Drinfeld modular forms of arbitrary rank

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To Elsabe, Lukas, Mira, Oliver and Thomas

Abstract

This monograph provides a foundation for the theory of Drinfeld modular forms of arbitrary rank r and is subdivided into three parts. In the first part, we develop the analytic theory. Most of the work goes into defining and studying the u-expansion of a weak Drinfeld modular form, whose coefficients are weak Drinfeld modular forms of rank $r-1$. Based on that we give a precise definition of when a weak Drinfeld modular form is holomorphic at infinity and thus a Drinfeld modular form in the proper sense.

In the second part, we compare the analytic theory with the algebraic one that was begun in a paper of the third author. For any arithmetic congruence subgroup and any integral weight we establish an isomorphism between the space of analytic modular forms with the space of algebraic modular forms defined in terms of the Satake compactification. From this we deduce the important result that this space is finite dimensional.

In the third part, we construct and study some examples of Drinfeld modular forms. In particular, we define Eisenstein series, as well as the action of Hecke operators upon them, coefficient forms and discriminant forms. In the special case $A = \mathbb{F}_q[t]$ we show that all modular forms for $\mathrm{GL}_r(\Gamma(t))$ are generated by certain weight one Eisenstein series, and all modular forms for $\mathrm{GL}_r(A)$ and $\mathrm{SL}_r(A)$ are generated by certain coefficient forms and discriminant forms. We also compute the dimensions of the spaces of such modular forms.

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Contents

Introduction

In [Dr74], Drinfeld introduced elliptic modules, now called *Drinfeld modules*, in order to prove a special case of the Langlands conjectures for GL_2 over function fields. These objects share many properties with elliptic curves, though their rank can be an arbitrary integer $r \geqslant 1$. In particular, Drinfeld constructed a moduli space of Drinfeld modules of rank r with a suitable level structure, both as an algebraic variety and with an analytic uniformisation as a quotient of an $r-1$ dimensional symmetric space Ω^r. This Ω^r is a rigid analytic space over a field $\mathbb{C}_\infty$ of positive characteric and plays the role of the complex upper half plane. In the case $r = 2$ Drinfeld [Dr77] used automorphic forms on Ω^r with values in $\mathbb{Q}_\ell$ to prove a case of the Langlands conjectures for the associated automorphic representations on GL_2.

But there is also a natural definition of *Drinfeld modular forms* on Ω^r with values in the field $\mathbb{C}_\infty$ of positive characteristic. Goss [Go80b] was the first to explicitly refer to these, defining them both algebraically, in the way Katz did in [Ka73], and analytically as (rigid analytic) holomorphic functions on Ω^r. In the case $r = 2$, where these are functions of one variable, it was relatively straightforward to write down the necessary condition of *holomorphy at infinity*. This led to the development of a theory of Drinfeld modular forms of rank 2, for instance by Gekeler [Ge86]; see [Ge99b] for a survey.

For $r \geqslant 3$ the situation concerning holomorphy at infinity is more subtle. In the related case of Siegel modular forms of genus $\geqslant 2$ the problem disappears, because the necessary condition at infinity holds automatically by the Köcher principle. One explanation for this is the fact that in the Satake compactification of the Siegel moduli space of abelian varieties the boundary has codimension $\geqslant 2$. By contrast, the moduli space of Drinfeld modules is always affine, so in any compactification as an algebraic variety the boundary has codimension 1; hence a condition at infinity is always required.

That condition is important for several reasons. On the one hand many relevant modular forms that one can construct naturally, such as Eisenstein series, satisfy it automatically. On the other hand a condition at infinity is necessary for one of the main structural results, the fact that the space of modular forms of given level and weight is finite dimensional.

The condition at infinity can be expressed in two quite different ways. The analytic way says that the u-expansion (which is a kind of Fourier expansion) of a modular form consists only of terms with non-negative index. For the algebraic way one identifies the analytic modular forms with sections of an invertible sheaf on a moduli space. Then one requires a compactification of this moduli space as a projective algebraic variety over $\mathbb{C}_\infty$ together with an extension of the invertible sheaf. The crucial step is to prove that a modular form satisfies the analytic condition at infinity if and only if the corresponding section on the moduli space extends to a section on that compactification. The finite dimensionality is then a direct consequence of the fact that the space of sections of a coherent sheaf on a projective algebraic variety is always finite dimensional. Using the *Satake compactification* of a Drinfeld moduli space, the third author [Pi13] has already established much of the necessary algebro-geometric theory for this.

The present monograph aims to provide the rest of the theory and thereby a foundation for the theory of Drinfeld modular forms of arbitrary rank. It is subdivided into three parts, corresponding to three preprints released in 2018. Part I develops the basic analytic theory, including u-expansions and holomorphy at infinity. Part II identifies the analytic modular forms discussed here with the algebraic modular forms defined in [Pi13] and deduces qualitative consequences such as the finite dimensionality of the space of modular forms of given level and weight. Part III illustrates the general theory by constructing and studying some important families of modular forms.

For a discussion on the history of Drinfeld modular forms of higher rank, see [BB17, §7].

We briefly mention here some recent developments. In a series of papers [Ge19a, Ge17, Ge19b, Ge18, Ge19c, Ge21] Gekeler constructs the building map from Ω^r to the Bruhat-Tits building of GL_r and uses this to study the growth and vanishing behaviours of important families of modular forms for $\mathrm{GL}_r(\mathbb{F}_q[t])$. This is a valuable complement to the theory presented in this monograph.

In [Su18], Sugiyama studies integrality of Drinfeld modular forms for $\mathrm{GL}_r(\mathbb{F}_q[t])$.

In a recent preprint [HY20], Hartl and Yu develop an arithmetic Satake compactification of Drinfeld moduli schemes and study arithmetic Drinfeld modular forms of arbitrary rank.

An approach to higher rank Drinfeld modular forms via lattices is treated in the Ph.D. thesis of Baker [Ba20].

In another direction [CG21], Chen and Gezmiş define and study the weight 2 "false Eisenstein series", a first example of a Drinfeld quasi-modular form in arbitrary rank.

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Part I
Analytic Theory

Outline of Part I

In Section 1 we introduce our notation and define the Drinfeld period domain Ω^r with its action by $\mathrm{GL}_r(F)$ for a global function field F. Weak modular forms for an arithmetic subgroup $\Gamma < \mathrm{GL}_r(F)$ are defined as holomorphic functions from Ω^r to $\mathbb{C}_\infty$ satisfying the functional equation (1.5) linking $f(\gamma(\omega))$ to $f(\omega)$ for every $\gamma \in \Gamma$.

Further preparations are made in the next two sections. In Section 2 we collect basic properties of exponential functions associated to strongly discrete subgroups of $\mathbb{C}_\infty$, and we outline the rigid analytic structure of Ω^r in Section 3.

Based on our choice of coordinates on Ω^r, we identify a *standard boundary component*, whose translates by $\mathrm{GL}_r(F)$ form all boundary components of codimension 1. Thus a weak modular form is holomorphic at all boundary components if and only if all its translates by $\mathrm{GL}_r(F)$ are holomorphic at the standard boundary component. The holomorphy at the standard boundary component is tested using the expansion with respect to a certain parameter u.

This parameter is defined in Section 4: We decompose elements $\omega \in \Omega^r$ as $\omega = \binom{\omega_1}{\omega'}$, where $\omega_1 \in \mathbb{C}_\infty$ is the first coordinate of ω and $\omega' \in \Omega^{r-1}$ consists of the remaining coordinates. Next, we assign to Γ a subgroup $\Lambda' \subset F^{r-1}$ isomorphic to the subgroup $\Gamma_U < \Gamma$ of translations which fix ω'. Then $\Lambda'\omega' \subset \mathbb{C}_\infty$ is a strongly discrete subgroup and we can form the associated exponential function $e_{\Lambda'\omega'}$. Now $e_{\Lambda'\omega'}(\omega_1)$ is invariant under the translations Γ_U and we define our parameter as its reciprocal $u \coloneqq u_{\omega'}(\omega_1) = e_{\Lambda'\omega'}(\omega_1)^{-1}$ in (4.14).

In Definition 4.12 we define neighbourhoods of infinity in Ω^r, then Theorem 4.16 states that the map $\binom{\omega_1}{\omega'} \mapsto \binom{u}{\omega'}$ induces rigid analytic isomorphisms from quotients of neighbourhoods of infinity by Γ_U to so-called pierced tubular neighbourhoods in $\mathbb{C}_\infty^\times \times \Omega^{r-1}$.

This allows us to show in Section 5 that every weak modular form f admits a u-expansion

$$f(\omega) = \sum_{n\in\mathbb{Z}} f_n(\omega') u_{\omega'}(\omega_1)^n$$

converging on a neighbourhood of infinity (Proposition 5.4), whose coefficients f_n are themselves weak modular forms on Ω^{r-1} (Theorem 5.9). These are the main results of Part I.

Finally, we define modular forms in Section 6 as weak modular forms all of whose translates by elements of $\mathrm{GL}_r(F)$ admit u-expansions with terms of non-negative index. It follows from Propositions 6.2 and 6.3 that this condition only needs to be tested for finitely many elements of $\mathrm{GL}_r(F)$. It will be shown in Part II of this monograph that this definition agrees with the algebraic definition of modular forms in [Pi13].

1 Weak modular forms

Throughout this monograph we fix a global function field F of characteristic $p > 0$, with exact field of constants $\mathbb{F}_q$ of cardinality q. We fix a place ∞ of F and let A denote the ring of elements of F which are regular away from ∞. This is a Dedekind domain with finite class group and group of units $A^\times = \mathbb{F}_q^\times$. Let $\pi \in F$ be a uniformising parameter at ∞, so that $|\pi| = q^{-\deg\infty}$. Let $F_\infty \cong \mathbb{F}_{q^{\deg\infty}}((\pi))$ denote the completion of F at ∞, and $\mathbb{C}_\infty$ the completion of an algebraic closure of F_∞.

We fix an unspecified non-zero constant $\xi \in \mathbb{C}_\infty^\times$, whose value can be set for normalisation purposes. For example, if $F = \mathbb{F}_q(t)$ and $A = \mathbb{F}_q[t]$, there are certain advantages in letting ξ be a period of the Carlitz module. For more general function fields F, a natural choice of ξ is a period of a certain sign-normalised rank-one Drinfeld module, see [Ge86, Chapter IV (2.14) and (5.1)]. However, we will not explicitly need the normalisation in this monograph, so the reader loses nothing by assuming that $\xi = 1$.

The *Drinfeld period domain of rank $r \geqslant 1$ over F_∞* is usually defined as the set of points $(\omega_1 : \ldots : \omega_r) \in \mathbb{P}^{r-1}(\mathbb{C}_\infty)$ for which $\omega_1, \ldots, \omega_r$ are F_∞-linearly independent. Any such point possesses a unique representative with $\omega_r = \xi$. We shall only work with these representatives, so we identify Ω^r with the following subset of $\mathbb{C}_\infty^r$:

$$\Omega^r := \big\{(\omega_1, \ldots, \omega_r)^T \in \mathbb{C}_\infty^r \,\big|\, \omega_1, \ldots, \omega_r \ F_\infty\text{-linearly independent and } \omega_r = \xi\big\}. \tag{1.1}$$

We write the elements of Ω^r as $r \times 1$ matrices, i.e. column vectors.

For any point $\omega \in \Omega^r$ and any matrix $\gamma \in \mathrm{GL}_r(F_\infty)$, the matrix product $\gamma\omega$ is again a column vector with F_∞-linearly independent entries. In particular its last entry is non-zero. Defining

$$j(\gamma, \omega) := \xi^{-1} \cdot (\text{last entry of } \gamma\omega) \in \mathbb{C}_\infty^\times, \tag{1.2}$$

we thus find that

$$\gamma(\omega) := j(\gamma, \omega)^{-1}\gamma\omega \tag{1.3}$$

again lies in Ω^r. This defines a left action of $\mathrm{GL}_r(F_\infty)$ on Ω^r. Also, for any γ, $\delta \in \mathrm{GL}_r(F_\infty)$ a direct calculation shows that

$$j(\gamma\delta, \omega) = j\big(\gamma, \delta(\omega)\big)j(\delta, \omega). \tag{1.4}$$

For any function $f : \Omega^r \to \mathbb{C}_\infty$ and any integers k and m we define the function $f|_{k,m}\gamma : \Omega^r \to \mathbb{C}_\infty$ by

$$(f|_{k,m}\gamma)(\omega) := \det(\gamma)^m j(\gamma, \omega)^{-k} f\big(\gamma(\omega)\big). \tag{1.5}$$

By direct calculation we deduce from (1.4) that

$$(f|_{k,m}\gamma\delta)(\omega) = ((f|_{k,m}\gamma)|_{k,m}\delta)(\omega). \tag{1.6}$$

Thus (1.5) defines a right action of $\mathrm{GL}_r(F_\infty)$ on the space of all functions $f:\Omega^r\to\mathbb{C}_\infty$.

For later use note also that, if $\gamma = a\cdot\mathrm{Id}_r$ for the identity matrix $\mathrm{Id}_r\in\mathrm{GL}_r(F)$, then $j(\gamma,\omega)=a$ and $\gamma(\omega)=\omega$ and $\det(\gamma)=a^r$; and hence

$$f|_{k,m}(a\cdot\mathrm{Id}_r) = a^{rm-k}f. \tag{1.7}$$

Remark 1.8 There are different conventions about whether Ω^r consists of row or column vectors and about how $\mathrm{GL}_r(F_\infty)$ acts on it. For instance, the first and third authors [Ba14], [Pi13] like Drinfeld [Dr74] use row vectors and the action $(\gamma,\omega)\mapsto\omega\gamma^{-1}$, whereas Kapranov [Ka87] uses column vectors and the action by left multiplication $(\gamma,\omega)\mapsto\gamma\omega$. These conventions differ not only by transposition, but also by the outer automorphism $\gamma\mapsto(\gamma^T)^{-1}$ of GL_r. The present monograph uses column vectors and left multiplication in order to make things compatible with the existing literature on rank 2 Drinfeld modular forms.

The set Ω^r can be endowed with the structure of a rigid analytic space. Experts may be content with the fact that Ω^r is an admissible open subset of $\mathbb{P}^{r-1}(\mathbb{C}_\infty)$ and inherits its rigid analytic structure, while others may consult Section 3 for more details. A holomorphic function on Ω^r is a global section of the structure sheaf of Ω^r, but a more useful characterisation is that a function $f:\Omega^r\to\mathbb{C}_\infty$ is holomorphic if and only if it is a uniform limit of rational functions on $\mathbb{P}^{r-1}(\mathbb{C}_\infty)$ whose poles all lie outside Ω^r.

Definition 1.9 *Consider integers k and m and an arithmetic subgroup $\Gamma<\mathrm{GL}_r(F)$. A* weak modular form of weight k and type m for Γ *is a holomorphic function $f:\Omega^r\to\mathbb{C}_\infty$ which for all $\gamma\in\Gamma$ satisfies*

$$f|_{k,m}\gamma = f.$$

The space of these functions is denoted by $\mathcal{W}_{k,m}(\Gamma)$.

Since Γ is an arithmetic subgroup of $\mathrm{GL}_r(F)$, its determinant $\det(\Gamma)$ is a finite subgroup of $F^\times$ and therefore contained in the multiplicative group of the field of constants $\mathbb{F}_q^\times$. Thus its order is a divisor of $q-1$, and the definition depends only on m modulo this divisor; in other words we have

$$\mathcal{W}_{k,m}(\Gamma) = \mathcal{W}_{k,m'}(\Gamma) \text{ whenever } m\equiv m' \text{ modulo } |\det(\Gamma)|. \tag{1.10}$$

On the other hand, for any $\alpha\in\mathbb{F}_q^\times$ we have $f|_{k,m}(\alpha\cdot\mathrm{Id}_r)=\alpha^{rm-k}f$ by (1.7). Thus

$$\mathcal{W}_{k,m}(\Gamma) = 0 \text{ unless } k\equiv rm \text{ modulo } \big|\Gamma\cap\{\text{scalars}\}\big|. \tag{1.11}$$

In the case $m=0$ we will suppress all mention of m and abbreviate $f|_k\gamma := f|_{k,m}\gamma$ and $\mathcal{W}_k(\Gamma) := \mathcal{W}_{k,m}(\Gamma)$. By (1.10) we may always do this when $\Gamma<\mathrm{SL}_r(F)$.

For later use we note the following direct consequence of (1.6):

Proposition 1.12 *For any $\delta\in\mathrm{GL}_r(F)$ we have $f\in\mathcal{W}_{k,m}(\Gamma)$ if and only if $f|_{k,m}\delta\in\mathcal{W}_{k,m}(\delta^{-1}\Gamma\delta)$.*

In general the space $\mathcal{W}_{k,m}(\Gamma)$ is infinite dimensional. A finite dimensional subspace of 'non-weak' modular forms will be characterised by conditions at infinity. The formulation of these conditions requires some preparation in the next two sections.

2 Exponential functions

A subgroup $H \subset \mathbb{C}_\infty$ is called *strongly discrete* if its intersection with every ball of finite radius is finite. For any such subgroup, any $z \in \mathbb{C}_\infty$, and any $\varepsilon > 0$, there are at most finitely many elements $h \in H \smallsetminus \{0\}$ with $\left|\frac{z}{h}\right| \geqslant \varepsilon$. In this case the product

$$e_H(z) := z \cdot \prod_{h \in H \smallsetminus \{0\}} \left(1 - \frac{z}{h}\right) \tag{2.1}$$

converges in $\mathbb{C}_\infty$, defining the *exponential function* $e_H : \mathbb{C}_\infty \to \mathbb{C}_\infty$ *associated to* H.

Proposition 2.2 *For any strongly discrete subgroup $H \subset \mathbb{C}_\infty$, the function $e_H : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is holomorphic, surjective, and has simple zeros at the points in H and no other zeros. It induces an isomorphism of additive groups and rigid analytic spaces*

$$\mathbb{C}_\infty / H \xrightarrow{\sim} \mathbb{C}_\infty.$$

The function e_H possesses an everywhere convergent power series expansion

$$e_H(z) = \sum_{i=0}^{\infty} e_{H,p^i} z^{p^i}$$

with $e_{H,p^i} \in \mathbb{C}_\infty$ and $e_{H,1} = 1$. If H is an $\mathbb{F}_q$-subspace, the expansion has the form

$$e_H(z) = \sum_{j=0}^{\infty} e_{H,q^j} z^{q^j}.$$

If H is finite, then $e_H(z)$ is a polynomial of degree $|H|$ in z.

Proof. When $H \subset \mathbb{C}_\infty$ is an A-lattice (see below), this is proved in [Go96, §4.2] and [Go80b, Prop. 1.27]. The case where $H \subset \mathbb{C}_\infty$ is merely a strongly discrete subgroup follows in exactly the same way. □

Proposition 2.3 *(a) For any two strongly discrete subgroups $H' \subset H \subset \mathbb{C}_\infty$, the subgroup $e_{H'}(H) \subset \mathbb{C}_\infty$ is strongly discrete and isomorphic to H/H', and we have*

$$e_H = e_{e_{H'}(H)} \circ e_{H'}.$$

(b) For any strongly discrete subgroup $H \subset \mathbb{C}_\infty$ and any $a \in \mathbb{C}_\infty^\times$, the subgroup $aH \subset \mathbb{C}_\infty$ is strongly discrete, and we have

$$e_{aH}(az) = a e_H(z).$$

Proof. For (a) see [Ge88b, (1.12)], and (b) follows immediately from the definition. □

An *A-lattice of rank r in $\mathbb{C}_\infty$* is a strongly discrete projective A-submodule $\Lambda \subset \mathbb{C}_\infty$ of rank r.

Proposition 2.4 *Let $H \subset \mathbb{C}_\infty$ be an A-lattice of rank r. Then for any $a \in A$ there exists a unique $\mathbb{F}_q$-linear polynomial $\varphi_a^H(z)$ of degree $|a|^r$ satisfying*

$$\varphi_a^H(e_H(z)) = e_H(az)$$

for all $z \in \mathbb{C}_\infty$. The map $\varphi^H : a \mapsto \varphi_a^H$ is a Drinfeld A-module of rank r.

Proof. [Go96, Thm. 4.3.1] □

3 The rigid analytic structure of Ω^r

Throughout the following we denote by $B(0,\rho) := \{z \in \mathbb{C}_\infty : |z| \leqslant \rho\}$ the closed disk of radius $\rho > 0$ centred at 0, and by $B(0,\rho)' = B(0,\rho) \smallsetminus \{0\}$ the associated punctured disk. We will also consider the annulus $D(0,\sigma,\rho) := \{z \in \mathbb{C}_\infty \mid \sigma \leqslant |z| \leqslant \rho\}$. Note that $B(0,\rho)$ and $D(0,\sigma,\rho)$ are affinoid whenever $\sigma, \rho \in |\mathbb{C}_\infty^\times|$.

We will describe the rigid analytic structure of Ω^r by covering it by suitable affinoid subspaces. Two such coverings already appear in [Dr74], and one of them is described in more detail in [SS91]. We follow the approach in [SS91], but adapt it to our convention that $\omega_r = \xi$.

We say that a linear form $F_\infty^r \to F_\infty$ is *unimodular* if its largest coefficient has absolute value 1. For any F_∞-rational hyperplane $H \subset \mathbb{P}^{r-1}(\mathbb{C}_\infty)$, we choose a unimodular linear form ℓ_H that defines it. Then $|\ell_H(\omega)|$ is well-defined and non-zero for any $\omega \in \Omega^r$. Using the standard norm $|\omega| := \max_{1 \leqslant i \leqslant r} |\omega_i|$ on $\mathbb{C}_\infty^r$, we set

$$h(\omega) \;:=\; \tfrac{1}{|\omega|} \cdot \inf\{|\ell_H(\omega)| : H \text{ an } F_\infty \text{ hyperplane}\}, \tag{3.1}$$

which measures the distance from $\omega \in \Omega^r$ to all boundary components combined. For any $n \in \mathbb{Z}^{>0}$ we also define

$$\Omega_n^r := \{\omega \in \Omega^r : h(\omega) \geqslant |\pi|^n\}. \tag{3.2}$$

Since $|\pi| < 1$, these subsets satisfy $\Omega_1^r \subset \Omega_2^r \subset \ldots$ and their union is Ω^r.

Lemma 3.3 *Every $\omega \in \Omega_n^r$ satisfies $|\xi| \leqslant |\omega| \leqslant |\xi||\pi|^{-n}$.*

Proof. The first inequality follows from $\omega_r = \xi$. Next, since $\omega \mapsto \omega_r$ is a unimodular F_∞-linear form, (3.1) implies that $|\omega| h(\omega) \leqslant |\xi|$, from which the second inequality follows. □

Proposition 3.4 *For each $n \in \mathbb{Z}^{>0}$, the set Ω_n^r is an affinoid subdomain of $\mathbb{P}^{r-1}(\mathbb{C}_\infty)$. Together they form an admissible covering of Ω^r, endowing it with the structure of an admissible open subset of $\mathbb{P}^{r-1}(\mathbb{C}_\infty)$.*

Proof. See version (C) of the proof of [SS91, Prop. 1]. □

Using the second (finer) covering in [Dr74, §6.2B], Drinfeld showed that, for any arithmetic subgroup $\Gamma < \mathrm{GL}_r(F)$, the quotient $\Gamma\backslash\Omega^r$ exists as a rigid analytic space.

For the following recall that a function $f : U \to \mathbb{C}_\infty$ on an admissible open subset $U \subset \Omega^r$ is called *holomorphic* if it is a section of the sheaf of functions on this space, or equivalently, if it is a uniform limit $f = \lim_{n\to\infty} f_n$ of rational functions $f_n : \mathbb{P}^{r-1}(\mathbb{C}_\infty) \dashrightarrow \mathbb{C}_\infty$ with no poles in U.

In the next section we shall need bounds on the values of certain exponential functions when we restrict to $\omega \in \Omega^r_n$. For this we require the following estimates:

Lemma 3.5 *For any $\gamma \in \mathrm{GL}_r(F_\infty)$ there exist positive constants c_1, c_2 and c_3 such that for every $\omega \in \Omega^r$ we have*

(a) $h(\omega) \leqslant c_1 |j(\gamma,\omega)||\omega|^{-1} \leqslant 1$;

(b) $|\gamma(\omega)| \leqslant c_2 h(\omega)^{-1}$; *and*

(c) $h(\gamma(\omega)) \geqslant c_3 h(\omega)$.

Proof. Let x be an entry of the last row of γ of maximal absolute value, and set $c_1 := |x^{-1}\xi| > 0$. Then by the definition (1.2) of $j(\gamma,\omega)$, the value $x^{-1}\xi j(\gamma,\omega)$ is a unimodular F_∞-linear combination of the ω_i's, so we obtain

$$h(\omega) \leqslant \frac{|x^{-1}\xi j(\gamma,\omega)|}{|\omega|} \leqslant 1.$$

This proves (a).

Next, let c_2' be the largest absolute value of an entry of γ. Then the matrix product satisfies $|\gamma\omega| \leqslant c_2'|\omega|$ and so $|\gamma(\omega)| = |j(\gamma,\omega)^{-1}||\gamma\omega| \leqslant |j(\gamma,\omega)^{-1}|c_2'|\omega| \leqslant c_1 c_2' h(\omega)^{-1}$, where the last inequality follows from (a). This proves (b) with $c_2 = c_1 c_2'$.

For (c) let c_3' denote the largest absolute value of an entry of γ^{-1}. Let ℓ be an arbitrary unimodular F_∞-linear form, which we write as a row vector, so that $\ell(\omega) = \ell\omega$. Choose $m_\ell \in \mathbb{C}_\infty^\times$ such that $\ell_\gamma := m_\ell \ell\gamma$ is a unimodular linear form. Then the entries of $m_\ell \ell = m_\ell \ell\gamma \cdot \gamma^{-1}$ have absolute value at most c_3'; hence $|m_\ell| \leqslant c_3'$. Since $\gamma(\omega) = j(\gamma,\omega)^{-1}\gamma\omega$, using the linearity of ℓ and the definition of $h(\omega)$ we find that

$$\frac{|\ell(\gamma(\omega))|}{|\gamma(\omega)|} = \frac{|\ell\gamma\omega|}{|\gamma\omega|} = \frac{|m_\ell|^{-1}|\ell_\gamma\omega|}{|\gamma\omega|} \geqslant \frac{c_3'^{-1}|\ell_\gamma\omega|}{c_2'|\omega|} \geqslant \frac{h(\omega)}{c_2'c_3'}.$$

Varying ℓ, the definition of $h(\gamma(\omega))$ now implies (c) with $c_3 := (c_2'c_3')^{-1}$. □

4 Neighbourhoods of infinity

From now on we assume that $r \geqslant 2$. Let U denote the algebraic subgroup of $\mathrm{GL}_{r,F}$ of matrices of the form

$$\left(\begin{array}{c|c} 1 & * \ \ldots \ * \\ \hline 0 & \\ \vdots & \mathrm{Id}_{r-1} \\ 0 & \end{array}\right) \tag{4.1}$$

where Id_{r-1} denotes the identity matrix of size $(r-1)\times(r-1)$. Fix an arithmetic subgroup $\Gamma < \mathrm{GL}_r(F)$ and set

$$\Gamma_U := \Gamma \cap U(F). \tag{4.2}$$

Then for all $\gamma \in \Gamma_U$ and $\omega \in \Omega^r$ we have $\det(\gamma) = j(\gamma,\omega) = 1$; hence every weak modular form for Γ is a Γ_U-invariant function on Ω^r.

Viewing elements of F^{r-1} as $1\times(r-1)$-matrices (row vectors), consider the isomorphism

$$\iota : F^{r-1} \xrightarrow{\sim} U(F), \quad v' \mapsto \begin{pmatrix} 1 & v' \\ 0 & \mathrm{Id}_{r-1} \end{pmatrix}. \tag{4.3}$$

Since Γ is commensurable with $\mathrm{GL}_r(A)$, the subgroup

$$\Lambda' := \iota^{-1}(\Gamma_U) \subset F^{r-1} \tag{4.4}$$

is commensurable with A^{r-1}. On the other hand, recall that Ω^r is the set of column vectors $\omega = (\omega_1,\ldots,\omega_r)^T \in \mathbb{C}_\infty^r$ with F_∞-linearly independent entries and $\omega_r = \xi$. For any such ω we have $\omega' := (\omega_2,\ldots,\omega_r)^T \in \Omega^{r-1}$, hence

$$\Omega^r \subset \mathbb{C}_\infty \times \Omega^{r-1}$$

inside $\mathbb{C}_\infty^r = \mathbb{C}_\infty \times \mathbb{C}_\infty^{r-1}$. Accordingly we write $\omega = \binom{\omega_1}{\omega'}$. The definition (3.1) then directly implies that $h(\omega) \leqslant h(\omega')$ and hence $\Omega_n^r \subset \mathbb{C}_\infty \times \Omega_n^{r-1}$.

For any element $\lambda' \in \Lambda'$ we can form the matrix product $\lambda'\omega' \in \mathbb{C}_\infty$. The definition (1.3) of the action on Ω^r then implies that

$$\iota(\lambda')\left(\binom{\omega_1}{\omega'}\right) = \binom{\omega_1+\lambda'\omega'}{\omega'}, \tag{4.5}$$

which extends the action to $\mathbb{C}_\infty \times \Omega^{r-1}$ by the same formula. For any $\omega' \in \Omega^{r-1}$ observe that $\Lambda'\omega' := \{\lambda'\omega' \mid \lambda' \in \Lambda'\}$ is a strongly discrete subgroup of $\mathbb{C}_\infty$, because Λ' is commensurable with A^{r-1} and the entries of ω' are F_∞-linearly independent. Thus the function

$$\mathbb{C}_\infty \times \Omega^{r-1} \to \mathbb{C}_\infty, \ \left[\binom{\omega_1}{\omega'}\right] \mapsto e_{\omega'\Lambda'}(\omega_1) \tag{4.6}$$

is well-defined and Γ_U-invariant.

