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THE DEMICLOSEDNESS PRINCIPLE FOR MAPPINGS OF ASYMPTOTICALY NONEXPANSIVE TYPE

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ABSTRACT. We establish the demiclosedness principle for mappings of asymptotically nonexpansive type on weakly compact convex subsets of a Banach space which satisfies Opial's condition and has the *GGLD property*

1. Introduction.

Let X be a real Banach space and let C be a nonempty closed convex subset of X. A mapping $T : C \longrightarrow X$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \text{ for all } x, y \in C,$$

and *asymptotically nonexpansive* [5] if there exists a sequence (k_n) of real numbers with $k_n \to 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \text{ for all } x,\ y \in C \text{ and } n \in \mathbf{N}.$$

More generally T is of *asymptotically nonexpansive type* [8] if

$$\limsup_n \left[\sup\{\|T^n x - T^n y\| - \|x - y\| \ :\ y \in C\}\right] \ \leq\ 0, \text{ for each } x \in C.$$

A mapping $f : C \longrightarrow X$ is *demiclosed* (at y) if $f(x) = y$ whenever $(x_n) \subset C$ with $x_n \overset{w}{\rightharpoonup} x$ and $f(x_n) \to y$.

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One of the fundamental results in the theory of nonexpansive mappings is Browder's demiclosedness principle [1], which states that if X is a uniformly convex Banach space, C is a closed convex set and $T : C \longrightarrow X$ is nonexpansive, then $I - T$ is demiclosed. This principle is also seen to be valid in spaces satisfying Opial's condition [10]:

If $x_n \overset{w}{\rightharpoonup} x_0$ and $x \neq x_0$ then $\limsup_n \|x_n - x_0\| < \limsup_n \|x_n - x\|$.

Given a Banach space X and sequence (x_n) in X let

$$r(c; x_n) := \inf \left\{ \liminf_n \|x_n - x\| - 1 \ : \ \|x\| \geq c \right\}.$$

We say X has the *locally uniform Opial condition* [9] if

$$r(c; x_n) > 0 \text{ whenever } c > 0, \liminf \|x_n\| \geq 1, \text{ and } x_n \overset{w}{\rightharpoonup} 0,$$

and the *uniform Opial condition* [11] if

$$r(c) := \inf \{ r(c; x_n) \ : \ \liminf \|x_n\| \geq 1 \text{ and } x_n \overset{w}{\rightharpoonup} 0 \} \ > \ 0, \text{ whenever } c > 0.$$

Recently the demiclosedness of $I - T$ at 0 for T of asymptotically nonexpansive type has been established by Xu [16] when X is uniformly convex and for asymptotically nonexpansive maps by Lin, Tan and Xu [9] when X is a Banach space with the locally uniform Opial condition, and hence when X is UKK (uniformly Kadec-Klee, [6]) with Opial's condition. Demiclosedness of $I - T$ at 0 when T is of asymptotically nonexpansive type and X satisfies a uniform Opial condition has been studied by Bruck, Kuczumow and Reich [2].

These results, while significant, leave open the important question of demiclosedness in spaces with Opial's condition. We too are unable to settle this. We will, however, establish demiclosedness of $I - T$ at 0 when T belongs to a general class of mappings of asymptotically nonexpansive type for a suite of spaces which is more extensive than those with the locally uniform Opial condition, or those which are UKK and have Opial's condition.

A Banach space X is said to have *property*−P [14] if whenever (x_n) is a nonconstant weak null sequence we have $\liminf_n \|x_n\| <$ diam $\{x_n\}$. We say X has *asymptotic*−P if whenever (x_n) is a weak null sequence which is not norm convergent we have $\liminf_n \|x_n\| < \operatorname{diam}_a\{x_n\}$, here $\operatorname{diam}_a\{x_n\} := \lim_n \operatorname{diam}\{x_k\}_{k=n}^{\infty}$ is the *asymptotic diameter* of the sequence (x_n).

A space X satisfies the *Generalized Gossez-Lami Dozo property* (GGLD) [7] if

$$\liminf_n \|x_n\| < \limsup_m \limsup_n \|x_m - x_n\|,$$

whenever (x_n) is a weak null sequence which is not norm convergent.

It is readily seen that both the locally uniform Opial condition and UKK are stronger properties than GGLD and that either GGLD, or Opial's condition, implies property$-$P. Examples showing that in general these implications are strict are considered in [12, 13].

In [12] property$-$P is seen to be the same as a condition introduced by Tingley [15] and subsequently known as WO, while asymptotic$-$P and GGLD are shown to be equivalent.

