INVERSE F$^{\underline{NS}}$

F

To understand inverse functions we must first consolidate the notion of a function.

FUNCTION:

a 'rule' f Eg. $f(x) = x^2 + 1$
'square x then add 1'

and

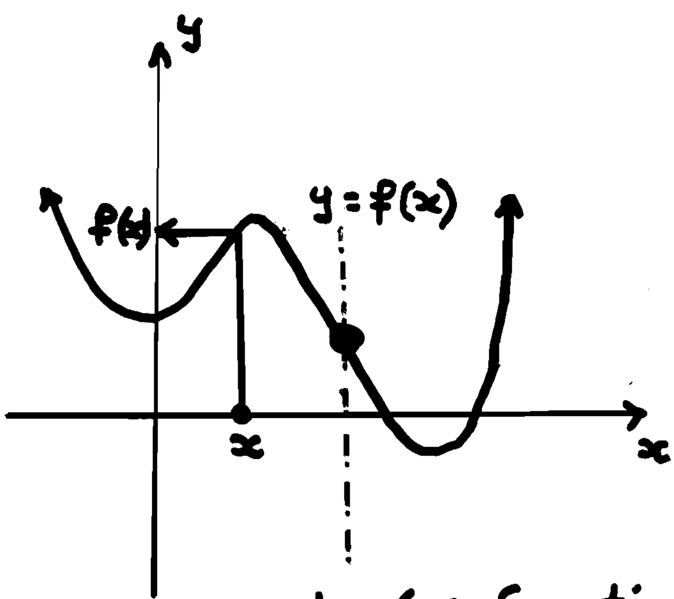
a 'domain' – the collection of objects on which the rule is allowed to act.

Eg. $-1 \leqslant x \leqslant 1$

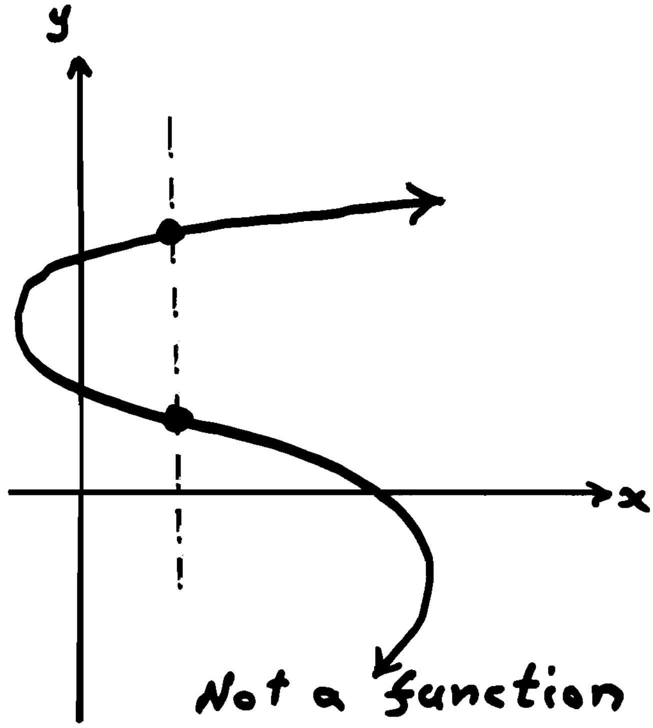
We require that the rule assign to each x in the domain a unique value $f(x)$.

Action of f;

$x \longmapsto f(x)$



Graph of a function

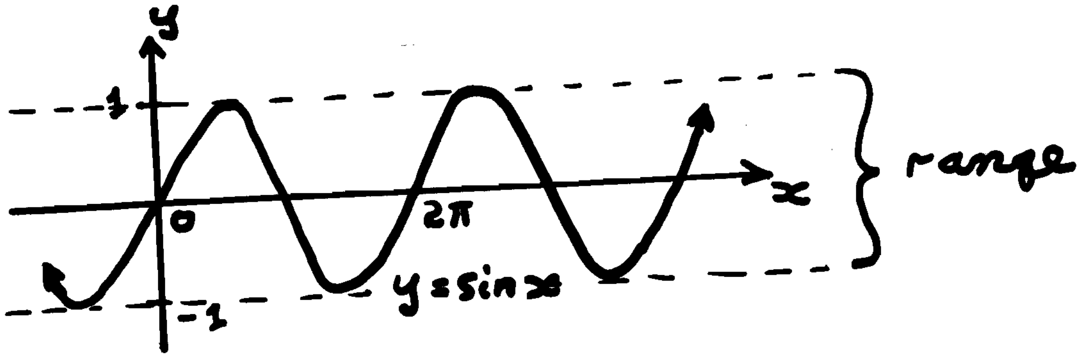


Not a function

"vertical line test"

The set of values a function assumes is its range.

