\ln x$ is $0 < x < \infty$.

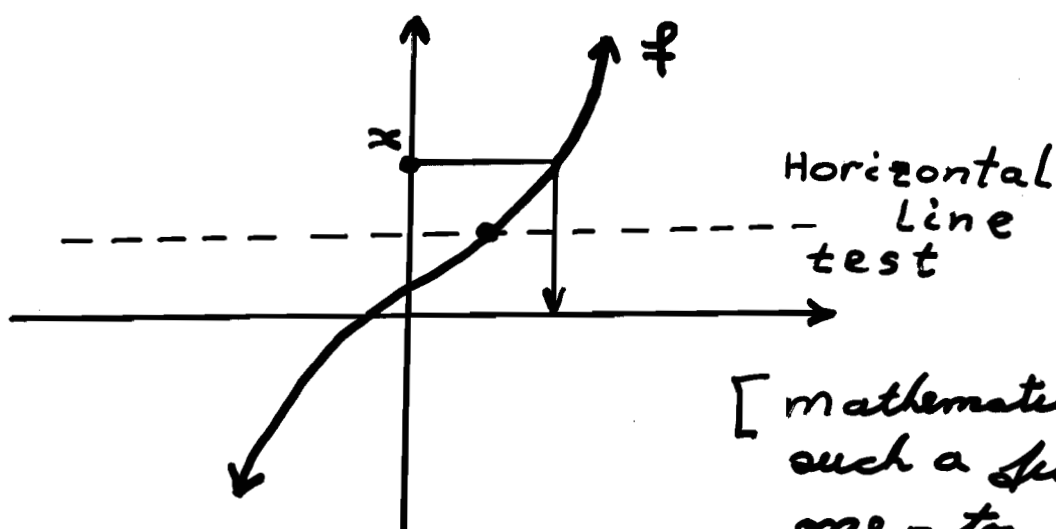


The range of $\ln x$ is all
real numbers.

Given a function f we can consider the rule which assigns to each x in the range of f those numbers at which f takes the value x .



For this to be a function there must be only one point in the domain of f where f takes the value x .



[mathematicians call such a function f one-to-one.]

When this is the case we call our new rule the inverse function of f and denote it by f^{-1} . It has the property:

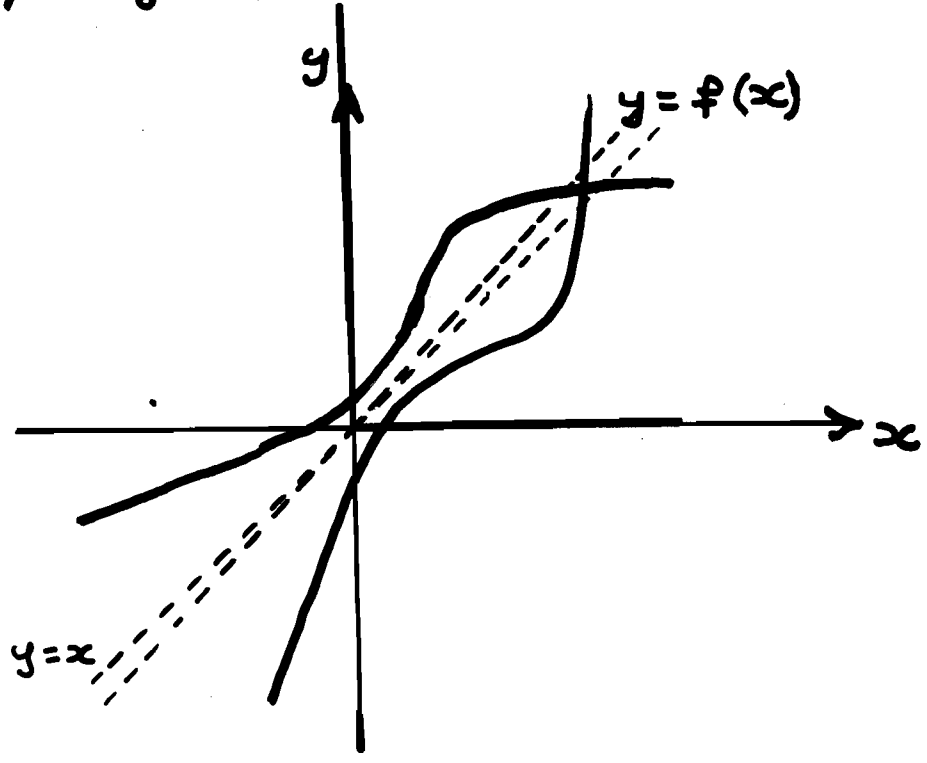
$$\text{if } y = f^{-1}(x) \text{ then } x = f(y).$$

