# Spaces of convex sets

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International Conference on Fixed Point Theory and Applications

20 – 24 July MMXV Galatasaray University, Istanbul, Turkey

CARMA

# Abstract

What we present here is based on recent joint work with one of our Honours students; **Theo Bendit**

*Abstract:* Let $C(X)$ denote the set of all non-empty closed bounded convex subsets of a normed linear space $X$. In 1952 Hans Rådström described how $C(X)$ equipped with the Hausdorff metric could be isometrically embedded in a normed lattice with the order an extension of set inclusion. We call this lattice the *Rådström* of $X$ and denote it by $R(X)$.

We will:

(a) outline Rådström's construction,

(b) survey the Banach space structure and properties of $R(X)$, including; completeness, density character, induced mappings, inherited subspace structure, reflexivity, and its dual space,

(c) explore possible synergies with metric fixed point theory.

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# Those involved


**Theo Bendit** **Brailey Sims**

# Basics

$X \equiv (X, \|\cdot\|)$ denotes a real normed linear space.

$B_X$ and $B_X^\circ$ the closed and open balls of $X$ respectively, and

$X^*$ the dual space of continuous linear functionals on $X$.

$\mathcal{C}(X)$ denotes the set of non-empty, closed, bounded, convex subsets of $X$.
For any $A, B \in \mathcal{C}(X)$, we define $\lambda A := \{\lambda a : a \in A\} \in \mathcal{C}(X)$ and their **Minkowski sum** $A + B$ to be
$A + B := \{a + b : a \in A, b \in B\}$.

While $A + B$ is non-empty, bounded, and convex. it may not be closed unless one of the sets is weakly compact.

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While $A + B$ is non-empty, bounded, and convex. it may not be closed unless one of the sets is weakly compact.

**Example**

Let $X = l^1$, $A := \phi^{-1}\{1\} \cap 2B_X$, where $\phi \in X^*$ is such that $(a_n) \mapsto \sum_{n=1}^{\infty}(1-2^{-n})a_n$, and let $B = B_X$. Then $a_k := (1-2^{-k})^{-1}e_k \in A$ and $b_k := -e_k$ are sequences of elements of $A$ and $B$ respectively, with $a_k + b_k \to 0$, so $0 \in \overline{A+B}$, but calculation shows $0 \notin A + B$.

To overcome this, we introduce a new "addition" in $\mathcal{C}(X)$:

$$A \oplus B := \overline{A+B} \in \mathcal{C}(X).$$

Observe that: $A \oplus B = \overline{A} \oplus B$, and so $\oplus$ is associative. In addition $\{0\}$ is an identity for $\oplus$.

So, $(\mathcal{C}(X), \oplus)$ is a commutative monoid.

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The following perhaps surprising result is a key feature in our constructions.

**Proposition (Order Cancellation Law - Brunn, 1889)**

*If $A, B, C \in \mathcal{C}(X)$, and $A \oplus C \subseteq B \oplus C$, then $A \subseteq B$.*

In particular we have:
*If $A \oplus C = B \oplus C$, then $A = B$.*

Thus, $H := (\mathcal{C}(X), \oplus)$ is a commutative monoid with cancellation law, and so it can be embed into an abelian group $G$ (its Grothendieck group) as follows,

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# Rådström's construction - the Grothendieck group

Define an equivalence relation $\sim$ on $H \times H$ by

$$(A, B) \sim (C, D) \iff A \oplus D = C \oplus B.$$

Let $G$ be the set of all equivalence classes, and $[A, B]$ be the equivalence class of the pair $(A, B)$.

Then

$$[A, B] + [C, D] := [A + C, B + D]$$

is a well-defined binary operation on G, with respect to which $G$ is an abelian group; with identity $\mathbf{0} := [\{0\}, \{0\}]$ $(= [A, A])$ and inverses given by $-[A, B] = [B, A]$.

Further,

$$\phi : H \to G : A \mapsto [A, \{0\}]$$

is an injective homomorphism, that is, $G$ contains a copy of $H$, and $x \mapsto \phi(\{x\})$ provides an embedding of $(X, +)$ into $G$.

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**Remark:** The construction of $G$ from $H$ mirrors the construction of $(\mathbb{Z}, +)$ from $(\mathbb{N} \cup \{0\}, +)$, and the construction of $(\mathbb{Q} \setminus \{0\}, \times)$ from $\mathbb{Z} \setminus \{0\}, \times)$.

As in these cases, we will avoid the cumbersome notation of pairs by using $C$ to denote both a non-empty, closed, bounded, convex subset of $X$ and its image $[C, \{0\}]$ under the embedding homomorphism $\phi$.