Eg. The range of $y = \sin x$ is $-1 \leq y \leq 1$



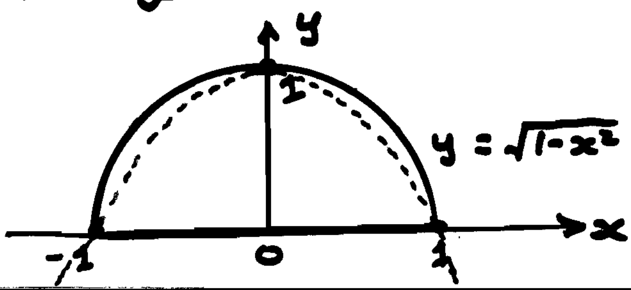
While it need not generally be the case, our functions will act on real numbers (ie. the domain will be a set of numbers) and the values assigned will also be numeric (ie. the range will be a set of numbers).

When the domain is not explicitly specified we take it to be all those numbers to which the rule f can meaningfully apply.

Eg.1) The function $f(x) = \sqrt{1-x^2}$

has domain $-1 \leq x \leq 1$,

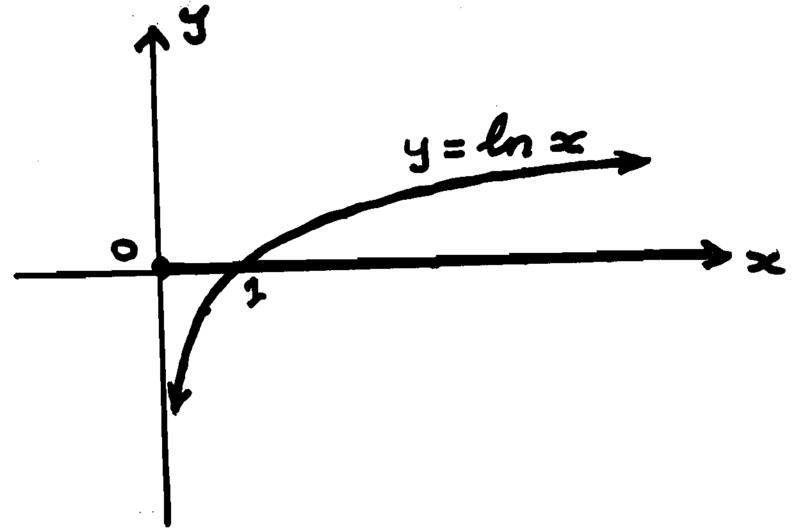
and range the interval from 0 to 1.



2) **ln** assigns to x the power, $\ln x$, to which e must be raised to obtain x.

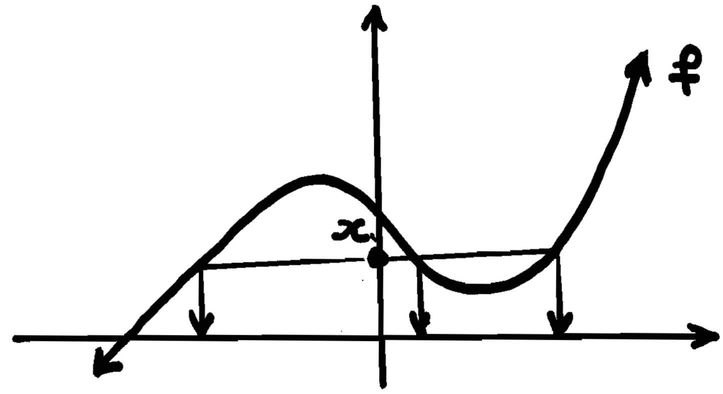
$$\boxed{\begin{array}{l}\ln x \text{ is the number such that} \\ \qquad e^{\ln x} = x\end{array}}$$

Since negative numbers do not arrise as any power of e we see that the domain of $\ln x$ is $0 < x < \infty$.

