ON THE STRONG ERGODIC THEOREM FOR COMMUTATIVE SEMIGROUPS OF NON-LIPSCHITZIAN MAPPINGS IN BANACH SPACES

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Abstract. Let C be a bounded closed convex subset of a uniformly convex Banach space X and let $\Im = \{T(t) : t \in G\}$ be a commutative semigroup of asymptotically nonexpansive in the intermediate mapping from C into itself. In this paper, we provide the strong mean ergodic convergence theorem for the almost-orbit of $\Im$.

1. Introduction

Let C be a nonempty closed convex subset of a real Banach space X. Then a mapping $T : C \mapsto C$ is called nonexpansive on C , if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We denote $F(T)$ the set of fixed points of a mapping T on C. Baillon [2] proved the first strong ergodic theorem for nonexpansive mapping in a Hilbert space: $\{T^n x\}$ is strongly almost convergent as $n \to \infty$ to a point of F(T) if X is Hilbert space and T is odd. and Bruck [6] obtained the same conclusion under the more general assumption that $\{T^n\}$ is "asymptotically isometric". The analogous results for nonexpansive and asymptotically nonexpansive (type) semigroups in Hilbert spaces were given by Bruck [6], Tan and Xu[27], Li[19], Li and Ma[20], and others.

On the other hand, Bruck's result has been extended by Kobayasi and Miyadera [17] to the case of uniformly convex Banach space. So far, much effort has devoted to studying nonlinear ergodic theory for (asymptotically) nonexpansive mappings

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and semigroups. See also [8, 9, 16, 24, 25 , 30]. for example, Kido and Takahashi[16] proved the strong ergodic theorem for commutative semigroup of nonexpansive mappings and Oka[25] proved the strong ergodic theorem for totally ordered commutative semigroups of asymptotically non-expansive mappings.

As we know, Bruck's Lemmas are essential tools in the proof of almost all mean ergodic theorem for asymptotically nonexpansive semigroup in a uniformly convex Banach spaces. However, Bruck's Lemmas do not extend beyond non-Lipschitzian mappings. It remains open whether the strong ergodic theorem is valid for non-Lipschitzian mappings in Banach space.

The purpose of this paper is to prove the strong ergodic theorems for commutative semigroup of asymptotically nonexpansive in the intermediate sense mappings in a uniformly convex Banach space. Our results enable us to handle simultaneously ergodic theorems for asymptotically non-expansive type mappings and semigroups in the intermediate sense, i. e., we can establish the strong almost convergence of $\{T^n x : n \geq 1\}(x \in C)$ and $\{T(t)x : t \geq 0\}(x \in C)$ in a unified way; See Section 5. Our results extend and unify many previously known results.

2. Preliminaries and Notations

Throughout this paper X denotes a uniformly convex Banach space, C a nonempty bounded closed convex subset of X, and G a commutative semigroup with the identity. The value of $x^* \in X^*$ (the dual space of X) at $x \in X$ will be denoted by (x, x^*). We denote by coM and by $\overline{co}M$ the convex hull and the closed convex hull of $M \subset X$, respectively. The closed ball centered at $0 \in X$ and of radius $r > 0$ is denoted by B_r. We also put

$$\triangle^n = \{\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) : 0 \leq \lambda_i \leq 1, 1 \leq i \leq n, \sum_{i=1}^{n} \lambda_i = 1\}$$

The duality mapping J(multivalued) from X into X^* will be defined by

$$J(x) = \{f \in X^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\},$$

for $x \in X$.

Let $m(G)$ be the Banach space of all bounded real valued functions on G with the supremum norm. Then, for each $s \in G$ and $f \in m(G)$, we can define $r_s f$ in $m(G)$ by $(r_s f)(t) = f(t+s)$. Let D be a subspace of $m(G)$ and let μ be an element of D^*, where D^* is the dual space of D. Then, we denote by $\mu(f)$ the value of μ at the element f of D. According to the time and circumstance, we write by $\mu_t(f(t))$ or $\int f(t) d\mu(t)$ the value $\mu(f)$. When D contains constants, a linear functional μ on

D is called a mean on D if $\|\mu\| = \mu(1) = 1$. Further, let D be invariant under every $r_s, s \in G$. Then a mean μ on D is called invariant if $\mu(r_s f) = \mu(f)$ for all $s \in G$ and $f \in D$. For $s \in G$, we can define a point evaluation δ_s by $\delta_s = f(s)$ for every $f \in m(G)$. A convex combination of point evaluations is called a finite mean on G.