The space c_0 equivalently renormed by

$$\|(\xi_n)\|' := \|(\xi_n)\|_\infty + \textstyle\sum_n 2^{-n}|\xi_n|,$$

considered by Jiménez-Melado [7], enjoys Opial's condition [3], but lacks asymptotic$-$P. Thus, while Opial's condition implies property$-$P it fails to imply GGLD.

2. Demiclosedness.

The following lemma is basic to our study. Variants have appeared in several of the references cited above. However, even in these more restricted cases our proof is more transparent.

Lemma 2.1. *Let X be a Banach space with Opial's condition, C a weakly compact convex subset of X, and $T : C \longrightarrow C$ a mapping of asymptotically nonexpansive type. If (x_n) is a sequence in C with $x_n \overset{w}{\rightharpoonup} x$ and $x_n - T^m x_n \rightarrow 0$, for each $m \in \mathbf{N}$, then $T^n x \overset{w}{\rightharpoonup} x$.*

Proof. Consider the *type* $\psi(x) := \limsup_n \|x - x_n\|$, and suppose that $(T^n x)$ is not weakly convergent to x, then there exists a subsequence $(T^{n_i} x)$ with $T^{n_i} x \overset{w}{\rightharpoonup} y \neq x$. Opial's condition implies $\psi(x) < \psi(y)$, so we can choose ϵ with $0 < \epsilon < \frac{1}{2}(\psi(y) - \psi(x))$. By the definition of asymptotic nonexpansive type, there exists $m_0 \in \mathbf{N}$ such that

$$\|T^k x - T^k x_n\| < \|x - x_n\| + \epsilon, \text{ for all } n \in \mathbf{N} \text{ and all } k \geq m_0.$$

Now, the weak-lower semi-continuity of ψ ensures that $\psi(y) \leq \liminf_i \psi(T^{n_i}x)$, so we can choose i_0 such that $n_{i_0} \geq m_0$ and $\psi(y) \leq \psi(T^{n_{i_0}}x) + \epsilon$. But then,

$$\begin{aligned}\psi(y) &\leq \psi(T^{n_{i_0}}x) + \epsilon =: \limsup_n \|T^{n_{i_0}}x - x_n\| + \epsilon \\ &= \limsup_n \|T^{n_{i_0}}x - T^{n_{i_0}}x_n\| + \epsilon, \text{ as } x_n - T^{n_{i_0}}x_n \to 0 \\ &\leq \limsup_n \|x - x_n\| + 2\epsilon, \text{ as } n_{i_0} \geq m_0 \\ &=: \psi(x) + 2\epsilon < \psi(y), \text{ by the choice of } \epsilon,\end{aligned}$$

a contradiction which establishes the result. □

Under the additional assumption of GGLD the conclusion of lemma 2.1 can be strengthened.

Lemma 2.2. *Let X be a Banach space with GGLD and Opial's condition, and suppose that T, C, (x_n) and x satisfy the conditions of lemma 2.1. Then $T^n x \to x$.*

Proof. By lemma 2.1 we know that $T^n x \xrightarrow{w} x$. Suppose that $\|x - T^n x\| \not\to 0$ then by GGLD

$$0 < B := \limsup_n \|T^n x - x\| < \limsup_n \limsup_k \|T^n x - T^k x\|$$

Thus, we may choose $\epsilon > 0$ so that

$$B + \epsilon < \limsup_n \limsup_k \|T^n x - T^k x\|. \tag{1}$$

Further, since T is of asymptotically nonexpansive type, there exists n_0 such that

$$\|T^m x - T^m y\| \leq \|x - y\| + \epsilon/2, \text{ for all } m \geq n_0 \text{ and all } y \in C. \tag{2}$$

Now, from (1) we can select $n > n_0$ and a sequence $n < k_1 < k_2 < \ldots$ such that

$$B + \epsilon \leq \|T^n x - T^{k_i}x\| = \|T^n x - T^n(T^{k_i - n}x)\|.$$

Taking $y = T^{k_i - n}x$ in (2) yields

$$B + \epsilon \leq \|x - T^{k_i - n}x\| + \epsilon/2, \text{ for } i = 1, , 2, \ldots.$$

Thus

$$\begin{aligned} B + \epsilon &\leq \limsup_i \|x - T^{k_i - n}x\| + \epsilon/2 \\ &\leq \limsup_n \|x - T^n x\| + \epsilon/2 = B + \epsilon/2, \end{aligned}$$

a contradiction. It follows that $B = 0$, and the result is established. □

To proceed further a continuity requirement for T seems necessary.