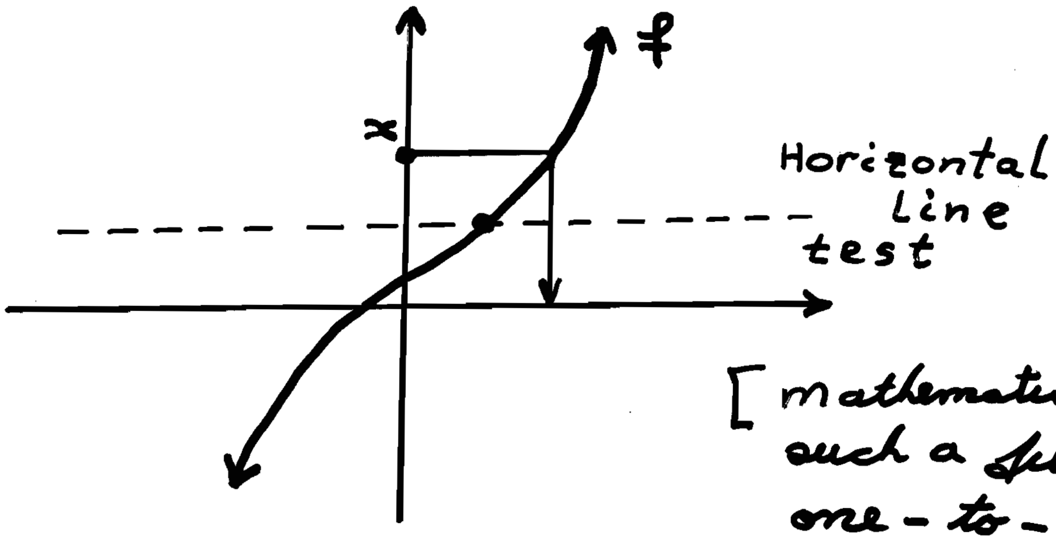


The range of $\ln x$ is all real numbers.

Given a function f we can consider the rule which assigns to each x in the range of f those numbers at which f takes the value x.



For this to be a function there must be only <u>one</u> point in the domain of f where f takes the value x.



[mathematicians call such a function f one-to-one.]

When this is the case we call our new rule the *inverse function* of f and denote it by f^{-1}. It has the property:

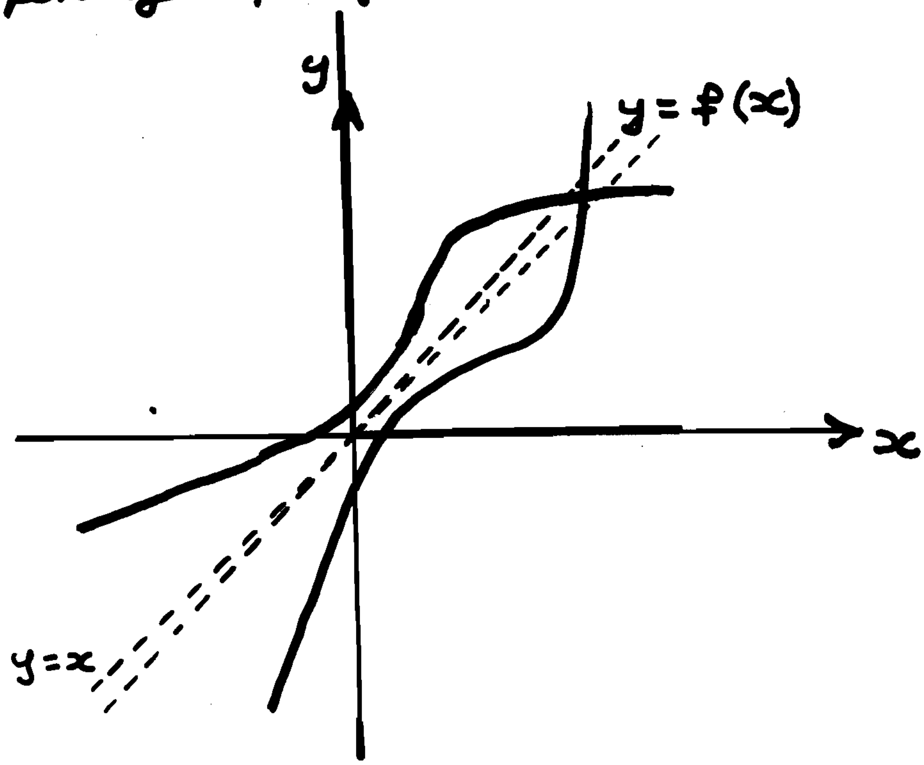
$$\text{if } y = f^{-1}(x) \text{ then } x = f(y).$$

$$\text{So } f(f^{-1}(x)) = x \text{ and } f^{-1}(f(x)) = x.$$

Eg. The inverse of e^x is $\ln x$.

The domain of f^{-1} is the range of f and the range of f^{-1} is the domain of f. (e^x and $\ln x$)

Reflecting the diagram (below) in the line $y = x$ so that the y-axis becomes horizontal and the x-axis vertical we obtain the graph of f^{-1}.



Throughout intervals where an invertible function f is continuous it must be either strictly increasing or strictly decreasing.

Derivatives of inverse functions.

If $y = f^{-1}(x)$, from $\frac{dy}{dx} = 1 \Big/ \frac{dx}{dy}$

and $x = f(y)$ we can find the derivative of f^{-1} from that of f.

