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UNIFORM NORMAL STRUCTURE AND RELATED NOTIONS

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Dedicated to Professor Ky Fan on his 85th birthday

ABSTRACT. Let X be a Banach space, let ϕ denote the usual Kuratowski measure of noncompactness, and let $k_X(\varepsilon) = \sup r(D)$ where $r(D)$ is the Chebyshev radius of D and the supremum is taken over all closed convex subsets D of X for which $diam(D) = 1$ and $\phi(D) \geq \varepsilon$. The space X is said to have ϕ-uniform normal structure if $k_X(\varepsilon) < 1$ for each $\varepsilon \in (0,1)$. It is shown that this concept, which lies strictly between normal structure and uniform normal structure, implies reflexivity. Hence such spaces have the fixed point property for nonexpansive mappings. Related concepts in metric spaces are also discussed.

1. INTRODUCTION

Our objective in this note is to introduce a 'noncompact' extension of the concept of uniform normal structure and discuss some of its properties and related notions. We begin some standard definitions and a brief review of the general topic.

Let X be a Banach space; let $\mathcal{C}$ denote the collection of all bounded closed convex subsets of X; let $\mathcal{C}_w$ denote the collection of all weakly compact convex subsets of X; let $\mathcal{A}$ denote the collection of all admissible subsets of X. Thus $\mathcal{A}$ is the collection of all sets of the form

$$B = \cap_{i \in I} B(x_i; r_i)$$

where $B(x_i; r_i)$ denotes a closed ball centered at $x_i \in X$ with radius $r_i \geq 0$, and I is some index set.

The *Chebyshev radius* $r(K)$ of $K \in \mathcal{C}$ is the number

$$r(K) = \inf_{y \in K} \{\sup\{\|x - y\| : x \in K\}\}.$$

A Banach space is said to have *normal structure* if $r(K)/diam(K) < 1$ whenever $K \in \mathcal{C}$ and $diam(K) > 0$. It is well-known that if a weakly compact convex subset of a Banach space has normal structure, then every nonexpansive mapping $T : K \to K$ has a fixed point. (T is *nonexpansive* if $\|T(x) - T(y)\| \leq \|x - y\|$ for each $x, y \in K$.) Thus Banach spaces which have normal structure have the weak fixed point property (weak-FPP). If the space is reflexive we refer to this as the FPP.

We now list the standard normal structure coefficients of X. The first was introduced by Bynum in 1980 [8]. These are called, respectively, the *normal structure coefficient*, the *weak normal structure coefficient*, and the *admissible normal structure coefficient* of X.

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$$N(X) = \sup\{r(K)/diam(K) : K \in \mathcal{C} \text{ and } diam(K) > 0\};$$

$$N_w(X) = \sup\{r(K)/diam(K) : K \in \mathcal{C}_w \text{ and } diam(K) > 0\};$$

$$N_a(X) = \sup\{r(K)/diam(K) : K \in \mathcal{A} \text{ and } diam(K) > 0\}.$$

X is said to have *uniform normal structure* (UNS) if $N(X) < 1$. This concept, which is a strengthening of the concept of normal structure, was introduced by Gillespie and Williams in 1979 ([10]) and it serves as the basis for this entire discussion. Notice that *weak uniform normal structure* (w-UNS) and *admissible uniform normal structure* (a-UNS) can be defined analogously. Gillespie and Williams proved that if a Banach space X has UNS then every bounded closed convex subset of X has the fixed point property for nonexpansive mappings, and they raised the question of whether every such space is reflexive. This question was answered affirmatively, and independently, by Bae [3] and Maluta [17].

While the above coefficients are natural in a Banach space environment, the third requires only a metric setting. It is well known (e.g., [4]) that for any hyperconvex metric space H,

$$N_a(H) = 1/2.$$

We now list some well-known facts about normal structure coefficients. Throughout we only consider the case $\dim X = \infty$.

1. It is easy to see that in general, $1/\sqrt{2} \leq N(X) \leq 1 - \delta_X(1)$, where δ_X is the usual modulus of convexity of X ([17]).

2. $N(\ell_p) = N(L_p) = \max\{2^{-1/p}, 2^{(1-p)/p}\}$ for $1 < p < \infty$; in particular $N(\ell_2) = N(L_2) = 1/\sqrt{2}$.

