MEAN LIPSCHITZIAN MAPPINGS

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Abstract. Lipschitz self mappings of metric spaces appear in many branches of mathematics. In this paper we introduce a modification of the Lipschitz condition which takes into account not only the mapping itself but also the behaviour a finite number of its iterates. We refer to such mappings as *mean Lipschitzian*. The study of this new class of mappings seems potentially interesting and leads to some new results in metric fixed point theory.

1. Introduction.

Lipschitz conditions have significant ramifications in many branches of mathematics. In particular, they often reflect regularity of self-mappings of metric space.

Let (M,ϱ) be a metric space and let $T : M \to M$ be a mapping. We say that T satisfies a *Lipschitz condition with constant* $k \geq 0$ if for all $x, y \in M$

$$\varrho(Tx, Ty) \leq k\varrho(x, y). \tag{1.1}$$

Obviously such a mapping is uniformly continuous on M. Mappings satisfying (1.1) with $k < 1$ are called *strict (or Banach) contractions* and those with $k = 1$ are said to be *nonexpansive*.Mappings satisfying (1.1) with any k are generally called *lipschitzian.* We shall also refer to these as *mappings of class* $L(k)$ or more generally of class L if k is not specified.

For any Lipschitzian mapping T there exists a smallest possible k such that (1.1) holds. This smallest k is refereed to as the *Lipschitz constant* for T and in what follows will be denoted by $k_\varrho(T)$, or simply $k(T)$ when the underlying metric is clear from the context.

Two metrics ϱ and d on M are said to be equivalent if there exists two constants $0 < a \leq b$ such that for all $x, y \in M$,

$$ad(x, y) \leq \varrho(x, y) \leq bd(x, y).$$

Clearly, any mapping T which is lipschitzian with respect to a given metric ϱ is also lipschitzian with respect to any equivalent metric d. The respective Lipschtz

Date: Februaary, 26 2008.
1991 *Mathematics Subject Classification.* Primary 47H10.
Key words and phrases. Lipschitzian mappings, contractions, nonexpansive mappings.

constants $k_\varrho(T)$ and $k_d(T)$ satisfy the relation

$$\frac{a}{b}k_\varrho(T) \le k_d(T) \le \frac{b}{a}k_\varrho(T). \tag{1.2}$$

For any two lipschitzian mappings $T, S : M \to M$ we have

$$k(T \circ S) \le k(T)\, k(S),$$

in particular, for the iterates of T; T^n, $n = 0, 1, 2, \cdots$, we have

$$k\left(T^{n+m}\right) \le k\left(T^n\right) k\left(T^m\right) \text{ and consequently } k\left(T^n\right) \le k(T)^n. \tag{1.3}$$

The inequalities (1.2) and (1.3) regulate the possible growth of the sequence of Lipschitz constants for the iterates of T. However, the Lipschitz constants of a mapping may behave in various ways. If T is a strict contraction then, $\lim_{n\to\infty} k\left(T^n\right) = 0$. If T is nonexpansive which means $k(T) \le 1$ then its powers, T^n, are also all nonexpansive. Even when $k(T)$ is large, it may happen that $k\left(T^n\right) < 1$ for certain $n > 1$. Also, the Lipschtz constants $k_d(T)$ relative to different equivalent metrics, d, may vary substantially within the limits set by (1.2). However, there exists a constant defined by

$$k_0(T) = \lim_{n\to\infty} \left(k_d\left(T^n\right)\right)^{\frac{1}{n}}$$

which, in view of (1.2), is independent of the selection of metric d within the class of all equivalent metrics to ϱ. One can show (see for example [GK]) that,

$$k_0(T) = \inf_n k_d(T^n)^{\frac{1}{n}} = \inf\left\{k_d(T) : d \text{ equivalent to } \varrho\right\}.$$

There are several subclasses of lipschitzian mappings that discussed in the literature. Of particular interest to us is the subclass of *uniformly lipschitzian mappings.* These are mappings characterized by the fact that

$$\sup\left\{k\left(T^n\right) : n = 1, 2, 3, \cdots\right\} < \infty. \tag{1.4}$$

