

# Property (M) and the weak fixed point property

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## Abstract

It is shown that in Banach spaces with the property (M) of Karlton, nonexpansive self mappings of nonempty weakly compact convex sets necessarily have fixed points. The stability of this conclusion under renormings is examined and conditions for such spaces to have weak normal structure are considered.

## 1. Introduction

Throughout  $X$  will denote a Banach space,  $B_X$  its unit ball;  $\{x \in X : \|x\| \leq 1\}$ ,  $S_X$  its unit sphere;  $\{x \in X : \|x\| = 1\}$ , and  $X^*$  the dual space of  $X$ .

A *weakly null type* on a Banach space  $X$  is a function of the form

$$\psi_{(x_n)}(x) = \limsup_n \|x - x_n\|,$$

where  $(x_n)$  is a weak null sequence. We say  $\psi_{(x_n)}$  is nontrivial if  $\psi_{(x_n)}(0) \neq 0$ ; that is, if  $\|x_n\| \not\rightarrow 0$ . If  $X$  is separable we may replace  $(x_n)$  by a subsequence so that  $\psi_{(x_n)}(x) = \lim_n \|x - x_n\|$ , for all  $x \in X$ .

Over the last decade an intimate connection has been established between the structure of certain weak null types and the geometry of the space, in particular the presence of weak normal structure, or the weak fixed point property. See, for example, Maurey [1980], and Sims [1992].

A Banach space  $X$  has *weak normal structure* if there are no nontrivial weakly compact convex diametral subsets. That is, if  $C$  is a weakly compact convex subset with  $\text{diam } C > 0$  then  $\inf_{y \in C} \sup_{x \in C} \|y - x\| < \text{diam } C$ .

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Weak normal structure is a sufficient condition for the *weak fixed point property* (w-fpp): Every nonexpansive self mapping of a nonempty weakly compact convex subset of  $X$  has a fixed point. Here  $T : C \rightarrow C$  nonexpansive means  $\|Tx - Ty\| \leq \|x - y\|$ , for all  $x, y \in C$ .

The reader is referred to the book by Goebel and Kirk [1990] for a full discussion of these properties and the connection between them.

Recently Kalton [1993] introduced *property (M)*: Weakly null types are constant on spheres about the origin. That is, for  $x_n \xrightarrow{w} 0$  the weakly null type  $\psi_{(x_n)}(x) = \limsup_n \|x - x_n\|$  is a function of  $\|x\|$  only.

Property (M) was an essential ingredient in Kalton's characterization of those separable Banach spaces  $X$  for which the compact operators  $\mathcal{K}(X)$  form an M-ideal in the algebra of all bounded linear operators,  $\mathcal{L}(X)$ . That is,

$$\mathcal{L}(X)^* = (\mathcal{K}(X)^\perp \oplus V)_1, \quad \text{for some closed subspace } V.$$

A link with the fixed point property for such spaces was observed in Sims [1982] where it was noted that Ásvad Lima [1982] had effectively shown that  $\mathcal{K}(X)$  an M-ideal in  $\mathcal{L}(X)$  implies that  $X^*$  is weak\*-uniformly Kadec-Klee (UKK\*) and hence has weak\* normal structure [van Dulst and Sims, 1983]. It therefore seems natural to inquire into connections between property (M), weak normal structure, and the w-fpp. This is particularly relevant since a detailed study of spaces with property (M), including stability under sums and renormings has been undertaken by Kalton [1993] and Kalton and Werner [1993], to which the reader is referred for details and examples.

## 2. Property (M) and weak normal structure

Property (M) relates to several properties previously introduced in connection with the w-fpp.

A Banach space  $X$  is said to have *WORTH* if every weakly null type satisfies  $\psi_{(x_n)}(-x) = \psi_{(x_n)}(x)$ , for all  $x \in X$  [Sims, 1988], and to satisfy the *non-strict Opial condition* if every weakly null type satisfies  $\psi_{(x_n)}(0) \leq \psi_{(x_n)}(x)$ , for all  $x \in X$ .