3. $N_a(L_\infty) = N_a(\ell_\infty) = 1/2$.

4. Fix $\lambda \geq 1$ and let X_λ denote the space ℓ_2 renormed as follows: For $x \in \ell_2$, set

$$|x|_\lambda = \max\{\|x\|_2, \lambda\|x\|_\infty\}.$$

Since

$$\|x\|_2 \leq |x|_\lambda \leq \lambda\|x\|_2$$

the spaces X_λ are reflexive, and it is easy to see that

$$N(X_\lambda) = \min\left\{1, \lambda/\sqrt{2}\right\}.$$

Thus

$$X_\lambda \text{ has UNS } \Leftrightarrow \lambda < \sqrt{2}.$$

Karlovitz [12] observed that while $X_{\sqrt{2}}$ fails even to have normal structure, it does have the FPP. Later Baillon-Schöneberg [5] proved that X_λ has 'asymptotic normal structure' $\Leftrightarrow \lambda < 2$; hence X_λ has the FPP in this case. Subsequently J. M. Borwein and B. Sims [7] showed that X_λ actually has the FPP, for all $\lambda \geq 1$. See also [16].

5. UNS $\Rightarrow$ Reflexivity ([3], [17]), see section 2 for a proof.

6. Every k-uniformly rotund Banach space has UNS ([8], also see [1]).

7. If $\rho'_X(0) := \lim_{\tau\to 0}\rho_X(\tau)/\tau < 1/2$ then X has UNS, where ρ_X is the usual modulus of smoothness of a Banach space. Since X is uniformly smooth $\Leftrightarrow \rho'_X(0) = 0$, this in particular implies that uniformly smooth Banach spaces have UNS. In fact, since it is known that $\rho'_X(0) < 1 \Leftrightarrow X^*$ is uniformly nonsquare $\Rightarrow X$ is superreflexive, it follows that $\rho'_X(0) < 1/2 \Rightarrow X$ is both reflexive and has UNS (see Prus [20] and Turett [21]).

8. It is also known that $N_w(X)$ is finitely determined for any Banach space X. That is, given $\varepsilon > 0$ there exists a finite subset F of X with the property

$$N_w(X) \geq r(conv(F))/diam(F) \geq (1-\varepsilon) N_w(X).$$

This gives rise to one of the fundamental open questions in the theory of Banach space geometry, namely: Is UNS a super-property? Equivalently, does UNS $\Rightarrow$ superreflexivity? (See [1] for details.)

9. Finally we mention that Maluta and Prus [18] have recently introduced a concept of k-uniform smoothness which is dual to k-uniform rotundity and shown that, although k-uniformly smooth spaces are superreflexive, they fail even to have normal structure.

2. UNS AND REFLEXIVITY

Here we give a proof that UNS implies reflexivity. This proof, which is based loosely on that given in [3], is found in [11]. We include the details because it is a modification of this approach provides the basis for our main result.

Suppose X has UNS and let $K_1^0 \supset K_2^0 \supset K_3^0 \supset \cdots$ be a sequence of nonempty bounded closed convex subsets of X. In view of Smulian's theorem we only need to show that this sequence has nonempty intersection.

By assumption

$$k_0 := \sup\{r(C)/diam(C) : C \in \mathcal{C} \text{ and } diam(C) > 0\} < 1.$$

Choose $k \in (k_0, 1)$, and for each $C \in \mathcal{C}$ let

$$\begin{aligned} A(C) &:= \{x \in C : \|x-y\| \leq k\, diam(C),\ \forall y \in C\} \\ &= [\cap_{y\in C} B(y; k\, diam(C))] \cap C. \end{aligned}$$

Thus $A(C)$ is a nonempty proper closed convex subset of C for each C with $diam(C) > 0$. In particular, $diam(A(C)) \leq k\, diam(C)$. Now set

$$\begin{aligned} K_1^1 &= \overline{conv} \cup_{i=1}^{\infty} A(K_i^0); \\ K_2^1 &= \overline{conv} \cup_{i=2}^{\infty} A(K_i^0); \\ &\vdots \\ K_n^1 &= \overline{conv} \cup_{i=n}^{\infty} A(K_i^0). \end{aligned}$$

