# BLAND'S ANTICYCLING RULE

In 1977 R. G. Bland showed that a remarkably simple modification to the simplex algorithm will prevent cycling. This result is perhaps not as widely known as it should be. Bland's original proof, and the simplified proofs found in more modern texts, while not difficult are often somewhat intricate. By working completely with the *tableau* we are able to give a particularly short and simple proof.

We will represent the linear programming problem in standard form:

$$\textit{minimize:} \quad f := \underline{c}\cdot(x_1,\cdots,x_n) + d$$

$$\textit{subject to:} \quad A\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix} + \begin{bmatrix}x_{n+1}\\ \vdots\\ x_{n+m}\end{bmatrix} = \underline{b}^T$$

$$x_1,\cdots,x_{n+m} \geq 0,$$

by the tableau

| $x_1$ *(non-basic variables)* | $x_j$ | $x_n$ | $-1$ | | |
|---|---|---|---|---|---|
| | A | | $\underline{b}^T$ | $-x_{n+1}$ | *basic variables (initially the slack variables)* |
| | | | | $-x_{n+i}$ | |
| | | | | $-x_{n+m}$ | |
| | $\underline{c}$ | | $-d$ | $f$ | |

We will further suppose that the problem has been brought to *feasible form* with $\underline{b} \geq 0$.

The general tableau resulting from a sequence of pivots will then be of the form

| $x_{\sigma(1)}$ | $x_{\sigma(j)}$ | $x_{\sigma(n)}$ | $-1$ | |
|---|---|---|---|---|
| | A' | | $\underline{b}'^T$ | $-x_{\sigma(n+1)}$ |
| | | | | $-x_{\sigma(n+i)}$ |
| | | | | $-x_{\sigma(n+m)}$ |
| | $\underline{c}'$ | | $-d'$ | $f$ |

Where $\sigma$ is some permutation of $(1,2,\cdots,n+m)$.

Note that, from the nature of the pivot operation, $(x_1,\cdots,x_{n+m})$ is a solution of the equality constraints of the standard form problem if and only if it satisfies the equivalent system of equations

$$A'\begin{bmatrix}x_{\sigma(1)}\\ \vdots\\ x_{\sigma(n)}\end{bmatrix} + \begin{bmatrix}x_{\sigma(n+1)}\\ \vdots\\ x_{\sigma(n+m)}\end{bmatrix} = \underline{b}'^{T},$$

and that at such a solution the value of the objective function is given by

$$f = c_1x_1 +\cdots+ c_nx_n + d$$
$$= c_1'x_{\sigma(1)} +\cdots+ c_n'x_{\sigma(n)} + d'.$$

Bland's modification to the simplex algorithm is to use the following criteria for the selection of a pivot point (I,J).

1) Choose J so that $\sigma(J) = \mathit{minimum}\{\sigma(j): c_j' < 0,\ j = 1,\cdots,n\}$. That is, from among those columns for which $c_j$ is strictly negative choose the one for which the associated non-basic variable has lowest index.

2) Choose I so that $\sigma(I) = \mathit{minimum}\{\sigma(i): i \in \mathfrak{M}\}$, where $\mathfrak{M}$ is the set of k such that $b_k/a_{kJ}$ is a minimal element of $\{b_p/a_{pJ}: a_{pJ} > 0\}$. That is, choose I in the usual way, but in the case of a tie, resolve it by selecting the row for which the associated basic variable has smallest index.

It is easily verified that a pivot selected in this way leads to a basic feasible solution at which the value of the objective function is not increased.

We now verify that when the simplex algorithm is modified by selecting pivots in the above way cycling is precluded.

To derive a contradiction, suppose that we have a tableau in feasible form for which the algorithm cycles. Delete from the tableau those rows and columns which do not contain pivots occuring in the cycle. In this way we obtain the tableau for a new problem which still cycles and is such that during the course of the cycle each of the initial non-basic variables becomes a basic variable and then again a non-basic variable and vice versa (as on completion of the cycle the tableau is returned to its original form).

For the algorithm to have cycled the objective function must have remained constant at d throughout the cycle. Since pivoting about the I,J'th position (where $c_J < 0$) gives $d' = d + b_Ic_J/a_{IJ}$ we must have $b_I = 0$, and consequently

$b'_i = b_i - b_I a_{iJ}/a_{IJ} = b_i$. Since each row contains a pivot point of the cycle we conclude that $\underline{b} = \underline{0}$.

Let N be the largest value subscript of any variable associated with the problem. The tableau from which $x_N$ is changed from a non-basic variable to a basic variable must have the form

| $x_{\sigma(1)}$ … $x_{\sigma(J-1)}$ | $x_{\sigma(J)}$ | $x_{\sigma(J+1)}$ … | $-1$ | |
|---|---|---|---|---|
| | | | $\underline{0}^T$ | $-x_{\sigma(n+i)}$ |
| *positive* | $c'_J < 0$ | *positive* | $-d$ | $f$ |

where $\sigma(J) = N$. (If any other component of $\underline{c}'$ besides the J'th were strictly negative we would have selected a column with $\sigma(j)$ less than N in which to pivot.)

The tableau from which $x_N$ is changed from a basic variable back to its original position as a non-basic variable must have the form

| $x_{\sigma'(1)}$ | $x_{\sigma'(J)}$ | $x_{\sigma'(n)}$ | $-1$ | |
|---|---|---|---|---|
| | *negative* | | | $-x_{\sigma'(n+1)}$ |
| | $a''_{IJ} > 0$ | | $\underline{0}^T$ | $-x_{\sigma'(n+I)}$ |
| | *negative* | | | $-x_{\sigma'(n+m)}$ |
| *positive* | $c''_J < 0$ | *positive* | $-d$ | $f$ |

where $\sigma'(n+I) = N$. (Since all the $b_i$'s are zero, all rows with $a''_{iJ} > 0$ are tied and so for the row associated with $x_N$ to have been chosen all other rows, being associated with variables of smaller index, must have $a''_{iJ} \leq 0$.)

From this last tableau it is easily verified that a solution (which is neither basic or feasible) to the equality constraints is:

$$x_{\sigma'(j)} = 1$$
$$x_{\sigma'(j)} = 0 \quad (j = 1,2,\cdots,n;\ j \neq J)$$
$$x_{\sigma'(n+i)} = -a''_{iJ}$$

all of which are positive except $x_N = x_{\sigma'(n+I)} = -a''_{IJ} < 0$.

For this solution the last tableau gives

$$f = d + c''_J < d \quad (\text{as } c''_J < 0).$$

On the other hand, using the previous tableau we have

$$f = d - c'_J x_N + \sum_{\substack{j=1 \\ j \neq J}}^{n} c'_j x_{\sigma(j)}$$

$$\geq d,$$

as $c'_J < 0$, $x_N < 0$, and for $j \neq J$ $c'_j \geq 0$ and $x_{\sigma(j)}$ $(\neq x_N) \geq 0$, giving the desired contradiction.

We conclude by observing that natural ways of coding the simplex algorithm often impliment by default a partial form of Bland's modification. This may contribute to the observed paucity of *real world* problems for which the simplex algorithm has been observed to cycle.

REFERENCES

Papadimitriou and Steightz *COMBINATORIAL OPTIMIZATION, Algorithms and Complexity*, Prentice-Hall