Lemma 2.3. *Under the conditions of lemma 2.2 suppose in addition that there exists an $N \in \mathbf{N}$ such that T^N is continuous at x, then $Tx = x$.*

Proof. From lemma 2.2 we know that $T^n x \to x$, and using the continuity of T^N we have that

$$T^{N+n}x \longrightarrow T^N x.$$

Taking the limit as $n \to \infty$ we obtain $x = T^N x$. But then $Tx = T^{nN+1}x$, for all n, and so, again taking the limit as $n \to \infty$, we have $Tx = x$. □

The last lemma almost immediately yields the following.

Theorem 2.4. *Let X be a Banach space with GGLD and Opial's condition. Let C be a weakly compact convex subset of X and let $T : C \longrightarrow C$ be a uniformly continuous mapping of asymptotically nonexpansive type. Then $I - T$ is demiclosed at* 0.

Proof. Suppose $(x_n) \subset C$ with $x_n \stackrel{w}{\rightharpoonup} x$ and $\|x_n - Tx_n\| \to 0$, we must show $x - Tx = 0$. Since T is uniformly continuous it follows from $x_n - Tx_n \to 0$ that $x_n - T^m x_n \to 0$ for each fixed $m \in \mathbf{N}$. The conclusion now follows from lemma 2.3. □

Corollary 2.5. *Let X be a Banach space with GGLD and Opial's condition, let C be a nonempty weakly compact convex subset of X, and let $T : C \longrightarrow C$ be an asymptotically nonexpansive mapping. Then $I - T$ is demiclosed at 0.*

3. Weak convergence of iterates.

In this section we derive some further corollaries to the basic results in section 2 under the additional assumption of asymptotic regularity.

Theorem 3.1. *Let X be a Banach space with GGLD and Opial's condition. Let C be a weakly compact convex subset of X and let $T : C \longrightarrow C$ be a mapping of asymptotically nonexpansive type which is asymptotically regular at the point $x \in C$ (that is, $T^{n+1}x - T^n x \to 0$), and for which T^N is continuous for some $N \in \mathbf{N}$. Then the iterates $(T^n x)$ converge weakly to a fixed point of T.*

Proof. Let $x_n := T^n x$, for $n = 1, 2, \ldots$, then (x_n) satisfies $x_n - T^m x_n \to 0$, for each m, as does every subsequence of (x_n). Consider $W(x)$ the set of all weak subsequential limits of (x_n).

$$W(x) := \{y \in X \ : \ y = \text{w}-\lim_j T^{n_j} x, \text{ for some increasing sequence } (n_j) \subseteq \mathbf{N}\}$$

Lemma 2.3 yields that $W(x) \subseteq \text{Fix}(T)$, the fixed point set of T.

It is sufficient to show:

$$\text{(1)} \qquad \text{For each } y \in W(x) \text{ the } \lim_n \|T^n x - y\| \text{ exists.}$$

As then $W(x) = \{y\}$ and $T^n x \overset{w}{\rightharpoonup} y \in \text{Fix}(T)$. Indeed, suppose $y = \text{w} - \lim_j T^{n_j} x$ and $z = \text{w} - \lim_j T^{m_j} x$ were two distinct points in $W(x)$, then we would obtain a contradiction as follows.

$$\begin{aligned}
\lim_n \|T^n x - y\| &= \lim_j \|T^{n_j} x - y\| \\
&< \lim_j \|T^{n_j} x - z\|, \text{ by Opial's condition} \\
&= \lim_n \|T^n x - z\| = \lim_j \|T^{m_j} x - z\| \\
&< \lim_j \|T^{m_j} x - y\|, \text{ by Opial's condition} \\
&= \lim_n \|T^n x - y\|.
\end{aligned}$$

It remains to establish (1). For $y \in W(x)$ and any $m \in \mathbf{N}$ we have

$$\begin{aligned}
\limsup_n \|T^n x - y\| &= \limsup_n \|T^{n+m} x - y\| \\
&= \limsup_n \|T^{n+m} x - T^n y\|, \text{ as } y \in \text{Fix}(T) \\
&\leq \|T^m x - y\|, T \text{ of asymptotically nonexpansive type.}
\end{aligned}$$

Thus, $\limsup_n \|T^n x - y\| \leq \liminf_m \|T^m x - y\|$, establishing the result. □

For completeness we have given a self contained proof of the last result, but it is well known that in the presence Opial's condition the sets $W(x)$ can contain at most one point, see for example the proof of Theorem 1 in [4].

Corollary 3.2. *Let X and C be as in theorem 3.1. If $T : C \longrightarrow C$ is asymptotically nonexpansive and T is asymptotically regular at $x \in C$ then $(T^n x)$ converges weakly to a fixed point of T.*

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