Claim. For $n = 1, 2, \cdots$ we have $diam\left(K_n^1\right) \leq k\, diam\left(K_n^0\right)$. To see this, let $x, y \in \cup_{i=n}^{\infty} A\left(K_i^0\right)$. Then $x \in A\left(K_p^0\right)$ and $y \in A\left(K_q^0\right)$ for, say, $n \leq p \leq q$. Since $K_q^0 \subseteq K_p^0$, both $x, y \in K_p^0$ so $\|x - y\| \leq k\, diam\left(K_p^0\right)$.

We now have:

$$\begin{array}{ccccccccccc} K_1^0 & \supset & K_2^0 & \supset & K_3^0 & \supset & \cdots & \supset & K_n^0 & \supset & \cdots \\ \cup & & \cup & & \cup & & & & \cup & & \\ K_1^1 & \supset & K_2^1 & \supset & K_3^1 & \supset & \cdots & \supset & K_n^1 & \supset & \cdots \end{array}$$

with $diam\left(K_n^1\right) \leq k\, diam\left(K_n^0\right),\ n = 1, 2, \cdots$.

By repeating the above construction step-by-step, we obtain sequences of nonempty bounded closed convex sets that are nested as follows:

$$\begin{array}{ccccccccccc} K_1^0 & \supset & K_2^0 & \supset & K_3^0 & \supset & \cdots & \supset & K_n^0 & \supset & \cdots \\ \cup & & \cup & & \cup & & & & \cup & & \\ K_1^1 & \supset & K_2^1 & \supset & K_3^1 & \supset & \cdots & \supset & K_n^1 & \supset & \cdots \\ \cup & & \cup & & \cup & & & & \cup & & \\ \vdots & & \vdots & & \vdots & & & & \vdots & & \\ \cup & & \cup & & \cup & & & & \cup & & \\ K_1^n & \supset & K_2^n & \supset & K_3^n & \supset & \cdots & \supset & K_n^n & \supset & \cdots \\ \cup & & \cup & & \cup & & & & \cup & & \\ \vdots & & \vdots & & \vdots & & & & \vdots & & \end{array}$$

Since $diam\left(K_i^n\right) \leq k\, diam\left(K_i^{n-1}\right) \leq \cdots \leq k^n\, diam\left(K_i^0\right) \to 0$, the diagonal sequence $\left\{K_{n+1}^n\right\}$ has nonempty intersection by Cantor's theorem. But since $K_{n+1}^n \subseteq K_{n+1}^0$, $n = 1, 2, \cdots$,

$$x \in \cap_{n=0}^{\infty} K_{n+1}^n \Rightarrow x \in \cap_{n=0}^{\infty} K_{n+1}^0.$$

Thus UNS implies reflexivity, but it is known that normal structure need not, see [9] where it is shown that every separable space can be equivalently renormed to have normal structure. In the next section we introduce notions that genuinely lie between UNS and normal structure and show that they entail reflexivity.

3. Non-compact UNS

We now introduce an extension of the concept of UNS. Let ϕ be the Kuratowski measure of noncompactness. Thus for a nonempty bounded subset A of X,

$$\phi(A) = \inf\left\{\varepsilon > 0 : A \subseteq \cup_{i=1}^n A_i \text{ with } diam(A_i) \leq \varepsilon\right\}.$$

In particular, $\phi(A) \leq diam(A)$ and satisfies the following properties. (Actually, property (iii) is not used in the sequel.) These properties also hold for other measures of noncompactness as well as that of Kuratowski (see, [6] and [2]).

(i) $\phi\left(\overline{A}\right) = \phi(A)$.

(ii) $\phi(A) \geq 0$, and $\phi(A) = 0 \Leftrightarrow \overline{A}$ is compact.

(iii) $\phi(conv(A)) = \phi(A)$.

(iv) If $A_1 \supset A_2 \supset A_3 \supset \cdots$ are nonempty, and if $\lim_n \phi(A_n) = 0$, then $\cap_{i=1}^{\infty} \overline{A_i} \neq \emptyset$.

