WEAKLY ORTHOGONAL BANACH LATTICES WITH UNIFORMLY MONOTONE NORM ARE UNIFORMLY OPIAL

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ABSTRACT. In 2005 Dalby and Sims showed that a weakly orthogonal Banach lattice with uniformly monotone norm satisfies Opial's condition. This short paper improves this result by showing that these Banach lattices satisfy the stronger uniform Opial's condition.

1. INTRODUCTION

A Banach space has the weak fixed point property if every nonexpansive mapping on every nonempty weak compact convex set has a fixed point. Weak orthogonality and the uniform Opial condition are two of a number of Banach space properties that have been investigated for their connection to the weak fixed point property. These properties have a common thread of linking, in one form or another, weakly convergent sequences and the norm topology.

Weak orthogonality was defined and used in [4] where the order properties of Banach lattices were shown to play a key role in fixed point theory. This property led to the definition of the closely related concept of WORTH in Banach spaces. WORTH, in conjunction with a number of geometric properties of Banach spaces, has been shown to imply the weak fixed point property.

There are three versions of Opial's condition; weak Opial, Opial and uniform Opial. Again, these conditions have played a role in fixed point theory. It is the latter, uniform Opial, that mainly concerns us here.

The concept of a uniform monotone norm in a Banach lattice, X, was used by Elton *et al.* [9] to show that such an X has the weak fixed point property provided that l_1 is not finitely represented in X. This result is a generalisation of Maurey's theorem [12] that reflexive subpaces of L_1 have the fixed point property.

In [7] Dalby and Sims showed that a weakly orthogonal Banach lattice with uniformly monotone norm satisfies Opial's condition. This paper improves this result by showing that such Banach lattices enjoy the stronger property of uniform Opial's

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condition. The proof is based partially on the proof of theorem 3.3 of Dhompongsa and Kaewcharoen [8] which is in turn based partially on a proof in Dalby [6].

Now for the definitions of these and related properties. Here X is an infinite dimensional separable real Banach space.

Definition 1.1 (Opial, 13)**.** A Banach space, X, has *Opial's condition* if (x_n) converges weakly to 0 ($x_n \rightharpoonup 0$), then

$$\liminf_{n\to\infty} \|x_n\| < \liminf_{n\to\infty} \|x_n - x\| \text{ for all } x \neq 0.$$

Definition 1.2 (Prus, 14)**.** A Banach space, X, has the *uniform Opial property* if for every $c > 0$ there is an $r > 0$ such that

$$1 + r \leq \liminf_{n\to\infty} \|x_n + x\|$$

for each $x \in X$ with $\|x\| \geq c$, and each sequence (x_n) with $x_n \rightharpoonup 0$, and $\liminf_{n\to\infty} \|x_n\| \geq 1$.

Definition 1.3 (Lin, Tan and Xu, 11)**.** *Opial's modulus* is

$$r_X(c) := \inf\{\liminf_{n\to\infty} \|x_n - x\| - 1 : c \geq 0, \|x\| \geq c, x_n \rightharpoonup 0 \text{ and } \liminf_{n\to\infty} \|x_n\| \geq 1\}.$$

That is, $r_X(c)$ is the infimum of the r that work in definition 1.2.

X has the *uniform Opial condition* if and only if $r(c) > 0$ for $c > 0$.

Definition 1.4 (Sims, 16)**.** A Banach space, X, has *WORTH* if for every weak null sequence and every $x \in X$

$$\lim_{n\to\infty} |\|x_n - x\| - |x_n + x\|| = 0.$$

This property was also introduced by Rosenthal [15] and Cowell and Kalton [5].

Definition 1.5 (Sims, 16)**.** A Banach lattice, X, is *weakly orthogonal* if whenever $x_n \rightharpoonup 0$ then

$$\lim_{n\to\infty} \||x_n| \wedge |x|\| = 0 \text{ for all } x \in X$$

Note that a slightly weaker version of this property was used by Borwein and Sims in [4].

Finally,

Definition 1.6 (Katznelson and Tzafriri, 10)**.** A Banach lattice, X, has *uniformly monotone norm* if there exists a strictly increasing continuous function δ on [0,1] with $\delta(0) = 0$ so that if $x, y \geq 0$ with $1 = \|y\| \geq \|x\|$ then $\|x + y\| \geq 1 + \delta(\|x\|)$.

Here we have used the definition used in [10]. The notion of a uniformly monotone norm was introduced in 1967 by Birkoff [3] using a slightly different, though equivalent (see Akcoglu and Suchestion [1]), formulation.

2. Results

First a small result to clear the way for the proof of the main theorem.

Proposition 2.1. *Let X be a weakly orthogonal Banach lattice then X has WORTH.*

Proof. Let (x_n) be a weak null sequence in X and $x \in X$. Then using weak orthogonality

$$\lim_{n\to\infty} \| |x_n| \wedge |x| \| = 0$$

In Banach lattices $2\,(|a| \wedge |b|) = ||a+b| - |a-b||$ so

$$\lim_{n\to\infty} \| |x_n + x| - |x_n - x| \| = 0,$$

from which WORTH follows. □

Theorem 2.2. *Let X be a weakly orthogonal Banach lattice with uniformly monotone norm then X satisfies the uniform Opial's condition.*

Proof. The uniformly monotone norm means that X has order continuous norm. See, for example, [2], [9] or [10].