$$\text{So } f(f^{-1}(x)) = x \text{ and } f^{-1}(f(x)) = x.$$

Eg. The inverse of e^x is $\ln x$.

The domain of f^{-1} is the range of f and the range of f^{-1} is the domain of f .
 (e^x and $\ln x$)

Reflecting the diagram (below) in the line $y = x$ so that the y -axis becomes horizontal and the x -axis vertical we obtain the graph of f^{-1} .



Throughout intervals where an invertible function f is continuous it must be either strictly increasing or strictly decreasing.

Derivatives of inverse functions.

7.

If $y = f^{-1}(x)$, from $\frac{dy}{dx} = 1 / \frac{dx}{dy}$

and $x = f(y)$ we can find the derivative of f^{-1} from that of f .

Eq. $y = \tan^{-1} x$

$$\frac{dy}{dx} = 1 / \frac{dx}{dy} \quad \text{where } x = \tan y = \frac{\sin y}{\cos y}$$

$$= 1 / \frac{\cos y \cos y - \sin y (-\sin y)}{\cos^2 y}$$

$$= 1 / \frac{\cos^2 y + \sin^2 y}{\cos^2 y}$$

$$= \cos^2 y$$

But want answer
in terms of x .

$$= \frac{1}{1 + (\tan y)^2}$$

$$s^2 + c^2 = 1$$

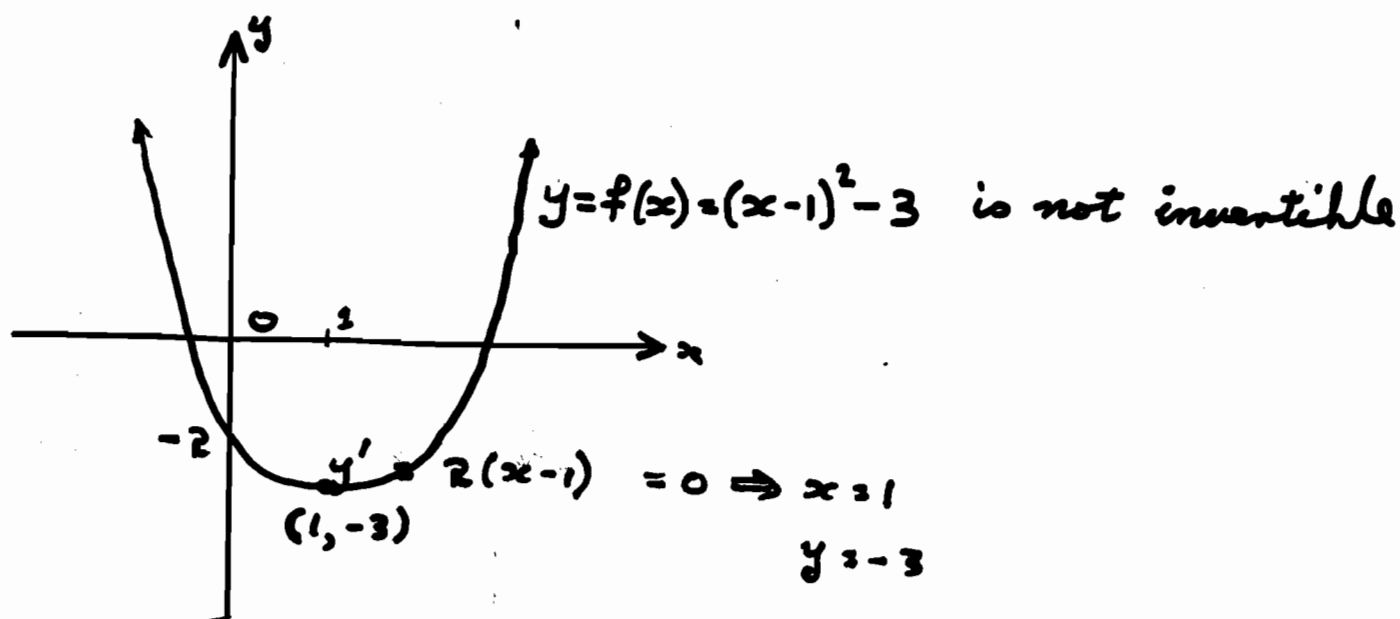
$$t^2 + 1 = \frac{1}{c^2}$$

$$= \frac{1}{1 + (\tan(\tan^{-1} x))^2}$$

$$= \frac{1}{1 + x^2}$$

By restricting the domain of a function which fails the 'horizontal line test' we may be able to obtain an invertible function.

Eg. [3/4 U, 1989, 6(a)]



$$y = 1 = \pm \sqrt{x+3}$$

- no good as $y \geq 1$ or $y - 1 \geq 0$.

$$(x-1)^2 - 3 = -2$$

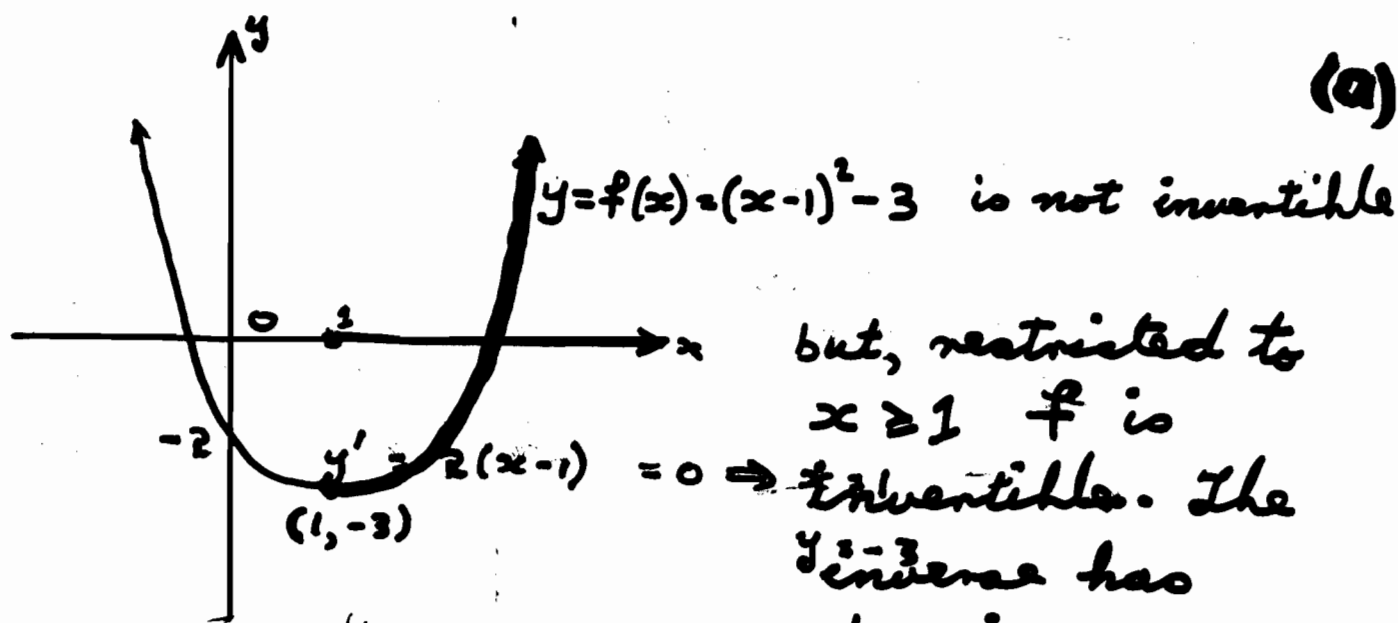
$$(x-1)^2 = 1$$

$$x-1 = \pm 1$$

$$x = 0 \text{ or } x = 2$$

By restricting the domain of a function which fails the 'horizontal line test' we may be able to obtain an invertible function.