Let $\Im = \{T(t) : t \in G\}$ be a family of mappings from C into itself. $\Im$ is said to be a commutative semigroup of asymptotically nonexpansive in the intermediate mappings on C if the following conditions are satisfied:

(a) $T(t+s)x = T(t)T(s)x$ for all $t, s \in G$ and $x \in C$:

(b) for each $t \in G$, there exists $\alpha(t) \geq 0$ such that

$$\|T(t)x - T(t)y\| \leq \|x - y\| + \alpha(t) \quad \text{for all} \quad x, y \in C$$

with

$$\lim_{t \in G} \alpha(t) = 0, \tag{2.1}$$

where $\lim_{t \in G} \alpha(t)$ denotes the limit of a net $\alpha(\cdot)$ on the directed system $(G, \leq)$ and the binary relation $\leq$ on G is defined by $a \leq b$ if and only if there is $c \in G$ with $a + c = b$. We denote by $F(\Im)$ the set $\{x \in C : T(t)x = x \text{ for all } t \in G\}$ of common fixed points of $T(t)$ in C.

As in [24], a function $u(\cdot) : G \mapsto C$ is said to be an almost-orbit of $\Im = \{T(t) : t \in G\}$ if

$$\lim_{t \in G}[\sup_{h \in G} \|u(h+t) - T(h)u(t)\|] = 0. \tag{2.2}$$

An almost-orbit $u(\cdot)$ is called asymptotically isometric, if it satisfies

$$\lim_{t \in G} \|u(t+h) - u(t+k)\| = \rho(h, k) \tag{2.3}$$

exists uniformly in $h, k \in G$. It is easily seen that if G is totally ordered, then (2.3) is equivalent to $\lim_{t \in G} \|u(t+h) - u(t)\|$ exists uniformly in $h \in G$.

Throughout the rest of this paper, $\Im = \{T(t) : t \in G\}$ is a commutative semigroup of asymptotically nonexpansive in the intermediate mappings on C such that each $T(t)$ is continuous, $u(\cdot)$ is an almost-orbit of $\Im$ and it is asymptotically isometric, and D is a subspace of $m(G)$ containing constant functions and invariant under r_s for every $s \in G$. Furthermore suppose for each $x^* \in X^*$, a function $h_{x^*} : t \mapsto \langle u(t), x^* \rangle$ is in D. Since X is reflexive, for any $\mu \in D^*$ there exists a unique element u_μ in X such that

$$\langle u_\mu, x^* \rangle = \int \langle u(t), x^* \rangle d\mu(t)$$

for all $x^* \in X^*$. We write u_μ by $\mu(t)\langle u(t)\rangle$ or $\int u(t)d\mu(t)$. If μ is a mean on D, then $\int u(t)d\mu(t)$ is contained in $\overline{co}\{u(t) : t \in G\}$. Also, if μ is a finite mean on G, say

$$\mu = \sum_{i=1}^{n} a_i \delta_{t_i} (t_i \in G, a_i \geq 0, i = 1, 2, \cdots, n, \sum_{i=1}^{n} a_i = 1),$$

then

$$\mu(t)\langle u(t)\rangle = \sum_{i=1}^{n} a_i u(t_i).$$

Denote by $\omega_w(u)$ the set of all weakly cluster points of the net $\{u(t) : t \in G\}$.

3. Lemmas and Proposition

In this section, we prove several lemmas which play a crucial role in the proof of our main theorems in the next section.