PROPOSITION 2.1: *For the following conditions on the Banach space  $X$  we have*

$$(i) \implies (ii) \implies (iii) \implies (iv).$$

(i)  $X$  has property (M).

(ii)  $X$  has WORTH.

(iii) If  $x_n \xrightarrow{w} 0$  then for each  $x \in X$  we have  $\psi_{(x_n)}(tx)$  is an increasing function of  $t$  on  $[0, \infty)$ .

(iv)  $X$  satisfies the non-strict Opial condition.

PROOF: All the implication are clear except for (ii)  $\implies$  (iii). To see this, note that for  $0 < t_1 < t_2$  there exists  $\beta \in (0, 1)$  such that  $t_1x = \beta(-t_2)x + (1 - \beta)t_2x$  and so, since  $\psi_{(x_n)}$  is convex and by WORTH  $\psi_{(x_n)}(-t_2x) = \psi_{(x_n)}(t_2x)$ , we have  $\psi_{(x_n)}(t_1x) \leq \beta\psi_{(x_n)}(-t_2x) + (1 - \beta)\psi_{(x_n)}(t_2x) = \psi_{(x_n)}(t_2x)$ . ■

An immediate consequence is lemma 2.1(3) of Kalton [1993].

COROLLARY 2.2: If  $X$  has property (M) any weakly null type  $\psi_{(x_n)}(x)$  is an increasing function of  $\|x\|$ .

It is well known [see for example, Goebel and Kirk, 1990] that if  $X$  fails to have weak normal structure then  $B_X$  contains a weak null sequence  $(x_n)$  satisfying

$$\lim_n \|x - x_n\| = 1, \quad \text{for all } x \in \overline{\text{co}}\{x_n\}_{n=1}^\infty.$$

In particular, since  $0 \in \overline{\text{co}}\{x_k\}_{k=1}^\infty$ , we have  $\|x_n\| \rightarrow 1$ . Thus we have the following.

PROPOSITION 2.3: Let  $X$  be a Banach space with property (M). If  $X$  fails to have weak normal structure then  $X$  admits a nontrivial weakly null type which is identically equal to 1 on  $B_X$ .

PROOF: Let  $(x_n)$  be a sequence in  $B_X$  such as described in the previous paragraph. Then  $\psi_{(x_n)}(0) = 1$  and  $\psi_{(x_n)}(x_m) = 1$ , for all  $m$ . Since  $\|x_m\| \rightarrow 1$ , it follows from corollary 2.2 and property (M) that  $\psi_{(x_n)}$  equals 1 on the open unit ball, and hence by continuity on  $B_X$ . ■

THEOREM 2.4 Let  $X$  be a Banach space with property (M). Then  $X$  has weak normal structure if there exists a point  $x_0 \in S_X$  such that whenever  $y_n \xrightarrow{w} x_0$  and  $\|y_n\| \rightarrow 1$  we have that the separation index  $\gamma(y_n) := \sup \inf_{k \neq m} \|y_{n_k} - y_{n_m}\| < 1$ , where the supremum is taken over all subsequences  $(y_{n_k})$  of  $(y_n)$ .

PROOF: Suppose  $X$  fails to have weak normal structure. Let  $\psi_{(x_n)}$  be the weakly null type of proposition 2.3. and let  $y_n := x_0 - x_n$ . Then,  $y_n \xrightarrow{w} x_0$  and  $\limsup_n \|y_n\| = \psi_{(x_n)}(x_0) = 1$ , so there is a subsequence with  $\gamma(y_{n_k}) < 1$ . But,  $\gamma(y_{n_k}) = \gamma(x_{n_k}) = 1$ , a contradiction. ■

COROLLARY 2.5: If  $X$  has property (M) and satisfies any of the following then  $X$  has weak normal structure.

