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REMARKS ON ORTHOGONAL CONVEXITY OF BANACH SPACES

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ABSTRACT. It is proved that orthogonal convexity defined by A. Jimenez-Melado and E. Llorens-Fuster implies the weak Banach-Saks property. Relations between orthogonal convexity and another geometric properties, such as nearly uniform smoothness and property (β), are studied.

Introduction.

Orthogonal convexity has been introduced by A. Jimenez-Melado and E. Llorens-Fuster (see [3] and [4]) as a geometric property of Banach spaces which implies the fixed point property for nonexpansive mappings. They have shown that various kinds of Banach spaces, such as c_0, uniformly convex spaces, spaces with the Schur property, the James space, among others have this property.

Let E be a Banach space. If A is a nonempty bounded subset of E, we put $|A| = \sup\{\|z\| : z \in A\}$. Moreover for $\lambda > 0$ and $x, y \in E$ let

$$M_\lambda(x,y) = \left\{z \in E : \max\{\|z-x\|, \|z-y\|\} \le \tfrac{1}{2}(1+\lambda)\|x-y\|\right\} .$$

Having a bounded sequence (x_n) in E, we use the following notation:

$$D[(x_n)] = \limsup_{n\to\infty}\left(\limsup_{m\to\infty}\|x_n - x_m\|\right)$$

$$A_\lambda[(x_n)] = \limsup_{n\to\infty}\left(\limsup_{m\to\infty}|M_\lambda(x_n,x_m)|\right) .$$

The Banach space E is called orthogonally convex if for each sequence (x_n) in E weakly convergent to zero, with $D[(x_n)] > 0$, there exists $\lambda > 0$ such that

$$A_\lambda[(x_n)] < D[(x_n)] \ .$$

In this paper we prove that orthogonal convexity implies the weak Banach-Saks property. Our second result concerns orthogonal convexity of some direct sums. The remaining part of the paper is devoted to the study of relations between orthogonal convexity and some infinite dimensional geometric properties.

Results. Let us recall that a Banach space E has the weak Banach-Saks property (also called the Banach-Saks-Rosenthal property) if every weakly null sequence (x_n) in E contains a subsequence (x'_n) such that the Cesaro means $n^{-1}\sum_{k=1}^{n} x'_k$ form a norm-convergent sequence.

Theorem 1. *Let E be an orthogonally convex Banach space. Then E has the weak Banach-Saks property.*

Proof. Assume the contrary, that is, there is a weakly null sequence (x_n) in E such that no subsequence has norm-converging Cesaro means. Using a method described in [1] one can find a constant c such that for a fixed integer m there exists a sequence $(y_n^m)_{n\geq 1}$ with the following properties.

1. There is a sequence of scalars (α_i) and an increasing sequence of integers (k_n) such that $m \leq k_1$,

$$y_n^m = \sum_{i=k_n+1}^{k_{n+1}} \alpha_i x_i \quad \text{and} \quad \sum_{i=k_n+1}^{k_{n+1}} |\alpha_i| \leq c$$

for all n.

2. If $n > 1$, then

$$\tfrac{m}{m+1}(|\alpha| + |\beta|) \leq \|\alpha y_1^m + \beta y_n^m\| \leq |\alpha| + |\beta|$$

for any scalars α, β.

Consider now a sequence (z_n) obtained by a rearrangement of the set $\{y_n^m\}_{m,n\geq 1}$. Since the sequence (x_n) is weakly null, from 1 it follows that also (z_n) converges weakly to zero.

By 2 we have $\|z_n\| \leq 1$ for all n. Conseqently $D[(z_n)] \leq 2$. On the other hand

$$D[(z_k)] \geq \limsup_{m\to\infty} (\limsup_{n\to\infty} \|y_1^m - y_n^m\|) \geq 2 \ .$$

Hence $D[(z_k)] = 2$ and from orthogonal convexity it follows that

$$A_\lambda[(z_n)] < 2 \tag{1}$$

for some $\lambda > 0$.