# Rådström's construction - extension to a linear space

Moreover, if we define scalar multiplication by,

$$\lambda[A, B] = \begin{cases} [\lambda A, \lambda B] & : \lambda \geq 0 \\ [-\lambda B, -\lambda A] & : \lambda < 0 \end{cases}$$

then, after some tedious verification, we have:

**Proposition**

*$G$ is a real linear space.*

This suggest defining,

$$\begin{aligned} A \ominus B := {} & A \oplus (-1B) \\ & = [A, \{0\}] \oplus [\{0\}, B] \\ & = [A, B] \end{aligned}$$

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# Rådström's construction - a new notation

Henceforth, we will mostly use the suggestive notation $A \ominus_X B$ for the equivalence class $[A, B]$.

**Comment:** The subscript $X$ is necessary to identify the space in which the elements of the equivalence class reside.
For example, if we have $Y$, a closed, strict subspace of $X$, then for any $A, B \in \mathcal{C}(Y) \subset \mathcal{C}(X)$, the class $A \ominus_Y B$ is a strict subset of $A \ominus_X B$.

However, when the space is clear from the context, we will simply write $A \ominus B$.

# Rådström's construction - adding order

Due to the order cancellation law, the subset partial order on $\mathcal{C}(X)$ can be extended to $G$ by defining,

$$A \ominus B \leq C \ominus D \iff A \oplus D \subseteq C \oplus B,$$

**Proposition**

*The relation $\leq$ on $G$ is well-defined, and makes $G$ a vector lattice.*

The positive cone is $G^+ = \{A \ominus B : A \supseteq B\}$.

**Note:** Despite the fact that $G = \mathcal{C}(X) \ominus \mathcal{C}(X)$, the positive cone and $\mathcal{C}(X)$ do not coincide.

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(1) $\mathbf{u} := B_X$ is an order unit for $G$, as $|A \ominus B| \leq n\mathbf{u}$, when $n$ is any integer larger than $\max_{a \in A} \|a\| + \max_{b \in B} \|b\|$.

(2) If $A \ominus B \leq \frac{1}{n}\mathbf{u}$ for all $n \in \mathbb{N}$, then $A \ominus B \leq \mathbf{0}$.

From these it follows that

$$\|A \ominus B\| := \inf \{\lambda \geq 0 : |A \ominus B| \leq \lambda \mathbf{u}\}$$

defines a lattice norm on G.

Further, calculation shows that,

$$\|A \ominus B\| = \mathcal{H}(A, B) := \max \left\{ \sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b) \right\},$$

here $\mathcal{H}$ is the **Hausdorff distance** on $\mathcal{C}(X)$.

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# The Rådström of $X$

$G$ with this norm is a normed linear lattice which we call the **Rådström of $\mathbf{X}$** and denote by $\mathcal{R}(X)$

[also known as a Minkowski-Rådström-Hörmander (MRH) space, or a Pinsker-Minkowski-Rådström-Hörmander (PMRH) lattice].

$\phi$ provides a monotone isometric embedding of $(\mathcal{C}(X), \mathcal{H})$ into $\mathcal{R}(X)$ and $x \mapsto \phi(\{x\})$ is a linear isometry from $X$ into $\mathcal{R}(X)$.

By the Krein-Kakutani theorem, there is a monotone linear isometry $\psi : \mathcal{R}(X) \to C(K)$ with $\psi(\mathcal{R}(X)))$ a dense subspace of $C(K)$ and $\psi(\mathbf{u})$ the constant function 1, where $K$ is a compact, Hausdorff topological space,
specifically, $K$ consists of the extreme points of the set of positive linear functionals in $B_{\mathcal{R}(X)^*}$ equipped with the the weak* topology, and for all $\mathbf{x} \in \mathcal{R}(X)$, $\psi(\mathbf{x}) = \hat{\mathbf{x}}|_K$,

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# Two examples of a Rådström

There are (precisely) two elementary examples of Rådström spaces, arising from the two simplest real normed linear spaces.