It is readily seen that a mapping, T, is uniformly lipschitzian if and only if it is nonexpansive with respect to some equivalent metric. Indeed, if for some equivalent metric d we have, $k_d(T) \le 1$ then $k_d\left(T^n\right) \le 1$ and (1.2) implies that $k_\varrho\left(T^n\right) \le \frac{b}{a}$ for $n = 1, 2, \cdots$. On the other hand if T is uniformly lipschitzian then it is nonexpansive with respect to the equivalent metric defined by

$$d(x, y) = \sup\left\{r\left(T^n x, T^n y\right) : n = 0, 1, 2, \cdots\right\}.$$

There are more classes and a variety of modifications of Lipschitz condition based on the following observation. For any two points $x, y \in M$ we have six distances involving the points and their images under T; $\varrho(x, y)$, $\varrho(Tx, Ty)$, $\varrho(Tx, x)$, $\varrho(y, Ty)$, $\varrho(Tx, y)$, $\varrho(x, Ty)$. The Lipschitz condition is an inequality between two of them; namely $\varrho(x, y)$ and $\varrho(Tx, Ty)$ which implies nice behavior of the mapping including uniform continuity. Many authors have proposed and considered other inequalities involving some or all of the six of the distances. Very often these conditions do not imply continuity of mapping under consideration and lead to artificial situations. Theese conditions are not the subject of this paper.

Basic facts and further details concerning lipschitzian mappings and related metric fixed point theory can be found in the following [GK1], [GKKWA], [KWASB]

The aim of this paper is to study a class of lipschitzian mappings described by the behaviour of a finite sequence of their iterates. We propose a definition of *mean lipschitzian mappings*, give examples and develop some basic theory, in particular their metric fixed point theory.

2. Basic definitions and facts.

Let $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ be a *multi-index* satisfying $\alpha_1 > 0$, $\alpha_n > 0$, $\alpha_i \geq 0, i = 2, ..., n-1$ and $\sum_{i=1}^{n} \alpha_i = 1$. The two coefficients α_1, α_n, which for technical reasons are assumed to be strictly positive, will be called the *initial* and the *final* indicies respectively, the number n shall be referred to as the *length* of the multi-index α.

Definition 1. *A mapping $T : M \to M$ is said to be $\alpha-$Lipschitzian for the constant $k \geq 0$ if for every $x, y \in M$ we have,*

$$\sum_{i=1}^{n} \alpha_i \varrho(T^i x, T^i y) \leq k\varrho(x, y). \tag{2.1}$$

When the multi-index α is not explicitly specified we will refer to such a mapping as a *mean Lipschitzian* mapping.

For a given α and k we will denote the class of all such mappings on M by $L(\alpha, k)$. The smallest constant k for which (2.1) holds will be called the $\alpha-$*Lipschitz constant* for T and denoted by $k(\alpha, T)$. Analogously to the classical case, we shall call T an $\alpha-$ *contraction* , or $\alpha-$*nonexpansive*, if (2.1) is satisfied with $k < 1$, or $k = 1$, respectively.

For the special case of when the multi-index has length $n = 2$, the formula (2.1) takes the form

$$\alpha_1 \varrho(Tx, Ty) + \alpha_2 \varrho(T^2 x, T^2 y) \leq k\varrho(x, y). \tag{2.2}$$

In what follows, for the sake of simplicity and clarity of argument, we often present results for multi-indices of length 2. In most cases, matching results for longer multi-indices follow in similar a way so we will simply state the extension and, where deemed necessary, provide hints concerning the proof.

Let us begin by listing some immediate consequences of the definition.

- Any $\alpha-$lipschitzian mapping is also lipschtzian in the classical sense with
$$k(T) \leq \frac{k(\alpha, T)}{\alpha_1}.$$
- For any $i = 1, 2, ..., n$ we have,
$$k\left(T^i\right) \leq \min\left[k(T)^i, \frac{k(\alpha, T)}{\alpha_i}\right].$$

Thus, for each α the class $L(\alpha,k)$ is contained in the class L. In practice, the above two remarks mean that if the mapping T satisfies (2.1) then we have,

$$\varrho\left(Tx,Ty\right)\leq\frac{k}{\alpha_1}\varrho\left(x,y\right)$$

and for $i=1,2,...,n$,

$$\varrho\left(T^ix,T^iy\right)\leq\min\left[\left(k\left(T\right)\right)^i,\frac{k}{\alpha_i}\right]\varrho\left(x,y\right).\leq\min\left[\left(\frac{k}{\alpha_1}\right)^i,\frac{k}{\alpha_i}\right]\varrho\left(x,y\right).$$