We take m_0 so that $\frac{1}{m_0} \leq \lambda$. Condition 2 shows that for every k

$$\|y_k^m\| \leq 1 \leq \tfrac{m}{m+1}(1+\lambda) \leq \tfrac{1}{2}(1+\lambda)\|y_1^m - y_n^m\| \ ,$$

whenever $m \geq m_0$ and $n > 1$. In particular this means that $y_1^m + y_n^m \in M_\lambda(y_1^m, y_n^m)$. Therefore applying 2 again, we obtain

$$\begin{aligned} A_\lambda[(z_n)] &\geq \limsup_{k\to\infty}\left(\limsup_{n\to\infty} |M_\lambda(y_1^{m_0+k}, y_n^{m_0+k})|\right) \\ &\geq \limsup_{k\to\infty}\left(\limsup_{n\to\infty} \|y_1^{m_0+k} + y_n^{m_0+k}\|\right) \geq 2 \ . \end{aligned}$$

This contradicts (1).

In [3] some statements on direct sums are given. Here we present another one. Let (X_i) be a sequence of spaces and Y be a space with a 1-unconditional basis (e_n) (i.e. (e_n) is an unconditionally monotone basis in the terminology of [2]).

$Y\{(X_i)\}$ denotes the space of all sequences $(x(i))$, where $x(i) \in X_i$ for each i, such that the series $\sum \|x(i)\|e_i$ converges in Y. The norm in $Y\{(X_i)\}$ is given by the formula

$$\|(x(i))\| = \left\|\sum_{i=1}^{\infty} \|x(i)\|e_i\right\|_Y \ .$$

Let us also recall that the basis (e_n) is shrinking if the coefficient functionals e_n^* form a basis of the dual space Y^* (see [2] p. 64).

Theorem 2. *Let (X_i) be a sequence of Banach spaces with the Schur property and Y be an orthogonally convex space with a 1-unconditional shrinking basis (e_i). Then the space $X = Y\{(X_i)\}$ is orthogonally convex.*

Proof. For any $x = (x(i)) \in X$ put $\bar{x} = \sum_{i=1}^{\infty} \|x(i)\|e_i \in Y$. By the definition $\|x\|_X = \|\bar{x}\|_Y$. Let (x_n) be an arbitrary weakly null sequence in X with $D[(x_n)] > 0$. Since all X_i have the Schur property, one can see that

$$\lim_{m\to\infty} \|x_m(i)\| = 0 \tag{2}$$

for each i.

But the basis of Y is shrinking. Therefore (2) implies that $(\bar{x}_n)$ converges weakly to zero in Y.

Fix an integer n. From (2) it is easy to see that

$$\lim_{m\to\infty} | \, \|x_n - x_m\|_X - \|\bar{x}_n - \bar{x}_m\|_Y | = 0 \, . \tag{3}$$

Consequently $D[(\bar{x}_n)] = D[(x_n)] > 0$. Since Y is orthogonally convex, there exists $\lambda > 0$ such that $A_\lambda[(\bar{x}_n)] < D[(\bar{x}_n)]$.

For a fixed n consider an arbitrary $z \in M_{\frac{\lambda}{2}}(x_n, x_m)$. Clearly

$$\max\{\|\bar{z} - \bar{x}_m\|, \|\bar{z} - \bar{x}_n\|\} \le \tfrac{1}{2}\left(1 + \tfrac{\lambda}{2}\right)\|x_m - x_n\|.$$

If $x_n \ne 0$, then it follows from (2) and (3) that

$$\left(1 + \tfrac{\lambda}{2}\right)\|x_m - x_n\| \le (1+\lambda)\|\bar{x}_m - \bar{x}_n\|$$

for m large enough. Obviously this inequality holds also in case $x_n = 0$. Therefore for large m we have $\bar{z} \in M_\lambda(\bar{x}_m, \bar{x}_n)$. Hence

$$A_{\frac{\lambda}{2}}[(x_n)] \le A_\lambda[(\bar{x}_n)] < D[(\bar{x}_n)] = D[(x_n)] \, ,$$

which means that X is orthogonally convex.