Definition 1. *For a Banach space X, the ϕ-normal structure coefficient is*

$$k_X(\varepsilon) := \sup\{r(D) : D \in \mathcal{C}, diam(D) = 1 \text{ and } \phi(D) \geq \varepsilon\}$$

where $\varepsilon \in [0,1]$.

Note that $k_X(\varepsilon)$ is a decreasing function of ε with $k_X(0) = N(X)$. Thus X has UNS if and only if $k_X(0) < 1$ and has normal structure if $k_X(1) < 1$. It will be useful to also note that if $diam(D) > 0$ and $\varepsilon \in [0,1]$,

$$k_X(\varepsilon) = \sup\{r(D)/diam(D) : D \in \mathcal{C} \text{ with } \phi(D) \geq \varepsilon\, diam(D)\}.$$

From the introductory discussion, if X fails the weak-FPP then X contains a diametral set K. That is, $r(K) = diam(K)$, hence $\phi(K) = diam(K)$ and we have the following.

Theorem 3.1. *If a Banach space X has $k_X(\varepsilon) < 1$ for some $\varepsilon \in (0,1)$, then X has the weak-FPP.*

Definition 2. *A Banach space is said to have ϕ-uniform normal structure (ϕ-UNS) if for each $\varepsilon \in (0,1)$, $k_X(\varepsilon) < 1$.*

Our main result is the following.

Theorem 3.2. *If a Banach space X has ϕ-UNS, then X is reflexive.*

Proof. Suppose X has ϕ-UNS and let $K_1^0 \supset K_2^0 \supset K_3^0 \supset \cdots$ be a sequence of nonempty bounded closed convex subsets of X. It is enough to show that there exists a subsequence, $(K_{n_k}^0)$, with $\cap_{k=1}^\infty K_{n_k}^0 \neq \emptyset$. As, if $x \in \cap_{k=1}^\infty K_{n_k}^0$, then for all k, $x \in K_{n_k}^0 \subseteq K_k^0$ so $x \in \cap_{k=1}^\infty K_k^0$ and, as before, the result follows by Smulian's theorem.

Let $\phi_0 := \lim_n \phi(K_n^0)$ and $d_0 := \lim_n diam(K_n^0)$. If $\phi_0 = 0$ we are done by property (iv) of ϕ. Now, assume that $\phi_0 > 0$ and necessarily $d_0 > 0$, then for all sufficiently large n we have $\phi(K_n^0) > \phi_0/2d_0\, diam(K_n^0)$. Let $k_0 := k_X(\phi_0/2d_0)$ and proceed to construct $\{K_n^1\}_{n=1}^\infty$ as in Section 2 but with k_0 in place of k. Then, as before, we have

$$diam(A(K_n^0)) \leq k_0\, diam(K_n^0) \text{ and } diam(K_n^1) \leq k_0\, diam(K_n^0),\ n = 1,2,\cdots.$$

Once again, if $\phi_1 := \lim_n \phi(K_n^1) = 0$ we are finished. So assume both ϕ_1 and so $d_1 := \lim_n diam(K_n^1)$ are strictly positive. Then for sufficiently large n, $\phi(K_n^1) > \phi_1/2d_1\, diam(K_n^1)$. Let $k_1 := k_X(\phi_1/2d_1)$ and proceed to construct $\{K_n^2\}_{n=1}^\infty$ as in Section 2 but with k_1 in place of k. Then,

$$diam(A(K_n^1)) \leq k_1\, diam(K_n^1) \text{ and } diam(K_n^2) \leq k_1\, diam(K_n^1),\ n = 1,2,\cdots$$

Continuing this process, either it terminates after a finite number of steps with one of the $\phi_j = 0$, in which case we are done, or we obtain, as in Section 2, a doubly infinite collection of closed convex subsets (K_n^j); $j = 0,1,2,\cdots$, $n = 1,2,\cdots$, that are nested as follows:

$$\begin{array}{ccccccccccc}
K_1^0 & \supset & K_2^0 & \supset & K_3^0 & \supset & \cdots & \supset & K_n^0 & \supset & \cdots \\
\cup & & \cup & & \cup & & & & \cup & & \\
K_1^1 & \supset & K_2^1 & \supset & K_3^1 & \supset & \cdots & \supset & K_n^1 & \supset & \cdots \\
\cup & & \cup & & \cup & & & & \cup & & \\
\vdots & & \vdots & & \vdots & & & & \vdots & & \\
\cup & & \cup & & \cup & & & & \cup & & \\
K_1^n & \supset & K_2^n & \supset & K_3^n & \supset & \cdots & \supset & K_n^n & \supset & \cdots \\
\cup & & \cup & & \cup & & & & \cup & & \\
\vdots & & \vdots & & \vdots & & & & \vdots & &
\end{array}$$

and with $\phi_j = \lim_n \phi\left(K_n^j\right) > 0$, and so $d_j = \lim_n diam\left(K_n^j\right) > 0$, for $j = 0, 1, 2, \cdots$. It suffices to show that $\lim_j \phi_j = 0$, as then we can extract $K_{n_j}^j$ with $\lim_j \phi\left(K_{n_j}^j\right) = 0$. Consequently, by property (iv) of ϕ, we have $\emptyset \neq \cap_{j=1}^{\infty} K_{n_j}^j \subseteq \cap_{j=1}^{\infty} K_{n_j}^0$ and we are done.

Now, suppose $\lim_j \phi_j > 0$, then necessarily $\lim_j d_j > 0$, and so $\alpha := \lim_j \phi_j / d_j > 0$. Then, since $\alpha \in (0, 1]$, for all sufficiently large j we have $\phi_j / d_j \geq \alpha/2$ and so $k_j \left(= k_X(\phi_j/2d_j)\right) \leq k := k_X(\alpha/4) < 1$. Starting from a sufficiently large j, we can therefore find $\left(K_{n_j}^j\right)$ such that

$$\begin{aligned}
diam\left(K_{n_{j+m}}^{j+m}\right) &\leq k_{j+m-1}\, diam\left(K_{n_{j+m}}^{j+m-1}\right) \\
&\leq k\, diam\left(K_{n_{j+m-1}}^{j+m-1}\right) \\
&\cdots \\
&\leq k^m diam\left(K_{n_j}^j\right).
\end{aligned}$$

Thus, $\lim_j diam\left(K_{n_j}^j\right) = 0$, and so $\lim_j d_j = 0$, a contradiction, as then we would have $\lim_j \phi_j = 0$. □

Corollary 1. *If a Banach space X has ϕ-UNS, then X has the FPP.*

Remark. An alternate definition for ϕ-UNS could be: a Banach space is said to have ϕ-uniform normal structure if for each $\varepsilon \in (0, 1)$,

$$k_\varepsilon := \sup\left\{r(D)/diam(D) : D \in \mathcal{C} \text{ and } \phi(D) \geq \varepsilon\right\} < 1.$$

However, this is equivalent to taking the supremum over all non-compact subsets in $\mathcal{C}$. Thus, though the results of section 3 remain valid, it provides a less sharp constant than the definition adopted. Nevertheless this alternative definition, which will be explored in section 5, does make sense in a metric space setting where scaling is not possible.

4. δ-UNIFORM NORMAL STRUCTURE

It is possible to formulate a concept which lies between UNS and ϕ-UNS. It is not clear that this concept has much significance in a Banach space context but it does offer another possibility when extended to metric spaces.

Definition 3. *A bounded convex subset K of a Banach space has δ-uniform normal structure (δ-UNS) if for each $\varepsilon > 0$,*

$$k_\varepsilon := \sup \{r(H)/diam(H) : H \subseteq K,\ H \ convex,\ diam(H) \geq \varepsilon\} < 1.$$

The following facts are fairly straightforward.

Proposition 1. *A Banach space X has δ-UNS if and only if it has UNS.*

Proposition 2. *Every compact convex subset of a Banach space has δ-UNS.*

Proposition 3. *If a bounded closed convex subset of a Banach space has δ-UNS, then it is weakly compact.*

The proof of Proposition 3 amounts to a routine re-working of the argument of the previous section.

5. METRIC SPACES

We begin with the relevant terminology and notation. Let (M, d) be a metric space, and for $A \subseteq M$ let

$$cov(A) = \cap \{B : B \text{ is a closed ball and } A \subseteq B\}.$$

Also let $\mathcal{A}(M) = \{D \subseteq M : D = cov(D)\}$. Thus $\mathcal{A}(M)$ denotes the collection of all *admissible* subsets of M.