Eg. [3/4u, 1989, 6(a)]



To find an expression for f^{-1} on this domain, suppose $y = f(x)$ then $y - 1 \geq 0$.

$x = f(y) = (y-1)^2 - 3 = -2$

So $(y-1)^2 = x + 3$, $(y-1)^2 = 1$

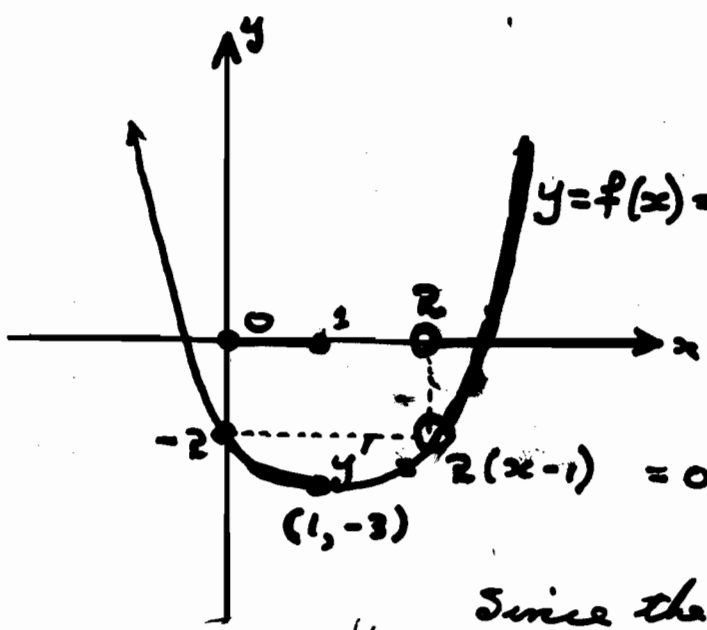
$y-1 = \sqrt{x+3}$ $x-1 = \pm 1$ $x = 0$ or $x = 2$

and $f^{-1}(x) = y = \sqrt{x+3} + 1$

By restricting the domain of a function which fails the 'horizontal line test' we may be able to obtain an invertible function.

Eg. [3/4 U, 1989, 6(a)]

(b)



$y = f(x) = (x-1)^2 - 3$ is not invertible

An alternative domain on which f is invertible is

$$\{0 \leq x \leq 1\} \cup \{x > 2\}$$

Since the question asked for the largest positive domain I don't know what the answer is, since I don't know how to compare these two domains and then ask others!

$$(x-1)^2 - 3 = -2$$

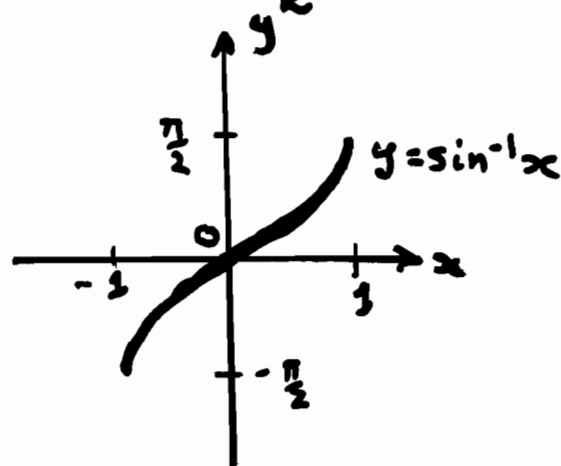
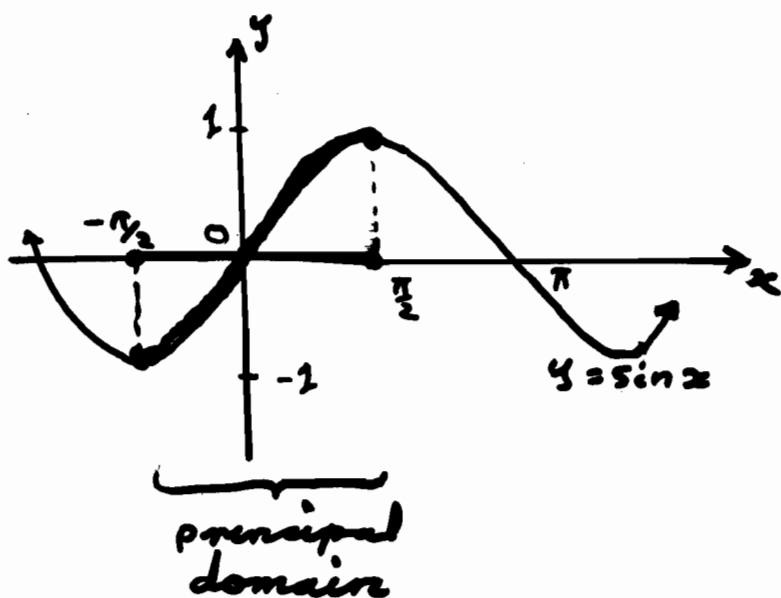
$$(x-1)^2 = 1$$