To simplify, in the following, for each $\varepsilon \in (0, 1]$, we define

$$a(\varepsilon) = \frac{\varepsilon^2}{10R}\delta(\frac{\varepsilon}{R}) \tag{3.1}$$

and

$$G_\varepsilon = \{h_\varepsilon \in G : \alpha(h + h_\varepsilon) < a(\varepsilon) \ for \ each \ h \in G\}, \tag{3.2}$$

where δ is the modulus of convexity of the norm, $d = 2\sup\{\|x\| : x \in C\}$, and $R = 4d + 1$. Noting that from (2.1), G_ε is nonempty for each $\epsilon > 0$, and if $h_\varepsilon \in G_\varepsilon$, then $h + h_\varepsilon \in G_\varepsilon$ for each $h \in G$. In the following, we write $a^{(0)}(\epsilon) = \epsilon$, $a^{(2)}(\epsilon) = a(a(\epsilon))$, and $a^{(n)}(\epsilon) = a^{(n-1)}(a(\epsilon))$ for each $n \geq 1$.

Lemma 3.1. *Let $\epsilon \in (0, 1)$, and $h \in G_{a(\epsilon)}$. Suppose x_1, x_2 are in C such that $\|x_1 - x_2\| - \|T(h)x_1 - T(h)x_2\| \leq 2a(\epsilon)$, then for each $\alpha \in (0, 1)$,*

$$\|T(h)(\alpha x_1 + (1 - \alpha)x_2) - \alpha T(h)x_1 - (1 - \alpha)T(h)x_2\| < \frac{\epsilon}{4} \tag{3.3}$$

Proof. Put $x = (1 - \lambda)(T(h)(\lambda x_1 + (1 - \lambda)x_2) - T(h)x_2)$ and $y = \lambda(T(h)x_1 - T(h)(\lambda x_1 + (1 - \lambda)x_2))$, then $\|x\| \leq (1 - \lambda)\alpha(h) + \lambda(1 - \lambda)\|x_1 - x_2\|$, and $\|y\| \leq \lambda\alpha(h) + \lambda(1 - \lambda)\|x_1 - x_2\|$, it then follows from the lemma in [27] that

$$\|\lambda x + (1 - \lambda)y\| \leq (\alpha(h) + \lambda(1 - \lambda)\|x - y\|)(1 - 2\lambda(1 - \lambda)\delta(\frac{\|x - y\|}{d}))$$

This implies that

$$2\lambda^2(1-\lambda)^2\|x_1 - x_2\|\delta(\frac{\|T(h)(\lambda x_1 + (1-\lambda)x_2) - \lambda T(h)x_1 - (1-\lambda)T(h)x_2\|}{d})$$
$$\leq \alpha(h) + \lambda(1-\lambda)(\|x_1 - x_2\| - \|T(h)x_1 - T(h)x_2\|) \tag{3.4}$$

Suppose that $\|x - y\| \geq \epsilon/4$. Then we shall give a contradiction in the following two cases:

Case 1: If $4\lambda(1-\lambda)\|x_1 - x_2\| \leq \epsilon$, then

$$\|x - y\| \leq \|x\| + \|y\| < \alpha(h) + \lambda(1-\lambda)\|x_1 - x_2\| < \epsilon/4$$

This is a contradiction.

Case 2: If $4\lambda(1-\lambda)\|x_1 - x_2\| > \epsilon$, then we have $R\lambda(1-\lambda) > \epsilon$. It then follows frow (3.4) that

$$\frac{\epsilon^2}{2R}\delta(\frac{\epsilon}{4d}) \leq 2a(\epsilon)$$

and hence, $5a(\epsilon) \leq 2a(\epsilon)$. This is a contradiction. The proof is completed. □

For each $\varepsilon > 0$ and $h \in G$, we set

$$F_\varepsilon(T(h)) = \{x \in C : \|T(h)x - x\| \leq \varepsilon\}.$$

Lemma 3.2. *For each $0 < \epsilon < 1$, there exist $\epsilon_0 > 0$ and $h_0 \in G$ such that*

$$\overline{co}F_{\epsilon_0}(T(h)) \subset F_\epsilon(T(h))$$

for each $h \geq h_0$.