The *Chebyshev radius* $r(D)$ of $D \in \mathcal{A}(M)$ is the number

$$r(D) = \inf_{y \in D} \{\sup \{d(x,y) : x \in D\}\}.$$

The family $\mathcal{A}(M)$ is said to have *normal structure* (or to be normal) if for each $D \in \mathcal{A}(M)$ with $diam(D) > 0$ it is the case that

$$r(D) < diam(D).$$

If there exists a constant $c \in (0,1)$ for which

$$r(D) < c\, diam(D)$$

for each $D \in \mathcal{A}(M)$ with $diam(D) > 0$ then $\mathcal{A}(M)$ is said to have *uniform normal structure.*

Finally, $\mathcal{A}(M)$ is said to be *compact* [resp., *countably compact*] if every family [resp., countable family] of nonempty sets in $\mathcal{A}(M)$ which has the finite intersection property has nonempty intersection.

In this context the fundamental fixed point result for nonexpansive mappings is the following (see [19], [14]).

Theorem 5.1. *Suppose M is a bounded metric space and suppose $\mathcal{A}(M)$ is compact and has normal structure. Then every nonexpansive $T : M \longrightarrow M$ has a fixed point.*

Using admissible sets it is possible to give metric space analogs of all the foregoing concepts. As before we use

Definition 4. *A bounded metric space M is said to have δ-UNS if for each $\varepsilon > 0$,*

$$k_\varepsilon := \sup\{r(D)/diam(D) : D \in \mathcal{A}(M) \text{ and } diam(D) \geq \varepsilon\} < 1.$$

Definition 5. *A bounded metric space M is said to have ϕ-UNS if for each $\varepsilon > 0$,*

$$k_\varepsilon := \sup\{r(D)/diam(D) : D \in \mathcal{A}(M) \text{ and } \phi(D) \geq \varepsilon\} < 1.$$

The principale results of this section are the following.

Theorem 5.2. *Suppose M is a bounded and complete metric space for which $\mathcal{A}(M)$ has ϕ-UNS. Then $\mathcal{A}(M)$ is countably compact.*

Theorem 5.3. *Suppose M is a bounded and complete metric space for which $\mathcal{A}(M)$ has δ-UNS. Then $\mathcal{A}(M)$ is compact.*

Proof. By Theorem 5.2 $\mathcal{A}(M)$ is countably compact, and it is clear that if $\mathcal{A}(M)$ has δ-UNS then $\mathcal{A}(M)$ has normal structure. However it is known (Kulesza-Lim [15]) that if $\mathcal{A}(M)$ is countably compact and has normal structure, then $\mathcal{A}(M)$ is in fact compact. □

Proof of Theorem 5.2. The approach is similar to that of Theorem 3.1. Suppose M has ϕ-UNS and let $D_1^0 \supset D_2^0 \supset D_3^0 \supset \cdots$ be a sequence of nonempty sets in $\mathcal{A}(M)$, and let $d_1 = \lim_n \phi(D_n^0)$. We only need to show that $\cap_{i=1}^\infty D_i^0 \neq \emptyset$. Since M is complete, if $d_1 = 0$ this follows from Cantor's theorem. Otherwise $\phi(D_n^0) \geq d_1 > 0$ for each n and by definition $k_{d_1} < 1$. Let

$$k_1 = \frac{1}{2}(1 + k_{d_1})$$

and define

$$A(D_i^0) = \left[\cap_{y \in K_i^0} B(y; k_1\, diam(D_i^0))\right] \cap D_i^0,\ i = 1, 2, \cdots.$$

Now let

$$D_1^1 = cov \cup_{i=1}^\infty A(D_i^0), D_2^1 = cov \cup_{i=2}^\infty A(D_i^0), \cdots, D_n^1 = cov \cup_{i=n}^\infty A(D_i^0).$$

As before $diam(D_n^1) \leq k_1\, diam(D_n^0)$, $n = 1, 2, \cdots$. Now proceed to construct $\{D_n^1\}_{n=1}^\infty$ as in Section 2 by replacing K with D and k with k_1. This gives

$$diam(D_n^1) \leq k_1\, diam(D_n^0),\ n = 1, 2, \cdots$$

where $k_{d_1} < k_1 < 1$. Now define $d_2 = \lim_n \phi(D_n^1)$.