As usual in a metric space, for any point $z \in \mathbb{C}_\infty$ and any subset $X \subset \mathbb{C}_\infty$ we write $d(z,X) := \inf\{|z-x| : x \in X\}$. Then we have:

Proposition 4.7 *(a) The function (4.6) is holomorphic.*

(b) For any $n \in \mathbb{Z}^{>0}$ and $c > 0$ there exists a constant $c_n > 0$, such that for any $\omega' \in \Omega_n^{r-1}$ and any $\omega_1 \in \mathbb{C}_\infty$ with $|\omega_1| < c$ we have $|e_{\Lambda'\omega'}(\omega_1)| < c_n$.

(c) For any $n \in \mathbb{Z}^{>0}$ and $R_n > 0$ there exists a constant $c_n > 0$, such that for any $\omega' \in \Omega_n^{r-1}$ and any $\omega_1 \in \mathbb{C}_\infty$ with $d(\omega_1, F_\infty^{r-1}\omega') < R_n$ we have $|e_{\Lambda'\omega'}(\omega_1)| < c_n$.

(d) For any $\omega' \in \Omega^{r-1}$ and $\omega_1 \in \mathbb{C}_\infty$ we have $|e_{\Lambda'\omega'}(\omega_1)| \geqslant d(\omega_1, F_\infty^{r-1}\omega')$.

Proof. The function is defined by the product $e_{\omega'\Lambda'}(\omega_1) = \omega_1 \cdot \prod_{0\neq\lambda'\in\Lambda'}(1 - \frac{\omega_1}{\lambda'\omega'})$, whose factors we examine in turn. First, as $\Lambda' \subset F^{r-1}$ is commensurable with A^{r-1}, there exists a constant $a \in A \smallsetminus \{0\}$ with $\Lambda' \subset a^{-1}A^{r-1}$. Let $0 \neq \lambda' \in \Lambda'$. Recall that λ' determines an F_∞ linear map $\mathbb{C}_\infty^{r-1} \to \mathbb{C}_\infty$ by matrix multiplication $v \mapsto \lambda' v$, and denote by $\ell_{\lambda'}$ the associated unimodular F_∞-linear map. For any $\omega' \in \Omega_n^{r-1}$ it follows that

$$(4.8)\qquad |\lambda'\omega'| \;=\; |\lambda'| \cdot \big|\ell_{\lambda'}(\omega')\big| \overset{(3.1)}{\geqslant} |\lambda'| \cdot h(\omega') \cdot |\omega'| \overset{(3.2)}{\geqslant} |\lambda'| \cdot |\pi^n| \cdot |\omega'| \overset{3.3}{\geqslant} |\lambda'| \cdot |\pi^n| \cdot |\xi|.$$

As λ' runs through $\Lambda' \smallsetminus \{0\}$, the value $|\lambda'\omega'|$ thus goes to ∞ uniformly over Ω_n^{r-1}. Varying n this implies that the function is holomorphic, proving (a).

To prove (b) observe next that all factors $1 - \frac{\omega_1}{\lambda'\omega'}$ with $|\lambda'\omega'| \geqslant |\omega_1|$ have absolute value less than or equal to 1. Since now $|\omega_1| < c$, we deduce that

$$(4.9)\qquad |e_{\omega'\Lambda'}(\omega_1)| \;<\; c \cdot \prod_{0<|\lambda'\omega'|<c} \frac{c}{|\lambda'\omega'|}.$$

Since $\Lambda' \subset a^{-1}A^{r-1}$, for any $\lambda' \in \Lambda' \smallsetminus \{0\}$ we have $|\lambda'| \geqslant \frac{1}{|a|}$. From (4.8) we thus deduce that $|\lambda'\omega'| \geqslant \frac{|\pi^n\xi|}{|a|}$. In particular each factor in the product (4.9) satisfies $\frac{c}{|\lambda'\omega'|} \leqslant \frac{c|a|}{|\pi^n\xi|}$; hence it is bounded by a constant that is independent of ω'. On the other hand, if $|\lambda'\omega'| < c$, the inequality (4.8) implies that $|\lambda'| < \frac{c}{|\pi^n\xi|}$. Thus each coefficient of $a\lambda' \in A^{r-1}$ has absolute value $< \frac{c|a|}{|\pi^n\xi|}$, the number of possibilities for which is bounded independently of ω'. The number of factors in (4.9) is thus also bounded independently of ω', and so is therefore the total value of the product, proving (b).

To prove (c) write $\omega_1 = x\omega' + y$, where $x \in F_\infty^{r-1}$ and $y \in \mathbb{C}_\infty$ with $|y| < R_n$. Since $\Lambda' \subset F^{r-1}$ is commensurable with A^{r-1}, the factor group F_∞^{r-1}/Λ' is compact. Thus there exists a constant $\alpha > 0$ depending only on A and Λ', such that every $x \in F_\infty^{r-1}$ can be written in the form $x = \lambda' + x_0$ for some $\lambda' \in \Lambda'$ and $x_0 \in \mathbb{C}_\infty$ with $|x_0| < \alpha$. Together we then have $\omega_1 = \lambda'\omega' + (x_0\omega' + y)$ with $|x_0\omega'| < \alpha|\omega'| \leqslant \alpha|\xi\pi^{-n}|$ by Lemma 3.3 and hence $|x_0\omega' + y| < \max\{\alpha|\xi\pi^{-n}|, R_n\}$. By part (b) this implies that $|e_{\Lambda'\omega'}(\omega_1)| = |e_{\Lambda'\omega'}(x_0\omega' + y)| < c_n$ for some constant $c_n > 0$ that is independent of ω_1 and ω', proving (c).

To prove (d) write $\omega_1 = \lambda'_0\omega' + y$ with $\lambda'_0 \in \Lambda'$ and $y \in \mathbb{C}_\infty$ such that $|y|$ is minimal. Then for all $\lambda' \in \Lambda'$ we have $|y - \lambda'\omega'| \geqslant |y|$. If $|y| \geqslant |\lambda'\omega'|$, this implies that $|y - \lambda'\omega'| \geqslant |\lambda'\omega'|$ and hence $\big|1 - \frac{y}{\lambda'\omega'}\big| \geqslant 1$. If $|y| < |\lambda'\omega'|$, we directly deduce that $\big|1 - \frac{y}{\lambda'\omega'}\big| = 1$. Writing $e_{\omega'\Lambda'}(\omega_1) = e_{\omega'\Lambda'}(y) = y\prod_{0\neq\lambda'\in\Lambda'}(1 - \frac{y}{\lambda'\omega'})$, we conclude that all factors in the product satisfy $\big|1 - \frac{y}{\lambda'\omega'}\big| \geqslant 1$. Thus it follows that $|e_{\omega'\Lambda'}(\omega_1)| \geqslant |y| \geqslant d(\omega_1, F_\infty^{r-1}\omega')$, proving (d). □

Proposition 4.10 *The quotient* $\Gamma_U \backslash (\mathbb{C}_\infty \times \Omega^{r-1})$ *exists as a rigid analytic space. Moreover we have an isomorphism of rigid analytic spaces*

$$\mathcal{E}: \ \Gamma_U \backslash (\mathbb{C}_\infty \times \Omega^{r-1}) \longrightarrow \mathbb{C}_\infty \times \Omega^{r-1}, \quad \left[\begin{pmatrix} \omega_1 \\ \omega' \end{pmatrix}\right] \mapsto \begin{pmatrix} e_{\Lambda'\omega'}(\omega_1) \\ \omega' \end{pmatrix}.$$

Proof. The existence of $\Gamma_U \backslash (\mathbb{C}_\infty \times \Omega^{r-1})$ as a rigid analytic space is shown in Simon Häberli's thesis, [Hä21].

By Proposition 2.2 we obtain a well-defined bijective and holomorphic map $\mathcal{E}$. As the derivative of $e_{\Lambda'\omega'}(X)$ with respect to X is identically 1, the morphism is also étale. By Proposition 4.11 below it is therefore an isomorphism. □

Proposition 4.11 *Let* $f : X \to Y$ *be a morphism of rigid analytic spaces defined over an algebraically closed field* K *which is étale and bijective. Then* f *is an isomorphism.*

Proof. (The proof is based on the analogous argument for schemes at [Stacks, Tag 02LC].) First we show that f is universally injective, i.e., for any morphism $g : Y' \to Y$ the morphism $f' : X' := X \times_Y Y' \to Y'$ is injective. So consider any points $x', x'' \in X'$ mapping to the same point $y' \in Y'$. Then they also map to the same point $y \in Y$, and by the bijectivity of f they therefore also map to the same point $x \in X$. Thus x' and x'' lie in the fiber product $x \times_y y'$ which, since all these points have the same residue field K, is $\mathrm{Sp}(K \otimes_K K) \cong \mathrm{Sp}(K)$ and therefore consists of a single point. This proves that $x' = x''$, as desired.

In particular, taking $Y' = X$, the projection $f_X : X \times_Y X \to X$ is injective, and hence the diagonal morphism $\Delta : X \to X \times_Y X$ is surjective (since $f_X \circ \Delta$ is the identity on X). On the other hand Δ is an open immersion, because f is étale. It follows that Δ and hence f_X are isomorphisms. On the other hand f is flat by étaleness and even faithfully flat by surjectivity. Since being an isomorphism is local for the étale topology, and f_X is an isomorphism, it follows that f is an isomorphism, as desired. □

Now we look at the situation near the standard boundary component.

Definition 4.12 *For any* $n \in \mathbb{Z}^{>0}$ *and* $R_n > 0$ *consider the* Γ_U*-invariant subset*

$$I(n, R_n) \ := \ \left\{\omega = \begin{pmatrix} \omega_1 \\ \omega' \end{pmatrix} \in \Omega^r \ \middle| \ \omega' \in \Omega_n^{r-1}, \ d(\omega_1, F_\infty^{r-1}\omega') \geqslant R_n \right\}.$$

An arbitrary Γ_U*-invariant admissible open subset* $\mathcal{N} \subset \Omega^r$*, such that for each* $n > 0$ *there exists an* $R_n > 0$ *with* $I(n, R_n) \subset \mathcal{N}$*, will be called a* neighbourhood of infinity.

Note that every subset of the form $I(n, R_n)$ is contained in Ω^r by construction; hence neighbourhoods of infinity exist and Ω^r is itself one.

Definition 4.13 *Any subset of* $\mathbb{C}_\infty \times \Omega^{r-1}$ *of the form*

$$\mathcal{T} \ = \ \bigcup_{n \geqslant 1} B(0, r_n) \times \Omega_n^{r-1}$$

for numbers $r_n \in |\mathbb{C}_\infty^\times|$ *will be called a* tubular neighbourhood of $\{0\} \times \Omega^{r-1}$*, or just a* tubular neighbourhood *for the sake of brevity. The intersection of a tubular neighbourhood with* $\mathbb{C}_\infty^\times \times \Omega^{r-1}$ *will be called a* pierced tubular neighbourhood.

Any tubular neighbourhood is an admissible subset, because it is the union of affinoid sets of the form $B(0,\rho)\times\Omega_n^{r-1}$ for $\rho\in|\mathbb{C}_\infty^\times|$ and the intersection of any two such affinoid sets is again of this form. The same holds for pierced tubular neighbourhoods, but in this case we must use affinoids of the form $D(0,\sigma,\rho)\times\Omega_n^{r-1}$.

Next recall that $e_{\Lambda'\omega'}(\omega_1)\neq 0$ whenever $\omega_1\notin\Lambda'\omega'$. In particular this holds for any $\omega=\binom{\omega_1}{\omega'}\in\Omega^r$, and so

$$u_{\omega'}(\omega_1) \;:=\; \frac{1}{e_{\Lambda'\omega'}(\omega_1)} \;\in\; \mathbb{C}_\infty^\times \tag{4.14}$$

is well-defined for all $\omega=\binom{\omega_1}{\omega'}\in\Omega^r$.

Example 4.15 Suppose that $A=\mathbb{F}_q[t]$, $r=2$, $\Lambda=A^2$ and $\xi=\bar\pi$ is a period of the Carlitz module. Then for $\omega=\binom{\omega_1}{\xi}\in\Omega^2$ we have

$$u_{\omega'}(\omega_1)=\frac{1}{e_{\xi A}(\omega_1)}=\frac{1}{\bar\pi e_A(z)},$$

where $z=\omega_1/\xi\in\mathbb{C}_\infty\smallsetminus F_\infty$ is the usual parameter at infinity in the rank 2 literature (see, e.g., [Ge88a]).

Theorem 4.16 *(a) The morphism*

$$\vartheta:\ \Gamma_U\backslash\Omega^r\longrightarrow\mathbb{C}_\infty^\times\times\Omega^{r-1},\quad \left[\binom{\omega_1}{\omega'}\right]\longmapsto\binom{u_{\omega'}(\omega_1)}{\omega'}$$

induces an isomorphism of rigid analytic spaces from $\Gamma_U\backslash\Omega^r$ to an admissible open subset of $\mathbb{C}_\infty^\times\times\Omega^{r-1}$.

(b) For any neighbourhood of infinity $\mathcal{N}\subset\Omega^r$, the image $\vartheta(\Gamma_U\backslash\mathcal{N})$ contains a pierced tubular neighbourhood.

(c) For any pierced tubular neighbourhood $\mathcal{T}'\subset\mathbb{C}_\infty^\times\times\Omega^{r-1}$ contained in the image of ϑ, there is a neighbourhood of infinity $\mathcal{N}\subset\Omega^r$ such that $\vartheta(\Gamma_U\backslash\mathcal{N})=\mathcal{T}'$, and ϑ induces an isomorphism

$$\Gamma_U\backslash\mathcal{N}\xrightarrow{\sim}\mathcal{T}'.$$

Proof. Part (a) is a direct consequence of Proposition 4.10. To prove (b) we must show that for any $n>0$ and $R_n>0$ there exists $r_n>0$ such that

$$B(0,r_n)'\times\Omega_n^{r-1}\;\subset\;\vartheta\big(\Gamma_U\backslash I(n,R_n)\big).$$

For this let c_n be the constant from Proposition 4.7 (c) and set $r_n:=c_n^{-1}$. Consider any point $\binom{z}{\omega'}\in B(0,r_n)'\times\Omega_n^{r-1}$. As the map $e_{\Lambda'\omega'}:\mathbb{C}_\infty\smallsetminus\Lambda'\omega'\to\mathbb{C}_\infty^\times$ is surjective by Proposition 2.2, and $u_{\omega'}=e_{\Lambda'\omega'}^{-1}$ by definition, there exists a point $\omega_1\in\mathbb{C}_\infty\smallsetminus\Lambda'\omega'$ with $u_{\omega'}(\omega_1)=z$. Since $z\in B(0,r_n)'$, we then have $|e_{\Lambda'\omega'}(\omega_1)|\geqslant c_n$. By Proposition 4.7 (c) we thus have

$d(\omega_1, F_\infty^{r-1}\omega') \geqslant R_n$, and so $\binom{\omega_1}{\omega'} \in I(n, R_n)$. Therefore $\binom{z}{\omega'} = \vartheta([\binom{\omega_1}{\omega'}]) \in \vartheta(\Gamma_U \backslash I(n, R_n))$, proving (b).

To prove (c), let $\mathcal{N} \subset \Omega^r$ denote the preimage of $\mathcal{T}'$, this is an admissible subset of Ω^r since $\mathcal{T}'$ is an admissible subset of $\mathbb{C}_\infty^\times \times \Omega^{r-1}$. It remains to show that $\mathcal{N}$ is a neighborhood of infinity. We must show that for any $n > 0$ and $r_n > 0$ there exists $R_n > 0$ such that

$$\vartheta\big(\Gamma_U \backslash I(n, R_n)\big) \;\subset\; B(0, r_n)' \times \Omega_n^{r-1}.$$

For this set $R_n := r_n^{-1}$ and consider any point $\binom{\omega_1}{\omega'} \in I(n, R_n)$. Then by Proposition 4.7 (d) we have $|e_{\Lambda'\omega'}(\omega_1)| \geqslant d(\omega_1, F_\infty^{r-1}\omega') \geqslant R_n$ and hence $|u_{\omega'}(\omega_1)| \leqslant r_n$. Therefore $\vartheta([\binom{\omega_1}{\omega'}]) \in B(0, r_n)' \times \Omega_n^{r-1}$, as desired. The isomorphy $\Gamma_U \backslash \mathcal{N} \xrightarrow{\sim} \mathcal{T}'$ then follows from (a). □

5 Expansion at infinity

In this section we show that every Γ_U-invariant holomorphic function admits a Laurent series expansion in $u_{\omega'}(\omega_1)$ which converges near infinity. As usual, we measure the size of a holomorphic function $g : \Omega_n^{r-1} \to \mathbb{C}_\infty$ by the supremum norm

$$\|g\|_n := \sup\{|g(\omega')| : \omega' \in \Omega_n^{r-1}\}.$$

Note that any rational function is bounded outside of a neighbourhood of its poles. In particular, a rational function with no poles on Ω^r is bounded on Ω_n^r. Since g is a *uniform* limit of rational functions on Ω_n^r, the supremum defined above will always be finite.

Lemma 5.1 *Let $n \in \mathbb{Z}^{>0}$ and $\rho \in |\mathbb{C}_\infty|$. Any holomorphic function $f : B(0, \rho)' \times \Omega_n^{r-1} \to \mathbb{C}_\infty$ has a unique Laurent series expansion*

$$f(z, \omega') = \sum_{k \in \mathbb{Z}} f_k(\omega') z^k, \tag{5.2}$$

which converges uniformly on every affinoid subset, where the functions $f_k : \Omega_n^{r-1} \to \mathbb{C}_\infty$ are holomorphic and satisfy the conditions

$$\limsup_{k \to \infty} \sqrt[k]{\|f_k\|_n} \;\leqslant\; \rho^{-1} \quad \textit{and} \quad \lim_{k \to -\infty} \sqrt[-k]{\|f_k\|_n} \;=\; 0.$$

Proof. Write $\rho = q^a$ with $a \in \mathbb{Q}$. Then the punctured disk $B(0, \rho)'$ is the union of the affinoid annuli

$$D(0, \sigma, \rho) \;=\; \{z \in \mathbb{C}_\infty \mid \sigma \leqslant |z| \leqslant \rho\} \;=\; \mathrm{Spm}\, \mathbb{C}_\infty\langle \tfrac{X}{\pi^a}, \tfrac{\pi^b}{X}\rangle$$

for all $\sigma = q^b < \rho$ with $b \in \mathbb{Q}$. Since Ω_n^{r-1} is also affinoid, say $\Omega_n^{r-1} = \mathrm{Spm}\, A_n^{r-1}$, the product is affinoid and more precisely

$$D(0, \sigma, \rho) \times \Omega_n^{r-1} \;=\; \mathrm{Spm}\, A_n^{r-1}\langle \tfrac{X}{\pi^a}, \tfrac{\pi^b}{X}\rangle.$$

Thus the restriction of f to $D(0,\sigma,\rho)\times\Omega_n^{r-1}$ has a uniformly convergent expansion of the form (5.2) with unique holomorphic functions $f_k:\Omega_n^{r-1}\to\mathbb{C}_\infty$ that satisfy

$$\limsup_{k\to\infty}\sqrt[k]{\|f_k\|_n}\ \leqslant\ \rho^{-1}\quad\text{and}\quad\limsup_{k\to-\infty}\sqrt[-k]{\|f_k\|_n}\ \leqslant\ \sigma.$$

By uniqueness, the functions f_k are independent of σ, so the proposition follows by letting σ go to 0. □

Lemma 5.3 *For any pierced tubular neighbourhood $\mathcal{T}'\subset\mathbb{C}_\infty^\times\times\Omega^{r-1}$, any holomorphic function $f:\mathcal{T}'\to\mathbb{C}_\infty$ has a unique Laurent series expansion*

$$f(z,\omega')=\sum_{k\in\mathbb{Z}}f_k(\omega')z^k$$

with holomorphic functions $f_k:\Omega^{r-1}\to\mathbb{C}_\infty$, which converges uniformly on every affinoid subset of $\mathcal{T}'$.

Proof. Suppose that $\mathcal{T}'=\bigcup_{n\geqslant1}B(0,r_n)'\times\Omega_n^{r-1}$ with $r_n\in|\mathbb{C}_\infty^\times|$. By Lemma 5.1, for any $n\geqslant1$ the restriction of f to $B(0,r_n)'\times\Omega_n^{r-1}$ admits a unique Laurent series expansion $\sum_{k\in\mathbb{Z}}f_k^{(n)}z^k$ with holomorphic functions $f_k^{(n)}:\Omega_n^{r-1}\to\mathbb{C}_\infty$ which converges uniformly on every affinoid subset. For any $n>m\geqslant1$, the uniqueness in Lemma 5.1 for the restriction of f to $B(0,\min\{r_m,r_n\})'\times\Omega_m^{r-1}$ implies that $f_k^{(n)}|\Omega_m^{r-1}=f_k^{(m)}$. By the sheaf property for admissible coverings, there are therefore unique holomorphic functions $f_k:\Omega^{r-1}\to\mathbb{C}_\infty$ with $f_k|\Omega_n^{r-1}=f_k^{(n)}$ for all n, and they satisfy the desired conditions. □

Proposition 5.4 *For any Γ_U-invariant holomorphic function $f:\Omega^r\to\mathbb{C}_\infty$ there exist unique holomorphic functions $f_n:\Omega^{r-1}\to\mathbb{C}_\infty$, such that the series*

$$\sum_{n\in\mathbb{Z}}f_n(\omega')\cdot u_{\omega'}(\omega_1)^n$$

converges to $f(\binom{\omega_1}{\omega'})$ on some neighbourhood of infinity, and uniformly on every affinoid subset thereof.

Proof. Being Γ_U-invariant f corresponds to a function $\bar f:\Gamma_U\backslash\Omega^r\to\mathbb{C}_\infty$. By Theorem 4.16 (c) the function $\bar f\circ\vartheta^{-1}$ then induces a holomorphic function on a pierced tubular neighbourhood $\mathcal{T}'\subset\mathbb{C}_\infty^\times\times\Omega^{r-1}$, where $\mathcal{T}'=\vartheta(\Gamma_U\backslash\mathcal{N})\subset\mathbb{C}_\infty^\times\times\Omega^{r-1}$ for a neighbourhood of infinity $\mathcal{N}\subset\Omega^r$. By Lemma 5.3 that function has a unique expansion of the form

$$\bar f\circ\vartheta^{-1}(\tbinom{z}{\omega'})\ =\ \sum_{n\in\mathbb{Z}}f_n(\omega')z^n.$$

By the definition of ϑ this yields a unique expansion

$$f(\tbinom{\omega_1}{\omega'})\ =\ \sum_{n\in\mathbb{Z}}f_n(\omega')\cdot u_{\omega'}(\omega_1)^n$$

on $\mathcal{N}$, which again converges uniformly on every affinoid subset, as desired. □

Remark 5.5 The series in Proposition 5.4 does not necessarily converge on all of Ω^r. For example, in [Ge99, Corollary 2.2], Gekeler shows that the u-expansion of the rank 2 Drinfeld discriminant function has the radius of convergence $q^{-1/(q-1)}$ only. This is in contrast with the classical case, where the q-expansion of a modular form converges on the entire upper half plane.

Any weak modular form for the group Γ is a Γ_U-invariant function; hence it possesses a u-expansion as in Proposition 5.4. Our next aim is to study its behaviour under conjugation by the "stabiliser of the standard boundary component". For this consider the algebraic subgroups

$$P := \left(\begin{array}{c|ccc} * & * & \dots & * \\ \hline 0 & * & \dots & * \\ \vdots & \vdots & & \vdots \\ 0 & * & \dots & * \end{array}\right), \quad \text{and} \quad M := \left(\begin{array}{c|ccc} * & 0 & \dots & 0 \\ \hline 0 & * & \dots & * \\ \vdots & \vdots & & \vdots \\ 0 & * & \dots & * \end{array}\right). \tag{5.6}$$

of $\mathrm{GL}_{r,F}$, so that $P = U \rtimes M$ is parabolic with unipotent radical U and Levi subgroup M.

Lemma 5.7 *Consider any element of the form* $\gamma = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \gamma' \end{smallmatrix}\right) \in M(F)$ *with* $\alpha \in F^\times$ *and* $\gamma \in \mathrm{GL}_{r-1}(F)$ *and any point* $\omega = \left(\begin{smallmatrix} \omega_1 \\ \omega' \end{smallmatrix}\right) \in \Omega^r$. *Then:*

(a) $\eta := j(\gamma,\omega) = j(\gamma',\omega')$ *and* $\gamma(\omega) = \left(\begin{smallmatrix} \eta^{-1}\alpha\omega_1 \\ \gamma'(\omega') \end{smallmatrix}\right)$.

(b) $\Lambda'_\gamma := \iota^{-1}(\gamma^{-1}\Gamma_U\gamma) = \alpha^{-1}\Lambda'\gamma'$.

(c) $u_{\gamma,\omega'}(\omega_1) := e_{\Lambda'_\gamma\omega'}(\omega_1)^{-1} = \eta^{-1}\alpha \cdot u_{\gamma'(\omega')}(\eta^{-1}\alpha\omega_1)$.

(d) There exist constants $k \geqslant 0$ *and* $c_4 > 0$ *such that for all* $n > 0$ *and* $R > 0$ *we have*

$$\gamma\,(I(n,R)) \;\subset\; I(n+k, c_4R).$$

(e) For any neighbourhood of infinity $\mathcal{N} \subset \Omega^r$ *the subset* $\gamma^{-1}(\mathcal{N})$ *is also a neighbourhood of infinity.*

(f) For any Γ_U*-invariant holomorphic function* $f : \Omega^r \to \mathbb{C}_\infty$ *with the expansion in Proposition 5.4 on* $\mathcal{N}$ *and any integers* k *and* m *we have the following expansion on* $\gamma^{-1}(\mathcal{N})$*:*

$$(f|_{k,m}\gamma)(\left(\begin{smallmatrix} \omega_1 \\ \omega' \end{smallmatrix}\right)) \;=\; \sum_{n\in\mathbb{Z}} \alpha^{m-n}(f_n|_{k-n,m}\gamma')(\omega') \cdot u_{\gamma,\omega'}(\omega_1)^n.$$

Proof. Assertion (a) follows directly from the definitions (1.2) and (1.3), with $\gamma'(\omega') = \eta^{-1}\gamma'\omega'$. Assertion (b) follows by direct calculation from the definition (4.3) of ι. Using (b) and Proposition 2.3 (b) we deduce that

$$e_{\Lambda'_\gamma\omega'}(\omega_1) \;=\; e_{\alpha^{-1}\Lambda'\gamma'\omega'}(\omega_1) \;=\; e_{\alpha^{-1}\eta\Lambda'\cdot\gamma'(\omega')}(\omega_1) \;=\; \alpha^{-1}\eta \cdot e_{\Lambda'\cdot\gamma'(\omega')}(\eta^{-1}\alpha\omega_1)$$

Taking reciprocals thus shows (c).

To prove (d) consider any $n > 0$ and $\omega' \in \Omega_n^{r-1}$. Then by definition (3.2) and Lemma 3.5 (c), both with $r-1$ in place of r, we have $h(\omega') \geqslant |\pi|^n$ and $h(\gamma'(\omega')) \geqslant c_3 h(\omega')$ for some constant c_3 depending only on γ'. Together we deduce that $h(\gamma'(\omega')) \geqslant |\pi|^{n+k}$ for some $k \geqslant 0$ depending only on γ'. By the definition (3.2) again this means that $\gamma'(\omega') \in \Omega_{n+k}^{r-1}$. Next, by Lemmas 3.5 (a) and 3.3, again with $r-1$ in place of r, we have $|\eta| = |j(\gamma', \omega')| \leqslant |\omega'| c_1^{-1} \leqslant |\xi||\pi|^{-n} c_1^{-1}$ for another constant c_1 depending only on γ'. Note also that, since $\gamma'(\omega') = \eta^{-1}\gamma'\omega'$, the associated F_∞-vector space is $F_\infty^{r-1}\gamma'(\omega') = \eta^{-1}F_\infty^{r-1}\omega'$. For any $\omega_1 \in \mathbb{C}_\infty$ we therefore have

$$\begin{aligned} d(\eta^{-1}\alpha\omega_1, F_\infty^{r-1}\gamma'(\omega')) &= d(\eta^{-1}\alpha\omega_1, \eta^{-1}\alpha F_\infty^{r-1}\omega') \\ &= |\eta^{-1}\alpha| \cdot d(\omega_1, F_\infty^{r-1}\omega') \;\geqslant\; |\alpha\pi^n\xi^{-1}| c_1 \cdot d(\omega_1, F_\infty^{r-1}\omega'). \end{aligned}$$

In view of Definition 4.12 this implies (d) with $c_4 := |\alpha\pi^n\xi^{-1}|c_1$.

To deduce (e) choose $R_n > 0$ such that $\bigcup_{n>0} I(n, R_n) \subset \mathcal{N}$. Then (d) implies that

$$\gamma\big(\textstyle\bigcup_{n>0} I(n, c_4^{-1}R_{n+k})\big) \;\subset\; \textstyle\bigcup_{n>0} I(n+k, R_{n+k}) \;\subset\; \mathcal{N},$$

and hence $\bigcup_{n>0} I(n, c_4^{-1}R_{n+k}) \subset \gamma^{-1}(\mathcal{N})$. Thus $\gamma^{-1}(\mathcal{N})$ is a neighbourhood of infinity, proving (e).