Proof. Since X is uniformly convex, by [5, Theorem 1.1], for given $\epsilon > 0$ we can choose a positive integer p such that for each $M \subset C$,

$$\text{co}M \subset \text{co}_pM + B_{\epsilon/4}, \tag{3.5}$$

where co_pM denotes the set of sums $\lambda_1 x_1 + \cdots + \lambda_p x_p$ with $0 \leq \lambda_i \leq 1, x_i \in M, 1 \leq i \leq p$, and $\sum_{i=1}^p \lambda_i = 1$. We first claim that

$$\text{co}_2F_{a(\frac{\epsilon}{4})}(T(h)) \subset F_{\frac{\epsilon}{4}}(T(h)), \tag{3.6}$$

for each $h \in G_{a(\frac{\epsilon}{4})}$, where $a(\frac{\epsilon}{4})$ and $G_{a(\frac{\epsilon}{4})}$ are defined in (3.1) and (3.2). In fact, let $x_0, x_1 \in F_{a(\frac{\epsilon}{4})}(T(h))$ and $x_t = tx_0 + (1-t)x_1$ for some $0 < t < 1$. Since

$$\|x_0 - x_1\| - \|T(h)x_0 - T(h)x_1\| \leq 2a(\frac{\epsilon}{4})$$

we have from Lemma 3.1 that

$$\|T(h)x_t - tT(h)x_0 - (1-t)T(h)x_1\| \leq \frac{\epsilon}{16}$$

This implies that

$$\|T(h)x_t - x_t\| \leq \frac{\epsilon}{4}$$

This shows (3.6) holds. By induction, we also have

$$\mathrm{co}_p F_{\epsilon_0}(T(h)) \subset F_{\frac{\epsilon}{4}}(T(h)) \tag{3.7}$$

for $\epsilon_0 = a^{(p-1)}(\epsilon/4)$ and $h \in G_{a^{(p-1)}(\epsilon/4)}$. From (3.5) and (3.7), we get

$$\mathrm{co} F_{\epsilon_0}(T(h)) \subset F_{\frac{\epsilon}{4}}(T(h)) + B_{\frac{\epsilon}{4}}.$$

But

$$C \cap (F_{\frac{\epsilon}{4}}(T(h)) + B_{\frac{\epsilon}{4}}) \subset F_{\epsilon}(T(h)),$$

because

$$\begin{aligned}\|T(h)x - x\| &\leq \|x-y\| + \|y - T(h)y\| + \|T(h)y - T(h)x\| \\ &\leq 2\|x-y\| + \|y - T(h)y\| + \alpha(h).\end{aligned}$$

Finally, noting that $F_\epsilon(T(h))$ is closed we get the desire result. □

Lemma 3.3. *Given $\varepsilon \in (0,1)$ and a positive integer p, there exists $t_\varepsilon \in G$ such that*

$$\|T(h)\sum_{i=1}^{p} a_i u(t+s_i) - \sum_{i=1}^{p} a_i u(t+s_i+h)\| < \varepsilon \tag{3.8}$$

for each $t \geq t_\varepsilon$, $h \in G_{a(\epsilon)}$ and $(a_1, a_2 \cdots, a_p) \in \triangle^p, s_i \in G, 1 \leq i \leq p$.

Proof. Put

$$\varphi(t) = \sup_{h \in G} \|u(h+t) - T(h)u(t)\|.$$

We shall prove the Lemma by mathematical induction.

If $p = 1$, then the assertion follows from the definition of almost-orbit. Now suppose that the assertion holds for $p = n-1$,

By the inductive assumption, there exists $t_{n-1} \in G_{a(\epsilon/4)}$ such that

$$\varphi(t) < \frac{1}{4} a^{(p)}(\frac{\epsilon}{4}) \tag{3.9}$$

$$||u(s+t)-u(k+t)\|-\rho(s,k)| \leq \frac{1}{4}a(\frac{\epsilon}{4}) \tag{3.10}$$

for each $t \geq t_{n-1}$ and s,k in G, and

$$\|T(h)\sum_{i=1}^{n-1} a_i u(t+s_i) - \sum_{i=1}^{n} a_i u(t+s_i+h)\| < \frac{1}{4}a(\frac{\varepsilon}{4}) \tag{3.11}$$