By following the steps of the proof of Theorem 3.2 it is possible to conclude that $\cap_{i=1}^\infty D_i^0 \neq \emptyset$ either via Cantor's theorem ($\inf d_j > 0$) or by an application of property (iv) of ϕ. □

The approach of this section does not lead to the conclusion that ϕ-UNS of $\mathcal{A}(M)$ implies compactness of $\mathcal{A}(M)$ because it is not clear that ϕ-UNS implies normal structure of $\mathcal{A}(M)$. Indeed, compact sets in $\mathcal{A}(M)$ may consist entirely of diametral points.

References

[1] D. Amir, On Jung's constant and related constants in normed linear spaces, Pacific J. Math. **118** (1985), 1–15.

[2] J. M. Ayerbe Toledano, T. Dominguez Benavides and G. López Acedo, Measures of noncompactness in metric fixed point theory, Operator Theory: Advances and Applications **99**, Birkhäuser Verlag, 1997.

[3] J. S. Bae, Reflexivity of a Banach space with a uniformly normal structure, Proc. Amer. Math. Soc. **90** (1984), 269–270.

[4] J. B. Baillon, Nonexpansive mappings and hyperconvex spaces, Contemp. Math. **72** (1988), 11-19.

[5] J. B. Baillon and R. Schöneberg, Asymptotic normal structure and fixed points of nonexpansive mappings, Proc. Amer. Math. Soc. **81** (1981), 257-264.

[6] J. Banas and K. Goebel, Measures of noncompactness in Banach spaces, Marcel Dekker, New York and Basel, 1980.

[7] J. M. Borwein and B. Sims, Non-expansive mappings on Banach lattices and related topics, Houston J. Math. **10** (1984), 339-356.

[8] W. L. Bynum, Normal structure coefficients for Banach spaces, Pacific J. Math. **86** (1980), 427-426.

[9] M. M. Day, R. C. James, and S. Swaminathan, Normed linear spaces that are uniformly convex in every direction, Can. J. Math. **XXIII** (1971), 1051-1059.

[10] A. A. Gillespie and B. B. Williams, Fixed point theorem for non-expansive mappings on Banach spaces with uniform normal structure, Appl. Anal. **9** (1979), 121-124.

[11] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge Univ. Press, Cambridge, 1990.

[12] L. Karlovitz, Existence of a fixed point for a nonexpansive map in a space without normal structure, Pacific J. Math. **66** (1976), 153-159.

[13] M. A. Khamsi, On metric spaces with uniform normal structure, Proc. Amer. Math. Soc. **106** (1989), 723-726.

[14] W. A. Kirk, Nonexpansive mappings in metric and Banach spaces, Rend. Sem. Mat. Fis. Milano **LI** (1981), 133-144.

[15] J. Kulesza and T. C. Lim, On weak compactness and countable weak compactness in fixed point theory, Proc. Amer. Math. Soc. **124** (1996), 3345-3349.

[16] P. K. Lin, Unconditional bases and fixed points of nonexpansive mappings, Pacific J. Math. **116** (1985), 69-76.

[17] E. Maluta, Uniformly normal structure and related coefficients, Pacific J. Math. **111** (1984), 357-369.

[18] E. Maluta and S. Prus, Banach spaces which are dual to k-uniformly convex spaces, J. Math. Anal. Appl. **209** (1997), 479–491.

[19] J. P. Penot, Fixed point theorems without convexity, Analyse Nonconvexe (1977), Pau), Bull. Math. Soc. France, Memoire **60** (1979), 129-152.

[20] S. Prus, Some estimates for the normal structure coefficient in Banach spaces, Rend. Circ. Mat. Palermo **40** (1991), 128-135.

[21] B. Turett, A dual view of a theorem of Baillon, *Nonlinear Analysis and Applications* (S. P. Singh and J. H. Burry, eds.), Marcel Dekker, New York, 1982, pp. 279-286.

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