Now we turn to the study of relations between orthogonal convexity and another geometric properties. Let us recall a definition introduced in [7].

A Banach space X is *nearly uniformly smooth* (NUS in short) provided for every $\epsilon > 0$ there exists $\eta > 0$ such that if $0 < t < \eta$ and (x_n) is a basic sequence in the unit ball B_X of X, then there is $k > 1$ for which

$$\|x_1 + tx_k\| \le 1 + \epsilon t \, .$$

We shall say that X is *weakly NUS* (WNUS in short) if it satisfies the condition obtained from the above definition by replacing "for every $\epsilon > 0$" by "for some ϵ in $(0,1)$".

In [7] it was proved that uniform smoothness implies NUS which in turn implies reflexivity. A slight modification of the proof of Proposition 2.3 [7] shows that WNUS spaces are reflexive.

Proposition 3. *A Banach space X is* WNUS *if and only if there exists a constant $c \in (0,1)$ such that for every basic sequence (x_n) in B_X there is $k > 1$ for which*

$$\|x_1 + x_k\| \leq 2 - c \,.$$

Proof. Assume that X is a WNUS space. From the definition we obtain some $\epsilon > 0$ and $\eta > 0$. Now if $t = \frac{1}{2}\min\{\eta, 1\}$ and (x_n) is a basic sequence in B_X, then there exists $k > 1$ such that

$$\begin{aligned} \|x_1 + x_k\| &\leq \|x_1 + tx_k\| + (1-t)\|x_k\| \\ &\leq 1 + \epsilon t + 1 - t = 2 - t(1-\epsilon) \,. \end{aligned}$$

Assume in turn that there exists $c \in (0,1)$ such that if (x_n) is a basic sequence in B_X, then $\|x_1 + x_k\| \leq 2 - c$ for some $k > 1$. For $t \in (0,1)$ we have

$$\begin{aligned} \|x_1 + tx_k\| &\leq t\|x_1 + x_k\| + (1-t)\|x_1\| \\ &\leq t(2-c) + 1 - t = 1 + t(1-c) \,, \end{aligned}$$

which shows that X is WNUS.

We also need the next definition.

A Banach space X has the *weak Opial property* provided for every weakly null sequence (x_n) in X and every $x \in X$

$$\liminf_{n\to\infty} \|x_n\| \leq \liminf_{n\to\infty} \|x + x_n\| \,.$$

Theorem 4. *If X is a* WNUS *Banach space with the weak Opial property, then X is orthogonally convex.*

Proof. Let X be a WNUS space with the weak Opial property. From Proposition 3 we obtain a constant $c \in (0,1)$. Take a positive $\lambda < \frac{c}{2-c}$. Let now (x_n) be a weakly null sequence in X with $D[(x_n)] > 0$. For a fixed n there exists a sequence of integers (m_k) such that

$$\lim_{k\to\infty} |M_\lambda(x_n, x_{m_k})| = \limsup_{m\to\infty} |M_\lambda(x_n, x_m)| \,.$$

Denote this limit by a_n. There is a sequence (z_k) with $z_k \in M_\lambda(x_n, x_{m_k})$, for all k, and $\lim_{k\to\infty} \|z_k\| = a_n$. Since X is reflexive, we can assume that (z_k) converges weakly to some z.