- If T is lipschitzian then for any α,T is $\alpha-$lipschitzian with
$$k\left(\alpha,T\right)\leq\sum_{i=1}^{n}\alpha_ik\left(T^i\right).$$
- Each class $L\left(\alpha,k\right)$ contains all the lipschitzian mappings T such that
$$\sum_{i=1}^{n}\alpha_ik\left(T^i\right)\leq k.$$
- If T is uniformly lipschitzian with $\sup\left\{k\left(T^i\right):i=0,1,2,...\right\}\leq k$, then for any α,T is $\alpha-$lipschitzian with $k\left(\alpha,T\right)\leq k$.

The above straightforward evaluations are sufficient for estimating $\alpha-$Lipschitz constants in simple situations, but do not exhaust all possible cases. Here are three examples.

Example 1. *Let* $\alpha=\left(\frac{1}{2},\frac{1}{2}\right),k=2$ *and consider the class* $\mathrm{L}\left(\left(\frac{1}{2},\frac{1}{2}\right),2\right)$. *This class contains all lipschitzian mappings satisfying*

$$\frac{1}{2}k\left(T\right)+\frac{1}{2}k\left(T^2\right)\leq2$$

but since $k\left(T^2\right)\leq k\left(T\right)^2$ *it contains all lipschitzian mappings with* $k\left(T\right)$ *satisfying*

$$k\left(T\right)+k\left(T\right)^2\leq4,$$

implying $k\left(T\right)\leq\frac{1}{2}\left(\sqrt{17}-1\right)$. *In other words* $L\left(\frac{1}{2}\left(\sqrt{17}-1\right)\right)\subset L\left(\left(\frac{1}{2},\frac{1}{2}\right),2\right)$

Example 2. *Let* $\alpha=(\alpha_1,\alpha_2)$ *and let* T *be* $\alpha-$*nonexpansive. Then,*

$$\alpha_1\varrho(Tx,Ty)+\alpha_2\varrho(T^2x,T^2y)\leq\varrho(x,y)\tag{2.3}$$

The above remarks give that

$$\varrho(Tx,Ty)\leq\frac{1}{\alpha_1}\varrho(x,y)\text{ and }\varrho(T^2x,T^2y)\leq\min\left[\frac{1}{\alpha_1^2},\frac{1}{\alpha_2}\right]\varrho(x,y)\tag{2.4}$$

However, from (2.3) we also have,

$$\alpha_1\varrho(T^2x,T^2y)+\alpha_2\varrho(T^3x,T^3y)\leq\varrho(Tx,Ty)$$

Multiplying both sides of this by α_1 *and adding* $\alpha_2\varrho\left(T^2x,T^2y\right)$ *to both sides, yields* $\left(\alpha_1^2+\alpha_2\right)\varrho(T^2x,T^2y)+\alpha_1\alpha_2\varrho(T^3x,T^3y)\leq\alpha_1\varrho(Tx,Ty)+\alpha_2\varrho(T^2x,T^2y)\leq\varrho(x,y).$ *But,* $\left(\alpha_1^2+\alpha_2\right)=\alpha_1\left(1-\alpha_2\right)+\alpha_2=1-\alpha_1\alpha_2$. *Consequently, we get*

$$\varrho(T^2x,T^2y)\leq\frac{1}{1-\alpha_1\alpha_2}\varrho(x,y),$$

which is sharper then the basic estimate from 2.4. For example if $\alpha = \left(\frac{1}{2}, \frac{1}{2}\right)$ *we get*

$$\varrho(T^2x, T^2y) \leq \frac{4}{3}\varrho(x, y)$$

better then the estimate $\varrho(T^2x, T^2y) \leq 2\varrho(x, y)$ *given by 2.4.*

Example 3. *Consider as a metric space the unit ball* B *in the* ℓ^1 *space of all absolutely summable sequences,* $x = (x_1, x_2, x_3, ...)$*, with the metric induced from the usual norm* $\|x\|_1 = \sum_{n=1}^{\infty} |x_n|$*.*