for each $t \geq t_{n-1}$, $h \geq t_{n-1}$, and $(a_1, a_2 \cdots, a_{n-1}) \in \triangle^{n-1}, s_i \in G, 1 \leq i \leq n-1$. Let$\lambda = \sum_{i=1}^{n} a_i\delta_{s_i}$, where, $(a_1, a_2, \cdots, a-n) \in \triangle^n, s_i \in G, 1 \leq i \leq n$. Put $\mu_1 = \sum_{i=1}^{n-2} a_i\delta_{s_i} + (a_{n-1}+a_n)\delta_{s_{n-1}}$, and $\mu_2 = \sum_{i=1}^{n-2} a_i\delta_{s_i} + (a_{n-1}+a_n)\delta_{s_n}$, then

$$\lambda = \frac{a_{n-1}}{a_{n-1}+a_n}\mu_1 + \frac{a_n}{a_{n-1}+a_n}\mu_2.$$

Put $t_n = 2t_{n-1} = t_{n-1} + t_{n-1}$, it then follows frow (3.11) that

$$\|T(k)\mu_1(s)\langle u(s+t_{n-1})\rangle - \mu_1(s)\langle u(s+t_{n-1}+k)\rangle\| \leq \frac{1}{4}a(\frac{\epsilon}{4})$$

and

$$\|T(k)\mu_2(s)\langle u(s+t_{n-1})\rangle - \mu_2(s)\langle u(s+t_{n-1}+k)\rangle\| \leq \frac{1}{4}a(\frac{\epsilon}{4})$$

for all $k \geq t_{n-1}$. Since for each $k \geq t_{n-1}$,

$$\|\mu_1(s)\langle u(s+t_{n-1})\rangle - \mu_2(s)\langle u(s+t_{n-1})\rangle\| = (a_{n-1}+a_n)\|u(s_{n-1}+t_{n-1}) - u(s_n+t_{n-1})\|$$

and

$$\begin{aligned}&\|T(k)\mu_1(s)\langle u(s+t_{n-1})\rangle - T(k)\mu_2(s)\langle u(s+t_{n-1})\rangle\| \\ &\qquad \geq (a_{n-1}+a_n)\|u(s_{n-1}+t_{n-1}+k) - u(s_n+t_{n-1}+k)\| - \frac{1}{2}a(\epsilon/4)\end{aligned}$$

By (3.10) and Lemma 3.1, we have

$$\|T(k)\lambda(s)\langle u(s+t_{n-1})\rangle - \lambda(s)\langle u(s+t_{n-1}+k)\rangle\| \leq \frac{\epsilon}{4} \tag{3.12}$$

This implies that

$$\|T(h)\lambda(s)\langle u(s+t_n)\rangle - T(h+t_{n-1})\lambda(s)\langle u(s+t_{n-1})\rangle\| \leq \alpha(h) + \frac{\epsilon}{4}$$

and

$$\|T(h+t_{n-1})\lambda(s)\langle u(s+t_{n-1})\rangle - \lambda(s)\langle u(s+t_n+h)\rangle\| \leq \frac{\epsilon}{4}$$

It follows that

$$\|T(h)\lambda(s)\langle u(s+t_n)\rangle - \lambda(s)\langle u(s+t_n+h)\rangle\| \leq \alpha(h) + \frac{\epsilon}{2} \leq \epsilon$$

This completed the proof □

Since G is commutative semigroup, there exists a net $\{\lambda_\alpha : \alpha \in A\}$ of finite means on G such that

$$\lim_{\alpha\in I} \|\lambda_\alpha - r_s^*\lambda_\alpha\| = 0 \tag{3.13}$$

for every $s \in G$, where I is a directed set and r_s^* is the conjugate operator of r_s (see [7]).

Lemma 3.4. *Let μ be an invariant mean on D, then*

$$u_\mu \in \cap_{s\in G}\overline{co}\{u(t) : t \geq s\} \cap F(S).$$

Proof. We only need to prove that $u_\mu \in F(\Im)$. Let $\epsilon > 0$, by Lemma 3.2, there exist $0 < \delta < \epsilon$ and $h_\epsilon \in G$ such that for each $h \geq h_\epsilon$, $\overline{clco}F_\delta(T(h)) \subset F_\epsilon(T(h))$. Now for fixed $h \geq h_\epsilon$, we have frow (3.13) that there exists $\alpha \in I$ such that