Eg. $y = \tan^{-1} x$

$$\frac{dy}{dx} = 1 \Big/ \frac{dx}{dy} \quad \text{where } x = \tan y = \frac{\sin y}{\cos y}$$

$$= 1 \Big/ \frac{\cos y \cos y - \sin y(-\sin y)}{\cos^2 y}$$

$$= 1 \Big/ \frac{\cos^2 y + \sin^2 y}{\cos^2 y}$$

$$= \cos^2 y$$

But want answer in terms of x.

$$= \frac{1}{1 + (\tan y)^2}$$

$s^2 + c^2 = 1$

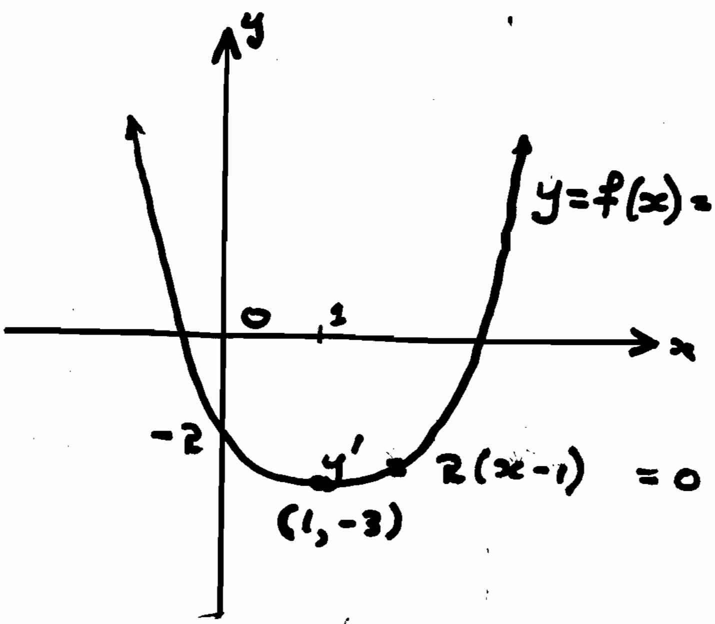
$t^2 + 1 = \frac{1}{c^2}$

$$= \frac{1}{1 + (\tan(\tan^{-1} x))^2}$$

$$= \frac{1}{1 + x^2}$$

By restricting the domain of a function which fails the "horizontal line test" we may be able to obtain an invertible function.

Eg. [3/4U, 1989, 6(a)]



$y = f(x) = (x-1)^2 - 3$ is not invertible

$y' = 2(x-1) = 0 \Rightarrow x = 1$

$y = -3$

$y - 1 = \pm\sqrt{x+3}$

— no good as $y \geq 1$ as $y - 1 \geq 0$.

$(x-1)^2 - 3 = -2$

$(x-1)^2 = 1$

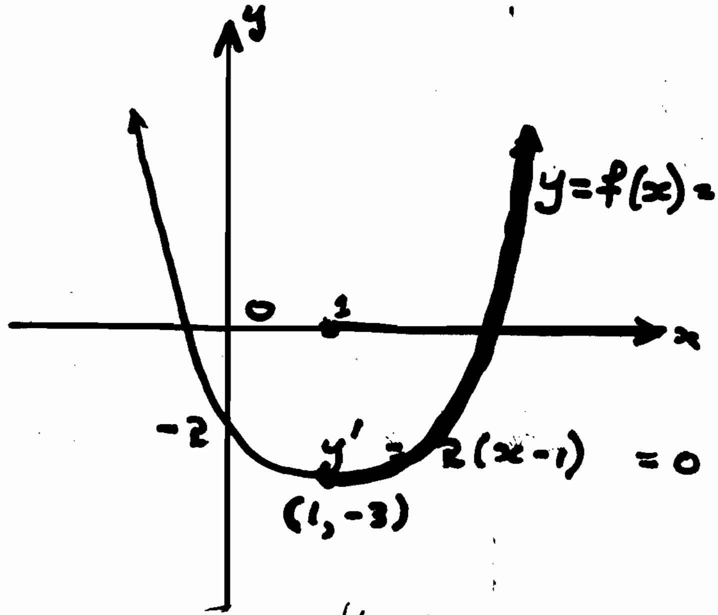
$x - 1 = \pm 1$

$x = 0$ or $x = 2$

By restricting the domain of a function which fails the 'horizontal line test' we may be able to obtain an invertible function.