Finally, using the definition (1.5), for any $\binom{\omega_1}{\omega'} \in \gamma^{-1}(\mathcal{N})$ we can now calculate

$$\begin{aligned} (f|_{k,m}\gamma)(\tbinom{\omega_1}{\omega'}) &\overset{(a)}{=} \eta^{-k}(\det\gamma)^m f(\tbinom{\eta^{-1}\alpha\omega_1}{\gamma'(\omega')}) \\ &\overset{5.4}{=} \eta^{-k}(\det\gamma)^m \cdot \sum_{n\in\mathbb{Z}} f_n(\gamma'(\omega')) \cdot u_{\gamma'(\omega')}(\eta^{-1}\alpha\omega_1)^n \\ &\overset{(c)}{=} \eta^{-k}(\alpha\det\gamma')^m \cdot \sum_{n\in\mathbb{Z}} f_n(\gamma'(\omega')) \cdot \big(\alpha^{-1}\eta u_{\gamma,\omega'}(\omega_1)\big)^n \\ &= \sum_{n\in\mathbb{Z}} \alpha^{m-n} \cdot \eta^{n-k}(\det\gamma')^m f_n(\gamma'(\omega')) \cdot u_{\gamma,\omega'}(\omega_1)^n \\ &= \sum_{n\in\mathbb{Z}} \alpha^{m-n} \cdot (f_n|_{k-n,m}\gamma')(\omega') \cdot u_{\gamma,\omega'}(\omega_1)^n, \end{aligned}$$

proving (f). □

For a first application consider the subgroup

$$\Gamma_M := \big\{\gamma' \in \mathrm{GL}_{r-1}(F) \;\big|\; \left(\begin{smallmatrix}1 & 0\\ 0 & \gamma'\end{smallmatrix}\right) \in \Gamma \cap M(F)\big\}. \tag{5.8}$$

Theorem 5.9 *Let f be a weak modular form of weight k and type m for the group Γ, and let f_n be its coefficients in the u-expansion from Proposition 5.4. Then, for each $n \in \mathbb{Z}$, the function f_n is a weak modular form of weight $k-n$ and type m for the group $\Gamma_M < \mathrm{GL}_{r-1}(F)$.*

Proof. Consider any $\gamma' \in \Gamma_M$ and set $\gamma := \left(\begin{smallmatrix}1 & 0\\ 0 & \gamma'\end{smallmatrix}\right)$, so that $\alpha = 1$ in the notation of Lemma 5.7. Since the subgroup Γ_U is normalised by γ, Lemma 5.7 (b) implies that $\Lambda'_\gamma = \Lambda'$ and hence $u_{\gamma,\omega'}(\omega_1) = u_{\omega'}(\omega_1)$. Let $\mathcal{N}$ be a neighbourhood of infinity on which the expansion from Proposition 5.4 converges. Then by Lemma 5.7 (e) the intersection $\mathcal{N} \cap \gamma^{-1}(\mathcal{N})$ is another neighbourhood of infinity. For any $\omega = \binom{\omega_1}{\omega'} \in \mathcal{N} \cap \gamma^{-1}(\mathcal{N})$ we can compare the expansions of $f(\omega) = (f|_{k,m}\gamma)(\omega)$ from Proposition 5.4 and 5.7 (f). Since $\alpha = 1$, by the uniqueness part of Proposition 5.4 we conclude that $f_n = f_n|_{k-n,m}\gamma'$ for all $n \in \mathbb{Z}$, proving the theorem. □

Corollary 5.10 *Let f be a weak modular form of weight k and type m for the group Γ. Then for any $n \in \mathbb{Z}$, the coefficient f_n in the u-expansion from Proposition 5.4 is identically zero unless*

$$n \equiv k - (r-1)m \quad \textit{modulo} \quad |\Gamma_M \cap \{\textit{scalars}\}|.$$

Proof. Combine Theorem 5.9 with (1.11) for $r-1$ in place of r. □

Lemma 5.11 *Consider any element of the form $\gamma = \begin{pmatrix} 1 & \beta \\ 0 & \mathrm{Id}_{r-1} \end{pmatrix} \in U(F)$ for some row vector $\beta \in F^{r-1}$ and any point $\omega = \binom{\omega_1}{\omega'} \in \Omega^r$. Then:*

(a) $j(\gamma,\omega) = \det(\gamma) = 1$ *and* $\gamma(\omega) = \binom{\omega_1+\beta\omega'}{\omega'}$.

(b) For any neighbourhood of infinity $\mathcal{N} \subset \Omega^r$ the subset

$$\mathcal{N}' \;:=\; \Big\{\binom{\omega_1}{\omega'} \in \gamma^{-1}(\mathcal{N}) \;\Big|\; |e_{\Lambda'\omega'}(\beta\omega') \cdot u_{\omega'}(\omega_1)| < 1\Big\}$$

is also a neighbourhood of infinity.

(c) For any Γ_U-invariant holomorphic function $f : \Omega^r \to \mathbb{C}_\infty$ with the expansion in Proposition 5.4 on $\mathcal{N}$ and any integers k and m we have the following expansion on $\mathcal{N}'$:

$$(f|_{k,m}\gamma)\big(\tbinom{\omega_1}{\omega'}\big) \;=\; \sum_{n\in\mathbb{Z}} \Big(\sum_{k\geqslant 0} \tbinom{k-n}{k} \cdot f_{n-k}(\omega') \cdot e_{\Lambda'\omega'}(\beta\omega')^k \Big) \cdot u_{\omega'}(\omega_1)^n.$$

Proof. Assertion (a) follows directly from the definitions (1.2) and (1.3).

To prove (b) choose $R_n > 0$ such that $\bigcup_{n>0} I(n,R_n) \subset \mathcal{N}$. Since $\beta\omega' \in F_\infty^{r-1}\omega'$, we have $d(\omega_1 + \beta\omega', F_\infty^{r-1}\omega') = d(\omega_1, F_\infty^{r-1}\omega')$ and therefore $\gamma^{-1}(I(n,R_n)) = I(n,R_n)$ by Definition 4.12. On the other hand we have $d(\beta\omega', F_\infty^{r-1}\omega') = 0$; applying Proposition 4.7 (c) thus yields constants $c_n > 0$, such that $|e_{\Lambda'\omega'}(\beta\omega')| < c_n$ for any $\omega' \in \Omega_n^{r-1}$. By Proposition 4.7 (d) and Definition 4.12, for any $\binom{\omega_1}{\omega'} \in I(n,c_n)$ we therefore have

$$|e_{\Lambda'\omega'}(\omega_1)| \;\geqslant\; d(\omega_1, F_\infty^{r-1}\omega') \;\geqslant\; c_n \;>\; |e_{\Lambda'\omega'}(\beta\omega')|.$$

By the definition of $u_{\omega'}(\omega_1)$ this implies that $|e_{\Lambda'\omega'}(\beta\omega') \cdot u_{\omega'}(\omega_1)| < 1$. Together this shows that $I(n, \max\{R_n, c_n\}) \subset \mathcal{N}'$. Varying n we conclude that $\mathcal{N}'$ is a neighbourhood of infinity, proving (b).

Next, by (a) and the definition (1.5), the expansion from Proposition 5.4 yields

$$(f|_{k,m}\gamma)\big(\tbinom{\omega_1}{\omega'}\big) \;=\; f\big(\tbinom{\omega_1+\beta\omega'}{\omega'}\big) \;=\; \sum_{n\in\mathbb{Z}} f_n(\omega') \cdot u_{\omega'}(\omega_1+\beta\omega')^n$$

Using the additivity of the exponential function we can rewrite

$$\begin{aligned} u_{\omega'}(\omega_1+\beta\omega')^n &= e_{\Lambda'\omega'}(\omega_1+\beta\omega')^{-n} \\ &= \big(e_{\Lambda'\omega'}(\omega_1)+e_{\Lambda'\omega'}(\beta\omega')\big)^{-n} \\ &= \big(1+e_{\Lambda'\omega'}(\beta\omega')u_{\omega'}(\omega_1)\big)^{-n}\cdot u_{\omega'}(\omega_1)^n. \end{aligned}$$

For $\binom{\omega_1}{\omega'}\in\mathcal{N}'$ we have $|e_{\Lambda'\omega'}(\beta\omega')\cdot u_{\omega'}(\omega_1)|<1$, so we can plug the binomial series into the above expansion and rearrange terms, yielding

$$\begin{aligned} (f|_{k,m}\gamma)(\tbinom{\omega_1}{\omega'}) &= \sum_{n\in\mathbb{Z}} f_n(\omega')\cdot\Big(\sum_{k\geqslant 0}\tbinom{-n}{k}\cdot e_{\Lambda'\omega'}(\beta\omega')^k\cdot u_{\omega'}(\omega_1)^{n+k}\Big) \\ &= \sum_{n\in\mathbb{Z}}\sum_{k\geqslant 0}\tbinom{-n}{k}\cdot f_n(\omega')\cdot e_{\Lambda'\omega'}(\beta\omega')^k\cdot u_{\omega'}(\omega_1)^{n+k} \\ &= \sum_{n'\in\mathbb{Z}}\Big(\sum_{k\geqslant 0}\tbinom{k-n'}{k}\cdot f_{n'-k}(\omega')\cdot e_{\Lambda'\omega'}(\beta\omega')^k\Big)\cdot u_{\omega'}(\omega_1)^{n'} \end{aligned}$$

with the substitution $n+k=n'$. Thus the stated expansion holds on $\mathcal{N}'$, proving (c). □

Definition 5.12 *Let $f:\Omega^r\to\mathbb{C}_\infty$ be a Γ_U-invariant holomorphic function and let f_n be its coefficients in the u-expansion from Proposition 5.4. Then the* order at infinity *of f is*

$$\mathrm{ord}_{\Gamma_U}(f) \;:=\; \inf\big\{n\in\mathbb{Z}\;\big|\; f_n(\omega')\neq 0 \textit{ for some } \omega'\in\Omega^{r-1}\big\} \;\in\; \mathbb{Z}\cup\{\pm\infty\}.$$

The function f is called meromorphic at infinity *if* $\mathrm{ord}_{\Gamma_U}(f)>-\infty$*, that is, if f_n is identically zero for all $n\ll 0$. It is called* holomorphic at infinity *if* $\mathrm{ord}_{\Gamma_U}(f)\geqslant 0$*, that is, if f_n is identically zero for all $n<0$.*

Proposition 5.13 *Consider a Γ_U-invariant holomorphic function $f:\Omega^r\to\mathbb{C}_\infty$ and an element $\gamma\in P(F)$. Then $f|_{k,m}\gamma$ is invariant under $\Gamma_{\gamma,U}:=(\gamma^{-1}\Gamma\gamma)\cap U(F)$, and we have*

$$\mathrm{ord}_{\Gamma_U}(f)=\mathrm{ord}_{\Gamma_{\gamma,U}}(f|_{k,m}\gamma).$$

In particular f is meromorphic, respectively holomorphic at infinity if and only if $f|_{k,m}\gamma$ has the corresponding property.

Proof. Since $P=U\rtimes M$, it suffices to prove this separately for elements of $M(F)$ and $U(F)$. In both cases the $\Gamma_{\gamma,U}$-invariance follows by direct calculation from the formula (1.6). The rest follows from the expansion in Lemma 5.7 for $\gamma\in M(F)$, respectively by close inspection of the expansion in Lemma 5.11 for $\gamma\in U(F)$. □

Proposition 5.14 *Let $\Gamma_1<\Gamma$ and hence $\Gamma_{1,U}:=\Gamma_1\cap U(F)<\Gamma_U$ be subgroups of finite index. Then for any Γ_U-invariant holomorphic function f we have*

$$\mathrm{ord}_{\Gamma_{1,U}}(f)=\mathrm{ord}_{\Gamma_U}(f)\cdot[\Gamma_U:\Gamma_{1,U}].$$

In particular f is meromorphic, respectively holomorphic at infinity with respect to Γ_U if and only if it is so with respect to $\Gamma_{1,U}$.

Proof. The lattice associated to $\Gamma_{1,U}$ is $\Lambda'_1 := \iota^{-1}(\Gamma_{1,U}) \subset \Lambda' = \iota^{-1}(\Gamma_U)$, so that $[\Lambda' : \Lambda'_1] = [\Gamma_U : \Gamma_{1,U}] = p^d$ for an integer $d \geqslant 0$. For any $\omega' \in \Omega^{r-1}$ we then also have $[\Lambda'\omega' : \Lambda'_1\omega'] = p^d$. Let B be a set of representatives for $\Lambda' \smallsetminus \Lambda'_1$ modulo Λ'_1. By Proposition 2.3 (a) we then have

$$e_{\Lambda'\omega'}(\omega_1) \;=\; e_{\Lambda'_1\omega'}(\omega_1) \cdot \prod_{\beta \in B} \left(1 - \frac{e_{\Lambda'_1\omega'}(\omega_1)}{e_{\Lambda'_1\omega'}(\beta\omega')}\right).$$

Taking reciprocals, we can therefore express the expansion parameter $u_{\omega'}(\omega_1) := e_{\Lambda'\omega'}(\omega_1)^{-1}$ with respect to Λ' in terms of the expansion parameter $u_{1,\omega'}(\omega_1) := e_{\Lambda'_1\omega'}(\omega_1)^{-1}$ with respect to Λ'_1 by the formula

$$u_{\omega'}(\omega_1) \;=\; u_{1,\omega'}(\omega_1)^{p^d} \cdot \prod_{\beta \in B} \frac{e_{\Lambda'_1\omega'}(\beta\omega')}{e_{\Lambda'_1\omega'}(\beta\omega')u_{1,\omega'}(\omega_1) - 1}.$$

The expansion from Proposition 5.4 thus yields

$$f\left(\left(\begin{smallmatrix}\omega_1\\ \omega'\end{smallmatrix}\right)\right) \;=\; \sum_{n \in \mathbb{Z}} f_n(\omega') \cdot u_{\omega'}(\omega_1)^n \;=\; \sum_{n \in \mathbb{Z}} f_n(\omega') \cdot u_{1,\omega'}(\omega_1)^{np^d} \cdot \prod_{\beta \in B} \left(\frac{e_{\Lambda'_1\omega'}(\beta\omega')}{e_{\Lambda'_1\omega'}(\beta\omega')u_{1,\omega'}(\omega_1) - 1}\right)^n$$

for all points $\left(\begin{smallmatrix}\omega_1\\ \omega'\end{smallmatrix}\right)$ in some neighbourhood of infinity. By Lemma 5.11 (b) with $\Gamma_{1,U}$ in place of Γ_U, for each $\beta \in B$ we have $\left|e_{\Lambda'_1\omega'}(\beta\omega')u_{1,\omega'}(\omega_1)\right| < 1$ on some neighbourhood of infinity. On the intersection of these neighbourhoods, we can plug the binomial series into the above expansion and rearrange terms. We conclude that the expansion with respect to $u_{\omega'}(\omega_1)$ has the first non-zero term $f_n(\omega') \cdot u_{\omega'}(\omega_1)^n$ if and only if the expansion with respect to $u_{1,\omega'}(\omega_1)$ has the first non-zero term

$$f_n(\omega') \cdot u_{1,\omega'}(\omega_1)^{np^d} \cdot \prod_{\beta \in B} \left(-e_{\Lambda'_1\omega'}(\beta\omega')\right)^n.$$

Then $\mathrm{ord}_{\Gamma_{1,U}}(f) = np^d = \mathrm{ord}_{\Gamma_U}(f) \cdot [\Gamma_U : \Gamma_{1,U}]$, and the proposition follows. □

Next, we restate holomorphy at infinity (Definition 5.12) in terms of boundedness criteria in certain neighborhoods of infinity. This is a natural consideration and, though not used elsewhere in this monograph, may be useful for future work.

We call a subset $X \subset \Omega^{r-1}$ *analytically Zariski-dense* if any holomorphic $f : \Omega^{r-1} \to \mathbb{C}_\infty$ that vanishes on X also vanishes identically on Ω^{r-1}.

Definition 5.15 *Let $X \subset \Omega^{r-1}$ be a subset.*

We say that f is bounded on vertical lines supported on *X if for every $\omega' \in X$ there exist constants $N, R > 0$ such that if $d(\omega_1, F_\infty^{r-1}\omega') > R$, then $|f(\left(\begin{smallmatrix}\omega_1\\ \omega'\end{smallmatrix}\right))| < N$. If for every $\omega' \in X$ and every $N > 0$, there exists an $R > 0$ with this property, we say that f* tends to 0 on vertical lines supported on *X.*

We say that f is bounded *(resp.* tends to 0*) on vertical strips supported on X if for every $z' \in X$ there exists an admissible neighbourhood $U \subset X$ of z' and constants $N, R > 0$ such that if $d(\omega_1, F_\infty^{r-1}\omega') > R$ and $\omega' \in U$, then $|f(\left(\begin{smallmatrix}\omega_1\\ \omega'\end{smallmatrix}\right))| < N$. (resp. if for all $N > 0$ there exists $R > 0$ with this property).*

Proposition 5.16 *Let $f : \Omega^r \to \mathbb{C}_\infty$ be a Γ_U-invariant holomorphic function. The following conditions are equivalent:*

1. *f is bounded on vertical strips supported on an analytically Zariski-dense set $X \subset \Omega^{r-1}$;*
2. *f is bounded on vertical lines supported on an analytically Zariski-dense set $X \subset \Omega^{r-1}$;*
3. *f is holomorphic at infinity.*

Moreover, $\mathrm{ord}_{\Gamma_U}(f) \geqslant 1$ *if and only if f tends to 0 on vertical lines (equivalently, vertical strips) supported on an analytically Zariski-dense set $X \subset \Omega^{r-1}$.*

Proof. By Proposition 5.4, the function f is given by its u-expansion

$$f(\binom{\omega_1}{\omega'}) = \sum_{k\in\mathbb{Z}} f_k(\omega')u_{\omega'}(\omega_1)^k, \tag{5.17}$$

which converges uniformly on any affinoid subset of a suitable neighborhood of infinity. By Theorem 4.16(b), this means that there exists a sequence $(r_n \in |\mathbb{C}_\infty^\times|)_{n\geqslant 1}$ such that (5.17) converges to a holomorphic function on

$$U_n := \{\binom{\omega_1}{\omega'} \in \Omega \mid (u_{\omega'}(\omega_1), \omega') \in B(0, r_n) \times \Omega_n^{r-1}\},$$

for each $n \geqslant 1$.

It is trivial that (1) ⇒ (2), so we proceed to prove that (2) ⇒ (3).

Let $X \subset \Omega^{r-1}$, $R > 0$ and $N > 0$ be the objects provided in the definition of (2), and let $\omega' \in X$.

Choose n sufficiently large that $\omega' \in \Omega_n^{r-1}$ and enlarge R, if necessary, so that $R \geqslant 1/r_n$.

Now let $\omega_1 \in \mathbb{C}_\infty^\times$ be such that $d(\omega_1, F_\infty^{r-1}\omega') > R$. By Proposition 4.7(d) $|u_{\omega'}(\omega_1)| < 1/R$, and so $\binom{\omega_1}{\omega'} \in U_n$. Furthermore $|f(\omega)| < N$ for all $\omega \in U_n$.

Now consider the Newton polygon of the series (5.17), that is the lower convex hull of the set of points $(k, -\log_q |f_k(\omega')|)$ in the Euclidean plane.

Lemma 5.1 implies that $\lim_{k\to-\infty} \|f_k\|_n^{-1/k} = 0$, and hence $\lim_{k\to-\infty} |f_k(\omega')|^{-1/k} = 0$. This means that the slopes of the Newton polygon tend to $-\infty$ as $k \to -\infty$, so either the series has a finite tail, or infinitely many points lie on the Newton polygon for negative k.

Consider the line $y = mx + c$ with slope $m = \log_q |u_\omega(\omega_1)|$ and tangent to the Newton polygon. By slightly perturbing ω_1, we may assume this line touches the Newton polygon in only one point $(k, -\log_q |f_k(\omega')|)$. The corresponding term in (5.17) then dominates the series, and the y-intercept of the line equals

$$c = -\log_q |f_k(\omega')u_{\omega'}(\omega_1)^k| = -\log_q |f(\omega)|.$$

Now, if there exist points on the Newton polygon with $k < 0$, then by choosing $m = \log_q |u_\omega(\omega_1)|$ sufficiently small (i.e. $d(\omega_1, F_\infty^{r-1}\omega')$ sufficiently large), we find that $|f(\omega)|$ can be made larger than the bound N, i.e. f is not bounded on the vertical line.

This contradiction shows that $f_k(\omega') = 0$ for all $k < 0$. Since this holds for all ω' in the analytically Zariski-dense set X, it follows that f_k vanishes identically on Ω^{r-1} for every $k < 0$, thus proving that f is holomorphic at infinity.

Furthermore, if there exists a point with $k = 0$, then the same argument shows that $|f(\omega)| \geqslant |f_0(\omega')|$, so f cannot vanish on the vertical line.

To prove that (3) ⇒ (1), suppose that f is holomorphic at infinity. Then the expansion (5.17) has no polar terms. Let $X = U = \Omega^{r-1}$, consider any $\omega' \in X$ and let $n \geqslant 1$ be such that $\omega' \in \Omega_n^{r-1}$. Let $R = 1/r_n$.

Since the u-expansion (5.17) converges to a holomorphic function on U_n, it follows from Lemma 5.1 that $\liminf_{k\to\infty} |f_k(\omega')|^{1/k} < R$. Now suppose that $d(\omega_1, F_\infty^{r-1}\omega') > R$. Then $|u_{\omega'}(\omega_1)| < 1/R$ by Proposition 4.7(d) and we obtain $\liminf_{k\to\infty} |f_k(\omega')u_{\omega'}(\omega_1)^k| < 1$, and so $f(\omega)$ is bounded by some $N > 0$. Thus f is bounded on vertical strips.

Lastly, if $f_0 = 0$, then we may write

$$|f(\omega)| = |u_{\omega'}(\omega_1)| \cdot |\sum_{k\geqslant 0} f_{k+1}(\omega')u_{\omega'}(\omega_1)^k|,$$

where the sum on the right is bounded as before, and $|u_{\omega'}(\omega_1)| \to 0$ as $R \to \infty$, so f vanishes on vertical strips. □

Remark 5.18 *Proposition 5.16 above gives three equivalent formulations of being holomorphic at infinity. Gekeler [Ge19b, (1.7) & Prop 1.8] also defines higher rank modular forms and provides another definition of being holomorphic at infinity. He defines a fundamental domain for Ω^r and defines f to be holomorphic at infinity if f is bounded on this fundamental domain. It is an interesting question whether Gekeler's definition is also equivalent to the ones in Proposition 5.16.*

6 Modular forms

Now we impose holomorphy conditions at all boundary components, not just the standard one. We achieve this by conjugating the standard boundary component by arbitrary elements $\delta \in \mathrm{GL}_r(F)$. Recall from Proposition 1.12 that for any weak modular form f of weight k and type m for Γ, and for any $\delta \in \mathrm{GL}_r(F)$, the function $f|_{k,m}\delta$ is a weak modular form of weight k and type m for the arithmetic subgroup $\delta^{-1}\Gamma\delta$. Determining the behaviour of f at all boundary components is equivalent to determining the behaviour of all conjugates $f|_{k,m}\delta$ at the standard boundary component.

Definition 6.1 *Let f be a weak modular form of weight k and type m for Γ.*

(a) If $\mathrm{ord}_{(\delta^{-1}\Gamma\delta)\cap U(F)}(f|_{k,m}\delta) \geqslant 0$ for all $\delta \in \mathrm{GL}_r(F)$, we call f a modular form.

(b) If $\mathrm{ord}_{(\delta^{-1}\Gamma\delta)\cap U(F)}(f|_{k,m}\delta) \geqslant 1$ for all $\delta \in \mathrm{GL}_r(F)$, we call f a cusp form.

In particular, a modular form *is a weak modular form f such that $f|_{k,m}\delta$ is holomorphic at infinity for all $\delta \in \mathrm{GL}_r(F)$. The space of these functions is denoted by $\mathcal{M}_{k,m}(\Gamma)$. The space of cusp forms is denoted by $\mathcal{S}_{k,m}(\Gamma)$. As with weak modular forms, we abbreviate $\mathcal{M}_k(\Gamma) := \mathcal{M}_{k,0}(\Gamma)$ and $\mathcal{S}_k(\Gamma) := \mathcal{S}_{k,0}(\Gamma)$.*

It may seem extravagant to impose conditions for infinitely many δ. However, the next two facts show that for fixed Γ, we only need to check these conditions for δ in a fixed finite set.

Proposition 6.2 *The numbers in Definition 6.1 depend only on the double coset $\Gamma\delta P(F)$.*

Proof. Since f is a weak modular form of weight k and type m for Γ, for any $\delta' = \gamma'\delta\gamma$ with $\gamma' \in \Gamma$ and $\gamma \in P(F)$ we have $f|_{k,m}\delta' = (f|_{k,m}\delta)|_{k,m}\gamma$ and hence $\mathrm{ord}_{(\delta'^{-1}\Gamma\delta')\cap U(F)}(f|_{k,m}\delta') = \mathrm{ord}_{(\delta^{-1}\Gamma\delta)\cap U(F)}(f|_{k,m}\delta)$ by Proposition 5.13. □

Proposition 6.3 *The double coset space $\Gamma\backslash\mathrm{GL}_r(F)/P(F)$ is finite. More precisely, let $\mathrm{Cl}(A)$ denote the class group of A. Then:*

(a) $\mathrm{GL}_r(A)\backslash\mathrm{GL}_r(F)/P(F)$ is in bijection with $\mathrm{Cl}(A)$.

(b) For any arithmetic subgroup $\Gamma < \mathrm{GL}_r(F)$, the set $\Gamma\backslash\mathrm{GL}_r(F)/P(F)$ has cardinality at most $|\mathrm{Cl}(A)|\cdot[\mathrm{GL}_r(A) : \mathrm{GL}_r(A)\cap\Gamma]$.

(c) If $\Gamma < \mathrm{GL}_r(A)$ then the double cosets of $\Gamma\backslash\mathrm{GL}_r(F)/P(F)$ can be represented by elements of $\mathrm{GL}_r(A)$ if and only if $\mathrm{Cl}(A) = \{1\}$.

Proof. By the orbit-stabiliser theorem the set $\mathrm{GL}_r(F)/P(F)$ is in bijection with the set of one-dimensional subspaces of F^r and hence with $\mathbb{P}^{r-1}(F)$. This bijection is equivariant under the left action of $\mathrm{GL}_r(F)$. To prove (a) it thus suffices to find a bijection between $\mathrm{GL}_r(A)\backslash\mathbb{P}^{r-1}(F)$ and $\mathrm{Cl}(A)$.

For this we associate to any column vector $x = (x_i)_i \in F^r \smallsetminus \{0\}$ the fractional ideal $I(x) := \sum_i Ax_i \subset F$. This ideal depends only on the $\mathrm{GL}_r(A)$-orbit of x, and its ideal class depends only on the corresponding point of $\mathbb{P}^{r-1}(F)$. Together we therefore obtain a well-defined map $\mathrm{GL}_r(A)\backslash\mathbb{P}^{r-1}(F) \to \mathrm{Cl}(A)$. This map is surjective, because $r \geqslant 2$ and every ideal of a Dedekind domain can be generated by 2 elements. We claim that it is also injective.

To see this we view A^r as a space of row vectors, so that right multiplication by x determines a surjective homomorphism of A-modules $p_x : A^r \to I(x)$. Since $I(x)$ is a projective A-module, the associated short exact sequence $0 \to \ker(p_x) \to A^r \to I(x) \to 0$ splits. Moreover, since the isomorphism class of a finitely generated projective A-module depends only on its rank and its highest exterior power, the isomorphism class of $\ker(p_x)$ is determined by that of $I(x)$.

Suppose now that two vectors $x, y \in F^r \smallsetminus \{0\}$ correspond to the same ideal class. Then $I(y) = u\cdot I(x)$ for some $u \in F^\times$, and by the preceding remarks there exists an isomorphism of A-modules $f : \ker(p_x) \to \ker(p_y)$. Combining these via suitable splittings we find an isomorphism of A-modules $g : A^r \to A^r$ making the following diagram commute:

$$\begin{CD}
0 @>>> \ker(p_x) @>>> A^r @>{p_x}>> I(x) @>>> 0 \\
@. @V{f}V{\wr}V @V{g}V{\wr}V @V{u\cdot}V{\wr}V @. \\
0 @>>> \ker(p_y) @>>> A^r @>{p_y}>> I(y) @>>> 0.
\end{CD}$$

Writing g as right multiplication by a matrix $\gamma \in \mathrm{GL}_r(A)$, the commutativity on the right hand side then means that $a\gamma y = axu$ for all $a \in A^r$. Thus $\gamma y = xu$ for some $\gamma \in \mathrm{GL}_r(A)$ and $u \in F^\times$, which is precisely the desired injectivity.

This finishes the proof of (a). Parts (b) and (c) are direct consequences of (a). □

Corollary 6.4 *Suppose that $\Gamma = \mathrm{GL}_r(A)$ for a principal ideal domain A. Then:*

(a) The condition in Definition 6.1 is independent of δ.

(b) If $m \not\equiv 0 \bmod (q-1)$, any modular form of weight k and type m for Γ is a cusp form.

Proof. Part (a) follows from Propositions 6.2 and 6.3 (a). To prove (b) let f be a modular form of weight k and type m for Γ, and let f_n be its coefficients in the u-expansion from Proposition 5.4, which are weak modular forms for the group $\Gamma_M = \mathrm{GL}_{r-1}(A)$. By assumption we then have $f_n = 0$ for all $n < 0$. If f is not a cusp form, then f_0 is not identically zero, so Corollary 5.10 implies that $k \equiv (r-1)m$ modulo $|\Gamma_M \cap \{\text{scalars}\}| = q-1$. But then f itself is also not identically zero, so (1.11) gives $k \equiv rm$ modulo $|\Gamma \cap \{\text{scalars}\}| = q-1$. Both congruences together imply that $m \equiv 0$ modulo $(q-1)$, contrary to the assumption. □

Remark 6.5 By Theorem 5.9 the coefficient f_n of the u-expansion of a modular form f is a weak modular form of weight $k-n$ for a subgroup $\Gamma_M < \mathrm{GL}_{r-1}(F)$. In contrast to the case of modular forms in characteristic zero, the weight $k-n$ here goes to $-\infty$ for $n \to \infty$. In Theorem 11.1 (b) of Part II we will see that any modular forms of weight < 0 for Γ_M must be zero if $r-1 \geqslant 2$. Thus for $r \geqslant 3$ and n large enough, the coefficient f_n will not be a modular form (only failing the holomorphic at infinity condition). However, one expects that there will be some rank $r-1$ discriminant function Δ_a and integer N for which $\Delta_a^N f_n$ will be holomorphic at infinity. It may be interesting to find bounds on N in terms of n.