- (i)  $X$  has the Kadec-Klee property (the relative weak and norm topologies agree on  $S_X$ ).
- (ii)  $X$  is reflexive.
- (iii)  $X$  has the Radon-Nikodym property.
- (iv)  $X$  has the point of continuity property: for every weakly closed bounded subset  $A$ , the identity map  $(A, \text{weak})$  to  $(A, \text{norm})$  has at least one point of continuity, see [Edgar and Wheeler, 1984] for details.
- (v)  $S_X$  contains at least one point at which the relative weak and norm topologies agree.

PROOF: (i)  $\implies$  (v), (ii)  $\implies$  (iii)  $\implies$  (iv)  $\implies$  (v), and (v) implies the condition of theorem 2.4. ■

### 3. Property (M) implies the weak fixed point property

For a weakly null type  $\psi_{(x_n)}$  on the Banach space  $X$  define

$$\lambda_{(x_n)} := \sup \psi_{(x_n)}(B_X).$$

From the weak lower semi-continuity of the norm, we have

$$1 \leq \lambda_{(x_n)} \leq 1 + \limsup_n \|x_n\|,$$

and if  $X$  has property (M) then

$$\begin{aligned} \lambda_{(x_n)} &= \psi_{(x_n)}(x), \quad \text{for any } x \in S_X, \\ &= \lim_m \psi_{(x_n)}(y_m), \quad \text{whenever } \|y_m\| \rightarrow 1. \end{aligned}$$

LEMMA 3.1: *If  $X$  has property (M) and  $\psi_{(x_n)}$  is a weakly null type with  $\psi_{(x_n)}(0) = 1$  then  $\lambda_{(x_n)} = D(x_n)$ , where  $D(x_n) := \limsup_m \limsup_n \|x_m - x_n\|$ .*

PROOF:  $\psi_{(x_n)}(0) = 1$  implies that the limsup of the norms of any subsequence of  $(x_n)$  is at most 1, and that there exists a subsequence  $(x_{n_k})$  with  $\|x_{n_k}\| \rightarrow 1$ .

Then

$$\begin{aligned} D(x_n) &\leq \lambda_{(x_n)} = \lim_k \psi_{(x_n)}(x_{n_k}) \\ &= \lim_k \limsup_n \|x_{n_k} - x_n\| \\ &\leq D(x_n). \end{aligned}$$
■

THEOREM 3.2: Let  $X_0 := (X, \|\cdot\|_0)$  be a Banach space having property (M) and let  $\|\cdot\|_1$  be an equivalent norm on  $X$  satisfying

$$\|x\|_0 \leq \|x\|_1 \leq b\|x\|_0, \quad \text{for all } x \in X.$$

If  $b < (1 + \sqrt{5})/2$ , then  $X_1 := (X, \|\cdot\|_1)$  has the w-fpp .

PROOF: Suppose  $X_1$  fails the w-fpp . Then, by standard arguments [see Goebel and Kirk, 1990, for example], there exists a weak compact convex subset  $K$  of  $X$  with  $\text{diam}_1 K = 1$  and a fixed point free  $\|\cdot\|_1$ -nonexpansive map  $T : K \rightarrow K$  with respect to which  $K$  is a minimal nonempty weak compact convex invariant subset which contains an approximate fixed point sequence  $(a_n)$  with  $a_n \xrightarrow{w} 0$  and, by the Goebel–Karlovitc lemma,  $\lim_n \|x - a_n\|_1 = \text{diam}_1 K = 1$  for all  $x \in K$ .

Let  $[X_1] := \ell_\infty(X_1)/c_0(X_1)$  with the quotient norm given canonically by  $\|[x_n]\| = \limsup_n \|x_n\|_1$ . Let  $[K] := \{[x_n] : x_n \in K, \text{ for } n = 1, 2, \dots\}$ . Then  $[T][x_n] := [Tx_n]$  is a well defined nonexpansive self mapping of  $[K]$ .