$$x-1 = \pm 1$$

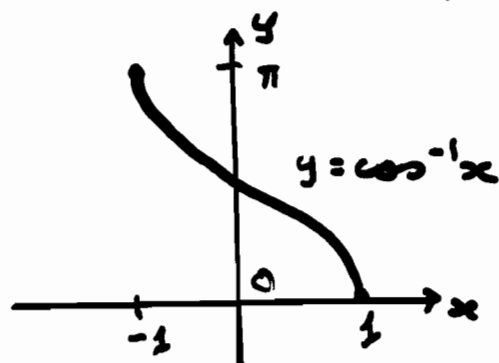
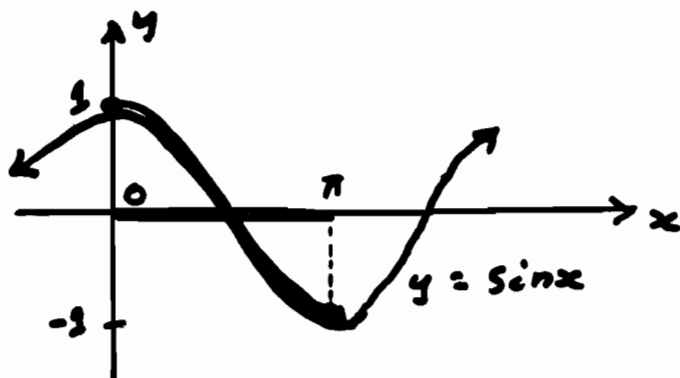
$$x = 0 \text{ or } x = 2$$

To obtain inverses for the trigonometric functions we need to restrict their domains.