Let $d_n = \limsup_{k\to\infty} \|x_n - x_{m_k}\|$. We shall show that

$$a_n \le Bd_n \ , \tag{4}$$

where $B = (1 - \frac{c}{2})(1 + \lambda) < 1$. If (z_k) converges to z in norm, then

$$a_n = \|z\| \le \liminf_{k\to\infty} \|z_k - x_{m_k}\| \le \tfrac{1}{2}(1 + \lambda)d_n \le Bd_n \ .$$

Let us now consider the case when (z_k) does not converge in norm. Passing to a subsequence, we can assume that the elements $z, z_1 - z, z_2 - z, \ldots$ form a basic sequence (see [2] p. 107). Proposition 3 gives us a further subsequence, which we still denote by (z_k), such that

$$a_n = \lim_{k\to\infty} \|z + (z_k - z)\| \le (2 - c) \max\left\{\|z\|, \liminf_{k\to\infty} \|z_k - z\|\right\} . \tag{5}$$

But

$$\|z\| \le \liminf_{k\to\infty} \|z_k - x_{m_k}\| \le \tfrac{1}{2}(1 + \lambda)d_n \ .$$

Moreover by the weak Opial property

$$\liminf_{k\to\infty} \|z_k - z\| \le \liminf_{k\to\infty} \|z_k - x_n\| \le \tfrac{1}{2}(1 + \lambda)d_n \ .$$

Therefore (5) shows that $a_n \le \frac{1}{2}(2 - c)(1 + \lambda)d_n = Bd_n$.

Having established (4), we conclude that

$$A_\lambda[(x_k)] = \limsup_{n\to\infty} a_n \le B \limsup_{n\to\infty} d_n \le BD[(x_k)] \ .$$

Let us now mention another geometric properties. In [8] the notion of the *uniform Opial property* was introduced.

A Banach space X has the uniform Opial property provided for every $c > 0$ there exists $r > 0$ such that if $\|x\| \ge c$ and (x_n) is a weakly null sequence in X with $\|x_n\| \ge 1$ for all n, then

$$1 + r \le \liminf_{n\to\infty} \|x + x_n\| \ .$$

It was proved that if X is a reflexive space with the uniform Opial property, then X^* has the fixed point property for nonexpansive mappings. But Lemma 2.2 [8] actually shows that the space X^* is WNUS and has the weak Opial property. Therefore we obtain the following corollary of Theorem 4.

Corollary 5. *If X is a reflexive space with the uniform Opial property, then X^* is orthogonally convex.*

The next property related to WNUS was defined in [9] as a generalization of uniform convexity. It was called property (β), but instead of quoting its definition we prefer to recall the following result (see [5]).

A Banach space X has property (β) if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that for each element $x \in B_X$ and each sequence (x_n) in B_X with $\inf\{\|x_m - x_n\| : m \neq n\} \geq \epsilon$ there is an index k for which

$$\tfrac{1}{2}\|x + x_k\| \leq 1 - \delta \ .$$

We shall say that X has *property* $w(\beta)$ provided it satisfies the condition obtained from the above one by replacing "for every $\epsilon > 0$" by "for some $\epsilon \in (0,1)$". In [6] it was observed that spaces with property $\mathrm{w}(\beta)$ are reflexive.

Spaces dual to those with property (β) satisfy so called *property* (β^*), which is known to imply NUS (see [6]). The same argument shows in fact that if a space X has property $\mathrm{w}(\beta)$, then X^* is WNUS. We shall prove that the space X is WNUS too.

Let (x_n) be a basic sequence in the unit ball B_X of a space X with property $\mathrm{w}(\beta)$. Since X is reflexive, the sequence (x_n) converges weakly to zero (see [2] p. 67).

From the definition of property $\mathrm{w}(\beta)$ we obtain some constants $\epsilon \in (0,1)$ and $\delta > 0$.

If $\|x_n\| > \epsilon$ for all $n > 1$, then passing to a subsequence, we can assume that $\inf\{\|x_m - x_n\| : m \neq n\} \geq \epsilon$. Therefore there is an index $k > 1$ such that

$$\|x_1 + x_k\| \leq 2(1 - \delta) \ .$$

If $\|x_k\| \leq \epsilon$ for some $k > 1$, then

$$\|x_1 + x_k\| \leq 1 + \epsilon \ .$$

In light of Proposition 3 this shows that the space X is WNUS.