Proposition 6.6 *For any $\delta \in \mathrm{GL}_r(F)$ we have $f \in \mathcal{M}_{k,m}(\Gamma)$ if and only if $f|_{k,m}\delta \in \mathcal{M}_{k,m}(\delta^{-1}\Gamma\delta)$.*

Proof. Direct consequence of Proposition 1.12 and the formula (1.6). □

In particular, whenever $\Gamma_1 \triangleleft \Gamma$ is a normal subgroup of finite index, the map $f \mapsto f|_{k,m}\gamma$ for all $\gamma \in \Gamma$ defines a right action of Γ on $\mathcal{M}_{k,m}(\Gamma_1)$. As a direct consequence of Definition 6.1 and Proposition 5.14 the subspace of invariants is then

$$\mathcal{M}_{k,m}(\Gamma_1)^{\Gamma} = \mathcal{M}_{k,m}(\Gamma). \tag{6.7}$$

Moreover, (1.10) and (1.11) imply that

$$\mathcal{M}_{k,m}(\Gamma) = \mathcal{M}_{k,m'}(\Gamma) \text{ whenever } m \equiv m' \text{ modulo } |\det(\Gamma)|, \text{ and} \tag{6.8}$$

$$\mathcal{M}_{k,m}(\Gamma) = 0 \text{ unless } k \equiv rm \text{ modulo } \big|\Gamma \cap \{\text{scalars}\}\big|. \tag{6.9}$$

As a direct consequence of the definitions we also have

$$\mathcal{M}_{k,m}(\Gamma) \cdot \mathcal{M}_{k',m'}(\Gamma) \subset \mathcal{M}_{k+k',m+m'}(\Gamma) \tag{6.10}$$

for all k, k', m, m'. In particular we can form the *graded ring of modular forms*

$$\mathcal{M}_*(\Gamma) := \bigoplus_{k \geqslant 0} \mathcal{M}_k(\Gamma). \tag{6.11}$$

Part II
Comparison with the algebraic theory

Introduction

In this part, we identify the analytic modular forms from Part I with the algebraic modular forms defined in [Pi13] and deduce qualitative consequences such as the finite dimensionality of the space of modular forms of given level and weight.

By definition, *weak Drinfeld modular forms* of weight k are holomorphic functions on the rigid analytic Drinfeld period domain Ω^r that satisfy a certain twisted transformation law under the action of an arithmetic congruence subgroup $\Gamma < \mathrm{GL}_r(F)$. *Drinfeld modular forms* are weak Drinfeld modular forms that are holomorphic at infinity after transformation by all elements of $\mathrm{GL}_r(F)$. By construction these seem to be purely analytic objects, but in this part we identify them with objects from algebraic geometry, as follows.

Roughly speaking, the quotient $\Gamma\backslash\Omega^r$ is the set of $\mathbb{C}_\infty$-valued points of a certain moduli space of Drinfeld modules M, which is an algebraic variety over $\mathbb{C}_\infty$. The transformation law means that weak modular forms of weight k can be interpreted as holomorphic sections of $\mathcal{L}^k$ for a certain invertible sheaf $\mathcal{L}$ on M, at least if Γ is sufficiently small. Here $\mathcal{L}$ is the dual of the relative Lie algebra of the universal Drinfeld module over M. Since M is affine of dimension $r-1$, for $r \geqslant 2$ there is an abundance of non-algebraic holomorphic sections of $\mathcal{L}^k$. (So the analogue of the Köcher principle for Siegel modular forms does not hold.)

To algebraise Drinfeld modular forms, we translate the condition at infinity into a condition on a compactification $\overline{M}$ of the moduli space M. For this we use the Satake compactification that was constructed analytically by Kapranov [Ka87] in the special case $A = \mathbb{F}_q[t]$ and by Häberli [Hä21] in general, and algebraically by the third author in [Pi13]. By [Pi13] the sheaf $\mathcal{L}$ extends naturally to an invertible sheaf on $\overline{M}$, again denoted $\mathcal{L}$, which is constructed as the dual of the relative Lie algebra of the unique generalised Drinfeld module over $\overline{M}$ that extends the universal Drinfeld module over M.

The main result of Part II, Theorem 10.9, states that the analytic Drinfeld modular forms of weight k correspond precisely to the sections of $\mathcal{L}^k$ over $\overline{M}$. Since $\overline{M}$ is a projective algebraic variety, it follows that the space of modular forms of each weight k is finite dimensional, and that the graded ring of modular forms of all weights for fixed Γ is a normal integral domain that is finitely generated as a $\mathbb{C}_\infty$-algebra: see Theorem 11.1. In the case $r = 2$ all this was done in Goss's thesis [Go80b].

Establishing these results with adequate precision requires a fair amount of technical details. For later use we also discuss the action of $\mathrm{GL}_r(F)$ as well as Hecke operators.

Outline of Part II

As a preparation for the modular interpretation of $\Gamma\backslash\Omega^r$, in Section 7 we construct the universal family of Drinfeld modules over Ω^r and its level structures. We also study its behaviour at the standard boundary component. In Proposition 7.16 we show that the universal family descends to a family over $\Gamma_U\backslash\Omega^r$ which extends naturally to a generalised Drinfeld module over the larger domain $\mathcal{U}$ obtained by adjoining a copy of Ω^{r-1}.

In Section 8 we construct the precise identification of $\Gamma\backslash\Omega^r$ with a moduli space of Drinfeld modules. This requires working with the ring of finite adèles $\mathbb{A}_F^{\mathrm{f}}$ of F and identifying $\Gamma\backslash\Omega^r$ with a connected component of a double quotient of the form

$$\mathrm{GL}_r(F)\Big\backslash\big(\Omega^r\times\mathrm{GL}_r(\mathbb{A}_F^{\mathrm{f}})/K\big)$$

for an open compact subgroup $K<\mathrm{GL}_r(\hat{A})$. That in turn can be identified naturally with the space of $\mathbb{C}_\infty$-valued points $M^r_{A,K}(\mathbb{C}_\infty)$ on a certain algebraic moduli space of Drinfeld modules $M^r_{A,K}$. This identification requires a precise description of the universal family and its level structure. Working adèlically also entails that $M^r_{A,K}$ is an algebraic variety over the given global field F itself, which eventually shows that the space of modular forms for Γ comes from a vector space over a certain finite abelian extension of F instead of $\mathbb{C}_\infty$.

As explained in Remark 1.8, there are different conventions about whether Ω^r consists of row or column vectors and about how $\mathrm{GL}_r(F_\infty)$ acts on it. In this monograph we have chosen to use column vectors and left multiplication. This affects the way that the universal family of Drinfeld modules on $\mathrm{GL}_r(F)\backslash(\Omega^r\times\mathrm{GL}_r(\mathbb{A}_F^{\mathrm{f}})/K)$ must be described. As our convention differs from that of [Pi13], several formulas from there have to be transformed to be used here. For instance, in the isomorphism (8.1) a double coset $[(\omega,g)]$ now corresponds to a point on the moduli space that was represented by the double coset $[(\omega^T,(g^T)^{-1})]$ in [Pi13]. The change in convention also affects the functoriality in Proposition 8.16, in whose proof the precise relationship is indicated. We wish to apologise for the resulting inconvenience.

In Section 9 we review the relevant facts about the Satake compactification of $\overline{M}^r_{A,K}$ of $M^r_{A,K}$. The crucial properties in Proposition 9.3 are that the composite map $\Gamma_U\backslash\Omega^r\twoheadrightarrow\Gamma\backslash\Omega^r\hookrightarrow M^r_{A,K}(\mathbb{C}_\infty)$ extends to an étale morphism $\mathcal{U}\to\overline{M}^r_{A,K}(\mathbb{C}_\infty)$ for the larger domain $\mathcal{U}$ from Section 7 and that, repeating this after transformation by arbitrary elements of $\mathrm{GL}_r(\mathbb{A}_F^{\mathrm{f}})$, the images of these maps cover a Zariski open subset $M^{r,+}_{A,K}(\mathbb{C}_\infty)$ of $\overline{M}^r_{A,K}(\mathbb{C}_\infty)$ whose closed complement has codimension $\geqslant 2$. Using this map we can identify the pullback of the generalised Drinfeld module on $\overline{M}^r_{A,K}$ with that constructed over $\mathcal{U}$ in Section 7.

In Section 10 we use these facts to show that an analytic modular form is holomorphic at infinity if and only if the corresponding section of $\mathcal{L}^k$ over $M^r_{A,K}(\mathbb{C}_\infty)$ extends holomorphically to a section over $M^{r,+}_{A,K}(\mathbb{C}_\infty)$. By rigid analytic analogues of the Hartogs principle and of GAGA the latter condition is equivalent to being the restriction of a section of $\mathcal{L}^k$ over $\overline{M}^r_{A,K}(\mathbb{C}_\infty)$ in the algebro-geometric sense, thereby establishing our first main result, Theorem 10.9.

This earns us our piece of cake in Section 11, where we deduce that the space of modular forms of each weight k is finite dimensional, and that the graded ring of modular forms of all weights for fixed Γ is a normal integral domain that is finitely generated as a $\mathbb{C}_\infty$-algebra.

The final Section 12 explains how the comparison isomorphism between analytic and algebraic modular forms behaves under Hecke operators on both sides.

7 Universal family of Drinfeld modules

As a preparation for the following sections, we construct the universal family of Drinfeld modules on $\Gamma\backslash\Omega^r$ associated to an A-lattice $L \subset F^r$ and study its behaviour at the standard boundary component. We first review the necessary details about Drinfeld modules and generalised Drinfeld modules.

Consider any scheme S over F. For any line bundle E on S, let $\mathrm{End}_{\mathbb{F}_q}(E)$ denote the ring of $\mathbb{F}_q$-linear endomorphisms of the group scheme underlying E. (These endomorphisms need not commute with scalar multiplication by $\mathcal{O}_S$.) By [Dr74, §5], any such endomorphism is a finite sum $\sum_i b_i\tau^i$ for sections $b_i \in H^0(S, E^{1-q^i})$, where $\tau : E \to E^q$, $x \mapsto x^q$ denotes the q-power Frobenius morphism. Set $\deg(a) := \dim_{\mathbb{F}_q}(A/(a))$ for any $a \in A \smallsetminus \{0\}$ and $\deg(0) := -\infty$.

Recall that a *Drinfeld A-module of rank r over S* is a pair (E,φ) consisting of a line bundle E over S and a ring homomorphism

$$\varphi\colon\ A \to \mathrm{End}_{\mathbb{F}_q}(E),\quad a \mapsto \varphi_a = \sum_{i=0}^{r\deg(a)} \varphi_{a,i}\tau^i \tag{7.1}$$

with $\varphi_{a,i} \in H^0(S, E^{1-q^i})$ satisfying the two conditions:

(a) The derivative $d\varphi : a \mapsto \varphi_{a,0}$ is the structure homomorphism $A \hookrightarrow F \to H^0(S,\mathcal{O}_S)$.

(b) For any $a \in A \smallsetminus \{0\}$ the term $\varphi_{a,r\deg(a)}$ is a nowhere vanishing section of $E^{1-q^{r\deg(a)}}$.

If instead of (b) we require only:

(c) For any point $s \in S$ and any non-constant $a \in A$ there exists $i > 0$ with $\varphi_{a,i} \not\equiv 0$;

we obtain the notion of a *generalised Drinfeld A-module of rank $\leqslant r$ over S* from [Pi13, Def. 3.1]. Over any point $s \in S$, the map φ then defines a Drinfeld A-module of some rank r_s satisfying $1 \leqslant r_s \leqslant r$.

An *isomorphism* of (generalised or not) Drinfeld A-modules over S is an isomorphism of line bundles that is equivariant with respect to the action of A on both sides. Furthermore, following [Pi13, Def. 3.8], a generalised Drinfeld A-module (E,φ) over S is called *weakly separating* if, for any Drinfeld A-module (E',φ') over any field F' containing F, at most finitely many fibers of (E,φ) over F'-valued points of S are isomorphic to (E',φ').

The analogous notions are used over a rigid analytic base S.

For the following construction we fix a finitely generated projective A-submodule $L \subset F^r$ of rank r. Recall that elements of F^r are viewed as row vectors and points in Ω^r as column vectors. Any $\omega \in \Omega^r$ thus determines an A-lattice $L\omega \subset \mathbb{C}_\infty$ of rank r. Let $e_{L\omega}$ be the associated exponential function from (2.1). For any $a \in A \smallsetminus \{0\}$ we have an inclusion of A-lattices $L\omega \subset a^{-1}L\omega$ of finite index, so $e_{L\omega}(a^{-1}L\omega)$ is a finite $\mathbb{F}_q$-subspace of $\mathbb{C}_\infty$. Thus

$$\psi_a^{L\omega} \;:=\; a \cdot e_{e_{L\omega}(a^{-1}L\omega)} \tag{7.2}$$

is a polynomial in $\mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{\mathrm{a},\mathbb{C}_\infty})$ which by Proposition 2.3 (a) and (b) satisfies the functional equation $\psi_a^{L\omega}(e_{L\omega}(z)) = e_{L\omega}(az)$. Setting also $\psi_0^{L\omega} := 0$, we obtain the Drinfeld A-module $(\mathbb{G}_{\mathrm{a},\mathbb{C}_\infty}, \psi^{L\omega})$ over $\mathbb{C}_\infty$ that is uniformised by the lattice $L\omega$. As ω varies over Ω^r, the exponential function $e_{L\omega}(z)$ is holomorphic in $(z,\omega) \in \mathbb{C}_\infty \times \Omega^r$; hence $\psi_a^{L\omega}$ is holomorphic in $\omega \in \Omega^r$ for each $a \in A$. Together this therefore defines a Drinfeld A-module

$$(\mathbb{G}_{\mathrm{a},\Omega^r}, \psi^L) \tag{7.3}$$

of rank r over Ω^r.

Also, any element $\ell \in F^r$ determines a holomorphic section

$$\mu_\ell^L :\; \omega \mapsto e_{L\omega}(\ell\omega) \tag{7.4}$$

of $\mathbb{G}_{\mathrm{a},\Omega^r}$ which depends only on the residue class $\ell + L$. For any non-zero ideal $N \subset A$ with $N\ell \subset L$ this section lies in the N-torsion subgroup $\psi^L[N]$ of ψ^L. Varying $\ell + L$ over $N^{-1}L/L$ this endows the Drinfeld A-module $(\mathbb{G}_{\mathrm{a},\Omega^r}, \psi^L)$ with a full level structure of level N by mapping

$$N^{-1}L/L \longrightarrow \psi^L[N], \quad \ell + L \mapsto \mu_\ell^L. \tag{7.5}$$

Next consider an arbitrary element $\gamma \in \mathrm{GL}_r(F)$. Then for any $\omega \in \Omega^r$ we have $L\omega = L\gamma^{-1}\gamma\omega = j(\gamma,\omega) \cdot L\gamma^{-1} \cdot \gamma(\omega)$ by (1.3). Multiplication by $j(\gamma,\omega)^{-1}$ thus induces an isomorphism of Drinfeld A-modules

$$(\mathbb{G}_{\mathrm{a},\mathbb{C}_\infty}, \psi^{L\omega}) \;\overset{\sim}{\longrightarrow}\; (\mathbb{G}_{\mathrm{a},\mathbb{C}_\infty}, \psi^{L\gamma^{-1}\cdot\gamma(\omega)}). \tag{7.6}$$

Here the target is the pullback of the Drinfeld A-module $(\mathbb{G}_{\mathrm{a},\Omega^r}, \psi^{L\gamma^{-1}})$ via the isomorphism $\gamma : \Omega^r \to \Omega^r$, $\omega \mapsto \gamma(\omega)$, evaluated at ω. Multiplication by the holomorphic function $j(\gamma,\_)^{-1}$ thus induces an isomorphism of Drinfeld A-modules

$$(\mathbb{G}_{\mathrm{a},\Omega^r}, \psi^L) \;\overset{\sim}{\longrightarrow}\; \gamma^*(\mathbb{G}_{\mathrm{a},\Omega^r}, \psi^{L\gamma^{-1}}) \tag{7.7}$$

over Ω^r. Also, for any $\ell \in F^r$, using Proposition 2.3 (b) we can calculate

$$\begin{aligned} \mu_\ell^L(\omega) &= e_{L\omega}(\ell\omega) \\ &= e_{j(\gamma,\omega)\cdot L\gamma^{-1}\cdot\gamma(\omega)}\big(j(\gamma,\omega)\cdot \ell\gamma^{-1}\cdot\gamma(\omega)\big) \\ &= j(\gamma,\omega)\cdot e_{L\gamma^{-1}\cdot\gamma(\omega)}\big(\ell\gamma^{-1}\cdot\gamma(\omega)\big) \\ &= j(\gamma,\omega)\cdot \mu_{\ell\gamma^{-1}}^{L\gamma^{-1}}(\gamma(\omega)). \end{aligned} \tag{7.8}$$

Multiplication by $j(\gamma,\_)^{-1}$ thus also sends the level N structure $\ell + L \mapsto \mu_\ell^L$ of $(\mathbb{G}_{\mathrm{a},\Omega^r}, \psi^L)$ to the level N structure $\ell\gamma^{-1} + L\gamma^{-1} \mapsto \gamma^*\mu_{\ell\gamma^{-1}}^{L\gamma^{-1}}$ of $\gamma^*(\mathbb{G}_{\mathrm{a},\Omega^r}, \psi^{L\gamma^{-1}})$.

Now let $\Gamma < \mathrm{GL}_r(F)$ be an arithmetic subgroup whose right action on F^r normalises the lattice L. Recall from [Dr74, Prop. 6.2] that $\Gamma < \mathrm{GL}_r(F)$ acts discontinuously on Ω^r; hence the quotient $\Gamma\backslash\Omega^r$ exists as a rigid analytic space by [FvdP04, §6.4]. Let $\pi_\Gamma : \Omega^r \twoheadrightarrow \Gamma\backslash\Omega^r$ denote the projection morphism.

Assume that Γ acts freely on Ω^r. Then Γ also acts freely on $\mathbb{G}_{\mathrm{a},\Omega^r} = \mathbb{G}_{\mathrm{a}} \times \Omega^r$ through $\gamma(z,\omega) := (j(\gamma,\omega)^{-1}z, \gamma(\omega))$, so the quotient $E_\Gamma := \Gamma\backslash(\mathbb{G}_{\mathrm{a}} \times \Omega^r)$ exists and is a line bundle on $\Gamma\backslash\Omega^r$. By construction the space of its sections over any open subset $U \subset \Gamma\backslash\Omega^r$ is

$$(7.9)\quad E_\Gamma(U) := \left\{ f : \pi_\Gamma^{-1}(U) \to \mathbb{C}_\infty \text{ holomorphic } \,\middle|\, \forall \gamma \in \Gamma : \ f(\gamma(\omega)) = j(\gamma,\omega)^{-1} f(\omega) \right\}.$$

This line bundle comes with a natural isomorphism

$$(7.10)\qquad \pi_\Gamma^* E_\Gamma \xrightarrow{\sim} \mathbb{G}_{\mathrm{a},\Omega^r}.$$

For any $\gamma \in \Gamma$ the equality $\pi_\Gamma = \pi_\Gamma \circ \gamma$ induces a commutative diagram

$$(7.11)\qquad \begin{array}{ccccc} \pi_\Gamma^* E_\Gamma & & \xrightarrow[\sim]{(7.10)} & & \mathbb{G}_{\mathrm{a},\Omega^r} \\ \| & & & & \downarrow\wr \\ \gamma^*\pi_\Gamma^* E_\Gamma & \xrightarrow[\sim]{(7.10)} & \gamma^*\mathbb{G}_{\mathrm{a},\Omega^r} & = & \mathbb{G}_{\mathrm{a},\Omega^r}, \end{array}$$

where the vertical map on the right is multiplication by $j(\gamma,\_)^{-1}$. The isomorphism (7.7) for all $\gamma \in \Gamma$ implies that there is a unique Drinfeld A-module of the form $(E_\Gamma, \bar\psi^L)$ over $\Gamma\backslash\Omega^r$ such that (7.10) induces an isomorphism

$$(7.12)\qquad \pi_\Gamma^*(E_\Gamma, \bar\psi^L) \xrightarrow{\sim} (\mathbb{G}_{\mathrm{a},\Omega^r}, \psi^L).$$

Moreover, since Γ normalises L, it acts on $N^{-1}L/L$ for any non-zero ideal $N \subset A$. For any residue class $\ell + L$ that is fixed by Γ, the formula (7.8) implies that the associated torsion point μ_ℓ^L descends to a torsion point $\bar\mu_\ell^L$ of $(E_\Gamma, \bar\psi^L)$. In particular, if Γ acts trivially on $N^{-1}L/L$, the level N structure (7.5) descends to a unique level N structure of $(E_\Gamma, \bar\psi^L)$

$$(7.13)\qquad N^{-1}L/L \longrightarrow \bar\psi^L[N], \quad \ell + L \mapsto \bar\mu_\ell^L.$$

Now set $\Gamma_U := \Gamma \cap U(F)$ as in (4.2) and let $\Lambda' := \iota^{-1}(\Gamma_U) \subset F^{r-1}$ be the corresponding subgroup from (4.4), which is commensurable with A^{r-1}. Then by Theorem 4.16 there exist an admissible open subset $\mathcal{U} \subset \mathbb{C}_\infty \times \Omega^{r-1}$ containing $\{0\} \times \Omega^{r-1}$ and a holomorphic map

$$(7.14)\qquad \vartheta : \Gamma_U\backslash\Omega^r \longrightarrow \mathcal{U}, \quad \left[\binom{\omega_1}{\omega'}\right] \longmapsto \binom{e_{\Lambda'\omega'}(\omega_1)^{-1}}{\omega'}$$

which induces an isomorphism of rigid analytic spaces $\Gamma_U\backslash\Omega^r \xrightarrow{\sim} \mathcal{U} \cap (\mathbb{C}_\infty^\times \times \Omega^{r-1})$. Also π_Γ factors through projection morphisms

$$\Omega^r \xrightarrow{\pi_{\Gamma_U}} \Gamma_U\backslash\Omega^r \xrightarrow{\pi_\Gamma^{\Gamma_U}} \Gamma\backslash\Omega^r, \qquad \pi_\Gamma = \pi_\Gamma^{\Gamma_U} \circ \pi_{\Gamma_U}.$$

For all $\gamma \in \Gamma_U$, the definition (1.2) implies that $j(\gamma, \omega) = 1$ and hence $e_{L\gamma(\omega)} = e_{L\omega}$ and $\psi_a^{L\gamma(\omega)} = \psi_a^{L\omega}$. For ease of notation we denote the function on $\mathbb{G}_\mathrm{a} \times \Gamma_U \backslash \Omega^r$ induced by $\psi_a^{L\omega}$ again by $\psi_a^{L\omega}$. Then the Drinfeld A-module $(\mathbb{G}_{\mathrm{a},\Omega^r}, \psi^L)$ is the pullback under π_{Γ_U} of a unique Drinfeld A-module of the form $(\mathbb{G}_{\mathrm{a},\Gamma_U\backslash\Omega^r}, \psi^L)$ over $\Gamma_U\backslash\Omega^r$. Moreover the isomorphism (7.12) descends to a natural isomorphism

$$(7.15) \qquad (\pi_\Gamma^{\Gamma_U})^*(E_\Gamma, \bar{\psi}^L) \xrightarrow{\sim} (\mathbb{G}_{\mathrm{a},\Gamma_U\backslash\Omega^r}, \psi^L).$$

Proposition 7.16 *There exists a unique generalised Drinfeld A-module of the form* $(\mathbb{G}_{\mathrm{a},\mathcal{U}}, \tilde{\psi}^L)$ *over* $\mathcal{U}$ *such that*

$$(\mathbb{G}_{\mathrm{a},\Gamma_U\backslash\Omega^r}, \psi^L) \;=\; \vartheta^*(\mathbb{G}_{\mathrm{a},\mathcal{U}}, \tilde{\psi}^L).$$

Its restriction to $\{0\} \times \Omega^{r-1} \subset \mathcal{U}$ *is a Drinfeld A-module of constant rank* $r-1$.

Proof. Since ϑ defines an isomorphism between $\Gamma_U\backslash\Omega^r$ and its image $\mathcal{U}' := \mathcal{U} \cap (\mathbb{C}_\infty^\times \times \Omega^{r-1}$, it is trivial to transfer the rank r Drinfeld module ψ^L from $\Gamma_U\backslash\Omega^r$ to $\mathcal{U}$. The real content of the Proposition is that it extends to a generalised Drinfeld module on $\mathcal{U}$. The strategy of the proof is to start with the exponential function $\mathbb{C}_\infty \times (\mathcal{U} \cap (\mathbb{C}_\infty^\times \times \Omega^{r-1})) \longrightarrow \mathbb{C}_\infty$, $(z, \vartheta([\omega])) \mapsto e_{L\omega}(z)$ associated to the Drinfeld A-module ψ^L rather than the Drinfeld module ψ^L itself, because the Drinfeld module can always be reconstructed from the exponential function. In the first part of the proof, we translate the formula for the exponential function associated to ψ^L to $\mathcal{U}'$. More specifically, by writing $\omega = \binom{\omega_1}{\omega'}$ as before, we will express $e_{L\omega}(z)$ as an infinite product in the variables (z, u, ω') for $u = u_{\omega'}(\omega_1) := e_{\Lambda'\omega'}(\omega_1)^{-1}$.

For this we define subgroups L' and L_1 by the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F^{r-1} & \xrightarrow{\quad v' \mapsto (0,v') \quad} & F^r & \xrightarrow{\quad (v_1,v') \mapsto v_1 \quad} & F & \longrightarrow & 0 \\
 & & \cup & & \cup & & \cup & & \\
0 & \longrightarrow & L' & \longrightarrow & L & \longrightarrow & L_1 & \longrightarrow & 0.
\end{array}$$

Since L is commensurable with A^r, the subgroups L' and L_1 are commensurable with A^{r-1} and A, respectively. Next, for any $(\ell_1, v') \in L$ and any $\lambda' \in \Lambda'$ we have $\left(\begin{smallmatrix} 1 & \lambda' \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma_U$ and hence $(\ell_1, v')\left(\begin{smallmatrix} 1 & \lambda' \\ 0 & 1 \end{smallmatrix}\right) = (\ell_1, \ell_1\lambda' + v') \in L$. In particular this implies that $\ell_1\Lambda' \subset L'$. As both Λ' and L' are commensurable with A^{r-1}, this is an inclusion of finite index if $\ell_1 \neq 0$.

Next we fix a subgroup $\tilde{L}_1 \subset L$ which maps isomorphically to L_1 under the projection $F^r \twoheadrightarrow F$. Then for any $\omega = \binom{\omega_1}{\omega'} \in \Omega^r$ we have $L\omega = \tilde{L}_1\omega \oplus L'\omega'$. Using Proposition 2.3 (a) and the definition (2.1) of the exponential function, for any $z \in \mathbb{C}_\infty$ we thus have

$$(7.17) \qquad e_{L\omega}(z) \;=\; e_{e_{L'\omega'}(L\omega)}(e_{L'\omega'}(z)) \;=\; \tilde{z} \cdot \prod_{\ell \in \tilde{L}_1 \smallsetminus \{0\}} \Big(1 - \frac{\tilde{z}}{e_{L'\omega'}(\ell\omega)}\Big)$$

with $\tilde{z} = e_{L'\omega'}(z)$. To transform the denominator write $\ell \in \tilde{L}_1 \smallsetminus \{0\}$ in the form $\ell = (\ell_1, v')$ with $\ell_1 \in L_1 \smallsetminus \{0\}$ and $v' \in F^{r-1}$. Then we have an inclusion of lattices $\Lambda'\omega' \subset \ell_1^{-1}L'\omega'$, and by the F_∞-linear independence of the coefficients of ω' the index is precisely $[L' : \ell_1\Lambda'] < \infty$. By the additivity of the exponential function we have

$$e_{\Lambda'\omega'}(\ell_1^{-1}\ell\omega) \;=\; e_{\Lambda'\omega'}(\omega_1 + \ell_1^{-1}v'\omega') \;=\; u^{-1} + e_{\Lambda'\omega'}(\ell_1^{-1}v'\omega')$$

with $u = e_{\Lambda'\omega'}(\omega_1)^{-1}$. Using Proposition 2.3 again we deduce that

$$\begin{aligned} e_{L'\omega'}(\ell\omega) &= \ell_1 \cdot e_{\ell_1^{-1}L'\omega'}(\ell_1^{-1}\ell\omega) \\ &= \ell_1 \cdot e_{e_{\Lambda'\omega'}(\ell_1^{-1}L'\omega')}\big(e_{\Lambda'\omega'}(\ell_1^{-1}\ell\omega)\big) \\ &= \ell_1 \cdot e_{e_{\Lambda'\omega'}(\ell_1^{-1}L'\omega')}\big(u^{-1} + e_{\Lambda'\omega'}(\ell_1^{-1}v'\omega')\big). \end{aligned}$$

By the definition and the additivity of the exponential function this in turn yields

$$\begin{aligned} e_{L'\omega'}(\ell\omega) &= \ell_1 \cdot \big(u^{-1} + e_{\Lambda'\omega'}(\ell_1^{-1}v'\omega')\big) \cdot \prod_{\substack{\ell' \in L' \smallsetminus \ell_1\Lambda' \\ \text{modulo } \ell_1\Lambda'}} \left(1 - \frac{u^{-1} + e_{\Lambda'\omega'}(\ell_1^{-1}v'\omega')}{e_{\Lambda'\omega'}(\ell_1^{-1}\ell'\omega')}\right) \\ &= \ell_1 \cdot \big(u^{-1} + e_{\Lambda'\omega'}(\ell_1^{-1}v'\omega')\big) \cdot \prod_{\substack{\ell' \in L' \smallsetminus \ell_1\Lambda' \\ \text{modulo } \ell_1\Lambda'}} \frac{e_{\Lambda'\omega'}(\ell_1^{-1}(\ell'-v')\omega') - u^{-1}}{e_{\Lambda'\omega'}(\ell_1^{-1}\ell'\omega')} \\ &= \ell_1 \cdot \frac{1 + e_{\Lambda'\omega'}(\ell_1^{-1}v'\omega')\cdot u}{u^{[L':\ell_1\Lambda']}} \cdot \prod_{\substack{\ell' \in L' \smallsetminus \ell_1\Lambda' \\ \text{modulo } \ell_1\Lambda'}} \frac{e_{\Lambda'\omega'}(\ell_1^{-1}(\ell'-v')\omega')\cdot u - 1}{e_{\Lambda'\omega'}(\ell_1^{-1}\ell'\omega')} \\ &= \frac{\ell_1}{u^{[L':\ell_1\Lambda']}} \cdot \frac{\prod_{\ell' \in L' \bmod \ell_1\Lambda'} \big(1 - e_{\Lambda'\omega'}(\ell_1^{-1}(\ell'-v')\omega')\cdot u\big)}{\prod_{\ell' \in L' \smallsetminus \ell_1\Lambda' \bmod \ell_1\Lambda'} e_{\Lambda'\omega'}(\ell_1^{-1}\ell'\omega')}, \end{aligned}$$

where the last transformation uses the fact that $(-1)^{[L':\ell_1\Lambda']-1} = 1$ because $[L' : \ell_1\Lambda']$ is a power of q. Plugging this into the formula (7.17) we conclude that

$$(7.18) \qquad e_{L\omega}(z) = \tilde{z} \cdot \prod_{(\ell_1,v') \in \tilde{L}_1 \smallsetminus \{0\}} \left(1 - \tilde{z} \cdot \frac{u^{[L':\ell_1\Lambda']}}{\ell_1} \cdot \frac{\prod_{\ell' \in L' \smallsetminus \ell_1\Lambda' \bmod \ell_1\Lambda'} e_{\Lambda'\omega'}(\ell_1^{-1}\ell'\omega')}{\prod_{\ell' \in L' \bmod \ell_1\Lambda'} \big(1 - e_{\Lambda'\omega'}(\ell_1^{-1}(\ell'-v')\omega')\cdot u\big)}\right).$$

As (ℓ_1, ℓ') runs through $\tilde{L}_1 \smallsetminus \{0\}$, the index $[L' : \ell_1\Lambda']$ goes to infinity. Using the geometric series we can therefore expand the right hand side of (7.18) as a power series in u whose coefficients are functions of $(\tilde{z}, \omega_1)$.

