# THE RAINBOW ANGLE . . . . B. SIMS

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Appearing in the *Scientific American* April 1977 is an article by Professor Nussenzveig discussing "the theory of the Rainbow" [1]. In "The Amateur Scientist" section of the July 1977 issue of the same magazine a simple experimental set-up is described by means of which many of the phenomena considered in Nussenzveig's article can be investigated [2]. It occurred to me that these articles could form the basis of a *special project* for senior high school students, particularly if a more quantitative mathematical study is possible for some of the material. It is the purpose of this note to demonstrate that this is indeed the case, at least for the determination of the *rainbow angle* for the *primary bow* of a *monochromatic Cartesian rainbow*. For what follows the first section of Nussenzveig's article [pp 116-119] would be useful background.

Since the rainbow originates from the redirecting of the sun's rays by water droplets in the air (Fig. 0), we begin by examining the interaction
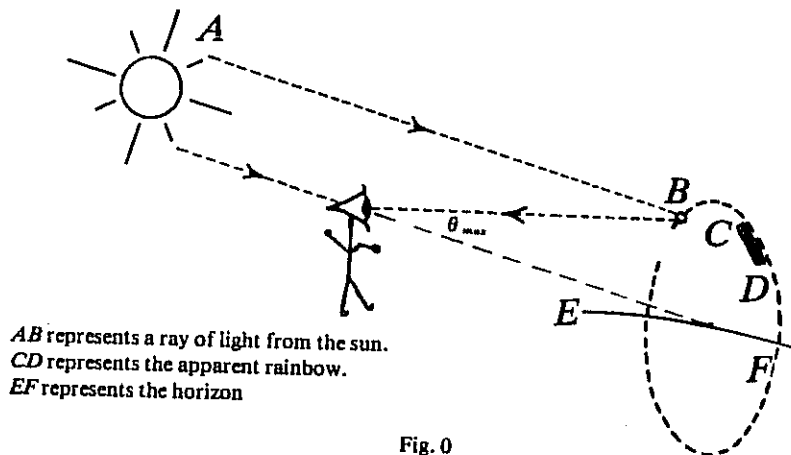
AB represents a ray of light from the sun.
CD represents the apparent rainbow.
EF represents the horizon

Fig. 0

of a ray of light with a single "spherical" droplet. Observing that the path of such a ray will lie in the plane containing the incident ray and the centre of the droplet, we see that the ray's path in this plane will be as illustrated in Fig. 1; and if we take, as our standard of length, the droplet's radius, the defining parameter of any particular ray is the *impact parameter p*, equal to the perpendicular distance from the droplet's centre O to the direction of the incident ray.

The object of our preliminary investigation is the angle θ_r as a function of p. Light of the "primary bow", due to an incident ray with this particular impact parameter, would come to an observer from a direction

making this angle with the direction of the incident light. Of course, the actually observed primary rainbow is the cumulative effect over all possible values of the impact parameter (that is, values of p between 0 and 1).

We assume that the refraction of a light ray entering or leaving the spherical droplet at a given point, and the reflection of a ray from a point on the droplet's surface, would be the same if the spherical air-water interface were replaced by a flat interface coincident with the sphere's tangent plane at that point. Since the tangent plane is perpendicular to the radius through the point of contact, this assumption makes the application of the laws of reflection and refraction particularly simple.
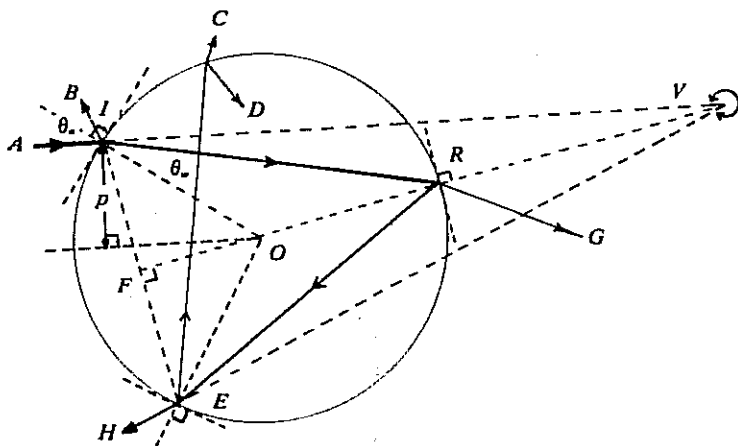
AI represents an incident ray.
RE represents an internally reflected ray.
EH represents an emergent ray: the incident ray's contribution to the primary rainbow.
IB represents a reflected ray.
RG represents a transmitted ray.
DC is related to a secondary rainbow.
The rainbow angle is shown at V.