Let $\tau : [-1, 1] \to [-1, 1]$ *be the function defined by*

$$\tau(t) = \begin{cases} 2t + 1, & if\ -1 \leq t \leq -\frac{1}{2} \\ 0, & if\ -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 2t - 1, & if\ \frac{1}{2} \leq t \leq 1 \end{cases}.$$

Obviously for all $t, s \in [-1, 1]$

$$|\tau(t) - \tau(s)| \leq 2|t - s| \ \text{ and } \ |\tau(t)| \leq |t|.$$

Define the mapping $T : B \to B$ *by,*

$$Tx = T(x_1, x_2, x_3, ...) = \left(\tau(x_2), \frac{2}{3}x_3, x_4, x_5, ...\right).$$

Then,

$$T^2x = T^2(x_1, x_2, x_3, ...) = \left(\tau\left(\frac{2}{3}x_3\right), \frac{2}{3}x_4, x_5, x_6, ...\right).$$

and

$$\begin{aligned} \|Tx - Ty\| &= |\tau(x_2) - \tau(y_2)| + \frac{2}{3}|x_3 - y_3| + \sum_{k=4}^{\infty} |x_k - y_k| \\ &\leq 2|x_2 - y_2| + \frac{2}{3}|x_3 - y_3| + \sum_{k=4}^{\infty} |x_k - y_k| \leq 2\|x - y\|. \end{aligned}$$

Similarly,

$$\begin{aligned} \left\|T^2x - T^2y\right\| &= \left|\tau\left(\frac{2}{3}x_3\right) - \tau\left(\frac{2}{3}y_3\right)\right| + \frac{2}{3}|x_4 - y_4| + \sum_{k=5}^{\infty} |x_k - y_k| \\ &\leq \frac{4}{3}|x_3 - y_3| + \frac{2}{3}|x_4 - y_4| + \sum_{k=5}^{\infty} |x_k - y_k| \leq \frac{4}{3}\|x - y\|. \end{aligned}$$

Observing that both estimates are sharp we see that $k(T) = 2$ *and* $k\left(T^2\right) = \frac{4}{3}$*. Also observe that from the definition of* T *repeating the argument above leads to the same sharp estimate* $k\left(T^i\right) = \frac{4}{3}$*, for all* $i \geq 2$*. Thus, all the iterations of* T *have a lipschitz constant greater than* 1*.*

Now let $\alpha = \left(\frac{1}{2}, \frac{1}{2}\right)$*. Using the above estimates we get*

$$\frac{1}{2}\|Tx - Ty\| + \frac{1}{2}\left\|T^2z - T^2y\right\| \leq |x_2 - y_2| + |x_3 - y_3| + \frac{5}{6}|x_4 - y_4| + \sum_{k=5}^{\infty} |x_k - y_k| \leq \|x - y\|.$$

Hence, despite having each iterate strictly expansive for some pairs of points in B*,* T *is* $\left(\frac{1}{2}, \frac{1}{2}\right)-$*nonexpansive.*

3. Mean contractions.

Mean contractions are mappings $T : M \to M$ satisfying (2.1) with a constant $k < 1$. The question is whether their behaviour is similar to that of classical contractions. Is the classical Banach Contraction Principle, valid for $\alpha-$contractions? The basic answer is given in the following:

Theorem 1. *Let (M, ϱ) be a metric space and suppose that $T : M \to M$ is an $\alpha-$contraction, then there exists a metric d equivalent to ϱ such that T is a classical contraction with respect to d.*

We present the (constructive) proof for multi-indices of lengths 2 and then give some hints for the general case which only differs from the case $n = 2$ in some of the technicalities.

Proof (for n=2). Let $\alpha = (\alpha_1, \alpha_2)$ and let $T : M \to M$ be an $\alpha-$ contraction with constant $k < 1$.

If $k < \alpha_1$ then, from section 2, $k_\varrho(T) \leq \frac{k}{\alpha_1}$, so T is already a contraction with respect to the original metric and we may take $d = \varrho$.