In the second part of the proof, we will show that this expression converges locally uniformly for all $\tilde{z} \in \mathbb{C}_\infty$ and all (u, ω_1) in a suitable tubular neighbourhood of $\{0\} \times \Omega^{r-1}$. In particular, it will also converge for all $z \in \mathbb{C}_\infty$ when $u = 0$ and thus extends to an exponential function on a tubular neighbourhood containing $\{0\} \times \Omega^{r-1}$.

For this take any $n > 0$. By Proposition 4.7 (c) there exists a constant $c_n > 0$, such that for any $\omega' \in \Omega_n^{r-1}$ and any $v' \in F_\infty^{r-1}$ we have $|e_{\Lambda'\omega'}(v'\omega')| < c_n$. In particular this inequality holds for $\ell_1^{-1}\ell'$ and $\ell_1^{-1}(\ell'-v')$ in place of v'. Thus if $|u| \leqslant r_n := (2c_n)^{-1}$, we have $|e_{\Lambda'\omega'}(\ell_1^{-1}(\ell'-v')\omega')\cdot u| < 2^{-1}$, so the geometric series for

$$\frac{1}{1 - e_{\Lambda'\omega'}(\ell_1^{-1}(\ell'-v')\omega')\cdot u}$$

converges uniformly to a value of norm 1. Combining the inequalities yields the bound

$$\left|\frac{u^{[L':\ell_1\Lambda']}}{\ell_1}\cdot\frac{\prod\limits_{\ell'\in L'\smallsetminus\ell_1\Lambda' \bmod \ell_1\Lambda'} e_{\Lambda'\omega'}(\ell_1^{-1}\ell'\omega')}{\prod\limits_{\ell'\in L' \bmod \ell_1\Lambda'}\big(1-e_{\Lambda'\omega'}(\ell_1^{-1}(\ell'-v')\omega')\cdot u\big)}\right| \;\leqslant\; \frac{r_n^{[L':\ell_1\Lambda']}c_n^{[L':\ell_1\Lambda']-1}}{|\ell_1|} \;=\; \frac{2^{-[L':\ell_1\Lambda']}}{|\ell_1|c_n}.$$

As both $|\ell_1|$ and $[L':\ell_1\Lambda']$ go to infinity with ℓ_1, for any $R>0$ this proves that the right hand side of (7.18) converges uniformly for all $(\tilde{z},u,\omega')\in B(0,R)\times B(0,r_n)\times\Omega_n^{r-1}$. Varying n and R it therefore converges locally uniformly on $\mathbb{C}_\infty\times\mathcal{T}$ for the tubular neighbourhood $\mathcal{T}:=\bigcup_{n\geqslant1}B(0,r_n)\times\Omega_n^{r-1}$ and the limit is a holomorphic function of $(\tilde{z},u,\omega')$. Substituting $\tilde{z}=e_{L'\omega'}(z)$, which is already a holomorphic function of $(z,\omega')\in\mathbb{C}_\infty\times\Omega^{r-1}$, thus yields a holomorphic function $E(z,u,\omega')$ on $\mathbb{C}_\infty\times\mathcal{T}$ such that

$$(7.19)\qquad e_{L\omega}(z) \;=\; E(z,e_{\Lambda'\omega'}(\omega_1)^{-1},\omega')$$

for all $z\in\mathbb{C}_\infty$ and $\omega=\binom{\omega_1}{\omega'}\in\Omega^r$ with $\vartheta([\omega])\in\mathcal{T}$.

In the third and final part of the proof, we show how this exponential function on $\mathbb{C}_\infty\times\mathcal{T}$ gives rise to a generalised Drinfeld module and do all the necessary checks to show that it is a generalised Drinfeld module whose restriction to $\{0\}\times\Omega^{r-1}$ has constant rank $r-1$.

Recall that for any $\omega\in\Omega^r$, the Drinfeld A-module $\psi^{L\omega}$ is characterised by the fact that for each $a\in A\smallsetminus\{0\}$ the function $\psi_a^{L\omega}$ is an $\mathbb{F}_q$-linear polynomial in $\mathbb{C}_\infty[z]$ satisfying the functional equation $\psi_a^{L\omega}(e_{L\omega}(z))=e_{L\omega}(az)$. Writing this as an identity of power series in z and observing that $e_{L\omega}(z)=z+(\text{higher terms})$, it follows that each coefficient of $\psi_a^{L\omega}$ is a certain polynomial with coefficients in A in finitely many coefficients of $e_{L\omega}(z)$. By what we have just proved, these coefficients, as functions of $(e_{\Lambda'\omega'}(\omega_1)^{-1},\omega')$, extend to holomorphic functions of $(u,\omega')\in\mathcal{T}$. Thus the same is true for the coefficients of $\psi_a^{L\omega}$. In other words, there is a unique holomorphic function $\tilde{\psi}_a^L$ on $\mathbb{C}_\infty\times\mathcal{T}$, which is an $\mathbb{F}_q$-linear polynomial of degree $\leqslant r\deg(a)$ in z, such that

$$(7.20)\qquad \psi_a^{L\omega}(z) \;=\; \tilde{\psi}_a^L(z,e_{\Lambda'\omega'}(\omega_1)^{-1},\omega')$$

for all $z\in\mathbb{C}_\infty$ and $\omega=\binom{\omega_1}{\omega'}\in\Omega^r$ with $\vartheta([\omega])\in\mathcal{T}$. Setting $\tilde{\psi}_0^L:=0$, the fact that $a\mapsto\psi_a^L$ is an $\mathbb{F}_q$-algebra homomorphism by continuity implies that $a\mapsto\tilde{\psi}_a^L$ is also $\mathbb{F}_q$-algebra homomorphism. Moreover, the fact that $\frac{\partial}{\partial z}\psi_a^L=a$ identically implies that $\frac{\partial}{\partial z}\tilde{\psi}_a^L=a$ identically as well. Furthermore, by continuity the functional equation $\psi_a^{L\omega}(e_{L\omega}(z))=e_{L\omega}(az)$ extends to a functional equation

$$(7.21)\qquad \tilde{\psi}_a^L\big(E(z,u,\omega'),u,\omega'\big) \;=\; E(az,u,\omega')$$

for all $z\in\mathbb{C}_\infty$ and $(u,\omega')\in\mathcal{T}$. If we substitute $u:=0$, the right hand side of (7.18) becomes just $\tilde{z}=e_{L'\omega'}(z)$; hence $E(z,0,\omega')=e_{L'\omega'}(z)$. Thus (7.21) reduces to the equation

$$(7.22)\qquad \tilde{\psi}_a^L\big(e_{L'\omega'}(z),0,\omega'\big) \;=\; e_{L'\omega'}(az).$$

For any $\omega' \in \Omega^{r-1}$ the map $a \mapsto \tilde{\psi}_a^L(\_, 0, \omega')$ is therefore the Drinfeld A-module of rank $r-1$ associated to the lattice $L'\omega' \subset \mathbb{C}_\infty$. All this together proves that $a \mapsto \tilde{\psi}_a^L$ constitutes a generalised Drinfeld A-module of rank $\leqslant r$ over $\mathcal{T}$, whose restriction to the locus $u = 0$ is a Drinfeld A-module of constant rank $r-1$.

We have thus proved the desired statement over $\mathcal{T}$. Since $\tilde{\psi}^L$ is already given over $\mathcal{U} \cap (\mathbb{C}_\infty^\times \times \Omega^{r-1})$, the existence and uniqueness also follows over $\mathcal{U}$, as desired. □

8 Drinfeld moduli spaces

Let $\hat{A} \cong \prod_\mathfrak{p} A_\mathfrak{p}$ be the profinite completion of A and $\mathbb{A}_F^\mathrm{f} = \hat{A} \otimes_A F$ the ring of finite adèles of F. For any open compact subgroup $K < \mathrm{GL}_r(\hat{A})$ let $M_{A,K}^r$ be the *Drinfeld modular variety of level* K, which is a normal integral affine algebraic variety over F. The associated rigid analytic space over $\mathbb{C}_\infty$ possesses a natural isomorphism

(8.1) $$\mathrm{GL}_r(F) \backslash \big(\Omega^r \times \mathrm{GL}_r(\mathbb{A}_F^\mathrm{f})/K\big) \xrightarrow{\sim} M_{A,K}^r(\mathbb{C}_\infty),$$

whose precise characterisation we shall describe below. For any $g \in \mathrm{GL}_r(\mathbb{A}_F^\mathrm{f})$ let π_g denote the composite morphism

(8.2) $$\begin{array}{ccccc} \Omega^r & \longrightarrow & \mathrm{GL}_r(F) \backslash \big(\Omega^r \times \mathrm{GL}_r(\mathbb{A}_F^\mathrm{f})/K\big) & \xrightarrow[\sim]{(8.1)} & M_{A,K}^r(\mathbb{C}_\infty), \\ [\omega] & \longmapsto & [(\omega, g)]. & & \end{array}$$

Consider the arithmetic subgroup

(8.3) $$\Gamma_g \ := \ \mathrm{GL}_r(F) \cap gKg^{-1}.$$

Then π_g factors through an isomorphism $\Gamma_g \backslash \Omega^r \xrightarrow{\sim} M_g(\mathbb{C}_\infty)$ for a unique connected component M_g of $M_{A,K}^r \times_{\mathrm{Spec}\, F} \mathrm{Spec}\, \mathbb{C}_\infty$. In other words we have a commutative diagram

(8.4) $$\begin{array}{ccc} \Omega^r & \xrightarrow{\ \pi_g\ } & M_{A,K}^r(\mathbb{C}_\infty) \\ \big\downarrow{\scriptstyle \pi_{\Gamma_g}} & \overset{i_g}{\nearrow} & \cup \\ \Gamma_g \backslash \Omega^r & \xrightarrow{\ \sim\ } & M_g(\mathbb{C}_\infty). \end{array}$$

For any $\gamma \in \mathrm{GL}_r(F)$ and $k \in K$ we have $[(\omega, g)] = [(\gamma(\omega), \gamma g k)]$ and hence

(8.5) $$\pi_g = \pi_{\gamma g k} \circ \gamma.$$

For any two elements g, $g' \in \mathrm{GL}_r(\mathbb{A}_F^\mathrm{f})$ we have $M_g = M_{g'}$ if and only if g and g' represent the same double coset in $\mathrm{GL}_r(F) \backslash \mathrm{GL}_r(\mathbb{A}_F^\mathrm{f})/K$. Thus for any choice of representatives $g_1, \ldots, g_n \in \mathrm{GL}_r(\mathbb{A}_F^\mathrm{f})$ we have

(8.6) $$M_{A,K}^r \times_{\mathrm{Spec}\, F} \mathrm{Spec}\, \mathbb{C}_\infty \ = \ \coprod_{i=1}^n M_{g_i}.$$

Since $M^r_{A,K}$ is integral, these connected components over $\mathbb{C}_\infty$ are Galois conjugate over F. Let F_K denote the field of constants of $M^r_{A,K}$ (which is a certain ray class field of F that can be characterised uniquely by abelian class field theory). Then the different connected components M_{g_i} are just the varieties obtained by base change $M^r_{A,K} \times_{\operatorname{Spec} F_K} \operatorname{Spec} \mathbb{C}_\infty$ for all F-linear embeddings $F_K \hookrightarrow \mathbb{C}_\infty$.

For later use we also record:

Proposition 8.7 *Elements $g_1, \ldots, g_n \in \mathrm{GL}_r(\mathbb{A}^{\mathrm{f}}_F)$ form representatives of the double quotient $\mathrm{GL}_r(F) \backslash \mathrm{GL}_r(\mathbb{A}^{\mathrm{f}}_F)/K$ if and only if their determinants $\det(g_1), \ldots, \det(g_n)$ form representatives of $F^\times \backslash (\mathbb{A}^{\mathrm{f}}_F)^\times / \det(K)$.*

Proof. Direct consequence of strong approximation [Ma91, (6.8)], [Pr77] for the simply connected reductive group SL_r to the effect that the closure of $\mathrm{SL}_r(F)$ in $\mathrm{GL}_r(\mathbb{A}^{\mathrm{f}}_F)$ is $\mathrm{SL}_r(\mathbb{A}^{\mathrm{f}}_F)$. ☐

Now assume that K is *fine*, which by [Pi13, Def. 1.4] means that the image of K in $\mathrm{GL}_r(A/\mathfrak{p})$ is unipotent for some maximal ideal $\mathfrak{p} \subset A$. Then by [Pi13, Prop. 1.5] there is a natural *universal family of Drinfeld A-modules* (E, φ) over $M^r_{A,K}$, using which one can interpret $M^r_{A,K}$ as a fine moduli space of Drinfeld A-modules with some generalised level structure. The pullback of (E, φ) under the morphism (8.1) can be described as follows. Viewing elements of F^r and $\hat{A}^r$ and $(\mathbb{A}^{\mathrm{f}}_F)^r$ as row vectors, for any $g \in \mathrm{GL}_r(\mathbb{A}^{\mathrm{f}}_F)$ we set

$$L_g \;:=\; \hat{A}^r g^{-1} \cap F^r \;\subset\; (\mathbb{A}^{\mathrm{f}}_F)^r, \tag{8.8}$$

which is a finitely generated projective A-module of rank r. Since $K < \mathrm{GL}_r(\hat{A})$, by construction the right action of Γ_g on F^r normalises L_g. Moreover, the assumption that K is fine implies that all torsion elements of Γ_g are unipotent; hence Γ_g acts freely on Ω^r. There is therefore a natural Drinfeld A-module $(E_{\Gamma_g}, \bar{\psi}^{L_g})$ over $\Gamma_g \backslash \Omega^r$ such that $\pi^*_{\Gamma_g}(E_{\Gamma_g}, \bar{\psi}^{L_g}) \cong (\mathbb{G}_{\mathrm{a},\Omega^r}, \psi^{L_g})$ by (7.12). For this there is a natural isomorphism

$$i_g^*(E, \varphi) \;\xrightarrow{\sim}\; (E_{\Gamma_g}, \bar{\psi}^{L_g}). \tag{8.9}$$

Moreover, suppose that K is the principal congruence subgroup of level N

$$K(N) \;:=\; \bigl\{ k \in \mathrm{GL}_r(\hat{A}) \bigm| k \equiv \mathrm{Id}_r \bmod N \bigr\}$$

for some non-zero ideal $N \subset A$. Then $M^r_{A,K(N)}$ represents the functor which to any scheme S over F associates the set of isomorphism classes of tuples (E, φ, μ) consisting of a Drinfeld A-module (E, φ) of rank r over S and a full level N structure $\mu : N^{-1}A^r/A^r \to \varphi[N]$. For any $g \in \mathrm{GL}_r(\mathbb{A}^{\mathrm{f}}_F)$ we then have

$$\Gamma_g \;=\; \bigl\{ \gamma \in \mathrm{GL}_r(F) \bigm| (\ell + L_g)\gamma = \ell + L_g \text{ for all } \ell \in N^{-1}L_g \bigr\}.$$

Thus the Drinfeld A-module $(E_{\Gamma_g}, \bar{\psi}^{L_g})$ on $\Gamma_g \backslash \Omega^r$ is endowed with a full level N structure $\bar{\mu}^{L_g} : N^{-1}L_g/L_g \to \bar{\psi}^{L_g}[N]$ by (7.13). To any coset $\ell + A^r \subset N^{-1}A^r$ associate the coset

$$\ell_g + L_g \;:=\; (\ell + \hat{A}^r)g^{-1} \cap F^r \;\subset\; N^{-1}L_g. \tag{8.10}$$

This induces an isomorphism $N^{-1}A^r/A^r \xrightarrow{\sim} N^{-1}L_g/L_g$. The isomorphism (8.9) sends the level N structure $\ell + A^r \mapsto i_g^*\mu(\ell + A^r)$ to the level N structure $\ell + A^r \mapsto \ell_g + L_g \mapsto \bar{\mu}_\ell^L$. In fact this characterises the isomorphism (8.9) uniquely. Moreover, since $M^r_{A,K(N)}$ is a fine moduli space for Drinfeld A-modules with a full level N structure, this also characterises the isomorphism (8.1) uniquely in this case.

For an arbitrary open compact subgroup K, choose any N such that $K(N) \triangleleft K$. Then the finite group $K/K(N)$ acts on $M^r_{A,K(N)}$ by transforming the level N structure, and the quotient is naturally isomorphic to $M^r_{A,K}$. The group $K/K(N)$ also acts by right multiplication on $\mathrm{GL}_r(F)\backslash(\Omega^r \times \mathrm{GL}_r(\mathbb{A}_F^{\mathrm{f}})/K(N))$, and the isomorphism (8.1) in the case of K is obtained from that in the case of $K(N)$ by taking quotients. In particular, the two instances of the map i_g from (8.4) for K and $K(N)$ form a commutative diagram with the projection $M^r_{A,K(N)} \twoheadrightarrow M^r_{A,K}$.

Similarly, if K is fine, in [Pi13, Prop. 1.5] the universal family on $M^r_{A,K}$ was constructed precisely so that its pullback is the given universal family over $M^r_{A,K(N)}$. The isomorphism (8.9) in the case of K is the unique one whose pullback yields the isomorphism (8.9) in the case of $K(N)$.

It is useful to know that isomorphisms of Drinfeld modules can be characterised uniquely by using just one torsion point. Since K is fine, by definition its image in $\mathrm{GL}_r(A/\mathfrak{p})$ is unipotent for some maximal ideal $\mathfrak{p} \subset A$, and so it fixes some non-zero coset $\ell + \hat{A}^r \subset \mathfrak{p}^{-1}\hat{A}^r$. For each $g \in \mathrm{GL}_r(\mathbb{A}_F^{\mathrm{f}})$ the subgroup Γ_g then fixes the corresponding coset $\ell_g + L_g \subset \mathfrak{p}^{-1}L_g$ defined by (8.10). The associated torsion point $\mu_{\ell_g}^{L_g}$ thus descends to a nowhere zero $\mathfrak{p}$-torsion point of $(E_{\Gamma_g}, \bar{\psi}^{L_g})$ over $\Gamma_g\backslash\Omega^r$. On the other hand, choosing $N \subset \mathfrak{p}$, the group $K/K(N)$ fixes the coset $\ell + \hat{A}^r$; hence the associated $\mathfrak{p}$-torsion point coming from the level N structure descends to a nowhere zero $\mathfrak{p}$-torsion point of the universal family (E,φ) over $M^r_{A,K}$. By construction the isomorphism (8.9) identifies the respective $\mathfrak{p}$-torsion points. As any isomorphism of Drinfeld modules is scalar and hence determined by the image of any non-zero point, it follows that the isomorphism is uniquely characterised by this.

In the following we care mostly about the composite isomorphism

$$\pi_g^*(E,\varphi) = \pi_{\Gamma_g}^* i_g^*(E,\varphi) \xrightarrow[\sim]{(8.9)} \pi_{\Gamma_g}^*(E_{\Gamma_g}, \bar{\psi}^{L_g}) \xrightarrow[\sim]{(7.12)} (\mathbb{G}_{\mathrm{a},\Omega^r}, \psi^{L_g}). \tag{8.11}$$

This changes with g as follows. Consider any $g \in \mathrm{GL}_r(\mathbb{A}_F^{\mathrm{f}})$ and $\gamma \in \mathrm{GL}_r(F)$ and $k \in K$. Since $K < \mathrm{GL}_r(\hat{A})$, from (8.8) we deduce that

$$L_{\gamma gk} = \hat{A}^r k^{-1}g^{-1}\gamma^{-1} \cap F^r = (\hat{A}^r g^{-1} \cap F^r)\gamma^{-1} = L_g\gamma^{-1}.$$

The isomorphisms from (8.11) for g and for γgk thus fit into a diagram

$$\begin{array}{ccc} \pi_g^*(E,\varphi) & \xrightarrow[\sim]{\text{(8.11) for } g} & (\mathbb{G}_{\mathrm{a},\Omega^r}, \psi^{L_g}) \\ \Big\| {\scriptstyle (8.5)} & & \wr\downarrow {\scriptstyle (7.7)} \\ \gamma^*\pi_{\gamma gk}^*(E,\varphi) & \xrightarrow[\sim]{\text{(8.11) for } \gamma gk} & \gamma^*(\mathbb{G}_{\mathrm{a},\Omega^r}, \psi^{L_{\gamma gk}}), \end{array} \tag{8.12}$$

where the vertical map on the right is multiplication by $j(\gamma,\_)^{-1}$. Using (7.8) one verifies that the isomorphisms preserve some nowhere vanishing torsion point. Thus the two composites must coincide; in other words the diagram (8.12) commutes.

We end this section by looking at functoriality. Consider a second open compact subgroup $K' < \mathrm{GL}_r(\hat{A})$ and an element $h \in \mathrm{GL}_r(\mathbb{A}_F^{\mathrm{f}})$ such that $hK'h^{-1} < K$. Then there is a well-defined map

$$\begin{array}{rccc} (8.13) & J_h : \mathrm{GL}_r(F)\big\backslash\big(\Omega^r \times \mathrm{GL}_r(\mathbb{A}_F^{\mathrm{f}})/K'\big) & \longrightarrow & \mathrm{GL}_r(F)\big\backslash\big(\Omega^r \times \mathrm{GL}_r(\mathbb{A}_F^{\mathrm{f}})/K\big), \\ & [(\omega, gh)] & \longmapsto & [(\omega, g)]. \end{array}$$

If h has coefficients in $\hat{A}$, we have $\hat{A}^r \subset \hat{A}^r h^{-1}$ and hence

$$L_g \;=\; \hat{A}^r g^{-1} \cap F^r \;\subset\; \hat{A}^r h^{-1} g^{-1} \cap F^r \;=\; L_{gh}$$

for any $g \in \mathrm{GL}_r(\mathbb{A}_F^{\mathrm{f}})$. Thus for any $\omega \in \Omega^r$ we have $L_g \cdot \omega \subset L_{gh} \cdot \omega$, and using Proposition 2.3 (a) we obtain an isogeny of Drinfeld modules

$$(8.14) \qquad \tilde{\eta}_h \;:=\; e_{e_{L_g\cdot\omega}(L_{gh}\cdot\omega)} :\; (\mathbb{G}_{\mathrm{a},\Omega^r}, \psi^{L_g}) \longrightarrow (\mathbb{G}_{\mathrm{a},\Omega^r}, \psi^{L_{gh}}).$$

By contrast, if h^{-1} has coefficients in $\hat{A}$, we have $\hat{A}^r h^{-1} \subset \hat{A}^r$ and hence $L_{gh} \subset L_g$, which yields an isogeny of Drinfeld modules

$$(8.15) \qquad \tilde{\xi}_h \;:=\; e_{e_{L_{gh}\cdot\omega}(L_g\cdot\omega)} :\; (\mathbb{G}_{\mathrm{a},\Omega^r}, \psi^{L_{gh}}) \longrightarrow (\mathbb{G}_{\mathrm{a},\Omega^r}, \psi^{L_g}).$$

By construction the isogenies $\tilde{\eta}_h$ and $\tilde{\xi}_h$ are mutually inverse isomorphisms if $h \in \mathrm{GL}_r(\hat{A})$. In analogy with (8.2) write

$$\begin{array}{rcccl} \pi'_{gh} : & \Omega^r & \longrightarrow & \mathrm{GL}_r(F)\big\backslash\big(\Omega^r \times \mathrm{GL}_r(\mathbb{A}_F^{\mathrm{f}})/K'\big) & \xrightarrow[\sim]{(8.1)} M^r_{A,K'}(\mathbb{C}_\infty), \\ & [\omega] & \longmapsto & [(\omega, gh)]. & \end{array}$$

Proposition 8.16 *(a) Via (8.1) the map J_h corresponds to a morphism of varieties*

$$J_h :\; M^r_{A,K'} \longrightarrow M^r_{A,K}.$$

(b) For every $g \in \mathrm{GL}_r(\mathbb{A}_F^{\mathrm{f}})$ we have $\pi_g = J_h \circ \pi'_{gh}$.

Now assume that K and K' are fine, and let (E,φ) and (E',φ') denote the respective universal families on $M^r_{A,K}$ and $M^r_{A,K'}$. Then:

(c) If h has coefficients in $\hat{A}$, there is a natural isogeny $\eta_h : J_h^*(E,\varphi) \to (E',\varphi')$ which for every $g \in \mathrm{GL}_r(\mathbb{A}_F^{\mathrm{f}})$ makes the following diagram commute:

$$\begin{array}{ccccc} \pi_g^*(E,\varphi) & \overset{(\mathrm{b})}{=\!=\!=} & \pi'^*_{gh} J_h^*(E,\varphi) & \xrightarrow{\pi'^*_{gh}\eta_h} & \pi'^*_{gh}(E',\varphi') \\ \wr\big\downarrow (8.11)\text{ for } g & & & & \wr\big\downarrow (8.11)\text{ for } gh \\ (\mathbb{G}_{\mathrm{a},\Omega^r}, \psi^{L_g}) & & \xrightarrow{\quad\tilde{\eta}_h\quad} & & (\mathbb{G}_{\mathrm{a},\Omega^r}, \psi^{L_{gh}}). \end{array}$$

(d) If h^{-1} has coefficients in $\hat{A}$, there is a natural isogeny $\xi_h : (E',\varphi') \to J_h^*(E,\varphi)$ which for every $g \in \mathrm{GL}_r(\mathbb{A}_F^{\mathrm{f}})$ makes the following diagram commute:

$$\begin{array}{ccccc}
\pi'^*_{gh}(E',\varphi') & \xrightarrow{\ \pi'^*_{gh}\xi_h\ } & \pi'^*_{gh}J_h^*(E,\varphi) & \overset{\text{(b)}}{=\!=\!=} & \pi_g^*(E,\varphi) \\
\wr\Big\downarrow \text{(8.11) for } gh & & & & \wr\Big\downarrow \text{(8.11) for } g \\
(\mathbb{G}_{\mathrm{a},\Omega^r},\psi^{L_{gh}}) & & \xrightarrow{\qquad\quad \tilde{\xi}_h \quad\qquad} & & (\mathbb{G}_{\mathrm{a},\Omega^r},\psi^{L_g}).
\end{array}$$

(e) For any $a \in A \smallsetminus \{0\}$ such that both h and ah^{-1} have coefficients in $\hat{A}$, we have $\eta_h \circ \xi_{a^{-1}h} = \varphi'_a$ and $\xi_{a^{-1}h} \circ \eta_h = J_h^*\varphi_a$.

(f) If $h \in \mathrm{GL}_r(\mathbb{A}_F^{\mathrm{f}})$ is a scalar matrix and $K = K'$, then J_h is the identity morphism. If in addition $h = a \cdot \mathrm{Id}_r$ for $a \in A \smallsetminus \{0\}$, then $\eta_h = \varphi_a$. If instead $h = a^{-1} \cdot \mathrm{Id}_r$ for $a \in A \smallsetminus \{0\}$, then $\xi_h = \varphi_a$.

Proof. (Sketch) The formulas in (b), (e), and (f) follow by direct calculation from the constructions in (8.13) and (8.14) and (8.15), once the remaining assertions are proved.

The constructions of J_h and ξ_h in (a) and (d) are those of [Pi13, Props. 2.6–7]. (Except that due to the change of convention explained in Remark 1.8 the present morphism J_h corresponds to the morphism $J_{(h^T)^{-1}}$ from [Pi13, Prop. 2.6], and the present isogeny ξ_h to the isogeny $\xi_{(h^T)^{-1}}$ from [Pi13, Prop. 2.7].) Roughly speaking, by taking invariants everything reduces to the case that $K = K(N)$ and $K' = K(N')$, where J_h and ξ_h can be described explicitly using the modular interpretation.

The construction of η_h in (c) is dual to that of ξ_h and follows the same principles. For an alternative construction observe that the formulas in (e) characterise η_h uniquely in terms of $\xi_{a^{-1}h}$. Noting that the endomorphism φ'_a of (E',φ') also factors through the isogeny $\xi_{a^{-1}h} : (E',\varphi') \to J_h^*(E,\varphi)$ constructed via the modular interpretation, one can construct η_h by the formula $\eta_h \circ \xi_{a^{-1}h} = \varphi'_a$ and deduce its properties from that. □

Proposition 8.17 *Consider open compact subgroups $K, K', K'' < \mathrm{GL}_r(\hat{A})$ and elements $h, h' \in \mathrm{GL}_r(\mathbb{A}_F^{\mathrm{f}})$ such that $hK'h^{-1} < K$ and $h'K''h'^{-1} < K'$. Then we have:*

(a) $J_{hh'} = J_h \circ J_{h'}$.

(b) $\eta_{hh'} = \eta_{h'} \circ J_{h'}^*\eta_h$ if K, K', K'' are fine and h, h' have coefficients in $\hat{A}$.