Fig. 1

Referring to Fig. 1, we note that the angle θ_a is related to θ_w by *Snell's law*,
$$\sin \theta_a / \sin \theta_w = r, \qquad \ldots (1)$$
where r, the *refractive index*, is a parameter depending only on the nature of the two media and the frequency of the light (for air-water and frequencies in the visible range $r \cong 1.333$). It is the variation of r with frequency (*dispersion*) which causes the bow for different frequencies to

appear in slightly different directions, giving the rainbow its spectacular colours.

Since $OI = OR$, $\angle ORI = \theta_w$. By the law of reflection: *angle of incidence equals angle of reflection,*

$$\angle ORE = \angle ORI = \theta_w;$$

therefore $\angle OER = \theta_w$ $(OE = OR)$

and so, applying Snell's law to the emergent ray at E we see that this ray makes an angle of θ_a with the line through O and E. Thus, the path of the ray is symmetric about the line through OR, on which the incident and emergent rays produced meet at V.

Now, $\theta_r = 2\angle OVI$ *(angles measured in degrees)*

$$= \; 2(90 - \angle VIF) \qquad \text{(triangle } IVF\text{)},$$

while $\angle VIF = \angle OIV + \angle OIF = \theta_a + \angle OIF$,

so, $\theta_r = 2(90 - \theta_a - \angle OIF)$.

Since, $\angle OIF + \theta_w = \angle RIF = 90 - \theta_w$ (triangle FIR),

we have, $\angle OIF = 90 - 2\theta_w$

and therefore $\theta_r = 2(2\theta_w - \theta_a)$.   . . . (2)

Combining (1) and (2), we find that

$$\theta_r = \; 2(2 \sin^{-1} (\sin (\theta_a/r) - \theta_a);$$

from the right-angle triangle IXO we have

$$\sin \theta_a = p, \qquad (OI = 1);$$

so, as a function of p we have

$$\theta_r (p) = 2(2 \sin^{-1} (p/r) - \sin^{-1} p). \qquad \text{. . . (3)}$$

For $r = 1.33$, a graph of θ_r versus p is given in Fig. 2.

The effect of the droplet in producing the primary rainbow is to turn the light ray's direction of travel through an angle of $(180 - \theta_r(p))$ known as the *rainbow angle*. Rays contributing to the primary rainbow reach an observer from those directions which make an angle with the direction of the incident light less than θ_{max}, the maximum value of $\theta_r (p)$, $0 \leqslant p \leqslant 1$.

Since $d\theta_r/dp = 2/\sqrt{(r^2 - p^2)} - 1/\sqrt{(1 - p^2)} = 0$,

when $p = p_m = \sqrt{(4 - r^2)} / \sqrt{3} \cong 0.86$, for $r = 1.33$ and $\theta_r(0.86) \cong 42.5$,

while at the two extremes $\theta(0) = 0$, and $\theta(1) \cong 14.9$,

we have $\theta_{max} \cong 42.5$.

The primary rainbow angle therefore lies between $180°$ and $137.5°$.

To determine the intensity of this backward directed light at each angle between 0 and θ_{max} (and hence the precise location of the primary rainbow) is a more difficult problem. Variations with p in the ratio of reflected to transmitted light at each of the interfaces on the ray's path, as well as the "density" of rays for a given angle of emergence, must be taken into account. A complete answer is only possible using wave-particle theories of light. See Nussenzveig's article for a qualitative account.

Neglecting the first effect, since each value of p corresponds to an incident light ray, the proportion of rays emerging at angles θ_r between θ_0 and $\theta_0 + \Delta\theta$ equals the "length" of the set

$$\{p : \theta_0 \leqslant \theta_r(p) \leqslant \theta_0 + \Delta\theta\}$$

where $\theta_r(p)$ is given by (3).

Thus, if $I(\theta_0)$ denotes the intensity of light emergent with angle θ_0, we have $I(\theta_0) \Delta\theta \cong k \times$ "length" $\{p : \theta_0 \leqslant \theta_r(p) \leqslant \theta_0 + \Delta\theta\}$, k being a proportionality constant, which for suitable units of intensity we can take equal to 1. This approximate identity becomes more nearly exact the smaller the magnitude of $\Delta\theta$.

Hence,

$$I(\theta_0) \Delta\theta \cong \begin{cases} p_1(\theta_0 + \Delta\theta) - p_1(\theta_0) & \text{for } 0 \leqslant \theta_0 < \theta_r(1) \\ p_1(\theta_0 + \Delta\theta) - p_1(\theta_0) + p_2(\theta_0) - p_2(\theta_0 + \Delta\theta_0) \end{cases}$$

for $\theta_r(1) \leqslant \theta_0 < \theta_{max}$,
where $p_1(\theta)$ is the inverse function of $\theta_r(p)$ on $0 \leqslant p \leqslant p_m (\cong 0.86)$, and $p_2(\theta)$ is the inverse function of $\theta_r(p)$ on $p_m \leqslant p \leqslant 1$.