**Example**

$$\mathcal{R}(\{0\}) \;=\; \{\mathbf{0}\}.$$

**Example**

Due to the simplistic nature of convex sets in $\mathbb{R}$,
$\mathcal{R}((\mathbb{R}, |\cdot|))$ is isometric to $\ell^2_\infty = (\mathbb{R}^2, \|\cdot\|_\infty)$,
under the surjective, linear isometry:

$$\iota : \mathcal{R}(\mathbb{R}) \to \mathbb{R}^2 : [a,b] \ominus [c,d]) \mapsto (a-c, b-d).$$

# On the other hand:

**Proposition**

*If* $\dim X \geq 2$*, then* $\mathcal{R}(X)$ *is infinite-dimensional.*

Further,

**Proposition**

$\mathcal{R}(X)$ *is separable if and only if* $X$ *is finite-dimensional.*

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# Rådströms as normed linear spaces

And, from the Blaschke Selection Principle, we have:

**Theorem**

*If* $\dim X \geq 2$*, then* $\mathcal{R}(X)$ *is incomplete.*

As a consequence, if $\dim X \geq 2$, then $\mathcal{R}(X)$ is not reflexive and, from the Krein-Kakutani represention, neither is its completion.

# Rådströms as normed linear spaces - Subspaces

We have already seen that $\mathcal{R}(X)$ contains a subspace isometric to $X$, and this is indeed the only subspace wholly contained in $\mathcal{C}(X)$.

Other subspace include,

$$\begin{aligned}
\mathcal{R}_{FD}(X) &:= \{A \ominus B \in \mathcal{R}(X) : \mathrm{span}(A), \mathrm{span}(B) \text{ are finite-dimensiona} \\
\mathcal{R}_{K}(X) &:= \{A \ominus B \in \mathcal{R}(X) : A, B \text{ are compact}\}, \\
\mathcal{R}_{wK}(X) &:= \{A \ominus B \in \mathcal{R}(X) : A, B \text{ are weakly compact}\}, \\
\mathcal{R}_{w^*K}(X^*) &:= \{A \ominus B \in \mathcal{R}(X^*) : A, B \text{ are weak* compact}\}.
\end{aligned}$$

The last 3 are closed subspaces and,

$$\overline{\mathcal{R}_{FD}(X)} = \mathcal{R}_K(X) \subseteq \mathcal{R}_{wK}(X), \quad \text{and } \mathcal{R}_{wK}(X^*) \subseteq \mathcal{R}_{w^*K}(X^*).$$

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Further, as might be expected the subspace structure of $X$ is mirrored in $\mathcal{R}(X)$, indeed,

**Theorem**

*Suppose $Y$ is a subspace of $X$, not necessarily closed, then $\mathcal{R}(Y)$ is isometrically isomorphic to a closed subspace of $\mathcal{R}(X)$.*

This is easily verified when $Y$ is closed and complemented, but for the general case it is non-trivial.

As a corollary we have:
For a normed linear space $X$, $\mathcal{R}(X) = \mathcal{R}(\widetilde{X})$, where $\widetilde{X}$ is the completion of $X$.

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# Induced operators

For $T : X \to Y$ a Lipschitz continuous map between normed linear spaces $X$ and $Y$ define,

$$\rho_T : \mathcal{C}(X) \to \mathcal{C}(Y) : C \mapsto \overline{\mathrm{co}} T(C),$$

then $C_L(\rho_T) = C_L(T)$

Further, if $T$ is linear then, taking the convex hull is superfluous, $\rho_T$ is additive and positive scalar-homogeneous, and we can extend it to a map from $\mathcal{R}(X)$ to $\mathcal{R}(Y)$ by defining

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# Induced linear operators

## Proposition

*Suppose $X$, $Y$ and $Z$ are normed spaces and $T : X \to Y$ and $S : Y \to Z$ are bounded linear operators, then:*

(1) $\rho_T$ *is a well defined linear operator,*

(2) $\rho_T$ *is bounded with $\|\rho_T\| = \|T\|$.*

(3) $\rho_T$ *is monotone.*

(4) *For any $k \in [0, \infty)$, $\rho_{kT} = k\rho_T$.*

(5) $\rho_{I_X} = I_{\mathcal{R}(X)}$

(6) *If $T$ is an isomorphism, then $\rho_T^{-1} = \rho_{T^{-1}}$.*

(7) *If $T$ is an isometry, then $\rho_T$ is an isometry.*

(8) $\rho_{ST} = \rho_S \rho_T$.

# Dual space of a Rådström - induced functionals

Each $f \in X^*$ induces a linear transformation $\rho_f : \mathcal{R}(X) \to \mathcal{R}(\mathbb{R}) = \ell^2_\infty$, so

$$\phi = \mathbf{v} \circ \rho_f \in \mathcal{R}(X)^*,$$

where $\mathbf{v} \in \ell^2_1 = \left(\ell^2_\infty\right)^*$.
We refer to $\phi$ as **a functional** (on $\mathcal{R}(X)$) **induced by** $\mathbf{f}$.