(c) $\xi_{hh'} = J_{h'}^*\xi_h \circ \xi_{h'}$ if K, K', K'' are fine and h^{-1}, h'^{-1} have coefficients in $\hat{A}$.

Proof. Direct calculation for the maps in (8.13) and (8.14) and (8.15). □

9 Satake compactification

According to [Pi13, Def. 4.1], any normal integral proper algebraic variety $\overline{M}^r_{A,K}$ over F which contains $M^r_{A,K}$ as an open dense subvariety, such that the universal family (E,φ) extends to a weakly separating generalised Drinfeld A-module $(\bar{E},\bar{\varphi})$ over $\overline{M}^r_{A,K}$, is called a *Satake compactification of* $M^r_{A,K}$. By [Pi13, Thm. 4.2], such a Satake compactification exists and is projective over F, and together with its "universal family" $(\bar{E},\bar{\varphi})$ it is uniquely determined up to unique isomorphism. The proof, however, tells us very little about what the boundary of this compactification looks like.

A rigid analytic construction of the same Satake compactification was given by Kapranov [Ka87] in the special case $A=\mathbb{F}_q[t]$ and by Häberli [Hä21] in general. They explicitly construct a rigid analytic space that is projective over $\mathbb{C}_\infty$ and has a natural stratification by finitely many rigid analytic spaces of the form $\Gamma'\backslash\Omega^{r'}$ for integers $1\leqslant r'\leqslant r$ and arithmetic subgroups $\Gamma'<\mathrm{GL}_{r'}(F)$. Häberli also proves that the result is naturally isomorphic to $\overline{M}^r_{A,K}(\mathbb{C}_\infty)$. What we need from this is an analytic description of $\overline{M}^r_{A,K}$ along all boundary strata of codimension 1, where the fibers of the universal family $(\bar{E},\bar{\varphi})$ are Drinfeld modules of rank $r-1$.

Since $\overline{M}^r_{A,K}$ is integral and contains $M^r_{A,K}$ as an open dense subvariety, each connected component M_g of $M^r_{A,K}\times_{\operatorname{Spec}F}\operatorname{Spec}\mathbb{C}_\infty$ is open and dense in a connected component $\overline{M}_g$ of $\overline{M}^r_{A,K}\times_{\operatorname{Spec}F}\operatorname{Spec}\mathbb{C}_\infty$, and the decomposition (8.6) extends to a decomposition

$$\overline{M}^r_{A,K}\times_{\operatorname{Spec}F}\operatorname{Spec}\mathbb{C}_\infty \;=\; \coprod_{i=1}^{n}\overline{M}_{g_i}. \tag{9.1}$$

Also, the field of constants of $\overline{M}^r_{A,K}$ is again F_K, and the connected components $\overline{M}_{g_i}$ are just the varieties obtained by base change $\overline{M}^r_{A,K}\times_{\operatorname{Spec}F_K}\operatorname{Spec}\mathbb{C}_\infty$ for all F-linear embeddings $F_K\hookrightarrow\mathbb{C}_\infty$.

Assume that K is fine. Consider any $g\in\mathrm{GL}_r(\mathbb{A}^{\mathrm{f}}_F)$, and set $\Gamma_{g,U}:=\Gamma_g\cap U(F)$ and $\Lambda'_g:=\iota^{-1}(\Gamma_{g,U})\subset F^{r-1}$ as in (4.2) and (4.4). By Theorem 4.16 there exist an admissible open subset $\mathcal{U}_g\subset\mathbb{C}_\infty\times\Omega^{r-1}$ containing $\{0\}\times\Omega^{r-1}$ and a holomorphic map

$$\vartheta_g\colon\ \Gamma_{g,U}\backslash\Omega^r\longrightarrow\mathcal{U}_g,\quad \left[\binom{\omega_1}{\omega'}\right]\longmapsto\binom{e_{\Lambda'_g\omega'}(\omega_1)^{-1}}{\omega'}, \tag{9.2}$$

which induces an isomorphism of rigid analytic spaces $\Gamma_{g,U}\backslash\Omega^r\xrightarrow{\sim}\mathcal{U}_g\cap(\mathbb{C}_\infty^\times\times\Omega^{r-1})$.

Proposition 9.3 *(a) There exists a unique morphism of rigid analytic spaces $\bar{\pi}_g:\mathcal{U}_g\to\overline{M}_g(\mathbb{C}_\infty)$ making the following diagram commute:*

$$\begin{array}{ccccc}
\Omega^r & \xrightarrow{\;\pi_{\Gamma_{g,U}}\;} & \Gamma_{g,U}\backslash\Omega^r & \xhookrightarrow{\;\vartheta_g\;} & \mathcal{U}_g \\
\big\downarrow{\scriptstyle\pi_g} & & & & \big\downarrow{\scriptstyle\bar{\pi}_g} \\
M^r_{A,K}(\mathbb{C}_\infty) & & \xhookrightarrow{\qquad\qquad} & & \overline{M}^r_{A,K}(\mathbb{C}_\infty).
\end{array}$$

(b) This morphism is étale and its image is a Zariski open subset of $\overline{M}^r_{A,K}(\mathbb{C}_\infty)$.

(c) *Varying* $g \in \mathrm{GL}_r(\mathbb{A}_F^{\mathrm{f}})$*, the union of the images of the different maps* $\bar\pi_g$ *is equal to* $M_{A,K}^{r,+}(\mathbb{C}_\infty)$ *for a certain Zariski open subset* $M_{A,K}^{r,+}$ *of* $\overline{M}{}^r_{A,K}$ *whose complement has codimension* $\geqslant 2$*.*

Proof. This is due to Kapranov [Ka87] in the special case $A = \mathbb{F}_q[t]$, and to Häberli [Hä21] in the general case. □

Remark 9.4 For our application of Proposition 9.3 in the proof of Lemma 10.7, it would suffice to have, for every g, an étale morphism on some arbitrarily small open subset $\mathcal{V}_g \subset \mathcal{U}_g$ that is not contained in $\mathbb{C}_\infty^\times \times \Omega^{r-1}$, such that every connected component of codimension 1 of $\overline{M}{}^r_{A,K}(\mathbb{C}_\infty) \smallsetminus M^r_{A,K}(\mathbb{C}_\infty)$ contains a point in the image of $\mathcal{V}_g$ for some g. It is probably possible to prove this without the explicit description of $\overline{M}{}^r_{A,K}(\mathbb{C}_\infty)$ by Kapranov and Häberli, using only the fact from [Pi13, Prop. 4.10] that the fiber of the universal family $(\bar E, \bar\varphi)$ over the generic point of any irreducible component of codimension 1 of $\overline{M}{}^r_{A,K} \smallsetminus M^r_{A,K}$ is a Drinfeld A-module of rank $r-1$. But it would be a shame not to use the wonderful results from [Ka87] and [Hä21] when they are available.

Next let $(\mathbb{G}_{\mathrm{a},\mathcal{U}_g}, \tilde\psi^{L_g})$ be the generalised Drinfeld A-module over $\mathcal{U}_g$ that is furnished by Proposition 7.16.

Proposition 9.5 *There exists a unique isomorphism of generalised Drinfeld modules over* $\mathcal{U}_g$

$$\bar\pi_g^*(\bar E, \bar\varphi) \ \xrightarrow{\ \sim\ }\ (\mathbb{G}_{\mathrm{a},\mathcal{U}_g}, \bar\psi^{L_g}),$$

whose pullback under $\vartheta_g \circ \pi_{\Gamma_{g,U}} : \Omega^r \to \mathcal{U}_g$ *is the isomorphism*

$$\pi_{\Gamma_{g,U}}^* \vartheta_g^* \bar\pi_g^*(\bar E, \bar\varphi) \overset{9.3\,(a)}{=\!=\!=} \pi_g^*(E, \varphi) \xrightarrow[\sim]{(8.11)} (\mathbb{G}_{\mathrm{a},\Omega^r}, \psi^{L_g}) \xrightarrow[\sim]{7.16} \pi_{\Gamma_{g,U}}^* \vartheta_g^* (\mathbb{G}_{\mathrm{a},\mathcal{U}_g}, \bar\psi^{L_g}).$$

Proof. Over $\mathcal{U}_g \cap (\mathbb{C}_\infty^\times \times \Omega^{r-1})$ the isomorphism is obtained from the construction preceding (7.15). The extension to $\mathcal{U}_g$ follows from analytic versions of [Pi13, Props. 3.7–8], which say that homomorphisms and isomorphisms of generalised Drinfeld modules extend uniquely under open dense embeddings of normal integral schemes, and whose proofs work equally well in the analytic setting. □

Proposition 9.6 *In the situation of Proposition 8.16 we have:*

(a) *The morphism* $J_h : M^r_{A,K'} \to M^r_{A,K}$ *extends uniquely to a morphism* $\bar J_h : \overline{M}{}^r_{A,K'} \to \overline{M}{}^r_{A,K}$*.*

Now assume that K *and* K' *are fine, and let* $(\bar E, \bar\varphi)$ *and* $(\bar E', \bar\varphi')$ *denote the respective universal families on* $\overline{M}{}^r_{A,K}$ *and* $\overline{M}{}^r_{A,K'}$*. Then:*

(b) *If* h *has coefficients in* $\hat A$*, the isogeny* $\eta_h : J_h^*(E, \varphi) \to (E', \varphi')$ *extends uniquely to an isogeny* $\bar\eta_h : \bar J_h^*(\bar E, \bar\varphi) \to (\bar E', \bar\varphi')$*.*

(c) If h^{-1} has coefficients in $\hat{A}$, the isogeny $\xi_h : (E', \varphi') \to J_h^*(E, \varphi)$ extends uniquely to an isogeny $\bar{\xi}_h : (\bar{E}', \bar{\varphi}') \to \bar{J}_h^*(\bar{E}, \bar{\varphi})$.

Proof. (Sketch) Assertions (a) and (c) are proved in [Pi13, Prop. 4.11]. The same kinds of arguments establish (b). □

Finally, the formulas in Proposition 8.16 (e), (f) and in Proposition 8.17 automatically extend to the respective Satake compactification, because the extended morphisms already exist and two morphisms on an integral scheme are equal if they coincide on an open dense subscheme.

10 Analytic versus algebraic modular forms

We keep the notation from the preceding section, and first we also assume that K is fine. Let $\mathrm{Lie}\,\bar{E}$ denote the Lie algebra of $\bar{E}$, which is an invertible coherent sheaf of modules on $\overline{M}^r_{A,K}$. (It is naturally isomorphic to the sheaf of sections of $\bar{E}$, but in the present context it is safer to view it as the Lie algebra.) Consider the dual invertible sheaf $\mathcal{L} := (\mathrm{Lie}\,\bar{E})^\vee$. By [Pi13, Thm. 5.3] this is ample. For any integer k we abbreviate $\mathcal{L}^k := \mathcal{L}^{\otimes k}$. Following [Pi13, Def. 5.4] we have:

Definition 10.1 *An* algebraic Drinfeld modular form of weight k and level K *is an element of the space*

$$\mathcal{M}_k^{\mathrm{alg}}(M^r_{A,K}) \;:=\; H^0(\overline{M}^r_{A,K}, \mathcal{L}^k).$$

Since $\overline{M}^r_{A,K}$ is a projective algebraic variety with field of constants F_K, this is a finite-dimensional vector space over F_K or, depending on one's point of view, over F. Our aim is to relate it with a space of analytic modular forms. Note that the decomposition (9.1) yields natural isomorphisms

$$(10.2)\qquad \mathcal{M}_k^{\mathrm{alg}}(M^r_{A,K}) \otimes_F \mathbb{C}_\infty \;\cong\; H^0(\overline{M}^r_{A,K} \times_{\mathrm{Spec}\,F} \mathrm{Spec}\,\mathbb{C}_\infty, \mathcal{L}^k) \;\cong\; \bigoplus_{i=1}^n H^0(\overline{M}_{g_i}, \mathcal{L}^k).$$

Also, any irreducible component $\overline{M}_g$ of $\overline{M}^r_{A,K} \times_{\mathrm{Spec}\,F} \mathrm{Spec}\,\mathbb{C}_\infty$ has field of definition F_K; hence pullback induces an isomorphism

$$(10.3)\qquad \mathcal{M}_k^{\mathrm{alg}}(M^r_{A,K}) \otimes_{F_K} \mathbb{C}_\infty \;\cong\; H^0(\overline{M}_g, \mathcal{L}^k).$$

Let $\mathcal{L}^{\mathrm{an}}$ denote the invertible sheaf on the rigid analytic space $\overline{M}^r_{A,K}(\mathbb{C}_\infty)$ obtained from $\mathcal{L}$. Its pullback $\pi_g^*\mathcal{L}^{\mathrm{an}}$ is an invertible sheaf on Ω^r, which must be trivial, because Ω^r is a Stein space ([SS91, Prop. 4]). In fact, we have an explicit trivialisation: The isomorphism of line bundles $\pi_g^* E \to \mathbb{G}_{\mathrm{a},\Omega^r}$ underlying the isomorphism of Drinfeld modules (8.11) induces an isomorphism for the dual of the sheaf of sections

$$(10.4)\qquad \pi_g^*\mathcal{L}^{\mathrm{an}} \;\xrightarrow{\sim}\; \mathcal{O}_{\Omega^r}.$$

Via this trivialisation, the pullback of any section $s \in H^0(M^r_{A,K}(\mathbb{C}_\infty), (\mathcal{L}^{\mathrm{an}})^k)$ becomes a holomorphic function $\pi_g^* s : \Omega^r \to \mathbb{C}_\infty$.

Lemma 10.5 *For any section $s \in H^0(M^r_{A,K}(\mathbb{C}_\infty), (\mathcal{L}^{\mathrm{an}})^k)$ and any $g \in \mathrm{GL}_r(\mathbb{A}^{\mathrm{f}}_F)$ and $\gamma \in \mathrm{GL}_r(F)$ and $k \in K$ we have*

$$\pi_g^* s \;=\; (\pi_{\gamma g k}^* s)|_k \gamma.$$

Proof. Since $\mathcal{L}$ is the dual of the invertible sheaf of sections of $\bar{E}$, the commutative diagram (8.12) yields a commutative diagram

$$\begin{array}{ccccc}
\pi_g^*(\mathcal{L}^{\mathrm{an}})^k & & \xrightarrow[\sim]{\text{(10.4) for } g} & & \mathcal{O}_{\Omega^r} \\
\Big\| {\scriptstyle (8.5)} & & & & \wr\Big\downarrow {\scriptstyle \text{multiplication by } j(\gamma,_)^k} \\
\gamma^*\pi_{\gamma g k}^*(\mathcal{L}^{\mathrm{an}})^k & \xrightarrow[\sim]{\text{(10.4) for } \gamma g k} & \gamma^*\mathcal{O}_{\Omega^r} & = & \mathcal{O}_{\Omega^r}.
\end{array}$$

For any $\omega \in \Omega^r$, evaluating s at the point $\pi_g(\omega) = \pi_{\gamma g k}(\gamma(\omega))$ therefore yields the equality

$$j(\gamma,\omega)^k \cdot (\pi_g^* s)(\omega) \;=\; (\pi_{\gamma g k}^* s)(\gamma(\omega)).$$

In view of (1.5) this implies that

$$(\pi_g^* s)(\omega) \;=\; j(\gamma,\omega)^{-k} \cdot (\pi_{\gamma g k}^* s)(\gamma(\omega)) \;=\; ((\pi_{\gamma g k}^* s)|_k \gamma)(\omega),$$

as desired. □

Lemma 10.6 *The map π_g^* induces an isomorphism*

$$H^0(M_g(\mathbb{C}_\infty), (\mathcal{L}^{\mathrm{an}})^k) \xrightarrow{\ \sim\ } \mathcal{W}_k(\Gamma_g).$$

Proof. By definition the pullback by π_g yields an isomorphism from $H^0(M_g(\mathbb{C}_\infty), (\mathcal{L}^{\mathrm{an}})^k)$ to the space of Γ_g-invariant sections in $H^0(\Omega^r, \pi_g^*(\mathcal{L}^{\mathrm{an}})^k)$. But for every $\gamma \in \Gamma_g$ we have $\pi_{\gamma g} = \pi_g \circ \gamma^{-1} = \pi_g$ by (8.5); so by Lemma 10.5 the γ-invariance translates into the formula $\pi_g^* s = (\pi_g^* s)|_k \gamma$. By Definition 1.9 the image of π_g^* is therefore just the space of weak modular forms $\mathcal{W}_k(\Gamma_g)$. □

Lemma 10.7 *The map π_g^* induces an isomorphism*

$$H^0(\overline{M}_g, \mathcal{L}^k) \xrightarrow{\ \sim\ } \mathcal{M}_k(\Gamma_g).$$

Proof. By rigid analytic GAGA due to Köpf [Kö74, Satz 4.7], analytification yields an isomorphism $H^0(\overline{M}_g, \mathcal{L}^k) \xrightarrow{\sim} H^0(\overline{M}_g(\mathbb{C}_\infty), (\mathcal{L}^{\mathrm{an}})^k)$. Next, set $M_g^+ := \overline{M}_g \cap M^{r,+}_{A,K}(\mathbb{C}_\infty)$ for the Zariski open subset $M^{r,+}_{A,K}$ of $\overline{M}^r_{A,K}$ from Proposition 9.3 (c). Since $\overline{M}_g$ is normal integral and the complement $\overline{M}_g \smallsetminus M_g^+$ has codimension $\geqslant 2$, by Bartenwerfer [Ba76, Satz 10] the restriction map induces an isomorphism $H^0(\overline{M}_g(\mathbb{C}_\infty), (\mathcal{L}^{\mathrm{an}})^k) \xrightarrow{\sim} H^0(M_g^+(\mathbb{C}_\infty), (\mathcal{L}^{\mathrm{an}})^k)$. By Lemma 10.6 any section $s \in H^0(M_g(\mathbb{C}_\infty), (\mathcal{L}^{\mathrm{an}})^k)$ corresponds to a weak modular form $\pi_g^* s \in \mathcal{W}_k(\Gamma_g)$. It remains to determine when s extends to a section in $H^0(M_g^+(\mathbb{C}_\infty), (\mathcal{L}^{\mathrm{an}})^k)$.

We first analyse when it extends to the image of the map $\bar{\pi}_g$ from Proposition 9.3 (a). Recall that $\mathcal{L}$ was defined as the dual of the invertible sheaf of sections of $\bar{E}$. Thus the isomorphism of generalised Drinfeld modules in Proposition 9.5 induces an isomorphism

$$\bar{\pi}_g^* \mathcal{L}^{\mathrm{an}} \;\cong\; \mathcal{O}_{\mathcal{U}_g}. \tag{10.8}$$

Let $\bar{\vartheta} : \Omega^r \to \mathcal{U}_g$ be the composite morphism in the top row of the diagram in Proposition 9.3 (a). Then by construction the pullback of the trivialisation (10.8) to Ω^r via $\bar{\vartheta}$ is just the trivialisation in (10.4). Thus s extends to a section of $(\mathcal{L}^{\mathrm{an}})^k$ over the image of $\bar{\pi}_g$ if and only if the function $\pi_g^* s : \Omega^r \to \mathbb{C}_\infty$ is the pullback via $\bar{\vartheta}$ of a holomorphic function $\mathcal{U}_g \to \mathbb{C}_\infty$. Here $\pi_g^* s$ is already a Γ_U-invariant function and therefore possesses a u-expansion by Proposition 5.4. Thus it is the pullback of a holomorphic function on $\mathcal{U}_g$ if and only if it is holomorphic at infinity in the sense of Definition 5.12.

Now recall that for any $g, g' \in \mathrm{GL}_r(\mathbb{A}_F^{\mathrm{f}})$ we have $M_g = M_{g'}$ if and only if $g' = \gamma g k$ for some $\gamma \in \mathrm{GL}_r(F)$ and $k \in K$. By Proposition 9.3 (c) the partial compactification M_g^+ is therefore the union of the images of the maps $\bar{\pi}_{\gamma g k}$ for all such γ and k. By the above argument for $\gamma g k$ in place of g, it follows that s extends to a section in $H^0(M_g^+(\mathbb{C}_\infty), (\mathcal{L}^{\mathrm{an}})^k)$ if and only if for all γ and k the pullback $\pi_{\gamma g k}^* s$ is holomorphic at infinity. But by Lemma 10.5 we have $\pi_{\gamma g k}^* s = (\pi_g^* s)|_k \gamma^{-1}$. Varying γ we thus conclude that π_g^* induces an isomorphism from $H^0(M_g^+(\mathbb{C}_\infty), (\mathcal{L}^{\mathrm{an}})^k)$ to the space of modular forms $\mathcal{M}_k(\Gamma_g)$. Combining everything yields the desired result. □

Theorem 10.9 *If K is fine, the maps π_g^* and the isomorphisms (10.3) respectively (10.2) induce isomorphisms*

$$\mathcal{M}_k^{\mathrm{alg}}(M_{A,K}^r) \otimes_{F_K} \mathbb{C}_\infty \;\xrightarrow{\sim}\; \mathcal{M}_k(\Gamma_g),$$
$$\mathcal{M}_k^{\mathrm{alg}}(M_{A,K}^r) \otimes_{F} \mathbb{C}_\infty \;\xrightarrow{\sim}\; \bigoplus_{i=1}^{n} \mathcal{M}_k(\Gamma_{g_i}).$$

Proof. Direct consequence of Lemma 10.7. □

The above isomorphisms are functorial in the following sense. Consider a second fine open compact subgroup $K' < \mathrm{GL}_r(\hat{A})$ and an element $h \in \mathrm{GL}_r(\mathbb{A}_F^{\mathrm{f}})$ such that $hK'h^{-1} < K$. By Proposition 9.6 (a) this data determines a morphism $\bar{J}_h : \overline{M}_{A,K'}^r \longrightarrow \overline{M}_{A,K}^r$. As before let $(\bar{E}', \bar{\varphi}')$ denote the universal generalised Drinfeld module on $\overline{M}_{A,K'}^r$. Let $\mathcal{L}'$ denote the dual of the invertible sheaf of sections of $\bar{E}'$.

With h fixed, consider any sufficiently divisible scalar $a \in A \smallsetminus \{0\}$, so that the element $ha \in \mathrm{GL}_r(\mathbb{A}_F^{\mathrm{f}})$ has coefficients in $\hat{A}$. As a consequence of Propositions 8.16 (f) and 8.17, we then have $\bar{J}_{ha} = \bar{J}_h$. The derivative of the isogeny $\bar{\eta}_{ha}$ in Proposition 9.6 (b) thus induces an isomorphism

$$(d\bar{\eta}_{ha})^\vee : \; \bar{J}_h^* \mathcal{L} = \bar{J}_{ha}^* \mathcal{L} \xrightarrow{\sim} \mathcal{L}'.$$

Lemma 10.10 *The isomorphism*

$$\rho_h := a \cdot (d\bar{\eta}_{ha})^\vee : \; \bar{J}_h^* \mathcal{L} \xrightarrow{\sim} \mathcal{L}'$$

is independent of the choice of a.

Proof. Consider a second element $b \in A \smallsetminus \{0\}$ such that hb has coefficients in $\hat{A}$. Then so does hab, and Propositions 8.17 (b) and 8.16 (f) imply that $\eta_{hab} = \eta_b \circ \eta_{ha} = \varphi'_b \circ \eta_{ha}$. Taking derivatives we deduce that $d\eta_{hab} = d\varphi'_b \circ d\eta_{ha} = b \cdot d\eta_{ha}$ and hence $ab \cdot (d\eta_{hab})^\vee = ab \cdot b^{-1} \cdot (d\eta_{ha})^\vee = a \cdot (d\eta_{ha})^\vee$. Interchanging a and b implies that $ab \cdot (d\eta_{hab})^\vee = b \cdot (d\eta_{hb})^\vee$ and hence $a \cdot (d\eta_{ha})^\vee = b \cdot (d\eta_{hb})^\vee$. Finally, this equality over the dense open subscheme $M^r_{A,K'}$ automatically extends to an equality over $\bar{M}^r_{A,K'}$. □

Using pullback and the isomorphism ρ_h we can now define a natural F-linear pullback map on modular forms, again denoted J_h^*, by the commutative diagram

$$\begin{array}{ccccc} J_h^*:\ \mathcal{M}_k^{\mathrm{alg}}(M^r_{A,K}) & \longrightarrow & & \longrightarrow & \mathcal{M}_k^{\mathrm{alg}}(M^r_{A,K'}) \\ \| & & & & \| \\ H^0(\bar{M}^r_{A,K},\mathcal{L}^k) & \xrightarrow{\bar{J}_h^*} & H^0(\bar{M}^r_{A,K'},\bar{J}_h^*\mathcal{L}^k) & \xrightarrow{\rho_h^k} & H^0(\bar{M}^r_{A,K'},\mathcal{L}'^k). \end{array} \tag{10.11}$$

To describe its behavior under the isomorphisms from Theorem 10.9, for any $g \in \mathrm{GL}_r(\mathbb{A}_F^{\mathrm{f}})$ consider the arithmetic subgroup $\Gamma'_{gh} := \mathrm{GL}_r(F) \cap ghK'(gh)^{-1}$, which by construction is contained in the arithmetic subgroup $\Gamma_g := \mathrm{GL}_r(F) \cap gKg^{-1}$.

Proposition 10.12 *For any $g \in \mathrm{GL}_r(\mathbb{A}_F^{\mathrm{f}})$ the diagram*

$$\begin{array}{ccc} \mathcal{M}_k^{\mathrm{alg}}(M^r_{A,K}) & \xrightarrow{J_h^*} & \mathcal{M}_k^{\mathrm{alg}}(M^r_{A,K'}) \\ \downarrow{\scriptstyle \pi_g^*} & & \downarrow{\scriptstyle \pi'^*_{gh}} \\ \mathcal{M}_k(\Gamma_g) & \hookrightarrow & \mathcal{M}_k(\Gamma'_{gh}) \end{array}$$

commutes, where the horizontal map on the bottom is the inclusion map.

Proof. Assume first that h has coefficients in $\hat{A}$. As $\mathcal{L}$ and $\mathcal{L}'$ are the duals of the invertible sheaves of sections of $\bar{E}$ and $\bar{E}'$, Proposition 8.16 (c) yields a commutative diagram

$$\begin{array}{ccc} \pi_g^*\mathcal{L}^{\mathrm{an}} = \pi'^*_{gh}J_h^*\mathcal{L}^{\mathrm{an}} & \xrightarrow{\pi'^*_{gh}\rho_h = \pi'^*_{gh}(d\eta_h)^\vee} & \pi'^*_{gh}\mathcal{L}'^{\mathrm{an}} \\ \wr\|\ (10.4)\text{ for } g & & \wr\|\ (10.4)\text{ for } gh \\ \mathcal{O}_{\Omega^r} & \xrightarrow{(d\tilde{\eta}_h)^\vee} & \mathcal{O}_{\Omega^r}. \end{array}$$

By the construction (8.14) of $\tilde{\eta}_h$ we have $d\tilde{\eta}_h = 1$. The desired commutativity thus follows from the definition of π_g^* and π'^*_{gh}.

In the general case take any $a \in A \smallsetminus \{0\}$ such that $ha \in \mathrm{GL}_r(\mathbb{A}_F^{\mathrm{f}})$ has coefficients in $\hat{A}$. Repeating the above calculation twice with (g,h) replaced by (g,ha) and (gh,a), respectively, and noting that $\pi'_{gha} = \pi'_{gh}$, yields a commutative diagram

$$\begin{array}{ccccc} \pi_g^*\mathcal{L}^{\mathrm{an}} & \xrightarrow{\pi'^*_{gha}(d\eta_{ha})^\vee} & \pi'^*_{gha}\mathcal{L}'^{\mathrm{an}} & \xleftarrow{\pi'^*_{gha}(d\eta_a)^\vee} & \pi'^*_{gh}\mathcal{L}'^{\mathrm{an}} \\ \wr\|\ (10.4)\text{ for } g & & \wr\|\ (10.4)\text{ for } gha & & \wr\|\ (10.4)\text{ for } gh \\ \mathcal{O}_{\Omega^r} & \xrightarrow{\mathrm{id}} & \mathcal{O}_{\Omega^r} & \xleftarrow{\mathrm{id}} & \mathcal{O}_{\Omega^r}. \end{array}$$

Here $d\eta_a = d\varphi'_a = a$ by Proposition 8.16 (f), hence the upper horizontal arrow on the right is multiplication by a^{-1}. Together we thus obtain the commutative diagram

$$\begin{array}{ccc} \pi_g^*\mathcal{L}^{\mathrm{an}} & \xrightarrow{\pi'^*_{gh}\rho_h \,=\, a\cdot\pi'^*_{gha}(d\eta_{ha})^\vee} & \pi'^*_{gh}\mathcal{L}'^{\mathrm{an}} \\ \wr\Big\|\text{(10.4) for } g & & \wr\Big\|\text{(10.4) for } gh \\ \mathcal{O}_{\Omega^r} & \xrightarrow{\mathrm{id}} & \mathcal{O}_{\Omega^r}, \end{array}$$

and again the desired commutativity follows from the definition of π_g^* and π'^*_{gh}. □

Proposition 10.13 *(a) If $K = K'$ and $h \in K$, then $J_h^* = \mathrm{id}$.*

(b) If $K = K'$ and $h = a \cdot \mathrm{Id}_r$ for $a \in A \smallsetminus \{0\}$ then $J_h^* = a^k \cdot \mathrm{id}$.

(c) For any fine open compact subgroups $K, K', K'' < \mathrm{GL}_r(\hat{A})$ and elements $h, h' \in \mathrm{GL}_r(\mathbb{A}_F^{\mathrm{f}})$ such that $hK'h^{-1} < K$ and $h'K''h'^{-1} < K'$, we have $J_{hh'}^* = J_{h'}^* \circ J_h^*$.