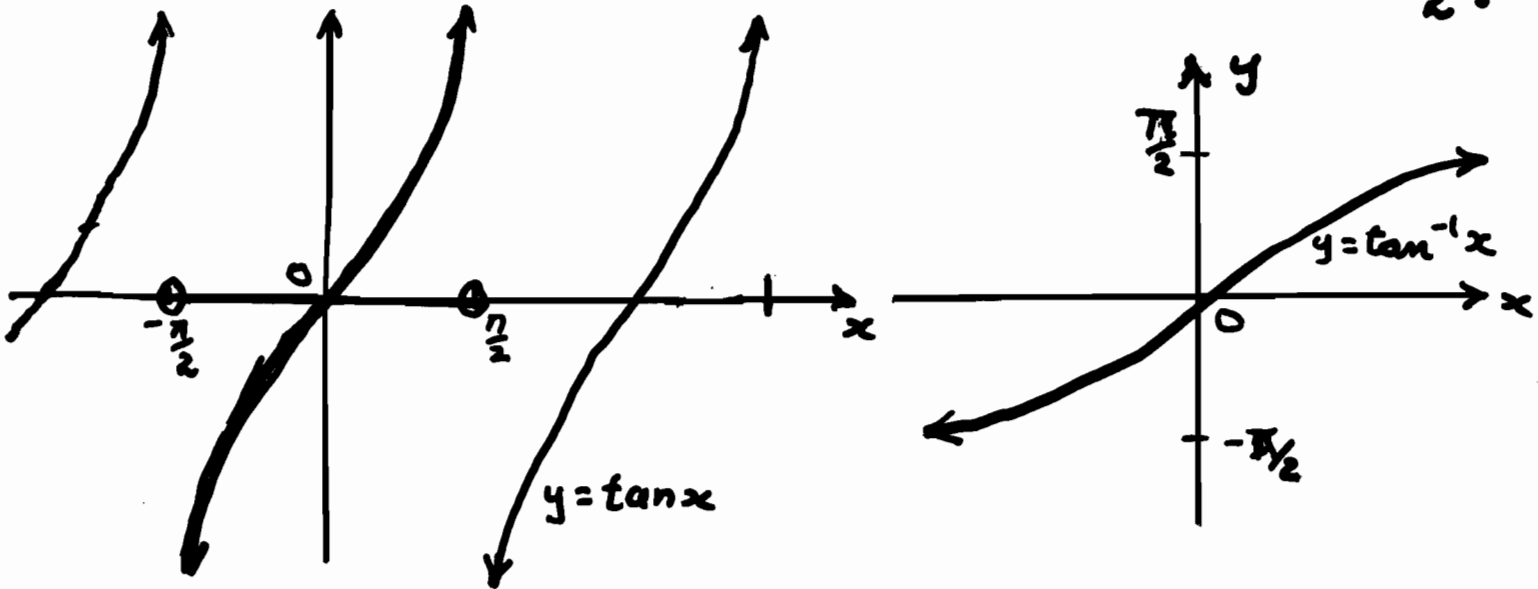
\sin^{-1} is obtained by restricting $\sin(x)$ to $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, so $\sin^{-1}(x)$ has domain $-1 \leq x \leq 1$ and range $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$



\cos^{-1} is obtained by restricting $\cos(x)$ to $0 \leq x \leq \pi$, so $\cos^{-1}(x)$ has domain $-1 \leq x \leq 1$ and range $0 \leq x \leq \pi$



\tan^{-1} is obtained by restricting $\tan x$ to $-\frac{\pi}{2} < x < \frac{\pi}{2}$, so $\tan^{-1} x$ has domain the whole x -axis and range $-\frac{\pi}{2} < y < \frac{\pi}{2}$.



Use:

$$f(x) = f_n \circ f_{n-1} \circ \dots \circ f_2 \circ f_1(x)$$

to "write" down

$$f^{-1}(x) = f_1^{-1} \circ f_2^{-1} \circ \dots \circ f_n^{-1}(x)$$

where f_1, f_2, \dots, f_n are invertible functions whose inverses are known

Prove

$$f(x) = \sqrt{x^3 + 1} = f_3 \circ f_2 \circ f_1(x) \quad [x \geq 0]$$

where

$$f_1(x) = x^3 \quad f_1^{-1}(x) = x^{\frac{1}{3}}$$

$$f_2(x) = x+1 \quad f_2^{-1}(x) = x-1$$

$$f_3(x) = \sqrt{x} \quad f_3^{-1}(x) = x^2 \quad (x \geq 0)$$

So

$$f^{-1}(x) = f_1^{-1} \circ f_2^{-1} \circ f_3^{-1}(x)$$

$$= (x^2 - 1)^{\frac{1}{3}}$$

* 3/4 u, 1993, 3(a)

** 3/4 u, 1989, 7(b) - see next page.

(4 u, 1986, 2 - see later)

* 3/4 u, 1991, 7(b)