Dividing the above expression by $\Delta\theta$, and then taking the limit as $\Delta\theta \to 0$, we have

$$I(\theta_0) = \begin{cases} dp_1/d\theta_0 & \text{for } 0 \leqslant \theta_0 < \theta_r(1) \\ dp_1/d\theta_0 - dp_2/d\theta_0 & \text{for } \theta_r(1) \leqslant \theta_0 < \theta_{max}. \end{cases}$$

While it is difficult to obtain an explicit expression for $I(\theta_0)$, we note that as $\theta_0 \to \theta_{max}$, $dp_1/d\theta_0 \to \infty$ and $dp_2/d\theta_0 \to -\infty$ so $I(\theta_0) \to \infty$. See Fig. 2.
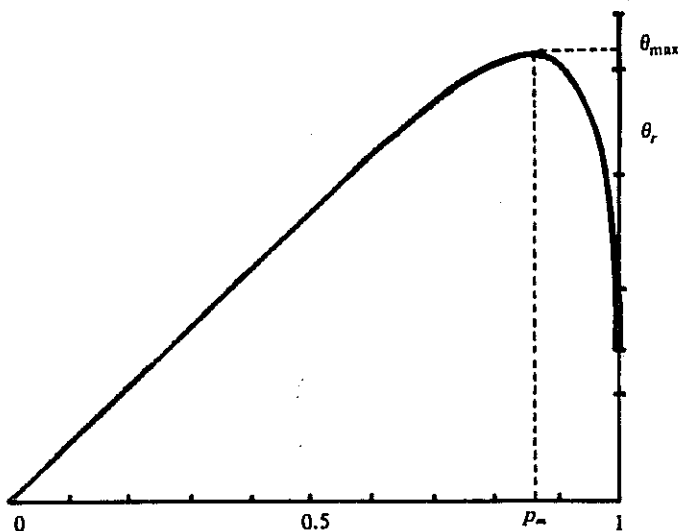

Fig. 2

The intensity is therefore "infinitely" greater at θ_{max} than at other angles and so the primary bow will be observed in a small angular region about θ_{max}.
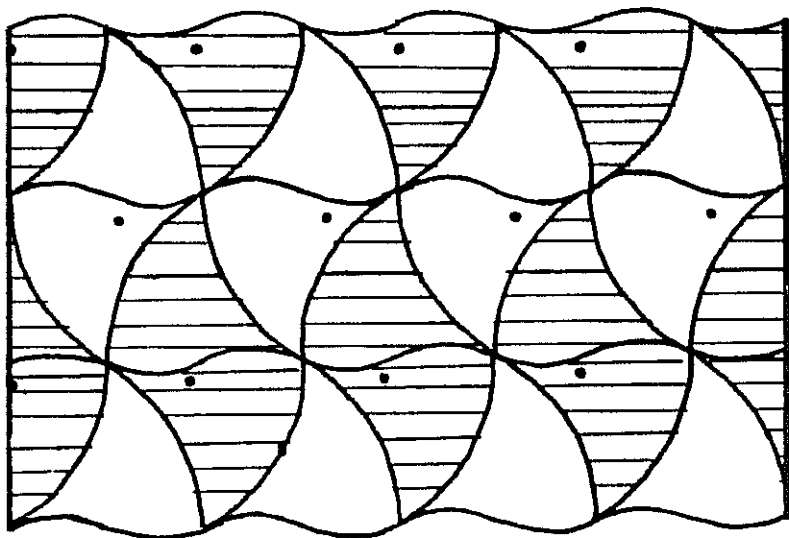
The rainbow angle for the primary bow is therefore $180 - \theta_{max} \cong 137.5$. The result is in good agreement with observation.

The secondary rainbow could be treated by similar, though more tedious, arguments; in which case it is the minimum, not the maximum, value of θ_r which proves significant.

REFERENCES

[1] H. Moyses Nussenzveig, "The Theory of the Rainbow", *Scientific American*, April 1977, pp. 116-127.
[2] Jearl Walker, "The Amateur Scientist — How to create and observe a dozen rainbows in a single drop of water", *Scientific American*, July 1977, pp. 138-144.

ADDENDUM. While repeating the experiments described in Walker's article [2], I tried replacing the water drop by a water filled, thin walled, glass "bubble" of about 2 cm. diameter blown at the end of a piece of glass tubing. This gave good results for the primary and secondary rainbows. In fact the bows could easily be projected onto a screen with a pin-hole in it through which the light beam is projected.

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