Proof. Direct computation using Proposition 8.17. □

Now recall that the elements $g_1, \ldots, g_n$ appearing in Theorem 10.9 are the representatives of the double quotient $\mathrm{GL}_r(F)\backslash \mathrm{GL}_r(\mathbb{A}_F^{\mathrm{f}})/K$ used in (8.6). Likewise choose representatives $g'_1, \ldots, g'_{n'}$ of the double quotient $\mathrm{GL}_r(F)\backslash \mathrm{GL}_r(\mathbb{A}_F^{\mathrm{f}})/K'$. For each $1 \leqslant j \leqslant n'$ consider the arithmetic subgroup $\Gamma'_{g'_j} := \mathrm{GL}_r(F) \cap g'_j K' g'^{-1}_j$, and choose $1 \leqslant i_j \leqslant n$ and $\gamma_j \in \mathrm{GL}_r(F)$ and $k_j \in K$ such that $\gamma_j g'_j h^{-1} k_j = g_{i_j}$. Then direct calculations show that $\gamma_j \Gamma'_{g'_j} \gamma_j^{-1} < \Gamma_{g_{i_j}}$ and that the following diagram commutes:

$$\begin{array}{ccccc} \coprod_{j=1}^{n'} \Gamma'_{g'_j}\backslash\Omega^r & \xrightarrow[\sim]{(\pi'_{g'_j})} & \mathrm{GL}_r(F)\big\backslash\big(\Omega^r \times \mathrm{GL}_r(\mathbb{A}_F^{\mathrm{f}})/K'\big) & \xrightarrow{\sim} & M^r_{A,K'}(\mathbb{C}_\infty) \\ \Big\downarrow & & \Big\downarrow & & \Big\downarrow J_h \\ \coprod_{i=1}^{n} \Gamma_{g_i}\backslash\Omega^r & \xrightarrow[\sim]{(\pi_{g_i})} & \mathrm{GL}_r(F)\big\backslash\big(\Omega^r \times \mathrm{GL}_r(\mathbb{A}_F^{\mathrm{f}})/K\big) & \xrightarrow{\sim} & M^r_{A,K}(\mathbb{C}_\infty), \end{array} \tag{10.14}$$

where the vertical map in the middle is $[(\omega, g)] \mapsto [(\omega, gh^{-1})]$ and the one on the left sends a coset $\Gamma'_{g'_j}\omega$ in the j-th subset to the coset $\Gamma_{g_{i_j}}\gamma_j(\omega)$ in the i_j-th subset.

Proposition 10.15 *If K and K' are fine, the map J_h^* from (10.11) and the isomorphisms from Theorem 10.9 for K' and K fit into a commutative diagram*

$$\begin{array}{ccc} \mathcal{M}_k^{\mathrm{alg}}(M^r_{A,K}) \otimes_F \mathbb{C}_\infty & \xrightarrow{J_h^*\otimes\mathrm{id}} & \mathcal{M}_k^{\mathrm{alg}}(M^r_{A,K'}) \otimes_F \mathbb{C}_\infty \\ \wr\Big\downarrow 10.9 & & \wr\Big\downarrow 10.9 \\ \bigoplus_{i=1}^{n} \mathcal{M}_k(\Gamma_{g_i}) & \longrightarrow & \bigoplus_{j=1}^{n'} \mathcal{M}_k(\Gamma'_{g'_j}) \\ (f_i)_{i=1}^n & \longmapsto & (f_{i_j}|_k\gamma_j)_{j=1}^{n'}. \end{array}$$

Proof. For each $1 \leqslant j \leqslant n'$ we have a commutative diagram

$$\begin{array}{ccccc} & \mathcal{M}_k^{\mathrm{alg}}(M^r_{A,K}) & \xrightarrow{\quad J_h^* \quad} & & \mathcal{M}_k^{\mathrm{alg}}(M^r_{A,K'}) \\ \swarrow{\scriptstyle \pi^*_{g_{i_j}}} & & \searrow{\scriptstyle \pi^*_{g'_j h^{-1}}} & & \downarrow{\scriptstyle \pi'^*_{g'_j}} \\ \mathcal{M}_k(\Gamma_{g_{i_j}}) & \xrightarrow{f \mapsto f|_k \gamma_j} & \mathcal{M}_k(\Gamma_{g'_j h^{-1}}) & \xhookrightarrow{\text{incl.}} & \mathcal{M}_k(\Gamma'_{g'_j}), \end{array}$$

which commutes on the left by the equation $\gamma_j g'_j h^{-1} k_j = g_{i_j}$ and Lemma 10.5, and on the right by Proposition 10.12 for $g = g'_j h^{-1}$. Summing over all j yields the desired formula. □

Finally consider an arbitrary open compact subgroup $K < \mathrm{GL}_r(\mathbb{A}^{\mathrm{f}}_F)$. Let $\tilde{K}$ be any open normal subgroup of K which is fine, for instance, the principal congruence subgroup $K(N)$ for a sufficiently divisible non-zero ideal $N \subsetneqq A$. Then by Proposition 10.13 the maps J_h^* for all $h \in K$ induce a right action of $K/\tilde{K}$ on the space of modular forms of level $\tilde{K}$. In [Pi13, Def. 5.4] we defined:

Definition 10.16 *The space of* algebraic Drinfeld modular forms of weight k and arbitrary level K *is the space of K-invariants*

$$\mathcal{M}_k^{\mathrm{alg}}(M^r_{A,K}) \;\coloneqq\; \mathcal{M}_k^{\mathrm{alg}}(M^r_{A,\tilde{K}})^K.$$

Once defined using one choice of $\tilde{K}$, the same equality then holds for arbitrary open compact subgroups $\tilde{K} \lhd K < \mathrm{GL}_r(\mathbb{A}^{\mathrm{f}}_F)$. This makes $\mathcal{M}_k^{\mathrm{alg}}(M^r_{A,K})$ independent of the choice of $\tilde{K}$. Moreover, for any $g \in \mathrm{GL}_r(\mathbb{A}^{\mathrm{f}}_F)$ we define the pullback map π_g^* on $\mathcal{M}_k^{\mathrm{alg}}(M^r_{A,K})$ as the restriction of the map π_g^* on $\mathcal{M}_k^{\mathrm{alg}}(M^r_{A,\tilde{K}})$. Using Proposition 10.12 in the case $h = \mathrm{Id}_r$ we find that this is again independent of the choice of $\tilde{K}$. Likewise we can define a map $J_h^* : \mathcal{M}_k^{\mathrm{alg}}(M^r_{A,K}) \to \mathcal{M}_k^{\mathrm{alg}}(M^r_{A,K'})$ for arbitrary h, K, K' as the restriction to K-, resp. K'-invariants from suitable smaller open compact subgroups. With this we can now conclude:

Proposition 10.17 *Theorem 10.9 and Propositions 10.12 and 10.13 and 10.15 hold for arbitrary open subgroups.*

Proof. (Sketch) For all $h \in K$ we have $hK'h^{-1} = K'$, so using Proposition 10.15 with K replaced by K' we can translate the right action of K/K' on $\mathcal{M}_k^{\mathrm{alg}}(M^r_{A,K'}) \otimes_F \mathbb{C}_\infty$ to the space $\bigoplus_{j=1}^{n'} \mathcal{M}_k(\Gamma'_{g'_j})$. This action interchanges the summands $\mathcal{M}_k(\Gamma'_{g'_j})$ whenever g'_j lies in the same coset $\mathrm{GL}_r(F) g_i K$, and the stabiliser of such a summand acts through the action of all $\gamma \in \Gamma_{g_i}$ by $f \mapsto f|_k \gamma$. But the space of invariants in $\mathcal{M}_k(\Gamma'_{g'_j})$ under this action is simply $\mathcal{M}_k(\Gamma_{g_i})$. Taking invariants we thus deduce the second isomorphism in Theorem 10.9 for the group K. The remaining statements follow in the same way by taking invariants in each case. □

11 Finiteness results

Theorem 11.1 *For any congruence subgroup $\Gamma < \mathrm{GL}_r(F)$ we have:*

(a) $\dim_{\mathbb{C}_\infty} \mathcal{M}_{k,m}(\Gamma) < \infty$ *for any integers k and m.*

(b) $\mathcal{M}_{k,m}(\Gamma) = 0$ *whenever $k < 0$ and $r \geqslant 2$.*

(c) The graded ring $\mathcal{M}_*(\Gamma) := \bigoplus_{k\geqslant 0} \mathcal{M}_k(\Gamma)$ is a normal integral domain that is finitely generated as a $\mathbb{C}_\infty$-algebra.

Proof. First assume that Γ is the principal congruence subgroup $\Gamma(N)$ associated to some level $0 \neq N \subsetneqq A$. Setting $K := K(N)$, for $g = 1$ the arithmetic subgroup Γ_g from (8.3) is then Γ. By Theorem 10.9 we thus have $H^0(\bar{M}^r_{A,K}, \mathcal{L}^k) \otimes_{F_K} \mathbb{C}_\infty \cong \mathcal{M}_k(\Gamma)$. As space of sections of a coherent sheaf on a projective algebraic variety it is therefore finite dimensional, proving (a). Moreover, since $\mathcal{L}$ is ample by [Pi13, Thm. 5.3], this space is zero if $k < 0$ and every irreducible component of the variety has dimension $\geqslant 1$, proving (b). Also, the ring $\bigoplus_{k\geqslant 0} H^0(\bar{M}^r_{A,K}, \mathcal{L}^k)$ is a normal integral domain that is finitely generated as an F-algebra by [Pi13, Thm. 5.6], from which (c) follows.

Next, for any two congruence subgroups $\Gamma' \lhd \Gamma$, the respective space or graded ring for Γ is obtained from that for Γ' by taking invariants under a certain action of the finite group Γ/Γ'. The statements for Γ thus follow from those for Γ'.

Finally, for an arbitrary congruence subgroup $\Gamma < \mathrm{GL}_r(F)$ consider the finitely generated A-submodule $L := \Gamma \cdot A^r \subset F^r$, and choose an ideal $0 \neq I \subsetneqq A$ such that $IL \subset A^r$. Let Γ' be the subgroup of elements of Γ that act trivially on L/IL. Then $\Gamma' \lhd \Gamma$ and $\Gamma' < \mathrm{GL}_r(A)$. Also Γ' is again a congruence subgroup, so it contains $\Gamma(N)$ for some level $0 \neq N \subsetneqq A$. As $\Gamma' < \mathrm{GL}_r(A)$, we then have $\Gamma(N) \lhd \Gamma' \lhd \Gamma$, and the statements for Γ follow from those for $\Gamma(N)$ by applying the above reduction step twice. □

Proposition 11.2 *Let $\Gamma < \mathrm{GL}_r(A)$ be a congruence subgroup whose image in $\mathrm{GL}_r(A/\mathfrak{p})$ is unipotent for some maximal ideal $\mathfrak{p} \subset A$. Then for every $k \gg 0$ there exists a non-zero cusp form of weight k for Γ.*

For an explicit construction of such cusp forms using Eisenstein series see Remark 16.11.

Proof. Choose a level $0 \neq N \subsetneqq A$ such that $\Gamma(N) < \Gamma$, and set $K := K(N) \cdot \Gamma < \mathrm{GL}_r(\hat{A})$. Then K is fine, and for $g = 1$ we have $\Gamma_g = K \cap \mathrm{GL}_r(A) = \Gamma$. Let ∞ denote the reduced divisor on $\bar{M}^r_{A,K}$ with support $\bar{M}^r_{A,K} \smallsetminus M^r_{A,K}$. By Theorem 10.9 and the definition of cusp forms we then have

$$H^0(\bar{M}^r_{A,K}, \mathcal{L}^k(-\infty)) \otimes_{F_K} \mathbb{C}_\infty \;\cong\; \mathcal{S}_k(\Gamma).$$

As $\mathcal{L}$ is ample, the left hand side is non-zero for all $k \gg 0$, as desired. □

12 Hecke operators

Consider any element $h \in \mathrm{GL}_r(\mathbb{A}_F^{\mathrm{f}})$ and any open compact subgroups $K,\ K' < \mathrm{GL}_r(\hat{A})$ such that $hK'h^{-1} < K$. Then by (10.11) and Proposition 10.17, there is a well-defined *pullback map*

$$J_h^*\colon\ \mathcal{M}_k^{\mathrm{alg}}(M_{A,K}^r) \longrightarrow \mathcal{M}_k^{\mathrm{alg}}(M_{A,K'}^r) \tag{12.1}$$

satisfying Proposition 10.13.

We can also construct a natural map in the other direction. Since J_h^* is an isomorphism if $hK'h^{-1} = K$, we restrict ourselves to the case that $h = \mathrm{Id}_r$ and $K' < K$. Choose an open subgroup $\tilde{K} < K'$ which is normal in K. Then by Definition 10.16 we have

$$\begin{array}{ccccc} \mathcal{M}_k^{\mathrm{alg}}(M_{A,K}^r) & \xhookrightarrow{J_{\mathrm{Id}_r}^*} & \mathcal{M}_k^{\mathrm{alg}}(M_{A,K'}^r) & & \\ \| & & \| & \overset{J_{\mathrm{Id}_r}^*}{\searrow} & \\ \mathcal{M}_k^{\mathrm{alg}}(M_{A,\tilde{K}}^r)^K & \underset{\text{trace}}{\overset{\hookrightarrow}{\leftarrow}} & \mathcal{M}_k^{\mathrm{alg}}(M_{A,\tilde{K}}^r)^{K'} & \hookrightarrow & \mathcal{M}_k^{\mathrm{alg}}(M_{A,\tilde{K}}^r). \end{array} \tag{12.2}$$

We define the dotted arrow by

$$f \longmapsto \mathrm{trace}(f) := \textstyle\sum_{h'} J_{h'}^* f, \tag{12.3}$$

where h' runs through a set of representatives of the quotient $K'\backslash K$. The composite of this trace map with the vertical isomorphisms in (12.2) is the *pushforward map*

$$J_{\mathrm{Id}_r,*}\colon\ \mathcal{M}_k^{\mathrm{alg}}(M_{A,K'}^r) \longrightarrow \mathcal{M}_k^{\mathrm{alg}}(M_{A,K}^r). \tag{12.4}$$

Now consider any element $h \in \mathrm{GL}_r(\mathbb{A}_F^{\mathrm{f}})$ and any open compact subgroup $K < \mathrm{GL}_r(\hat{A})$, bearing no particular relation with each other. Then we call the pair of morphisms

$$M_{A,K}^r \xleftarrow{J_h} M_{A,K\cap h^{-1}Kh}^r \xrightarrow{J_{\mathrm{Id}_r}} M_{A,K}^r \tag{12.5}$$

the *Hecke correspondence on $M_{A,K}^r$ associated to h*. The composite map

$$T_h\colon\ \mathcal{M}_k^{\mathrm{alg}}(M_{A,K}^r) \xrightarrow{J_h^*} \mathcal{M}_k^{\mathrm{alg}}(M_{A,K\cap h^{-1}Kh}^r) \xrightarrow{J_{\mathrm{Id}_r,*}} \mathcal{M}_k^{\mathrm{alg}}(M_{A,K}^r) \tag{12.6}$$

is called the *Hecke operator on $\mathcal{M}_k^{\mathrm{alg}}(M_{A,K}^r)$ associated to h*. It depends only on the double coset KhK.

The composites of Hecke operators are calculated as follows:

Proposition 12.7 *For any h, $h' \in \mathrm{GL}_r(\mathbb{A}_F^{\mathrm{f}})$ and any open compact subgroup $K < \mathrm{GL}_r(\hat{A})$ the Hecke operators on $\mathcal{M}_k^{\mathrm{alg}}(M_{A,K}^r)$ satisfy*

$$T_{h'} \circ T_h \;=\; \sum_{h''} \big[K \cap h''^{-1}Kh'' : K \cap h^{-1}Kh \cap h''^{-1}Kh''\big] \cdot T_{h''}$$

where h'' runs through a set of representatives of the double quotient

$$(hKh^{-1} \cap K)\backslash hKh'/(K \cap h'^{-1}Kh').$$

Proof. This is [Pi13, Prop. 6.10] with the change of conventions taken into account. □

In the rest of this section we work out how the maps $J_{\mathrm{Id}_r,*}$ and T_h translate under the isomorphism from Theorem 10.9.

Proposition 12.8 *Consider any open compact subgroups $K' < K < \mathrm{GL}_r(\hat{A})$ and any representatives $g_1, \ldots, g_n$ of the double quotient $\mathrm{GL}_r(F)\backslash \mathrm{GL}_r(\mathbb{A}_F^{\mathrm{f}})/K$ and representatives $g'_1, \ldots, g'_{n'}$ of the double quotient $\mathrm{GL}_r(F)\backslash \mathrm{GL}_r(\mathbb{A}_F^{\mathrm{f}})/K'$. For each $1 \leqslant i \leqslant n$ consider the arithmetic subgroup $\Gamma_{g_i} := \mathrm{GL}_r(F) \cap g_i K g_i^{-1}$ and for each $1 \leqslant j \leqslant n'$ the arithmetic subgroup $\Gamma'_{g'_j} := \mathrm{GL}_r(F) \cap g'_j K' g'^{-1}_j$. Then the map $J_{\mathrm{Id}_r,*}$ from (12.4) and the isomorphisms from Theorem 10.9 for K' and K fit into a commutative diagram*

$$\begin{array}{ccc}
\mathcal{M}_k^{\mathrm{alg}}(M_{A,K'}^r) \otimes_F \mathbb{C}_\infty & \xrightarrow{J_{\mathrm{Id}_r,*} \otimes \mathrm{id}} & \mathcal{M}_k^{\mathrm{alg}}(M_{A,K}^r) \otimes_F \mathbb{C}_\infty \\
\wr\big\downarrow 10.9 & & \wr\big\downarrow 10.9 \\
\bigoplus_{j=1}^{n'} \mathcal{M}_k(\Gamma'_{g'_j}) & \longrightarrow & \bigoplus_{i=1}^{n} \mathcal{M}_k(\Gamma_{g_i}), \\
(f_j)_{j=1}^{n'} & \longmapsto & \Big(\sum_{j,\gamma} f_j|_k\gamma\Big)_{i=1}^{n},
\end{array}$$

where, for each index i, the sum extends over all pairs of indices $1 \leqslant j \leqslant n'$ and elements $\gamma \in \mathrm{GL}_r(F) \cap g'_j K g_i^{-1}$ up to left multiplication by $\Gamma'_{g'_j}$.

Proof. Suppose first that $K' \lhd K$. Then for any $h \in K$ and any $1 \leqslant i \leqslant n$ there is an index $1 \leqslant j_{ih} \leqslant n'$ and an element $\gamma_{ih} \in \mathrm{GL}_r(F)$ such that $g'_{j_{ih}} \in \gamma_{ih} g_i h^{-1} K'$. By Propositions 10.15 and 10.17 the map $J_h^* \otimes \mathrm{id}$ thus corresponds to the map

$$(f_j)_{j=1}^{n'} \;\longmapsto\; (f_{j_{ih}}|_k\gamma_{ih})_{i=1}^{n}.$$

Next observe that j_{ih} is unique and γ_{ih} is unique up to multiplication on the left by $\Gamma'_{g'_{j_{ih}}}$, and both depend only on i and the coset $K'h$. Summing over all cosets $K'h \subset K$ thus shows that $J_{\mathrm{Id}_r,*} \otimes \mathrm{id}$ corresponds to the map

$$(f_j)_{j=1}^{n'} \;\longmapsto\; \sum_{K'h} (f_{j_{ih}}|_k\gamma_{ih})_{i=1}^{n} \;=\; \Big(\sum_{j,\gamma} f_j|_k\gamma\Big)_{i=1}^{n}$$

with the indicated summation over (j,γ). This proves the assertion in the case $K' \triangleleft K$.

In the general case, one must take an open compact subgroup $\tilde{K} < K'$ which is normal in K, choose representatives for $\mathrm{GL}_r(F)\backslash \mathrm{GL}_r(\mathbb{A}_F^{\mathrm{f}})/\tilde{K}$, write down the commutative diagrams from Proposition 10.15 for the maps $J_{\mathrm{Id}_r}^* : \mathcal{M}_k^{\mathrm{alg}}(M_{A,K}^r) \to \mathcal{M}_k^{\mathrm{alg}}(M_{A,\tilde{K}}^r)$ and $J_{\mathrm{Id}_r}^* : \mathcal{M}_k^{\mathrm{alg}}(M_{A,K'}^r) \to \mathcal{M}_k^{\mathrm{alg}}(M_{A,\tilde{K}}^r)$ and $J_h^* \colon \mathcal{M}_k^{\mathrm{alg}}(M_{A,\tilde{K}}^r) \longrightarrow \mathcal{M}_k^{\mathrm{alg}}(M_{A,\tilde{K}}^r)$ for all $h \in K$, and eliminate everything concerning $\tilde{K}$ from the resulting expression for $J_{\mathrm{Id}_r,*} \otimes \mathrm{id}$. We leave this direct and tedious calculation to the reader. □

Proposition 12.9 *Consider any element $h \in \mathrm{GL}_r(\mathbb{A}_F^{\mathrm{f}})$, any open compact subgroup $K < \mathrm{GL}_r(\hat{A})$ and any representatives $g_1,\ldots,g_n$ of the double quotient $\mathrm{GL}_r(F)\backslash \mathrm{GL}_r(\mathbb{A}_F^{\mathrm{f}})/K$. Then the Hecke operator T_h from (12.6) and the isomorphism from Theorem 10.9 fit into a commutative diagram*

$$\begin{array}{ccc}
\mathcal{M}_k^{\mathrm{alg}}(M_{A,K}^r) \otimes_F \mathbb{C}_\infty & \xrightarrow{T_h \otimes \mathrm{id}} & \mathcal{M}_k^{\mathrm{alg}}(M_{A,K}^r) \otimes_F \mathbb{C}_\infty \\
\wr\big\downarrow 10.9 & & \wr\big\downarrow 10.9 \\
\bigoplus_{i=1}^n \mathcal{M}_k(\Gamma_{g_i}) & \longrightarrow & \bigoplus_{i=1}^n \mathcal{M}_k(\Gamma_{g_i}), \\
(f_i)_{i=1}^n & \longmapsto & \Big(\sum_{i',\delta} f_{i'}|_k\,\delta\Big)_{i=1}^n,
\end{array}$$

where, for each index i, the sum extends over all pairs of indices $1 \leqslant i' \leqslant n$ and elements $\delta \in \mathrm{GL}_r(F) \cap g_{i'}KhKg_i^{-1}$ up to left multiplication by $\Gamma_{g_{i'}}$. Moreover, the index i' that actually occurs in the sum depends only i and h.

Proof. Set $K' := K \cap h^{-1}Kh$ and choose representatives $g_1',\ldots,g_{n'}'$ of the double quotient $\mathrm{GL}_r(F)\backslash \mathrm{GL}_r(\mathbb{A}_F^{\mathrm{f}})/K'$. For each $1 \leqslant j \leqslant n'$ select an index $1 \leqslant i_j \leqslant n$ and elements $\gamma_j \in \mathrm{GL}_r(F)$ and $k_j \in K$ such that $\gamma_j g_j' h^{-1} k_j = g_{i_j}$. Then by Propositions 10.15 and 12.8 we have a commutative diagram

$$\begin{array}{ccccc}
\mathcal{M}_k^{\mathrm{alg}}(M_{A,K}^r) \otimes_F \mathbb{C}_\infty & \xrightarrow{J_h^* \otimes \mathrm{id}} & \mathcal{M}_k^{\mathrm{alg}}(M_{A,K'}^r) \otimes_F \mathbb{C}_\infty & \xrightarrow{J_{\mathrm{Id}_r,*} \otimes \mathrm{id}} & \mathcal{M}_k^{\mathrm{alg}}(M_{A,K}^r) \otimes_F \mathbb{C}_\infty \\
\wr\big\downarrow 10.9 & & \wr\big\downarrow 10.9 & & \wr\big\downarrow 10.9 \\
\bigoplus_{i=1}^n \mathcal{M}_k(\Gamma_{g_i}) & \longrightarrow & \bigoplus_{j=1}^{n'} \mathcal{M}_k(\Gamma'_{g_j'}) & \longrightarrow & \bigoplus_{i=1}^n \mathcal{M}_k(\Gamma_{g_i}), \\
(f_i)_{i=1}^n & \longmapsto & (f_{i_j}|_k\gamma_j)_{j=1}^{n'} & \longmapsto & \Big(\sum_{j,\gamma} f_{i_j}|_k\gamma_j\,|_k\gamma\Big)_{i=1}^n,
\end{array}$$

where, for each index i, the sum extends over all pairs of indices $1 \leqslant j \leqslant n'$ and elements $\gamma \in \mathrm{GL}_r(F) \cap g_j'Kg_i^{-1}$ up to left multiplication by $\mathrm{GL}_r(F) \cap g_j'K'g_j'^{-1}$. Using the fact that $f_{i_j}|_k\gamma_j\,|_k\gamma = f_{i_j}|_k\,\gamma_j\gamma$ we can rewrite this as

$$(f_i)_{i=1}^n \longmapsto \Big(\sum_{j,\delta} f_{i_j}|_k\,\delta\Big)_{i=1}^n, \tag{12.10}$$

where, for each index i, the sum extends over all pairs of indices $1 \leqslant j \leqslant n'$ and elements $\delta \in \mathrm{GL}_r(F) \cap \gamma_j g'_j K g_i^{-1}$ up to left multiplication by $\mathrm{GL}_r(F) \cap \gamma_j g'_j K' g'^{-1}_j \gamma_j^{-1}$.

To analyse this sum note first that by construction we have $\gamma_j g'_j = g_{i_j} k_j^{-1} h$. For each j the element δ therefore runs through $\mathrm{GL}_r(F) \cap g_{i_j} k_j^{-1} h K g_i^{-1}$ up to left multiplication by $\mathrm{GL}_r(F) \cap g_{i_j} k_j^{-1} h K' h^{-1} k_j g_{i_j}^{-1}$.

For any j and δ that occur in the sum this shows that $\delta g_i \in g_{i_j} k_j^{-1} h K$. Taking determinants and using the fact that $k_j \in K$ we deduce that $\det(g_i)$ and $\det(g_{i_j} h)$ represent the same coset in $F^\times \backslash (\mathbb{A}_F^{\mathrm{f}})^\times / \det(K)$. The coset of $\det(g_{i_j})$ therefore depends only on i and h, but not on j. By Proposition 8.7 it follows that i_j depends only on i and h, but not on j. For the rest of the proof we therefore fix indices i and i' such that $\det(g_i)$ and $\det(g_{i'} h)$ represent the same coset in $F^\times \backslash (\mathbb{A}_F^{\mathrm{f}})^\times / \det(K)$, and we can restrict ourselves to indices j with $i_j = i'$.

Note that this already proves the last statement of the proposition. It also shows that δ lies in $\mathrm{GL}_r(F) \cap g_{i'} K h K g_i^{-1}$. Moreover, since $hK'h^{-1} < K$ and $k_j \in K$, we have $\mathrm{GL}_r(F) \cap g_{i'} k_j^{-1} h K' h^{-1} k_j g_{i'}^{-1} \subset \mathrm{GL}_r(F) \cap g_{i'} K g_{i'}^{-1} = \Gamma_{g_{i'}}$. Thus any equivalence class of pairs (j, δ) in the sum (12.10) determines a unique coset $\Gamma_{g_{i'}} \delta$.

Suppose that two pairs (j, δ) and (j', δ') determine the same coset $\Gamma_{g_{i'}} \delta = \Gamma_{g_{i'}} \delta'$. Write $\delta' = \varepsilon \delta$ with $\varepsilon \in \Gamma_{g_{i'}}$. Since $\delta \in g_{i'} k_j^{-1} h K g_i^{-1}$ and $\delta' \in g_{i'} k_{j'}^{-1} h K g_i^{-1}$, it follows that δ' lies in both $\varepsilon g_{i'} k_j^{-1} h K g_i^{-1}$ and $g_{i'} k_{j'}^{-1} h K g_i^{-1}$. Multiplying by g_i from the right we deduce that $\varepsilon g_{i'} k_j^{-1} h k = g_{i'} k_{j'}^{-1} h$ for some $k \in K$. By the definition of $\Gamma_{g_{i'}}$ we have $g_{i'}^{-1} \varepsilon^{-1} g_{i'} \in K$, and since k_j, $k_{j'} \in K$, we find that $k = h^{-1} k_j g_{i'}^{-1} \varepsilon^{-1} g_{i'} k_{j'}^{-1} h \in K \cap h^{-1} K h = K'$. The calculation $\varepsilon \gamma_j g'_j k = \varepsilon g_{i'} k_j^{-1} h k = g_{i'} k_{j'}^{-1} h = \gamma_{j'} g'_{j'}$ now implies that g'_j and $g'_{j'}$ represent the same double coset in $\mathrm{GL}_r(F) \backslash \mathrm{GL}_r(\mathbb{A}_F^{\mathrm{f}}) / K'$. By the choice of $g'_1, \ldots, g'_{n'}$ as representatives of these double cosets it follows that $j = j'$. Thus both δ and δ' lie in $\mathrm{GL}_r(F) \cap g_{i'} k_j^{-1} h K g_i^{-1}$, and hence $\varepsilon = \delta' \delta^{-1}$ lies in $\mathrm{GL}_r(F) \cap g_{i'} k_j^{-1} h K h^{-1} k_j g_{i'}^{-1}$. Since also $\varepsilon \in \Gamma_{g_{i'}} = \mathrm{GL}_r(F) \cap g_{i'} K g_{i'}^{-1}$ and $k_j \in K$ and $hKh^{-1} \cap K = hK'h^{-1}$, we then actually have $\varepsilon \in \mathrm{GL}_r(F) \cap g_{i'} k_j^{-1} h K' h^{-1} k_j g_{i'}^{-1}$. This shows that the map sending an equivalence class of pairs (j, δ) in the sum (12.10) to the coset $\Gamma_{g_{i'}} \delta$ is injective.