Given  $\epsilon \in (0, 1/2)$  let

$$W := \{[w_n] \in [K] : \|[w_n] - [a_n]\| \leq \frac{1}{2} - \epsilon \text{ and } D[w_n] \leq \frac{1}{2} + \epsilon\},$$

where  $D[w_n] := D_1(w_n) = \limsup_m \limsup_n \|w_m - w_n\|_1$  is well defined, since for  $(w_n - y_n) \in c_0(X_1)$  we have  $D_1(w_n) = D_1(y_n)$ .

Then  $W$  is  $[T]$  invariant, closed, convex and nonempty, as  $(\frac{1}{2} + \epsilon)[a_n] \in W$ . Thus, by Lin's [1985] extension of the Goebel–Karlovitc lemma,  $W$  contains elements of norm arbitrarily close to one.

On the other hand, for  $[w_n] \in W$  we may without loss of generality suppose that  $w_n \in K$ , for all  $n$ , and we may extract a subsequence  $(w_{n_k})$  such that

$$\lim_k \|w_{n_k}\|_1 = \|[w_n]\|,$$

$$(w_{n_k}) \text{ is weakly convergent to some } w_0 \in K,$$

and

$$d := \lim_k \|w_{n_k} - w_0\|_0 \text{ exists.}$$

Now,  $\|w_0\|_1 \leq \liminf_k \|w_{n_k} - a_{n_k}\|_1 \leq \|[w_n] - [a_n]\| \leq \frac{1}{2} - \epsilon$ . Thus, given any  $\eta \in (0, \epsilon)$ , if  $\liminf_k \|w_{n_k} - w_0\|_1 < \frac{1}{2} + \eta$  we have

$$\|[w_n]\| \leq \liminf_k \|w_{n_k} - w_0\|_1 + \|w_0\|_1 < 1 + \eta - \epsilon.$$

So in this case  $\|[w_n]\|_1$  is uniformly bounded away from one.

Thus, we need only consider the case when

$$\liminf_k \|w_{n_k} - w_0\|_1 \geq \frac{1}{2} + \eta.$$

For this case, provided  $b \leq (\frac{1}{2} + \eta)/(\frac{1}{2} - \epsilon)$  we have

$$\begin{aligned} d = \lim_k \|w_{n_k} - w_0\|_0 &\geq (1/b) \liminf_k \|w_{n_k} - w_0\|_1 \\ &\geq (\frac{1}{2} + \eta)/b \\ &\geq \frac{1}{2} - \epsilon \\ &\geq \|w_0\|_1 \\ &\geq \|w_0\|_0. \end{aligned}$$

Let  $y_k := (1/d)(w_0 - w_{n_k})$ , so  $\|y_k\|_0 \rightarrow 1$ . Then

$$\begin{aligned} \|[w_n]\| &= \lim_k \|w_{n_k}\|_1 \leq bd \limsup_k \|\frac{1}{d}w_0 - \frac{1}{d}(w_0 - w_{n_k})\|_0 \\ &= bd \psi_{(y_k)}((1/d)w_0) \\ &\leq bd \lambda_{(y_k)}, \quad \text{as } \|(1/d)w_0\|_0 \leq 1, \\ &= bd D_0(y_k), \quad \text{by lemma 3.1, as } \psi_{(y_k)}(\mathbf{0}) = 1, \\ &= b D_0(w_{n_k}) \\ &\leq b D_0(w_n) \\ &\leq b D[w_n] \\ &\leq b(\frac{1}{2} + \epsilon). \end{aligned}$$

Thus again  $\|[w_n]\|$  is uniformly bounded away from one provided  $b < 1/(\frac{1}{2} + \epsilon)$ .