Eg. [3/4U, 1989, 6(a)]

(a)



$y=f(x)=(x-1)^2-3$ is not invertible

but, restricted to $x \geq 1$ f is invertible. The inverse has domain: $x \geq -3$, range: $y \geq 1$

$y' = 2(x-1) = 0 \Rightarrow x = 1$ $\quad y = -3$

$y-1 = \pm\sqrt{x+3}$

To find an expression for f^{-1} on this domain, suppose $y = f^{-1}(x)$ ~~then~~ as $y-1 \geq 0$.

no good as $y-1 \geq 1$

$x = f(y) = (y-1)^2 - 3$ $\quad (x-1)^2 - 3 = -2$

So $(y-1)^2 = x+3$, $\quad (x-1)^2 = 1$

$y - 1 = \sqrt{x+3}$ $\quad x-1 = \pm 1$

$x = 0$ or $x = 2$

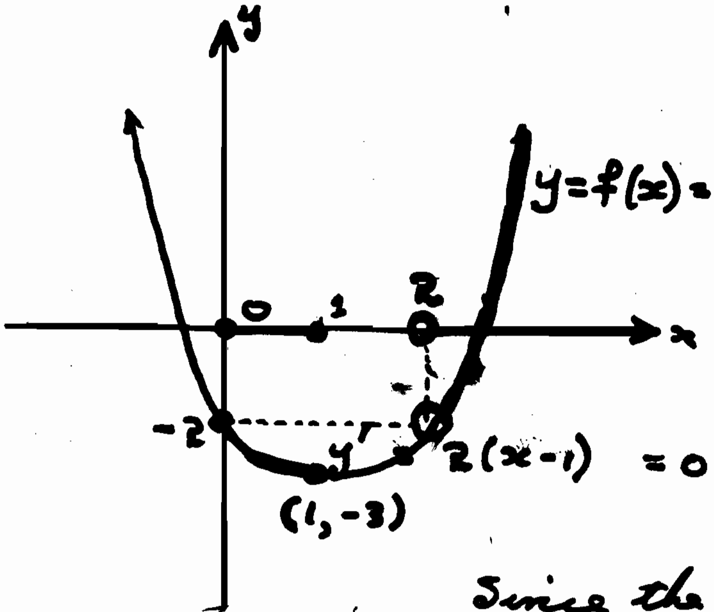
and

$f^{-1}(x) = y = \sqrt{x+3} + 1$

By restricting the domain of a function which fails the 'horizontal line test' we may be able to obtain an invertible function.

Eg. [3/4 U, 1989, 6(a)]

(b)



$y = f(x) = (x-1)^2 - 3$ is not invertible

An alternative domain on which f is invertible is

$2(x-1) = 0 \Rightarrow x = 1, \; y = -3$

$\{0 \leq x \leq 1\} \cup \{x > 2\}$

$y > 1$ Since the question asked for ~~the~~ $\pm$ $\sqrt{}$ ~~largest~~ positive domain I don't know what the answer is, since I ~~don't know how~~ to compare these two domains and those of others!