3/4 u, 1994 6 - done previous page

3/4 u 1989 6(a) - done on slides

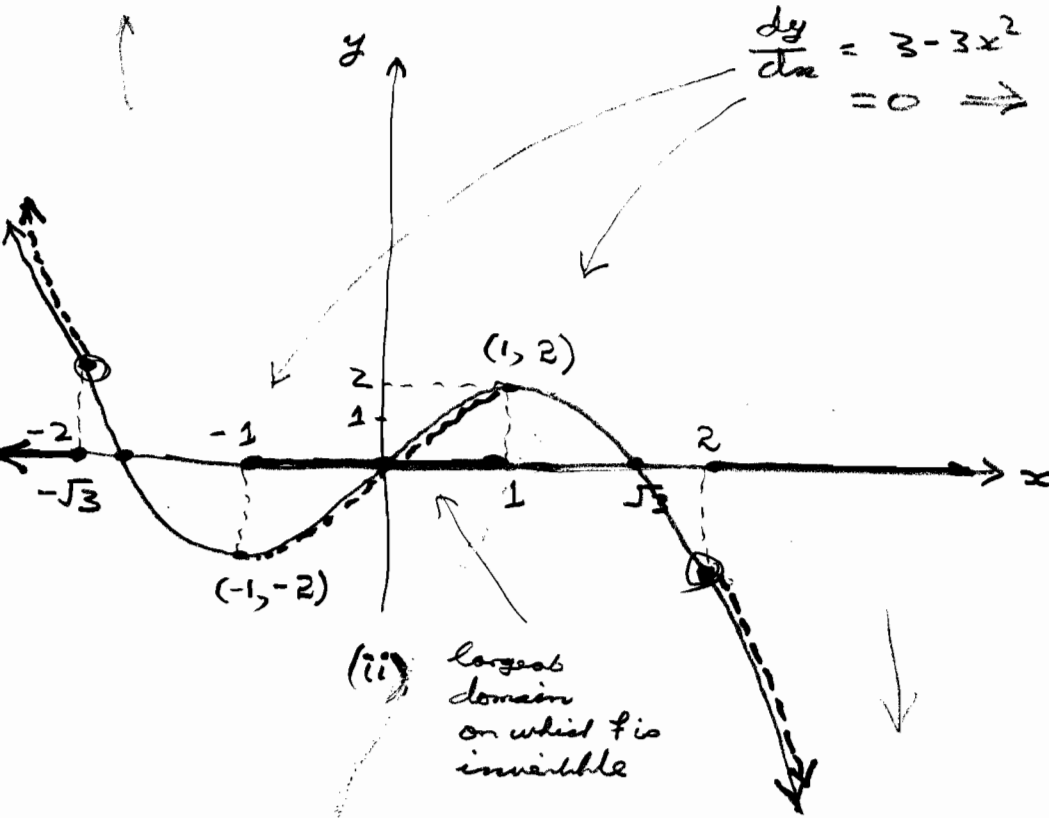
3/4 u 1995 ?

3/4 u, 1994, 6.

$$y = f(x) = 3x - x^3 \quad (\text{odd})$$

$$= 0 \Rightarrow x = 0 \text{ or } x^2 = 3, \text{ so } x = \pm\sqrt{3}$$

(i)



$$\frac{dy}{dx} = 3 - 3x^2$$

$$= 0 \Rightarrow x^2 = 1, \text{ so } x = \pm 1$$

$x^3 - 3x - 2 = 0$
 $x^3 - 3x - 2 = 0$
 $x^3 - 3x - 2 = 0$

$$3x - x^3 = -2$$

$$x^3 - 3x - 2 = 0$$

double root is $x = -1$

$$\therefore (x+1)^2(x-2) = 0$$

$$(x^2 + 2x + 1)(x-2) = 0$$

$$x = 2$$

(ii) largest domain on which f is invertible

(iii) domain of f^{-1} = range of f on $[-1, 1]$
 = $[-2, 2]$ OR $\{x < -2\} \cup [-1, 1] \cup \{x > 2\}$
 = whole line.

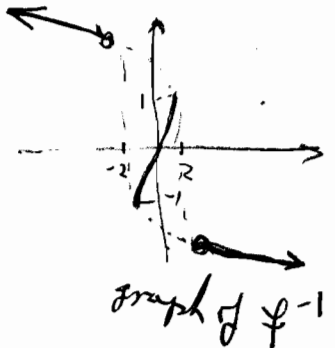
Let $y = f^{-1}(x)$ then $x = f(y) = 3y - y^3$

$$\text{and } \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{3 - 3y^2}$$

From graph $f^{-1}(0) = 0$

So at $x = 0$ $y = f^{-1}(0) = 0$

$$\therefore \left. \frac{dy}{dx} \right|_{x=0} = \frac{1}{3 - 3 \cdot 0^2} = \frac{1}{3}$$