In this way we arrive at a contradiction whenever

$$b < \min\left(\frac{\frac{1}{2} + \eta}{\frac{1}{2} - \epsilon}, \frac{1}{\frac{1}{2} + \epsilon}\right),$$

where  $0 < \eta < \epsilon < 1/2$ . That is, for  $b < (1 + \sqrt{5})/2$ . ■

**COROLLARY 3.3:** *Let  $X$  be a Banach space with property (M). If  $Y$  is a Banach space for which the Banach–Mazur distance  $d(Y, X) < (1 + \sqrt{5})/2$ , then  $Y$  has the w-fpp .*

**PROOF:** This follows directly from theorem 3.2 and the observation that both the w-fpp and property (M) are preserved if the norm is replaced by a strictly positive scalar multiple of itself. ■

REMARKS: (1) The constant  $(1 + \sqrt{5})/2$  is equal to that obtained by Jiménez-Melado and Lloréns-Fuster [1992] for  $X = \ell_2$ , and appears to be the best known for the  $\ell_p$  spaces with  $p$  near 2.

(2) Theorem 3.2 affords another proof that  $c_0$ , while failing to have weak normal structure, none-the-less enjoys the w-fpp . Indeed in the presence of property (M) the appearance of  $c_0$  in the space appears to be the main impediment to weak normal structure. To see this, suppose  $X$  has (M), but fails to have weak normal structure. Since weak normal structure is separably determined, by passing to a subspace if necessary we may assume that  $X$  is separable. Then, by the argument for proposition 2.3,  $X$  admits a nontrivial weakly null type

$$\psi_{(x_n)}(x) := \lim_n \|x - x_n\| \quad (x_n \xrightarrow{w} 0, \|x_n\| = 1)$$

which is constant on  $B_X$ . From Karlton [1993] Lemma 3.6 and the discussion preceding it there exist  $(y_n) \subset X$ , a basic subsequence of  $(x_n)$ , and constants  $k, K > 0$  such that

$$k \left\| \sum \xi_n y_n \right\| \leq \|(\xi_n)\|_F \leq K \left\| \sum \xi_n y_n \right\|,$$

for all finitely supported sequences  $(\xi_n)$ , where  $\|\cdot\|_F$  is the Orlicz norm arising from  $F(t) := \lim_n \|x - tx_n\| - 1$  whenever  $\|x\| = 1$ . But, for  $t \geq 1$  we readily see that  $F(t) = t - 1$ , in particular  $F$  is degenerate, so

$$k \left\| \sum \xi_n y_n \right\| \leq \|(\xi_n)\|_\infty \leq 2K \left\| \sum \xi_n y_n \right\|$$

and  $c_0 \hookrightarrow X$ .

Since the inclusion of  $c_0$  is an isomorphic (and hence almost isometric) one, this does not provide a characterization of weak normal structure in spaces with property (M). It does however give an alternative, albeit substantially less direct, proof for some of the necessary conditions for weak normal structure given in Corollary 2.5. In particular conditions (ii) and (iii). Whether or not it captures (i) and most importantly (v), or the result of theorem 2.4, is unclear, and leads to the question: does the Kadec-Klee property in the presence of (M) imply  $c_0 \not\hookrightarrow X$ ? In particular, since  $c_0$  itself admits an equivalent Kadec-Klee norm we ask is there such a renorming which also retains property (M)?

(3) A dual property to (M), property  $(M^*)$ , is defined in  $X^*$  by requiring that

$$\psi_{(f_n)} : X^* \longrightarrow \mathbf{R}^+ : f \longmapsto \limsup_n \|f - f_n\|$$

be a function of  $\|f\|$  only, whenever  $f_n \xrightarrow{w^*} 0$ . Kalton [1993] shows that if  $X^*$  has  $(M^*)$  then  $X$  has (M) and the natural embedding of  $X$  is an M-ideal in  $X^{**}$  and so by Lima [1982]  $X^*$  has the Radon-Nikodym property. Thus, if  $X^*$  has property  $(M^*)$ , then  $X^*$  has weak\* normal structure and  $X$  has the w-fpp .

(4) Since property (M) implies WORTH which in turn implies the non-strict Opial condition we are left with the question: does WORTH, or indeed the non-strict Opial condition, imply the w-fpp ?

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