$$\|\lambda_\alpha - r_h^*\lambda_\alpha\| \leq \frac{\delta}{R} \tag{3.14}$$

By Lemma 3.3, there exists $t_\alpha \in G$ such that

$$\|T(h)\lambda_\alpha(s)\langle u(s+t_\alpha+t)\rangle - \lambda_\alpha(s)\langle u(s+t_\alpha+t+h)\rangle\| \leq \frac{\delta}{2}$$

for each $t \in G$. This together with (3.14) imply that

$$\lambda_\alpha(s)\langle u(s+t_\alpha+t) \subset F_\delta(T(h))$$

for all $t \in G$. It follows that

$$\mu(t)\langle u(t)\rangle = \mu(t)\lambda_\alpha(s)\langle u(s+t_\alpha+t) \in \overline{co}F_\delta(T(h)) \subset F_\epsilon(T(h))$$

and hence $T(t)u_\mu \to u_\mu$ as $t \in G$. Therefore, $u_\mu \in F(\Im)$ by the continuity of $T(t)$. This completes the proof. □

4. Strong Ergodic theorem

In this section, using the lemmas in section 3 we concern with the strong ergodic theorems for almost-orbits of commutative semigroups of asymptotically nonexpansive in the intermediate mappings.

As in [11], a net $\{\mu_\alpha : \alpha \in A\}$ of continuous linear functionals on D is called strongly regular if it satisfies the following conditions:

(a) $\sup_{\alpha\in A} \|\mu_\alpha\| < +\infty$;
(b) $\lim_{\alpha\in A} \mu_\alpha(1) = 1$;
(c) $\lim_{\alpha\in A} \|\mu_\alpha - r_s^*\mu_\alpha\| = 0$ for each $s \in G$, where A is a directed set.

THEOREM 4.1. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space, $\Im = \{T(t) : t \in G\}$ a commutative semigroup of asymptotically nonexpansive in the intermediate mappings on C such that each $T(t)$ is continuous, Let D be a subspace of m(G) containing constant function and invariant under r_s for every $s \in G$. Let $u(\cdot)$ be an asymptotically isometric almost-orbit of $\Im$ such that the function $t \mapsto \langle u(t), x^*\rangle$ is in D for each $x^* \in X^*$. Let $\{\mu_\alpha : \alpha \in A\}$ be a strongly regular net of continuous linear functionals on D. Then $\int u(t+h)d\mu_\alpha(t)$ converges strongly to p, a unique point of the set $F(\Im)\bigcap \bigcap_{s\in G} \overline{co}\{u(t) : t \geq s\}$, uniformly in $h \in G$.*

Proof. By Lemma 3.4, there exists an element p in the set $F(\Im)\bigcap \bigcap_{s\in G} \overline{co}\{u(t) : t \geq s\}$. We shall show that $\int u(h+t)d\mu_\alpha(t)$ converges strongly to p uniformly in $h \in G$. To this end, let $\epsilon > 0$ and $r = R + 12\sup\{\|\mu_\alpha\| : \alpha \in A\}$. By [5, Theorem 1.1], we can choose a positive integer p such that for each $M \subset C$, $\text{co}M \subset \text{co}_pM + B_{\epsilon/r}$. Now one may choose $t_\epsilon \in G$ from Lemma 3.3 such that

$$\|T(h)\sum_{i=1}^{p} a_i u(t+s_i) - \sum_{i=1}^{p} a_i u(t+s_i+h)\| < \frac{\varepsilon}{r} \tag{4.1}$$

for each $t \geq t_\varepsilon$, $h \in G_a(\epsilon/r)$ and $(a_1, a_2 \cdots, a_p) \in \triangle^p, s_i \in G, 1 \leq i \leq p$. Since $p \in \overline{co}\{u(t) : t \geq t_\epsilon\} \subset \overline{co}_p\{u(t) : t \geq t_\epsilon\} + B_{\epsilon/r}$, there is $b = (b_1, b_2, \cdots, c_p) \in \Delta^p, s_i \geq t_\epsilon, 1 \leq i \leq p$ such that