Now, suppose that $k \geq \alpha_1$. We begin by observing that by adding $\alpha_2 \varrho(Tx, Ty)$ to both sides the basic inequality,

$$\alpha_1 \varrho(Tx, Ty) + \alpha_2 \varrho(T^2x, T^2y) \leq k\varrho(x, y)$$

can be rewritten in the form,

$$\varrho(Tx, Ty) + \alpha_2 \varrho(T^2x, T^2y) \leq \varrho(x, y) + \alpha_2 \varrho(Tx, Ty) - (1 - k)\, \varrho(x, y)\,. \tag{3.1}$$

Define d by,

$$d(x, y) = \varrho(x, y) + \alpha_2 \varrho(Tx, Ty)\,.$$

It is readily seen that d is a metric on M and that d is equivalent to ϱ with,

$$\varrho(x, y) \leq d(x, y) \leq \left(1 + \frac{\alpha_2 k}{\alpha_1}\right) \varrho(x, y) = \frac{\alpha_1 + \alpha_2 k}{\alpha_1} \varrho(x, y)\,.$$

Now, from (3.1) we have

$$\begin{aligned} d(Tx, Ty) &\leq d(x, y) - (1 - k)\, \varrho(x, y) \leq \\ &\leq d(x, y) - \frac{\alpha_1 (1 - k)}{\alpha_1 + \alpha_2 k} d(x, y) = \\ &= \frac{k}{\alpha_1 + \alpha_2 k} d(x, y)\,. \end{aligned}$$

Further, since $\alpha_1 \leq k < 1$, we have $\frac{k}{\alpha_1 + \alpha_2 k} < \frac{1}{1 + \alpha_2} < 1$ and so, T is $d-$ contraction. □

Hints for the case $n > 2$. Again, rewrite the basic inequality,

$$\sum_{i=1}^{n} \alpha_i \varrho(T^i x, T^i y) \leq k\varrho(x, y)$$

as,

$$d(Tx,Ty) \leq d(x,y) - (1-k)\,\varrho(x,y) \tag{3.2}$$

where the equivalent metric d is defined by

$$d(x,y) = \sum_{j=1}^{n} \left(\sum_{i=j}^{n} \alpha_i \right) \varrho(T^{j-1}x, T^{j-1}y). \tag{3.3}$$

Then follow the steps from the proof when $n = 2$. □

4. Mean lipschtzian mappings with constant greater than one.

First observe that formulas (3.2) and (3.3) define a metric d on M regardless of the size of the $\alpha-$Lipschitz constant k and that this metric is equivalent to ϱ with,

$$\varrho(x,y) \leq d(x,y) \leq b\varrho(x,y),$$

where

$$b = \left(1 + \left(\sum_{i=2}^{n} \alpha_i \right) k_\varrho(T) + \left(\sum_{i=3}^{n} \alpha_i \right) k_\varrho\left(T^2\right) + ... + \alpha_n k_\varrho\left(T^{n-1}\right) \right).$$

Since $\varrho(x,y) \leq d(x,y)$, for any $k \geq 1$ we have from (3.2) that,

$$d(Tx,Ty) \leq kd(x,y),$$

which implies the following.

Conclusion 1. *Any $\alpha-$lipschitzian mapping $T : M \to M$ with $k(\alpha,T) \geq 1$ is lipschitzian in the classic sense with respect to the equivalent metric d defined by (3.3) with $k_d(T) \leq k(\alpha,T)$.*

In the setting of general metric spaces considered above not much can be said regarding the existence of fixed points for mappings with a Lipschitz constant greater than one. The natural setting for such considerations are when M is a closed bounded convex subset of a Banach space and ϱ is the metric inherited from the norm $\varrho(x,y) = \|x-y\|$. From now on we shall concentrate only on this case. t; Thus, let X be a Banach space with norm $\|\cdot\|$ and let C be a nonempty closed bounded convex subset of X. If C is compact then the celebrated Schauder fixed point theorem ensures any continuous mapping $T : C \to C$ has a fixed point. If C is noncompact, then Schauder's theorem fails to hold in general.

For any mapping $T : C \to C$ let,

$$d(T) = \inf\{\|x - Tx\| : x \in C\},$$

we refer to $d(T)$ as the *the minimal displacement* of points under T.