$$(x-1)^2 - 3 = -2$$

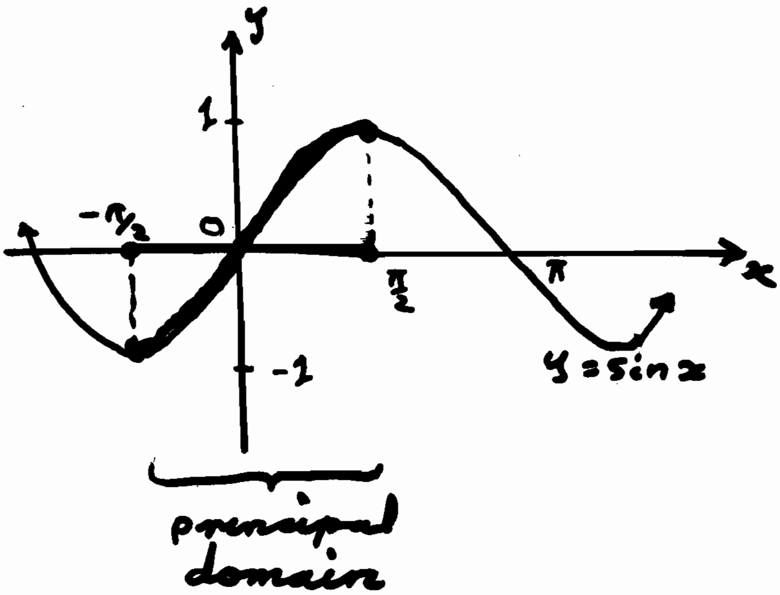
$$(x-1)^2 = 1$$

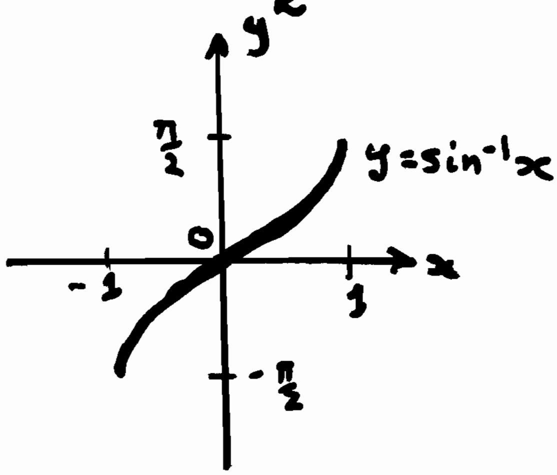
$$x - 1 = \pm 1$$

$$x = 0 \text{ or } x = 2$$

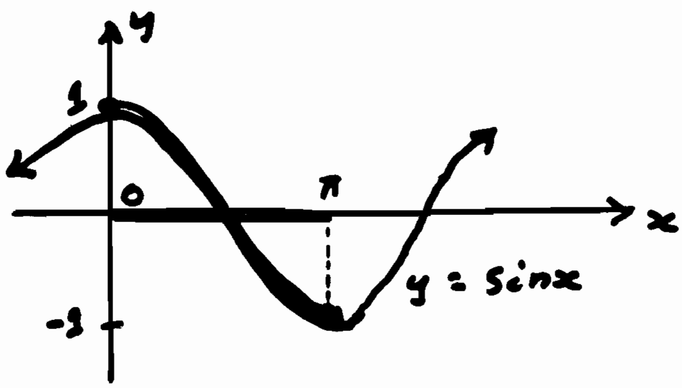
To obtain inverses for the trigonometric functions we need to restrict their domains.

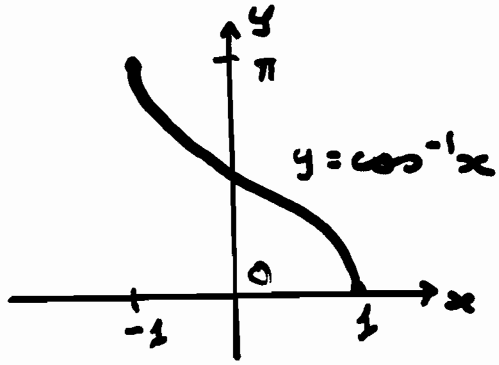
<u>$\sin^{-1}$</u> is obtained by restricting $\sin(x)$ to $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, so $\sin^{-1}(x)$ has domain $-1 \leq x \leq 1$ and range $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$



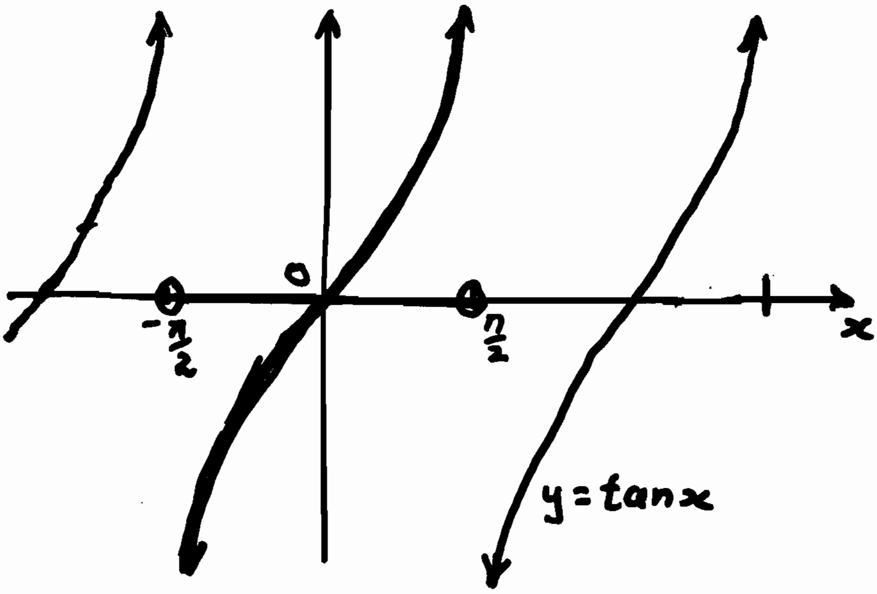


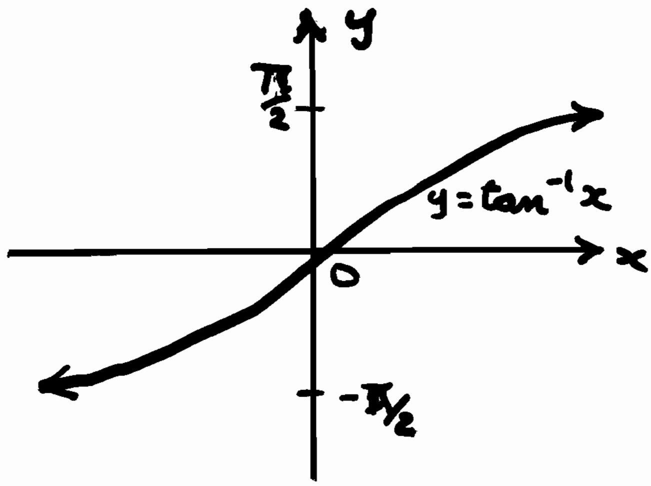
<u>$\cos^{-1}$</u> is obtained by restricting $\cos(x)$ to $0 \leq x \leq \pi$, so $\cos^{-1}(x)$ has domain $-1 \leq x \leq 1$ and range $0 \leq x \leq \pi$





<u>$\tan^{-1}$</u> is obtained by restricting $\tan x$ to $-\frac{\pi}{2} < x < \frac{\pi}{2}$, so $\tan^{-1} x$ has domain the whole x-axis and range $-\frac{\pi}{2} < y < \frac{\pi}{2}$.