Since for reflexive spaces the weak Opial property is self- dual, we thus obtain the next corollary of Theorem 4.

Corollary 6. *If a Banach space X has property* w(β) *and the weak Opial property, then X and X^* are orthogonally convex.*

Let us point out that the assumption of having the weak Opial property is essential in the above results.

Example. There exists a NUS Banach space X which is not orthogonally convex.

Let us consider the following norm in a two-dimensional space.

$$\|(\alpha,\beta)\|_0 = \max\left\{|0.9\alpha+\beta|, t\,|\alpha-0.1\beta|, \left(\alpha^2+(0.1\beta)^2\right)^{\frac{1}{2}}\right\},$$

where $t=\frac{216}{217}$ and α,β are real numbers.

For an element $x=(\alpha_i)\in l_2$ we write $S_n x=(\alpha_{n+1},\alpha_{n+2},\ldots)$.

Let now E denote the space l_2 with the equivalent norm given by the formula

$$\|x\| = \sup\{\left[\|(\alpha_1,\alpha_n)\|_0^2 + (0.1\|S_n x\|_2)^2\right]^{\frac{1}{2}} : n>1,\}$$

where $x=(\alpha_i)\in l_2$ and $\|\cdot\|_2$ is the initial norm of l_2.

It is easy to see that

$$0.1\|x\|_2 \le \|x\| \le 1.4\|x\|_2 \tag{6}$$

for every $x\in E$.

Our space X is the l_8-sum of countably many copies of E, so that $X=l_8\{(E_i)\}$, where $E_i=E$ for all i. Therefore each element $x\in X$ is of the form $x=(x(i))$ with $x(i)\in E$ for all i.

Let (e_n) be the natural basis of E. We consider elements $e_k^m\in X$ such that $e_k^m(i)=0$ if $i\neq m$ and $e_k^m(i)=e_k$ if $i=m$. The set $\{e_k^m\}_{m,k\geq 1}$ rearranged into a sequence (x_n) by the formula $x_n=e_k^m$ if $m+k=n+1$ gives us a basis of the space X.

Let (P_n) be the sequence of the natural projections associated to this basis, so that

$$P_n x = \sum_{j=1}^n \alpha_j x_j$$

for any $x=\sum_{j=1}^\infty \alpha_j x_j\in X$.

It is easy to see that the sequence (x_n) converges weakly to zero in X. Moreover $D[(x_n)] = \|(1,-1)\|_0 = 1.1t$.

We take scalars α, β such that

$$0.9\alpha + \beta = 1\ , \quad t(\alpha - 0.1\beta) = -1$$

and put $a = \frac{11}{20}t\alpha + 1$, $b = \frac{11}{20}t\beta$.

Let us now fix an arbitrary $\lambda > 0$. For any m, k with $k > 1$, straightforward computations show that

$$z = ae_1^m + be_k^m \in M_\lambda(e_1^m, e_k^m)\ .$$

Therefore $|M_\lambda(e_1^m, e_k^m)| \geq \|z\| = \|(a,b)\|_0 > D[(x_n)]$. Consequently

$$A_\lambda[(x_n)] \geq \limsup_{m\to\infty}\left(\limsup_{k\to\infty}|M_\lambda(e_1^m, e_k^m)|\right) > D[(x_n)]\ ,$$

which proves that X is not orthogonally convex.

In order to show that X is NUS let us consider elements $x, y \in X$ such that there exists n for which $P_n x = x$ and $P_n y = 0$. We shall check that

$$\|x+y\|^2 \leq \|x\|^2 + 6.25\|y\|^2. \tag{7}$$

Let us fix i. If $x(i) \neq 0$, then there are sequences of scalars (α_j), (β_j) and an integer $n \geq 1$ such that

$$x(i) = \textstyle\sum_{j=1}^{n} \alpha_j e_j\ , \quad y(i) = \sum_{j=n+1}^{\infty} \beta_j e_j.$$

To estimate $\|x(i)+y(i)\|^2$ it suffices to consider an expression

$$s = \|(\gamma_1, \gamma_m)\|_0^2 + 0.01 \textstyle\sum_{j=m+1}^{\infty} \gamma_j^2,$$

where $\gamma_j = \alpha_j$ if $1 \leq j \leq n$ and $\gamma_j = \beta_j$ if $j > n$.