Various examples of lipschitzian mappings having strictly positive minimal displacement may be found in [GK1]. The general result concerning the existence of such mappings is due to P.K. Lin and Y. Sternfeld [LPKSY]

Theorem 2. *For C as above and for any $k > 1$ there exists a mapping $T : C \to C$ of class* $\mathrm{L}(k)$ *such that $d(T) > 0$.*

We may formalize this in the following way. Define the *minimal displacement characteristic* for C to be,

$$\varphi_C(k) = \sup\{d(T) : \ T : C \to C, \ T \in L(k)\}.$$

Then, the above theorem states that $\varphi_C(k) > 0$ for $k > 0$.

For simplicity we will take $C = B$, the unit ball of X, and write φ for φ_B. Obviously φ is an increasing function for which it is known that: $\varphi(0) = 1$, $\lim_{k\to\infty} \varphi(k) = 1$, $\varphi(k) \leq 1 - \frac{1}{k}$ and there are spaces (referred to as *extremal* spaces) for which the last estimate is sharp (infra).

By analogue, for mappings in the class of $L(\alpha, k)$ we define

$$\varphi(\alpha, k) = \sup\{d(T) : \ T : B \to B, \ T \in L(\alpha, k)\}.$$

The following is readily obtained.

Theorem 3. *For any multi-index α of arbitrary length we have $\varphi(\alpha, k) > 0$, for $k > 1$ and $\lim_{k\to\infty} \varphi(\alpha, k) = 1$.*

Proof. For $k > 1$ and any α of length n, we have from section 2 that the class $L(\alpha, k)$ contains all classes $L(l)$ such that

$$\sum_{i=1}^{n} \alpha_i l^i \leq k$$

Let l_k be the largest such l. Directly we get $0 < \varphi(l_k) \leq \varphi(\alpha, k)$ and since l_k tends to infinity with $k \to \infty$ we see that $\lim_{k\to\infty} \varphi(\alpha, k) = 1$. □

Note, we have not claimed that $\varphi(\alpha, 1) = 0$. This will be discussed in the next section.

The above properties and estimates for $\varphi(\alpha, k)$ are not exact. It was shown in section 2 that the Lipschitz constants of all iterates of T can be greater than the constant with respect to α. At present, an exact formulas for the characteristic $\varphi(k)$ is only known for a few spaces (*infra*) all of which are extremal; that is, for which $\varphi(k) = 1 - \frac{1}{k}$. It is known for Hilbert and more generally uniformly convex spaces that $\varphi(k) < 1 - \frac{1}{k}$ but the exact formula remains elusive. Analogous estimates of $\varphi(\alpha, k)$ should depend not only on the space but also on the selected α. It seems that no results in this direction are known.

To illustrate some possibilities we end this section with an example.

Example 4. *Consider the space c_0 and its unit ball B. Let $\tau : \mathbf{R} \to \mathbf{R}$ be the function defined by $\tau(t) = \min\{1, |t|\}$. For any given $k > 1$, let us consider the mapping $T : B \to B$,*

$$Tx = T(x_1, x_2, x_3, ...) = (1, \tau(k|x_1|), \tau(k|x_2|), \tau(k|x_3|), ...).$$

Then, T is lipschitzian with $k(T) = k$ and for all $n = 1, 2, 3, ...$, also $k(T^n) = k^n$. Moreover for all $x \in B$ we have $\|x - Tx\| > 1 - \frac{1}{k}$. Indeed, were the opposite

inequality satisfied by any $x = (x_1, x_2, x_3, ...)$ *it would imply that* $x_i \geq \frac{1}{k}$ *for* $, i = 1, 2, 3, ...$ *which is impossible for* x *in* c_0. *This proves that for* c_0,

$$\varphi(k) = 1 - \frac{1}{k},$$

so it is an instance of an extremal space as mentioned earlier. The space c_0 *is isometric to the product of two* $c_0 \times c_0$ *with maximum norm. The unit ball in this setting is the product of two unit balls* B. *With this formulation, define the mapping* $F : B \times B \to B \times B$ *by*

$$F(x, y) = (y, Tx).$$

It is lipschitzian with $k(F) = k$, *but the consecutive iterates of* F *behave in a different manner than those of* T. *First we have*