Dalby and Sims [7; proposition 3.2] showed that a Banach lattice with order continuous norm is weakly orthogonal if and only if the lattice operations are weak sequentially continuous. This will be used below.

Assume that X does not satisfy the uniform Opial's condition then there exists a $c > 0$ with $r(c) = 0$ where $r(\cdot)$ is Opial's modulus.

Thus, given $\epsilon > 0$ there exists a weak null sequence (x_n) with $\liminf_{n\to\infty} \|x_n\| \geq 1$ and $x \in X$ with $\|x\| \geq c$ where $\liminf_{n\to\infty} \|x_n + x\| - 1 < \epsilon$.

Proposition 3.5 in Dalby and Sims [7] showed that a Banach lattice with these properties satisfies Opal's condition. So

$$\liminf_{n\to\infty} \|x_n\| < \liminf_{n\to\infty} \|x_n + x\|.$$

This leads to

$$1 \leq \liminf_{n\to\infty} \|x_n\| < \liminf_{n\to\infty} \|x_n + x\| \leq 1 + \epsilon.$$

Weak lower semicontiuity of the norm means that

$$c \leq \|x\| \leq \liminf_{n\to\infty} \|x_n + x\| \leq 1 + \epsilon.$$

Now, let δ be the function associated with the uniform monotone norm then

$$\left\| \frac{|x_n|}{\|x_n\|} + \frac{|x|}{1+\epsilon} \right\| \geq 1 + \delta\left(\frac{\|x\|}{1+\epsilon}\right) \text{ for all } n.$$

So

$$\liminf_{n\to\infty} \left\| \frac{|x_n|}{\|x_n\|} + \frac{|x|}{1+\epsilon} \right\| \geq 1 + \delta\left(\frac{\|x\|}{1+\epsilon}\right)$$

$$\liminf_{n\to\infty}\left\||x_n| + \|x_n\|\frac{|x|}{1+\epsilon}\right\| \geq \liminf_{n\to\infty}\|x_n\|\left(1+\delta\left(\frac{\|x\|}{1+\epsilon}\right)\right) \geq 1+\delta\left(\frac{c}{1+\epsilon}\right).$$

Now

$$\liminf_{n\to\infty}\left\||x_n| + \|x_n\|\frac{|x|}{1+\epsilon}\right\| \leq \liminf_{n\to\infty}\||x_n| + |x|\| + \limsup_{n\to\infty}\left\|-|x| + \|x_n\|\frac{|x|}{1+\epsilon}\right\|.$$

Consider $\liminf_n \||x_n| + |x|\|$. Because the lattice operations are weak sequentially continuous we have $|x_n| \rightharpoonup 0$ and weak orthogonality can be employed. By proposition 2.1 weak orthogonality implies WORTH which means

$$\liminf_n \||x_n| + |x|\| = \liminf_n \||x_n| - |x|\|.$$

Banach lattice properties can now used, namely $||a| - |b|| \leq |a-b| \leq |a| + |b|$, to show

$$\begin{aligned}\liminf_n \||x_n| + |x|\| &= \liminf_n \||x_n| - |x|\| \\ &\leq \liminf_n \|x_n - x\| \\ &\leq \liminf_n \||x_n| + |x|\|.\end{aligned}$$

Thus

$$\liminf_n \||x_n| + |x|\| = \liminf_n \|x_n - x\| = \liminf_n \|x_n + x\|,$$

where the last equality follows from property WORTH.

Finally, putting all this together.

$$\begin{aligned}1+\delta\left(\frac{c}{1+\epsilon}\right) &\leq \liminf_{n\to\infty}\left\||x_n| + \|x_n\|\frac{|x|}{1+\epsilon}\right\| \\ &\leq \liminf_{n\to\infty}\|x_n + x\| + \limsup_{n\to\infty}\left\|-|x| + \|x_n\|\frac{|x|}{1+\epsilon}\right\| \\ &\leq 1+\epsilon + \limsup_{n\to\infty}\left|-1 + \frac{\|x_n\|}{1+\epsilon}\right|\|x\| \\ &= 1+\epsilon + \limsup_{n\to\infty}\left(1 - \frac{\|x_n\|}{1+\epsilon}\right)\|x\| \\ &\leq 1+\epsilon + \limsup_{n\to\infty}\left(1 - \frac{\|x_n\|}{1+\epsilon}\right)(1+\epsilon) \\ &\leq 1+\epsilon+1+\epsilon - \liminf_{n\to\infty}\|x_n\| \\ &\leq 2+2\epsilon-1 \\ &= 1+2\epsilon\end{aligned}$$

So for all $\epsilon > 0$, $\delta\left(\frac{c}{1+\epsilon}\right) \leq 2\epsilon$.

Taking $\epsilon \to 0$ and using the continuity of δ we obtain $\delta(c) \leq 0$. This contradicts δ being strictly increasing and $\delta(0) = 0$.

Thus X enjoys uniform Opial's condition.

□

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