In particular we have,

$$\begin{aligned}\alpha_f(A \ominus B) &:= \max f(A) - \max f(B), \quad \text{here } \mathbf{v} = (1,0),\\ \omega_f(A \ominus B) &:= \min f(A) - \min f(B), \quad \text{here } \mathbf{v} = (0,1),\\ &= -\alpha_{-f}(A \ominus B).\end{aligned}$$

$\alpha_f \in \mathcal{R}(X)^*_+$, and every functional induced by $f$ is a linear combination of $\alpha_f$ and $\omega_f$.

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# Dual space of a Rådström - induced functionals

**Theorem**

*$\phi \in \mathcal{R}(X)^*_+$ is induced by $f \in X^*$ if $\phi(B_{\ker(f)}) = 0$.*

**Corollary**

*The set $\{\sigma_f : f \in S_{X^*}\}$ is a (lattice) orthogonal set.*

This yields an orthogonal, and hence linearly independent, subset of $\mathcal{R}(X)^*$ that is infinite when $\dim(X) > 1$,
giving an alternative proof that $\mathcal{R}(X)$ is infinite dimensional whenever $\dim(X) > 1$.

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# Structure of $\{\sigma_f : f \in S_{X^*}\}$

**Theorem**

$$\{\sigma_f : f \in S_{X^*}\} \subseteq Ext\left(\mathcal{R}(X)^*_+ \cap S_{\mathcal{R}(X)^*}\right),$$

*with equality if and only if $X$ is finite-dimensional.*

We introduce two subspaces of $\mathcal{R}(X)^*$

$$\Sigma := \left\{\sigma \in \mathcal{R}(X)^* : \sigma = \sum_{f \in S_{X^*}} c_f \sigma_f\right\},$$

where only countably many of the scalars, $c_f$, are non-zero, and

$$\Sigma^\perp := \{\psi \in \mathcal{R}(X)^* : \psi \text{ is orthogonal to } \sigma_f \text{ for all } f \in S_{X^*}\}.$$

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# Structure theorem for functionals

**Theorem**

$$\mathcal{R}(X)^* \;=\; \Sigma \;\oplus\; \Sigma^{\perp},$$

*where* $\oplus$ *denotes direct sum.*
*Moreover,* $\phi = \psi + \sum_{f \in S_{X^*}} c_f \sigma_f \geq 0$ *if and only if* $\psi \geq 0$ *and* $c_f \geq 0$ *for all* $f \in S_{X^*}$*.*

# Elements of $\Sigma_+^\perp$

**Proposition**

*For* $\phi \in \mathcal{R}(X)^*_+$,

$$\phi\left(\mathcal{R}_{FD}(X)\right) = \{0\} \implies \phi \in \Sigma^\perp.$$

However, the converse is demonstrably false; indeed, for $X = \mathbb{R}^n$, with $n \geq 2$, and $\mu$ Lebesuge measure,

$$\phi : \mathcal{R}(X) \to \mathbb{R} : A \ominus B \mapsto \int_{B_{X^*}} \sigma_f(A \ominus B)\, \mathrm{d}\mu(f),$$

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# Synergies with metric fixed point theory

For $C \in \mathcal{C}(X)$, we have seen how a nonexpansive map $T : C \to X$ induces a nonexpansive map

$$\rho_T : \mathcal{C}(C) \subset \mathcal{R}(X) \to \mathcal{R}(X),$$

where $\mathcal{C}(C) := \{A \in \mathcal{C}(X) : A \subseteq C\}$.

The fixed points of $\rho_T$ are the invariant sets for $T$ and the lattice minimal elements of $\mathrm{Fix}(\rho_T)$ are the minimal invariant sets of $T$.

Thereby, opening the possibility of transferring:

- the structure of fixed point sets to the family of (minimal) invariant sets of $T$,
- algorithms for approximating fixed points to ways of approximating invariant set,
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# Set valued mappings

A multifunction $\tau : C \in \mathcal{C}(X) \to 2^X$ taking nonempty closed bounded convex values can be regarded as a mapping

$$T : \mathbf{C} := \{\, \{x\} : x \in C\} \subset \mathcal{R}(X) \to \mathcal{R}(X) : \{x\} \mapsto \tau(x).$$

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# Some references


W. A. Coppel, *Foundations of Convex Geometry*, Cambridge University Press, 1998, xiv + 222.

Hans Rådström, An Embedding Theorem for Spaces of Convex Sets, *Proc. Amer. Math. Soc.*, **3** (1952), 165–169.

H. H. Schaefer, *Aspects of Banach Lattices*, in Studies in Functionl Analysis, R. G. Bartle (editor), Maths Association of America, Studies in Mathematics; Vol. 21, 1980, pp. 227