$$\begin{aligned} F(x,y) &= (y,Tx),\ F^2(x,y) = (Tx,Ty),\ F^3(x,y) = \left(Ty,T^2x\right), \\ F^4(x,y) &= \left(T^2x,T^2y\right),\ F^5(x,y) = \left(T^2y,T^3x\right), ... \end{aligned}$$

and the consecutive Lipschitz constants for F *are*

$$k, k, k^2, k^2, k^3, k^3, k^4, ...$$

Consequently, for any α *of length* 2, F *is of class* $L(\alpha, k)$. *The minimal displacement of* F *can be evaluated as follows:*

$$\begin{aligned} \|F(x,y) - (x,y)\| &= \|(y,Tx) - (x,y)\| = \max\{\|y-x\|, \|Tx-y\|\} \geq \\ &\geq \max\{\|y-x\|, \|Tx-x\| - \|y-x\|\} \geq \\ &\geq \max\left\{\|y-x\|, \left(1-\frac{1}{k}\right) - \|y-x\|\right\} \geq \\ &\geq \frac{1}{2}\left(1-\frac{1}{k}\right). \end{aligned}$$

Hence we conclude that for the space c_0 *and for all* α *of length* 2,

$$\varphi(\alpha, k) \geq \frac{1}{2}\varphi(k) = \frac{1}{2}\left(1-\frac{1}{k}\right).$$

5. Mean nonexpansive mappings on convex sets in Banach spaces.

The theory of nonexpansive mappings lies at the core of metric fixed point theory. As above, the most common setting is that of nonexpansive self mappings of nonempty closed bonded convex subsets of a Banach space.

In what follows X will be a Banach space and C a nonempty closed bonded convex subset of X

If $T : C \to C$ is nonexpansive that is,

$$\|Tx - Ty\| \leq \|x - y\|,$$

then for any $z \in C$ and any $\varepsilon > 0$, the mapping $T_\varepsilon = \varepsilon z + (1-\varepsilon)T$ is a contraction. Since, in this setting, all contractions have fixed points and $T = \lim_{\varepsilon \to 0} T_\varepsilon$ uniformly on C, we have,

$$d(T) = \inf\{\|x - Tx\| : x \in C\} = 0. \tag{5.1}$$

That is, all nonexpansive self mappings of nonempty closed bonded convex sets in a Banach space have zero minimal displacement.

Whether this minimal displacement is achieved and so the mapping have a fixed point depends on the (geometric) properties of the space X or the set C itself.

A substantial part of the theory is devoted to finding conditions under which the infimum in (5.1) is attained for all nonexpansive self mappings of C. In other words, conditions under which each nonexpansive self mapping has a fixed point. When this holds we say that C has the *fixed point property for nonexpansive mappings* (shortly *fpp)* and when this happens for all such C in X we say the space has the fpp.

Mean $\alpha-$nonexpansive mappings are defined by,

$$\sum_{i=1}^{n} \alpha_i \left\|T^i x - T^i y\right\| \leq \|x - y\| . \tag{5.2}$$

As was observed in the last section such mappings are nonexpansive with respect to the equivalent metric,

$$d(x,y) = \sum_{j=1}^{n} \left(\sum_{i=j}^{n} \alpha_i \right) \|(T^{j-1}x - T^{j-1}y\|, \tag{5.3}$$

that is,

$$d(Tx, Ty) \leq d(x,y) .$$

Several natural questions arise. Are $\alpha-$nonexpansive mapping a uniform (or pointwise) limit of mean contractions? Is $d(T) = 0$ for such mappings? The answers in general seem to be unknown. However, some partial results have been given in ([MJPGK]).

The first observation is that (5.2) implies that the mapping $T_\alpha = \sum_{i=0}^{n} \alpha_i T^i$ is nonexpansive. However, T_α being nonexpansive is much weaker than T being $\alpha-$nonexpansive, for instance, it does not entail the continuity of T. We leave finding examples demonstrating this as an exercise for the reader.

The observation that T_α is nonexpansive and so $d(T_\alpha) = 0$ does have some interesting consequences. To illustrate this we repeat an argument from ([MJPGK]) in the simplest case of multi-indices of length 2.