In the case when $n \geq m$, by (6) we have

$$s \leq \|x(i)\|^2 + 0.01\|y(i)\|_2^2 \leq \|x(i)\|^2 + \|y(i)\|^2\ .$$

If $n < m$, then

$$s \leq \alpha_1^2 + 6.25\beta_m^2 + 0.01 \sum_{j=m+1}^{\infty} \beta_j^2$$
$$\leq \|x(i)\|^2 + 6.25\|y(i)\|^2.$$

Therefore

$$\begin{aligned}\|x+y\|^2 &= \Big(\sum_{i=1}^{\infty} \|x(i)+y(i)\|^8\Big)^{\frac{1}{4}} \\ &\le \Big(\sum_{i=1}^{\infty}(\|x(i)\|^2 + (6.25\|y(i)\|)^2)^4\Big)^{\frac{1}{4}} \\ &\le \Big(\sum_{i=1}^{\infty} \|x(i)\|^8\Big)^{\frac{1}{4}} + \Big(\sum_{i=1}^{\infty}(6.25\|y(i)\|)^8\Big)^{\frac{1}{4}} \\ &= \|x\|^2 + 6.25\|y\|^2.\end{aligned}$$

Let now (x_n) be a weakly null sequence in the unit ball B_X and t be a positive number. Using (7) it is easy to see that there exists $k > 1$ such that

$$\|x_1 + tx_k\| \le [1 + 6.25t^2]^{\frac{1}{2}}$$

(compare to [7]). This implies that X is NUS.

Remark 7. *The spaces X and X^* have property (β).*

Proof. By Theorem 4 [4] it suffices to show that X satisfies the following uniform Kadec-Klee property.

For every $\epsilon > 0$ there exists $\delta > 0$ such that if (z_n) is a sequence in B_X converging weakly to z, with $\inf\{\|z_m - z_n\| : m \ne n\} \ge \epsilon$, then $\|z\| \le 1 - \delta$.

This will be proved once we show that

$$\|x\|^8 + [0.1\|y\|]^8 \le \|x+y\|^8$$

whenever $x, y \in X$ are such that $P_n x = x$ and $P_n y = 0$ for some $n \ge 1$ (see [7]). But if x, y satisfy the above condition, then

$$\|x(i)\|^2 + [0.1\|y(i)\|]^2 \le \|x(i) + y(i)\|^2$$

for every i. Hence

$$\begin{aligned}\|x+y\|^8 &= \sum_{i=1}^{\infty} \|x(i)+y(i)\|^8 \\ &\geq \sum_{i=1}^{\infty} [\|x(i)\|^2 + (0.1\|y(i)\|)^2]^4 \\ &\geq \sum_{i=1}^{\infty} \|x(i)\|^8 + \sum_{i=1}^{\infty} (0.1\|y(i)\|)^8 \\ &= \|x\|^8 + 10^{-8}\|y\|^8.\end{aligned}$$

We have proved that some direct sums are orthogonally convex. Our next remark shows that in general even l_p-sums of orthogonally convex spaces need not be orthogonally convex.

Remark 8. *The space E is orthogonally convex.*

Proof. Let (Q_n) be the sequence of the natural projections associated to the basis (e_n) of E. We put $R_n = Q_1 + I - Q_n$.

It is easy to see that $\|Q_n\| = \|R_n\| = 1$ for all n. Moreover if $\|Q_n x\| \leq 1$ and $\|R_n x\| \leq 1$, then $\|x\| \leq 2^{\frac{1}{2}}$. It follows that the assumptions of Theorem 2 [3] are satisfied. By this theorem E is orthogonally convex.

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