$$\|\sum_{i=1}^{p} b_i u(s_i) - p\| < \frac{2\epsilon}{r} \tag{4.2}$$

Then, noting that $p \in F(\Im)$, it follows from (4.1) and (4.2) that

$$
\begin{aligned}
\|\sum_{i=1}^{p} b_i u(t+h+s_i) - p\| &\le \|\sum_{i=1}^{p} b_i u(t+h+s_i) - T(t+h)\sum_{i=1}^{p} b_i u(s_i)\| \\
&\quad + \|T(t+h)\sum_{i=1}^{p} b_i u(s_i) - p\| \\
&\le \frac{4\epsilon}{r} \qquad (4.3)
\end{aligned}
$$

for all $t \in G$, and $h \in G_{a(\frac{\epsilon}{r})}$. Now put $h_0 \in G_{a(\frac{\epsilon}{r})}$, since $\{\mu_\alpha : \alpha \in A\}$ is strongly regular , there is $\alpha_0 \in A$ such that

$$
\|\mu_\alpha - r^*_{h_0+s_i}\mu_\alpha\| < \frac{\epsilon}{r} \qquad (4.4)
$$

for all $1 \le i \le p$, and

$$
\|1 - \mu_\alpha(1)\| < \frac{\epsilon}{r} \qquad (4.5)
$$

for all $\alpha \ge \alpha_0$. It then follows from (4.3)–(4.5) that

$$
\begin{aligned}
\|\int u(h+t)d\mu_\alpha(t) - p\| &\le \|\int u(h+t)d\mu_\alpha(t) - \int \sum_{i=1}^{p} b_i u(h+t+h_0+s_i)d\mu_\alpha(t)\| \\
&+ \|\int (\sum_{i=1}^{p} b_i u(h+t+h_0+s_i) - p)d\mu_\alpha(t)\| + |1-\mu_\alpha(1)|\cdot\|p\| \\
&\le d\sum_{i=1}^{p} b_i \|\mu_\alpha - r^*_{h_0+s_i}\mu_\alpha\| \\
&\quad + \sup_{\alpha\in A}\|\mu_\alpha\|.\sup_{t\in G}\|\sum_{i=1}^{p} b_i u(h+t+h_0+s_i) - p\| + \frac{\epsilon}{3} \\
&\le \epsilon
\end{aligned}
$$

for all $\alpha \ge \alpha_0$ and $h \in G$. The proof is completed . □

5. Applications.

In this section, using results in section 4, we provide nonlinear ergodic theorems for asymptotically nonexpansive in the intermediate mappings and semigroups in a uniformly convex Banach space.

Let T be an asymptotically nonexpansive in the intermediate mapping from C into itself and let it be continuous. Let $\{x_n\}$ be an almost-orbit of T, i.e.,

$$\lim_{n\to\infty}[\sup_{m\geq 0}\|x_{n+m}-T^m x_n\|]=0.$$

$\{x_n\}$ is said to be asymptotically isometric, if

$$\lim_{n\to\infty}\|x_n - x_{n+k}\| \quad \text{exists}$$

uniformly in $k \geq 1$.

Let $\Im=\{T(t): t\geq 0\}$ be an asymptotically nonexpansive in the intermediate semigroup on C such that each $T(t)$ is continuous and let $u(\cdot) : R^+ \to C$ be an almost-orbit of $\Im$, i.e.,

$$\lim_{s\to\infty}[\sup_{t\geq 0}\|u(t+s)-T(t)u(s)\|]=0.$$

$\{u(t)\}$ is said to be asymptotically isometric, if

$$\lim_{t\to\infty}\|u(t+h)-u(t)\| \quad \text{exists}$$

uniformly in $h \geq 0$.

Put $G = N, \Im = \{T^i : i \in G\}$, and $D = m(G)$ in theorem 4.1, We get the following theorem 5.1, and 5.2.