Lemma 1. *If $T : C \to C$ is $\alpha-$nonexpansive for $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1 \geq \frac{1}{2}$, then $d(T) = 0$.*

Proof. Let $\varepsilon > 0$. From the above observation, there exists a point $z \in C$ such that

$$\|z - T_\alpha z\| = \left\|z - \alpha_1 Tz - \alpha_2 T^2 z\right\| \leq \alpha_2 \varepsilon.$$

From the $\alpha-$nonexpansiveness of T we have

$$\begin{aligned}\alpha_1 \left\|Tz - T^2z\right\| + \alpha_2 \left\|T^2z - T^3z\right\| &\leq \|z - Tz\| \\ &\leq \|z - T_\alpha z\| + \|T_\alpha z - Tz\| \\ &\leq \alpha_2\varepsilon + \alpha_2 \left\|Tz - T^2z\right\| \\ &= \alpha_2\varepsilon + (1-\alpha_1)\left\|Tz - T^2z\right\|.\end{aligned}$$

Thus,

$$(2\alpha_1 - 1)\left\|Tz - T^2z\right\| + \alpha_2 \left\|T^2z - T^3z\right\| \leq \alpha_2\varepsilon$$

and, since $\alpha_1 \geq \frac{1}{2}$, taking $x = T^2z$ we see that $\|x - Tx\| \leq \varepsilon$, from which the conclusion follows. □

This elementary argument leaves open the question of what happens for $0 < \alpha_1 < \frac{1}{2}$; for instance, are there $(\alpha_1, \alpha_2)-$nonexpansive mappings with $\alpha_1 < \frac{1}{2}$ for which $d(T) > 0$,? However, the situation in the limiting case of $\alpha_1 = 0$ is clear. Here, T^2 is nonexpansive and no condition is imposed on T itself, in which case there are known examples of discontinuous mappings with $T^2 = I$ that have $d(T) > 0$.

Actually the paper ([MJPGK]) contains a more general result:

Theorem 4. *If $T : C \to C$ is $\alpha-$nonexpansive for $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ with $\alpha_1 \geq 2^{\frac{1}{1-n}}$, then $d(T) = 0$.*

In ([MJPGK]) it is also remarked that the evaluation based only on the value of the initial index, α_1, is not exact. The following is an intriguing question.

Problem 1. *For $n = 2, 3, ...$ determine the set of all multi-indices α of lengths n such that each $\alpha-$nonexpansive mapping $T : C \to C$ has $d(T) = 0$.*

Investigations of this type seem to only be in a preliminary stage.

In all of the above we did not impose any special geometrical properties on the Banach space under consideration. We shall conclude with some remarks concerning the case when X is uniformly convex. Let us recall that a space X is *uniformly convex* if its modulus of convexity

$$\delta_X(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon\right\}$$

is strictly positive for all $\varepsilon > 0$.

Perhaps the best known result comes from ([GKWAK1])

Theorem 5. *For any uniformly convex Banach space X, there exists a constant $\gamma_X > 1$ such that for every nonempty closed bounded convex subsets C, all uniformly lipschitzian mappings $T : C \to C$ satisfying $\sup\{k(T^n) : n = 1, 2, ...\} < \gamma_X$ have a fixed point in C.*

Exact values for γ_X are unknown, even for classical uniformly convex Banach spaces such as ℓ^p and L^p. When H is a Hilbert space it is only known that $\sqrt{2} \leq \gamma_H \leq \frac{\pi}{2}$. For a closer look at these considerations we refer the interested reader to ([KWASB]).

Since, any $\alpha-$nonexpansive mapping T is nonexpansive with respect to the metric d defined by (5.3) and d satisfies,

$$\|x-y\| \leq d(x,y) \leq b\|x-y\|,$$

where,

$$b(T) = \sum_{j=1}^{n}\left(\sum_{i=j}^{n}\alpha_i\right) k\left(T^{j-1}\right)$$

it follows that T is uniformly lipschitzian with $k(T^n) \leq b$. So, we have the following.

Conclusion 2. *If X is uniformly convex, then each $\alpha-$nonexpansive mapping $T : C \to C$ with $b(T) < \gamma_X$ has a fixed point.*

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