Use:

$f(\cancel{x}) = f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1 (x))$

to "write" down

$f^{-1}(x) = f_1^{-1} \circ f_2^{-1} \circ \cdots \circ f_n^{-1}(x)$

where $f_1, f_2, \cdots f_n$ are invertible functions whose inverses are known

Prove

$$f(x) = \sqrt{x^3+1} = f_3 \circ f_2 \circ f_1 (x) \quad [x \geq \bullet]$$

where

$f_1(x) = x^3 \qquad f_1^{-1}(x) = x^{\frac{1}{3}}$

$f_2(x) = x+1 \qquad f_2^{-1}(x) = x-1$

$f_3(x) = \sqrt{x} \qquad f_3^{-1}(x) = x^2 \quad (x \geq 0)$

So

$$f^{-1}(x) = f_1^{-1} \circ f_2^{-1} \circ f_3^{-1}(x)$$

$$= (x^2 - 1)^{\frac{1}{3}}$$

* 3/4 U, 1993, 3(a)

** 3/4 U, 1989, 7(b) — see next page.

(4 U, 1986, 2 — see later)

* 3/4 U, 1991, 7(b)

3/4 U, 1994 6 — done previous page

3/4 U 1989 6(a) — done on slides

3/4 U 1995 ?

3/4 u, 1994, 6.

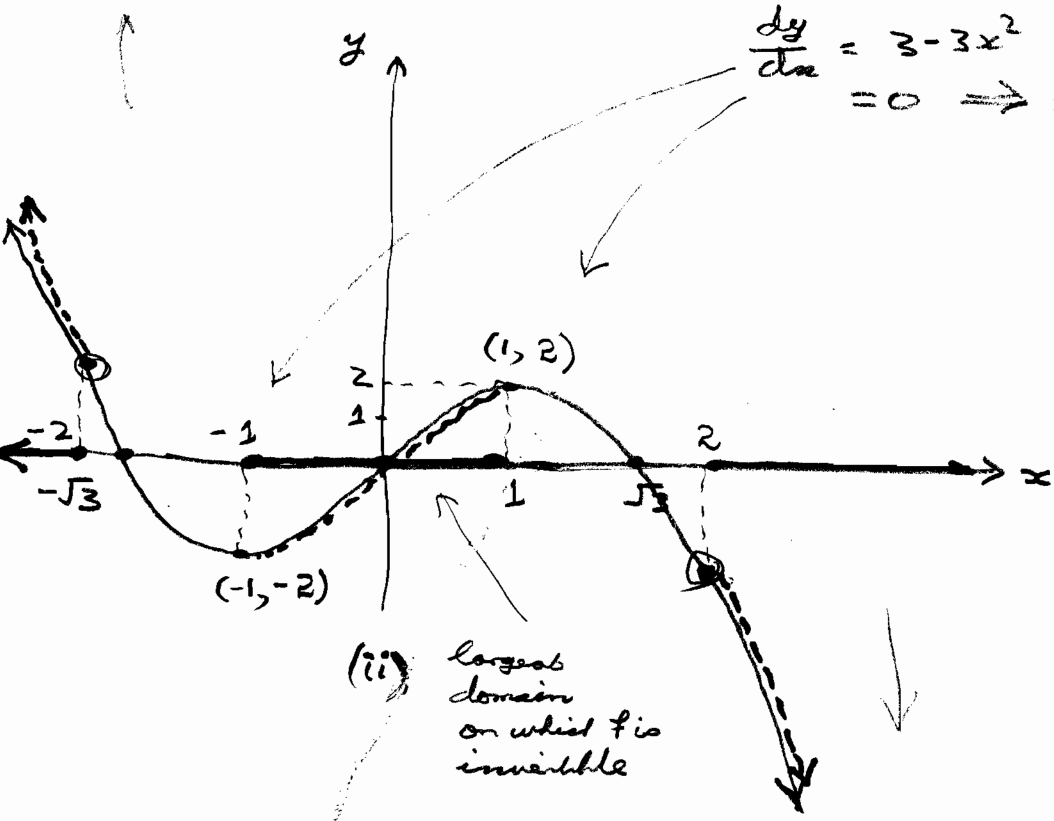
$y = f(x) = 3x - x^3$ (odd)