Theorem 5.1. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space X , T an asymptotically nonexpansive in the intermediate mappings on C such that each T is continuous, and $\{x_n\}$ be an asymptotically isometric almost-orbit of T. Then $n^{-1}\sum_{i=1}^{n-1} x_{i+k}$ converges strongly to some point p, a unique point of the set $F(T)\bigcap \bigcap_{m\geq 1}\overline{co}\{x_n : n\geq m\}$, as $n\to\infty$, uniformly in $k = 0, 1, 2, \cdots$.*

Proof. Put $\mu_n(f) = n^{-1}\sum_{i=1}^{n-1} f(i)$ for each $n \geq 1$ and $f \in D$. Then, $\{\mu_n : n \geq 1\}$ is a strongly regular net on D. □

Let $\mathbb{N} = \{0, 1, 2, \cdots\}$ and let $Q = \{q_{n,m}\}_{n,m\in N}$ be a matrix satisfying the following conditions :

(a) $\sup_{n\geq 0}\sum_{m=1}^{\infty}|q_{n,m}| < +\infty$,
(b) $\lim_{n\to\infty}\sum_{m=1}^{\infty} q_{n,m} = 1$, and
(c) $\lim_{n\to\infty}\sum_{m=1}^{\infty}|q_{n,m+1} - q_{n,m}| = 0$.

Then, Q is called a strongly regular matrix.

Theorem 5.2. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space X, T an asymptotically nonexpansive in the intermediate mappings on C such that each T is continuous, and $\{x_n\}$ be an asymptotically isometric almost-orbit of T. If $Q = \{q_{n,m}\}_{n,m\in\mathbb{N}}$ is strongly regular, then*
$\sum_{m=0}^{\infty} q_{n,m}\ x_{m+k}$ converges strongly to some point p, a unique point of the set $F(T) \bigcap \bigcap_{m\geq 1} \overline{co}\{x_n : n \geq m\}$, as $n \to \infty$ uniformly in $k = 0, 1, 2, \cdots$.

Proof. $\mu_n(f) = \sum_{m=0}^{\infty} q_{n,m} f(m)$ for each $n \geq 1$ and $f \in D$. Then $\{\mu_n : n \geq 1\}$ is a strongly regular net on D. □

Put $G = R^+$, $\Im = \{T(t) : t \in G\}$, and $D = \{f \in m(G) : f(\cdot)$ is a strongly measurable function on $G\}$ in Theorem 4.1. We get the following results.

Theorem 5.3. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space X , $\Im = \{T(t) : t \geq 0\}$ an asymptotically nonexpansive in the intermediate semigroup on C such that each $T(t)$ is continuous, and $u(\cdot)$ be an asymptotically isometric almost-orbit of $\Im$ and strongly measurable, then $s^{-1} \int_0^s u(t + h)dt$ converges strongly to some point p, a unique point of the set $F(\Im) \bigcap \bigcap_{s\in G} \overline{co}\{u(t) : t \geq s\}$, as $s \to \infty$ uniformly in $h \geq 0$.*

Proof. Put $\mu_s(f) = \frac{1}{s} \int_0^s f(t)dt$ for each $s > 0$ and $f \in D$. Then, $\{\mu_s : s > 0\}$ is a strongly regular net on D. □

Let $Q : R^+ \times R^+ \mapsto R$ be a function satisfying the following conditions:

(a) $\sup_{s\geq 0} \int_0^\infty |Q(s, t)|dt < +\infty$,
(b) $\lim_{s\to 0} \int_0^\infty Q(s, t)dt = 1$,
(c) $\lim_{s\to 0} \int_0^\infty |Q(s, t + h) - Q(s, t)|dt = 0$ for all $h \geq 0$.

Then $Q(\cdot, \cdot)$ is called a strongly regular kernel.

Theorem 5.4. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space X , $\Im = \{T(t) : t \geq 0\}$ an asymptotically nonexpansive in the intermediate semigroup on C such that each $T(t)$ is continuous, and $u(\cdot)$ be asymptotically isometric an almost-orbit of $\Im$. Suppose that $u(\cdot)$ is strongly measurable and $Q(\cdot, \cdot)$ is a strongly regular kernel, then $\int_0^\infty Q(s, t)u(t + h)dt$ converges strongly to some point p, a unique point of the set $F(\Im) \bigcap \bigcap_{s\in G} \overline{co}\{u(t) : t \geq s\}$, as $s \to \infty$ uniformly in $h \geq 0$.*

Proof. Put $\mu_s(f) = \int_0^\infty Q(s, t)f(t)dt$ for each $s > 0$ and $f \in D$. Then $\{\mu_s : s > 0\}$ is a strongly regular net on D. □

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