44, 86, Q2

$$\begin{aligned}
 S'' &= 3 \\
 (S^2 - S^2)' & \\
 &= 2S'S'' - 2SS' \\
 &= 0 \\
 \Rightarrow S^2 - S^2 &= C = 1 \quad \leftarrow x=0
 \end{aligned}$$

$$y = S(x) = \frac{e^x - e^{-x}}{2}$$

$$S'(x) = \frac{e^x + e^{-x}}{2} > 0 \quad \forall x$$

So $S(x)$ is strictly increasing

$\therefore S^{-1}(x)$ exists for all x . $\left\{ \begin{array}{l} x \rightarrow \infty \quad S(x) \rightarrow \infty \\ x \rightarrow -\infty \quad S(x) \rightarrow -\infty \end{array} \right.$
So range is \mathbb{R} .

$$y = S^{-1}(x) \quad x = S(y) = \frac{e^y + e^{-y}}{2}$$

$$\frac{dx}{dy} = \frac{e^y + e^{-y}}{2}$$

$$\frac{d}{dx} S^{-1}(x) = \frac{dy}{dx} = \frac{2}{e^y + e^{-y}} = \frac{1}{\frac{e^y + e^{-y}}{2}} = \frac{1}{C(y)} = \frac{1}{S'(y)}$$

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{1}{x + \sqrt{x^2 + 1}} \times \left(1 + \frac{x}{\sqrt{x^2 + 1}} \right) \\
 &= \frac{1}{\frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1}}} \\
 &= \frac{1}{\sqrt{x^2 + 1}} \\
 &= \frac{1}{\sqrt{1 + S(y)^2}} \\
 &= \frac{1}{\sqrt{1 + x^2}}
 \end{aligned}$$

$$y = S^{-1}(x) \Rightarrow$$

$$e^y - 2x - e^{-y} = 0$$

$$(e^y)^2$$

$$(e^y)^2 - 2xe^y - 1 = 0$$

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2}$$

$e^y > 0$
RHS < 0 if choose -
So

$$\text{So } y = \ln(x + \sqrt{x^2 + 1})$$

3/4 u, 1989, 7(b)

$$y = \tan^{-1} x$$

$$x = \tan y = \frac{\sin y}{\cos y}$$

$$\frac{dx}{dy} = \frac{1}{\cos^2 y}$$

$$= 1 + \tan^2 y$$

$$s^2 + c^2 = 1$$

$$t^2 + 1 = \frac{1}{c^2}$$

$$\text{so } \frac{dy}{dx} = \frac{1}{1+x^2}$$

$$y = \tan^{-1} \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{1}{1 + \frac{1}{x^2}} \times -\frac{1}{x^2} = \frac{-x^2}{1+x^2} \times \frac{1}{x^2} = -\frac{1}{1+x^2}$$

$$\left(\tan^{-1} \frac{1}{x}\right)' = -\left(\tan^{-1} x\right)'$$

$$\text{so } \tan^{-1} x + \tan^{-1} \frac{1}{x} = C = \frac{\pi}{2} \quad (x > 0)$$

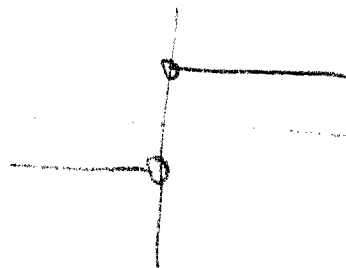
$$\tan\left(\tan^{-1} x + \tan^{-1} \frac{1}{x}\right) = \frac{x + \frac{1}{x}}{1 - x \cdot \frac{1}{x}} = \frac{x + \frac{1}{x}}{1 - 1} = \frac{x + \frac{1}{x}}{0} = \infty \quad x > 0$$

such as

5/8/89 - 2006

$\frac{t_a + t_b}{1 - t_a t_b}$

$\frac{t_a - t_b}{1 + t_a t_b}$



$-\infty \quad x < 0$