$= 0 \Rightarrow x = 0$ or $x^2 = 3$, so $x = \pm\sqrt{3}$

(i)

$\frac{dy}{dx} = 3 - 3x^2$

$= 0 \Rightarrow x^2 = 1$, so $x = \pm 1$



(ii) largest domain on which f is invertible

$3x - x^3 = -2$

$x^3 - 3x - 2 = 0$

double root is $x = -1$

ie $(x+1)^2(x - ?)$

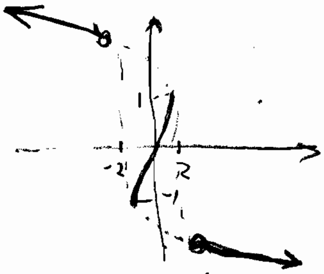
$(x^2+2x+1)(x-2) = 0$

$x = 2$

(iii) domain of f^{-1} = range of f on $[-1, 1]$

$= [-2, 2]$ or range of f on $\{x < -2\} \cup [-1,1] \cup \{x > 2\}$

= whole line.



graph of f^{-1}

Let $y = f^{-1}(x)$ then $x = f(y) = 3y - y^3$

and $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{3 - 3y^2}$

From graph $f^{-1}(0) = 0$

So at $x = 0$ $y = f^{-1}(0) = 0$

$\therefore \left.\frac{dy}{dx}\right|_{\text{at } x=0} = \frac{1}{3 - 3 \times 0^2} = \frac{1}{3}$.

4U, 86, Q2

$S'' = S$

$(S'^2 - S^2)' = 2S'S'' - 2SS' = 0$

ie $S'^2 - S^2 = C = 1$ ← $x = 0$

$$y = S(x) = \frac{e^x - e^{-x}}{2}$$

$$S'(x) = \frac{e^x + e^{-x}}{2} > 0 \ \forall x.$$

So $S(x) \nearrow$

$\therefore$ $S^{-1}(x)$ exists for all x.

$x \to \infty$ $S(x) \to \infty$

$x \to -\infty$ $S(x) \to -\infty$

So range is $\mathbb{R}$.

$y = S^{-1}(x)$ $\quad x = S(y) = \frac{e^y - e^{-y}}{2}$

$$\frac{dx}{dy} = \frac{e^y + e^{-y}}{2}$$

$$\frac{d}{dx}S^{-1}(x) = \frac{dy}{dx} = \frac{2}{e^y + e^{-y}} = \frac{1}{\frac{e^y + e^{-y}}{2}} = \frac{1}{C(y)} = \frac{1}{S'(y)}$$

$$= \frac{1}{\sqrt{1 + S(y)^2}} = \frac{1}{\sqrt{1 + x^2}}$$

$$\frac{dy}{dx} = \frac{1}{x + \sqrt{x^2+1}} \times \left(1 + \frac{x}{\sqrt{x^2+1}}\right)$$

$$= \quad '' \quad \frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1}}$$

$$= \frac{1}{\sqrt{x^2+1}}$$

$y = S^{-1}(x) \Rightarrow$

$$e^y - 2x - e^{-y} = 0$$

$(e^y)^{-2}$

$$(e^y)^2 - 2x e^y - 1 = 0$$

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2}$$

$$= x \pm \sqrt{x^2+1}$$

$e^y > 0$

RHS < 0 if choose $-$

So

$$\text{So } y = \ln\left(x + \sqrt{x^2+1}\right)$$

3/4 U, 1989, 7(b)

$y = \tan^{-1} x$

$x = \tan y = \frac{\sin y}{\cos y}$

$\frac{dx}{dy} = \frac{1}{\cos^2 y}$ $\qquad s^2 + c^2 = 1$

$= 1 + \tan^2 y$ $\qquad t^2 + 1 = \frac{1}{c^2}$

so $\frac{dy}{dx} = \frac{1}{1+x^2}$

$y = \tan^{-1} \frac{1}{x}$

$\frac{dy}{dx} = \frac{1}{1+\frac{1}{x^2}} \times -\frac{1}{x^2} = \frac{x^2}{1+x^2} \times \frac{1}{x^2} = -\frac{1}{1+x^2}$

$\left(\tan^{-1} \frac{1}{x}\right)' = -\left(\tan^{-1} x\right)'$

so $\tan^{-1} x + \tan^{-1} \frac{1}{x} = C = \frac{\pi}{2}$ $(x = 1)$

$\tan\left(\tan^{-1} x + \tan^{-1} \frac{1}{x}\right) = \frac{x + \frac{1}{x}}{1 - x \times \frac{1}{x}}$ = "∞" $x > 0$

$-\infty$ $x < 0$

$\frac{t_a + t_b}{1 - t_a t_b}$