Consider now an arbitrary element $\delta \in \mathrm{GL}_r(F) \cap g_{i'} K h K g_i^{-1}$. Choose $k \in K$ such that $\delta \in g_{i'} k^{-1} h K g_i^{-1}$. By the choice of $g'_1, \ldots, g'_{n'}$ there exists an index j with $\mathrm{GL}_r(F) g_{i'} k^{-1} h K' = \mathrm{GL}_r(F) g'_j K'$. Since $\gamma_j g'_j = g_{i_j} k_j^{-1} h$, we deduce that $\mathrm{GL}_r(F) g_{i'} k^{-1} h K' = \mathrm{GL}_r(F) g_{i_j} k_j^{-1} h K'$. By the same argument as above it follows that $i' = i_j$, and we can find an element $\varepsilon \in \mathrm{GL}_r(F)$ such that $\varepsilon g_{i'} k^{-1} h \in g_{i'} k_j^{-1} h K'$. Since $hK'h^{-1} < K$ and k_j, $k \in K$, we then have $\varepsilon \in \mathrm{GL}_r(F) \cap g_{i'} k_j^{-1} h K' h^{-1} k g_{i'}^{-1} < \mathrm{GL}_r(F) \cap g_{i'} K g_{i'}^{-1} = \Gamma_{g_{i'}}$. Thus $\varepsilon \delta \in \mathrm{GL}_r(F) \cap \varepsilon g_{i'} k^{-1} h K g_i^{-1} = \mathrm{GL}_r(F) \cap g_{i'} k_j^{-1} h K g_i^{-1}$, and so the coset $\Gamma_{g_{i'}} \delta$ arises from the pair $(j, \varepsilon \delta)$ in the sum (12.10). In other words the map sending an equivalence class of pairs (j, δ) in the sum (12.10) to the coset $\Gamma_{g_{i'}} \delta$ is surjective.

All this together shows that in (12.10) we can equivalently sum over all $\delta \in \mathrm{GL}_r(F) \cap g_{i'} K h K g_i^{-1}$ up to left multiplication by $\Gamma_{g_{i'}}$. Also, since $f_{i_j} = f_{i'} \in \mathcal{M}_k(\Gamma_{g_{i'}})$, the function $f_{i'}|_k \delta$ depends only on the coset $\Gamma_{g_{i'}} \delta$. This finishes the proof. $\square$

Finally, we define Hecke operators on analytic Drinfeld modular forms as follows:

Definition 12.11 *For any arithmetic subgroups* Γ, $\Gamma' < \mathrm{GL}_r(F)$ *and any element* $\delta \in \mathrm{GL}_r(F)$ *we define the associated* Hecke operator *by*

$$T_\delta : \mathcal{M}_k(\Gamma') \longrightarrow \mathcal{M}_k(\Gamma), \ f \longmapsto \textstyle\sum_\gamma f|_k \gamma,$$

where γ *runs through a set of representatives of the quotient* $\Gamma' \backslash \Gamma' \delta \Gamma$.

Using (1.6) and Proposition 6.6 one finds that this is well-defined, and by construction it depends only on the double coset $\Gamma'\delta\Gamma$. Also, since the action of $\mathrm{GL}_r(F)$ preserves cusp forms and $\mathcal{M}_k(\Gamma) \cap \mathcal{S}_k(\Gamma \cap \delta^{-1}\Gamma'\delta) = \mathcal{S}_k(\Gamma)$, the Hecke operator induces a map

$$T_\delta : \mathcal{S}_k(\Gamma') \longrightarrow \mathcal{S}_k(\Gamma). \tag{12.12}$$

We can now rewrite the formula in Proposition 12.9 as follows.

Theorem 12.13 *The map on the bottom in Proposition 12.9 is equal to*

$$(f_i)_{i=1}^n \longmapsto \Big(\sum_{i',\delta} T_\delta(f_{i'}) \Big)_{i=1}^n,$$

where, for each index i*, the sum extends over all pairs of indices* $1 \leqslant i' \leqslant n$ *and double cosets* $\Gamma_{g_{i'}} \delta \Gamma_{g_i} \subset \mathrm{GL}_r(F) \cap g_{i'} K h K g_i^{-1}$*. Again the index* i' *that actually occurs depends only on* i *and* h*.*

Proof. By construction the set $\mathrm{GL}_r(F) \cap g_{i'} K h K g_i^{-1}$ is invariant under left multiplication by $\Gamma_{g_{i'}} = \mathrm{GL}_r(F) \cap g_{i'} K g_{i'}^{-1}$ and right multiplication by $\Gamma_{g_i} = \mathrm{GL}_r(F) \cap g_i K g_i^{-1}$, and it is a finite disjoint union of double cosets $\Gamma_{g_{i'}} \delta \Gamma_{g_i}$. The formula results by direct computation from (1.6). □

Remark 12.14 In Theorem 12.13 it can happen that $\mathrm{GL}_r(F) \cap g_{i'} K h K g_i^{-1}$ decomposes into several double cosets. This is related to the fact that the algebraic Hecke operator T_h is by construction defined over F, whereas the analytic Hecke operator T_δ is only defined over $\mathbb{C}_\infty$. Thus if $M^r_{A,K\cap h^{-1}Kh}(\mathbb{C}_\infty)$ has more connected components than $M^r_{A,K}(\mathbb{C}_\infty)$, their common field of definition $F_{K\cap h^{-1}Kh}$ is a proper extension of the field of definition F_K of the connected components of $M^r_{A,K}(\mathbb{C}_\infty)$, and the algebraic Hecke operator T_h can be viewed as an analytic Hecke operator T_δ followed by a trace map with respect to $F_{K\cap h^{-1}Kh}/F_K$.

Part III
Examples

Introduction

In the present Part III we illustrate the general theory constructed in Parts I and II by constructing some important families of modular forms.

Let L be a finitely generated projective A-submodule of rank r of F^r, viewed as a set of row vectors. For any $\omega \in \Omega^r$ we thus obtain a strongly discrete A-lattice $L\omega \subset \mathbb{C}_\infty$ of rank r. Our convention on row vectors implies that $\mathrm{GL}_r(F)$ acts on F^r from the right. We denote the stabiliser of L by

$$\Gamma_L \;:=\; \bigl\{\gamma \in \mathrm{GL}_r(F) \bigm| L\gamma = L\bigr\}.$$

For $L = A^r$ we simply have $\Gamma_L = \mathrm{GL}_r(A)$. Note that for any non-zero ideal $N \subset A$, an element of $\mathrm{GL}_r(F)$ stabilises the lattice L if and only if it stabilises the lattice $N^{-1}L$; thus $\Gamma_{N^{-1}L} = \Gamma_L$. More generally, for any coset $v + L \subset F^r$ we consider the congruence subgroup

$$\Gamma_{v+L} \;:=\; \bigl\{\gamma \in \mathrm{GL}_r(F) \bigm| v\gamma + L\gamma = v + L\bigr\} \;<\; \Gamma_L.$$

Also, for any non-zero ideal $N \subset A$ we consider the principal congruence group

$$\Gamma_L(N) \;:=\; \bigcap_{v \in N^{-1}L} \Gamma_{v+L} \;=\; \ker\bigl(\Gamma_L \to \mathrm{Aut}(N^{-1}L/L)\bigr).$$

All these groups are arithmetic subgroups of $\mathrm{GL}_r(F)$.

Outline of Part III

In Section 13 we construct the Eisenstein series of all weights $k \geqslant 1$ associated to all cosets $v + L$ and compute their u-expansions in Proposition 13.10. In Theorem 13.16 we show that they are modular forms of weight k for the groups Γ_{v+L}.

In Section 14 we determine the action of Hecke operators (defined in Section 12) on Eisenstein series, restricting ourselves to Hecke operators that are supported away from the level of the Eisenstein series (see Assumption 14.1). In each case, Theorem 14.11 identifies the Hecke image of an Eisenstein series as a linear combination of Eisenstein series. In particular, we deduce that Eisenstein series are eigenforms under many Hecke operators.

Coefficient forms are defined in Section 15, they are modular forms for Γ_L which occur as coefficients of Drinfeld modules, isogenies or exponential functions associated to the lattice $L\omega$.

Section 16 deals with discriminant forms, which arise as highest coefficients of Drinfeld modules or as roots thereof. These are always cusp forms. Certain $(q-1)$-st roots are examples of modular forms with non-zero type m.

Lastly, we discuss the special case of $A = \mathbb{F}_q[t]$ and $L = A^r$ in Section 17. Here we exploit the explicit description of algebraic modular forms for $\Gamma(t)$ from [PS14] and [Pi13] together with our identification of analytic and algebraic modular forms from Part II. This allows us to prove in Theorem 17.1 that the graded ring $\mathcal{M}_*(\Gamma(t))$ of modular forms of all weights for $\Gamma(t)$ is generated over $\mathbb{C}_\infty$ by the weight one Eisenstein series $E_{1,v+L}$ for all $v \in t^{-1}L \smallsetminus L$. Using invariants, we then deduce that the rings $\mathcal{M}_*(\mathrm{GL}_r(A))$ and $\mathcal{M}_*(\mathrm{SL}_r(A))$ are generated by suitable algebraically independent coefficient forms. This generalises known results from the $r = 2$ case due to Cornelissen, Goss and Gekeler, respectively. Lastly, we give some dimension formulae in Theorem 17.11.

13 Eisenstein series

For any integer $k \geqslant 1$ and any vector $v \in F^r$ we define the *Eisenstein series of weight k associated to the coset $v + L$* by

$$E_{k,v+L}(\omega) \;:=\; \sum_{0 \neq x \in v+L} (x\omega)^{-k}. \tag{13.1}$$

Proposition 13.2 *This series defines a holomorphic function $\Omega^r \to \mathbb{C}_\infty$.*

Proof. By Proposition 3.4 it suffices to show that the series converges uniformly on the affinoid set Ω^r_n from (3.2) for every n. For this observe that any $x \in F^r \smallsetminus \{0\}$ determines a unimodular F_∞-linear form $\frac{x}{|x|}$ on F^r_∞. For any $\omega \in \Omega^r_n$ it follows that

$$|x\omega| \;=\; |x| \cdot \left|\tfrac{x}{|x|}\omega\right| \overset{(3.1)}{\geqslant} |x| \cdot h(\omega) \cdot |\omega| \overset{(3.2)}{\geqslant} |x| \cdot |\pi^n| \cdot |\omega| \overset{3.3}{\geqslant} |x| \cdot |\pi^n| \cdot |\xi|.$$

As x runs through $(v + L) \smallsetminus \{0\}$, the norm $|x|$ goes to infinity; hence $|x\omega|^{-k}$ goes to zero uniformly over Ω^r_n, as desired. □

Some basic transformation properties of Eisenstein series are:

Proposition 13.3 *(a) For every $\gamma \in \mathrm{GL}_r(F)$ we have $E_{k,v+L}|_k\gamma = E_{k,v\gamma+L\gamma}$.*

(b) In particular $E_{k,v+L}$ is a weak modular form of weight k for the group Γ_{v+L}.

(c) For any A-submodule of finite index $L' \subset L$ we have $E_{k,v+L} = \sum_{v'+L'} E_{k,v'+L'}$, where the sum extends over all L'-cosets $v' + L' \subset v + L$.

Proof. (a) results from the calculation

$$\begin{aligned} (E_{k,v+L}|_k\gamma)(\omega) &\overset{(1.5)}{=} j(\gamma,\omega)^{-k} \cdot \sum_{0 \neq x \in v+L} (x \cdot \gamma(\omega))^{-k} \\ &= \sum_{0 \neq x \in v+L} (j(\gamma,\omega) \cdot x \cdot \gamma(\omega))^{-k} \\ &\overset{(1.3)}{=} \sum_{0 \neq x \in v+L} (x\gamma\omega)^{-k} \\ &= E_{k,v\gamma+L\gamma}(\omega). \end{aligned}$$

(b) is a direct consequence of (a), and (c) is obvious from the definition (13.1). □

Our next goal is to determine the u-expansion of $E_{k,v+L}$, which requires some preparation. For any strongly discrete $\mathbb{F}_q$-subspace $H \subset \mathbb{C}_\infty$ consider the power series expansion of the exponential function

$$e_H(z) := z \cdot \prod_{h \in H \smallsetminus \{0\}} \Big(1 - \frac{z}{h}\Big) = \sum_{i=0}^{\infty} e_{H,q^i} z^{q^i} \tag{13.4}$$

with $e_{H,q^i} \in \mathbb{C}_\infty$ and $e_{H,1} = 1$ that is furnished by Proposition 2.2.

Proposition 13.5 *(a) For any strongly discrete $\mathbb{F}_q$-subspace $H \subset \mathbb{C}_\infty$ we have*

$$e_H(z)^{-1} = \sum_{h \in H} (z-h)^{-1}.$$

(b) For every $k \geqslant 1$, there exists a unique so-called Goss polynomial $G_k(X, Y_1, Y_2, \ldots)$ *with coefficients in $\mathbb{F}_p$ in the variables X and Y_i for all integers $1 \leqslant i < \log_q k$, such that for every strongly discrete $\mathbb{F}_q$-subspace $H \subset \mathbb{C}_\infty$ we have*

$$G_k\big(e_H(z)^{-1},\ e_{H,q},\ e_{H,q^2},\ \ldots\big) = \sum_{h \in H} (z-h)^{-k}.$$

(c) These polynomials further satisfy:

(i) G_k is monic of degree k in X and divisible by X.

(ii) $G_1 = X$ and $G_k = X\big(G_{k-1} + \sum_{1 \leqslant i < \log_q k} Y_i G_{k-q^i}\big)$ for all $k > 1$.

(iii) $G_{pk} = G_k^p$.

(iv) $X^2 \frac{\partial}{\partial X} G_k = k G_{k+1}$.

Proof. The existence of these polynomials was first obtained by Goss in [Go80c, Prop. 6.6], but in this generality see Gekeler [Ge13, Thm. 2.6]. □

Remark 13.6 We shall see in Proposition 13.13 that the vanishing order at infinity of the Eisenstein series $E_{k,v+L}$ is controlled by the vanishing order of the Goss polynomial G_k at $X = 0$. By part (i) of Proposition 13.5 (c) this vanishing order is $\geqslant 1$, and part (ii) implies that it is equal to k for all $k \leqslant q$. In [Ge13], Gekeler gives a formula for the order of the Goss polynomial at $X = 0$ in the case $A = \mathbb{F}_p[t]$ and $H = \bar{\pi} A$, where p is prime and $\bar{\pi}$ is the Carlitz period. This determines the vanishing order of the Eisenstein series in the rank 2 case for $A = \mathbb{F}_p[t]$.

Corollary 13.7 *For any $v \in F^r \smallsetminus L$ we have*

$$E_{1,v+L}(\omega) \;=\; e_{L\omega}(v\omega)^{-1}.$$

Proof. Direct computation using the substitution $x = v - \ell$ and Proposition 13.5 (a):

$$E_{1,v+L}(\omega) \;=\; \sum_{0 \neq x \in v+L} (x\omega)^{-1} \;=\; \sum_{\ell \in L} (v\omega - \ell\omega)^{-1} \;=\; e_{L\omega}(v\omega)^{-1}. \qquad \square$$

Now define A-submodules L' and L_1 by the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^{r-1} & \xrightarrow{\qquad\qquad} & F^r & \xrightarrow{\qquad\qquad} & F & \longrightarrow & 0 \\ & & \cup & x' \mapsto (0,x') & \cup & (x_1,x') \mapsto x_1 & \cup & & \\ 0 & \longrightarrow & L' & \xrightarrow{\qquad\qquad} & L & \xrightarrow{\qquad\qquad} & L_1 & \longrightarrow & 0. \end{array} \tag{13.8}$$

Since L is finitely generated projective of rank r, the A-modules L' and L_1 are finitely generated projective of ranks $r-1$ and 1, respectively. Also fix a subgroup $\tilde{L}_1 \subset L$ which maps isomorphically to L_1, so that $L = \tilde{L}_1 \oplus (\{0\} \times L')$. Write $v = (v_1, v') \in F^r = F \times F^{r-1}$.

Lemma 13.9 *The subgroup $\Lambda' \subset F^{r-1}$ from (4.4) that corresponds to $\Gamma_{v+L} \cap U(F)$ is the finitely generated A-submodule of rank $r-1$*

$$\Lambda' \;=\; \big\{ \lambda' \in F^{r-1} \,\big|\, (v_1 + L_1)\lambda' \subset L' \big\}.$$

Moreover, for any $x_1 \in (v_1 + L_1) \smallsetminus \{0\}$ the inclusion $x_1\Lambda' \subset L'$ has finite index.

Proof. For any $\lambda' \in F^{r-1}$ and $(x_1, x') \in F^r = F \times F^{r-1}$ we have $(x_1, x')\left(\begin{smallmatrix} 1 & \lambda' \\ 0 & 1 \end{smallmatrix}\right) = (x_1, x_1\lambda' + x')$. By the definition of Γ_{v+L} in the introduction it follows that $\lambda' \in \Lambda'$ if and only if for every $(x_1, x') \in v + L$ we have $(0, x_1\lambda') \in L$, or equivalently $x_1\lambda' \in L'$. As (x_1, x') runs through $v + L$, its first component x_1 runs through $v_1 + L_1$, so the formula for Λ' follows.

Since L' and L_1 are finitely generated A-modules of ranks $r-1$ and 1, respectively, the formula implies that Λ' is a finitely generated A-submodule of rank $r-1$. For $x_1 \in (v_1 + L_1) \smallsetminus \{0\}$ it follows that $x_1\Lambda' \subset L'$ is an inclusion of finitely generated A-modules of the same rank and hence of finite index. $\square$

As before we write $\omega = \left(\begin{smallmatrix} \omega_1 \\ \omega' \end{smallmatrix}\right) \in \Omega^r \subset \mathbb{C}_\infty \times \Omega^{r-1}$. Then the expansion parameter from (4.14) is the function $u := u_{\omega'}(\omega_1) := e_{\Lambda'\omega'}(\omega_1)^{-1}$.

Proposition 13.10 *We have*

$$E_{k,v+L}\left(\left(\begin{smallmatrix} \omega_1 \\ \omega' \end{smallmatrix}\right)\right) \;=\; \sum_{x=(x_1,x') \in v+\tilde{L}_1} \begin{cases} E_{k,x'+L'}(\omega') & \textit{if } x_1 = 0, \\ G_k\big(e_{L'\omega'}(x\omega)^{-1},\ e_{L'\omega',q},\ e_{L'\omega',q^2},\ \ldots\big) & \textit{if } x_1 \neq 0, \end{cases}$$

where G_k is the k-th Goss polynomial from Proposition 13.5 and in the second case

$$e_{L'\omega'}(x\omega)^{-1} \;=\; \frac{u^{[L':x_1\Lambda']}}{x_1} \cdot \frac{\displaystyle\prod_{\ell' \in L' \smallsetminus x_1\Lambda' \bmod x_1\Lambda'} e_{\Lambda'\omega'}(x_1^{-1}\ell'\omega')}{\displaystyle\prod_{\ell' \in L' \bmod x_1\Lambda'} \big(1 - e_{\Lambda'\omega'}(x_1^{-1}(\ell' - x')\omega') \cdot u\big)}.$$

Moreover, the right hand side converges locally uniformly for all (u, ω') in a suitable tubular neighbourhood of $\{0\} \times \Omega^{r-1}$.

Proof. Using the fact that $L = \tilde{L}_1 \oplus (\{0\} \times L')$, we break up the series defining $E_{k,v+L}$ as

$$E_{k,v+L}(\omega) = \sum_{0 \neq x \in v+L} (x\omega)^{-k} = \sum_{x \in v+\tilde{L}_1} \Bigg(\sum_{0 \neq y \in x+(\{0\}\times L')} (y\omega)^{-k} \Bigg). \tag{13.11}$$

Write $x = (x_1, x') \in F^r = F \times F^{r-1}$, and observe that for any $y = (y_1, y') \in F^r = F \times F^{r-1}$ we have $y\omega = y_1\omega_1 + y'\omega'$.

If $x_1 = 0$, the inner sum of (13.11) is just

$$\sum_{0 \neq y' \in x'+L'} (y'\omega')^{-k} = E_{k,x'+L'}(\omega').$$

Such a term occurs only if v lies in $L + (\{0\} \times F^{r-1})$, and then it occurs for a unique x.

If $x_1 \neq 0$, we write $y = x - (0, \ell')$, so that $y\omega = x\omega - \ell'\omega'$. By Proposition 13.5 (b) the inner sum of (13.11) then becomes

$$\sum_{\ell' \in L'} (x\omega - \ell'\omega')^{-k} = G_k\big(e_{L'\omega'}(x\omega)^{-1},\ e_{L'\omega',q},\ e_{L'\omega',q^2},\ \ldots\big).$$

To transform $e_{L'\omega'}(x\omega)$ we proceed as in the proof of Proposition 7.16. First, by Lemma 13.9 we have an inclusion of finite index $\Lambda'\omega' \subset x_1^{-1}L'\omega'$, and by the F_∞-linear independence of the coefficients of ω the index is precisely $[L' : x_1\Lambda']$. By the additivity of the exponential function we have

$$e_{\Lambda'\omega'}(x_1^{-1}x\omega) = e_{\Lambda'\omega'}(\omega_1 + x_1^{-1}x'\omega') = u^{-1} + e_{\Lambda'\omega'}(x_1^{-1}x'\omega')$$

with $u = e_{\Lambda'\omega'}(\omega_1)^{-1}$. Using Proposition 2.3 we deduce that

$$\begin{aligned} e_{L'\omega'}(x\omega) &= x_1 \cdot e_{x_1^{-1}L'\omega'}(x_1^{-1}x\omega) \\ &= x_1 \cdot e_{e_{\Lambda'\omega'}(x_1^{-1}L'\omega')}\big(e_{\Lambda'\omega'}(x_1^{-1}x\omega)\big) \\ &= x_1 \cdot e_{e_{\Lambda'\omega'}(x_1^{-1}L'\omega')}\big(u^{-1} + e_{\Lambda'\omega'}(x_1^{-1}x'\omega')\big). \end{aligned}$$

By the definition and the additivity of the exponential function this in turn yields

$$\begin{aligned} e_{L'\omega'}(x\omega) &= x_1 \cdot \big(u^{-1} + e_{\Lambda'\omega'}(x_1^{-1}x'\omega')\big) \cdot \prod_{\substack{\ell' \in L' \smallsetminus x_1\Lambda' \\ \text{modulo } x_1\Lambda'}} \left(1 - \frac{u^{-1} + e_{\Lambda'\omega'}(x_1^{-1}x'\omega')}{e_{\Lambda'\omega'}(x_1^{-1}\ell'\omega')}\right) \\ &= x_1 \cdot \big(u^{-1} + e_{\Lambda'\omega'}(x_1^{-1}x'\omega')\big) \cdot \prod_{\substack{\ell' \in L' \smallsetminus x_1\Lambda' \\ \text{modulo } x_1\Lambda'}} \frac{e_{\Lambda'\omega'}(x_1^{-1}(\ell' - x')\omega') - u^{-1}}{e_{\Lambda'\omega'}(x_1^{-1}\ell'\omega')} \\ &= x_1 \cdot \frac{1 + e_{\Lambda'\omega'}(x_1^{-1}x'\omega') \cdot u}{u^{[L':x_1\Lambda']}} \cdot \prod_{\substack{\ell' \in L' \smallsetminus x_1\Lambda' \\ \text{modulo } x_1\Lambda'}} \frac{e_{\Lambda'\omega'}(x_1^{-1}(\ell' - x')\omega') \cdot u - 1}{e_{\Lambda'\omega'}(x_1^{-1}\ell'\omega')} \\ &= \frac{x_1}{u^{[L':x_1\Lambda']}} \cdot \frac{\prod_{\ell' \in L' \bmod x_1\Lambda'} \big(1 - e_{\Lambda'\omega'}(x_1^{-1}(\ell' - x')\omega') \cdot u\big)}{\prod_{\ell' \in L' \smallsetminus x_1\Lambda' \bmod x_1\Lambda'} e_{\Lambda'\omega'}(x_1^{-1}\ell'\omega')}, \end{aligned}$$

where the last transformation uses the fact that $(-1)^{[L':x_1\Lambda']-1} = 1$ because $[L' : x_1\Lambda']$ is a power of q. Combining everything we obtain the desired formula.

For the convergence take any $n > 0$. By Proposition 4.7 (c) there exists a constant $c_n > 0$, such that for any $\omega' \in \Omega_n^{r-1}$ and any $x' \in F_\infty^{r-1}$ we have $|e_{\Lambda'\omega'}(x'\omega')| < c_n$. In particular this inequality holds for $x_1^{-1}\ell'$ and $x_1^{-1}(\ell' - x')$ in place of x'. Thus if $|u| \leqslant r_n := (2c_n)^{-1}$, we have $|e_{\Lambda'\omega'}(x_1^{-1}(\ell' - x')\omega') \cdot u| < 2^{-1}$, so the geometric series for

$$\frac{1}{1 - e_{\Lambda'\omega'}(x_1^{-1}(\ell' - x')\omega') \cdot u}$$

converges uniformly to a value of norm 1. Combining the inequalities yields the bound

$$\left| \frac{u^{[L':x_1\Lambda']}}{x_1} \cdot \frac{\prod\limits_{\ell' \in L' \smallsetminus x_1\Lambda' \bmod x_1\Lambda'} e_{\Lambda'\omega'}(x_1^{-1}\ell'\omega')}{\prod\limits_{\ell' \in L' \bmod x_1\Lambda'} \big(1 - e_{\Lambda'\omega'}(x_1^{-1}(\ell' - x')\omega') \cdot u\big)} \right| \leqslant \frac{r_n^{[L':\ell_1\Lambda']} c_n^{[L':\ell_1\Lambda']-1}}{|x_1|} = \frac{2^{-[L':\ell_1\Lambda']}}{|x_1| c_n}.$$

Also recall that G_k is a polynomial of fixed degree in X which is divisible by X, and the values $e_{L'\omega',q},\ e_{L'\omega',q^2},\ \ldots$ for the other variables are holomorphic functions on Ω^{r-1} and hence bounded on Ω_n^{r-1}. As both $|x_1|$ and $[L' : x_1\Lambda']$ go to infinity with x_1, this proves that the right hand side of the formula for $E_{k,x'+L'}(\omega')$ converges uniformly for all $(u, \omega') \in B(0, r_n) \times \Omega_n^{r-1}$. Varying n it therefore converges locally uniformly on the tubular neighbourhood $\bigcup_{n\geqslant 1} B(0, r_n) \times \Omega_n^{r-1}$. □

Remark 13.12 In principle, the u-expansion of $E_{k,v+L}$ in terms of powers of u can be computed from Proposition 13.10 by multiplying out the geometric series involved. As it stands, however, the sum is essentially a sum over a coset of $L_1 \subset F$, which is a fractional ideal of A. In the rank 2 case, Petrov [Pe13] has shown that there are many Drinfeld modular forms with such expansions and that they exhibit many desirable properties because of it. One may ask if there are other examples in the higher rank case.

Proposition 13.13 *(a) The u-expansion of $E_{k,v+L}(\omega)$ has constant term $E_{k,x'+L'}(\omega')$ if $v \in L + (0, x')$ for some $x' \in F^{r-1}$, and constant term 0 otherwise.*

(b) If $v \notin L + (\{0\} \times F^{r-1})$, the order at infinity of $E_{k,v+L}(\omega)$ with respect to the group $\Gamma_{v+L} \cap U(F)$ is at least

$$\mathrm{ord}_X(G_k) \cdot \min\big\{[L' : x_1\Lambda'] \,\big|\, x_1 \in v_1 + L_1\big\}.$$

Proof. Assertion (a) follows from Proposition 13.10 and the fact that the Goss polynomial G_k is divisible by X. In (b) let $d := \mathrm{ord}_X(G_k)$ denote the vanishing order at $X = 0$ of G_k as a polynomial in independent variables $X, Y_1, Y_2, \ldots$ and write $G_k = X^d H(Y_1, Y_2, \ldots) +$ (higher terms in X). Then each summand in Proposition 13.10 contributes

$$\left(u^{[L':x_1\Lambda']} \cdot \frac{\prod\limits_{\ell' \in L' \smallsetminus x_1\Lambda' \bmod x_1\Lambda'} e_{\Lambda'\omega'}(x_1^{-1}\ell'\omega')}{x_1} \right)^d \cdot H\big(e_{L'\omega',q},\ e_{L'\omega',q^2}, \ldots\big) + \big(\text{higher terms in } u\big)$$

to the u-expansion of $E_{k,v+L}(\omega)$. Recall that $v = (v_1, v')$, so that as $x = (x_1, x')$ runs through $v + \tilde{L}_1$, its first component x_1 runs through $v_1 + L_1$. Combining this yields the desired lower bound. □

Remark 13.14 For the purposes explained in Remark 16.8 below, one should hope that the inequality in Proposition 13.13 is always an equality in the case $k = 1$. By (16.2) this would yield a formula for the order at infinity of every discriminant form. For example we have:

Proposition 13.15 *If $A = \mathbb{F}_q[t]$, for any $v \in t^{-1}L \smallsetminus L$ the order at infinity of $E_{1,v+L}$ with respect to the group $\Gamma_{v+L} \cap U(F)$ is 0 if $v \in L + (\{0\} \times F^{r-1})$ and 1 otherwise.*

Proof. As above write $v = (v_1, v')$. If $v_1 \in L_1$, the u-expansion of $E_{1,v+L}$ has constant term $E_{1,v'+L'}$ by Proposition 13.13 (a), which is non-zero by Corollary 13.7; hence the order is 0 in this case.

Otherwise we have $t^{-1}L_1 = \mathbb{F}_q \cdot v_1 + L_1$ and this A-module is generated by a unique element $x_1 \in v_1 + L_1$. By Lemma 13.9 we deduce that $\Lambda' = x_1^{-1}L'$. This x_1 is then the unique element of the coset $v_1 + L_1$ that satisfies $[L' : x_1\Lambda'] = 1$. Since, moreover, $G_1(X) = X$ by Proposition 13.5 (b), Proposition 13.10 implies that $E_{1,v+L}(\omega) = \frac{u}{x_1} +$ (higher terms in u). The order is therefore 1 in that case. □

Theorem 13.16 *The Eisenstein series $E_{k,v+L}$ is a modular form of weight k for the group Γ_{v+L}.*

Proof. By Proposition 13.3 (b) it is already a weak modular form for Γ_{v+L}. Moreover, for every $\gamma \in \mathrm{GL}_r(F)$ we have $E_{k,v+L}|_k\gamma = E_{k,v\gamma+L\gamma}$ by Proposition 13.3 (a), and the latter is holomorphic at infinity by Proposition 13.10. □

14 Hecke action on Eisenstein series

For any coset $v + L$ the quotient $(Av + L)/L$ is a finite A-module that is generated by one element; hence it is isomorphic to A/N for a unique non-zero ideal N. Equivalently N is the largest ideal of A such that Γ_{v+L} contains the principal congruence subgroup $\Gamma_L(N)$. We can therefore view N as a kind of *level* of the Eisenstein series $E_{k,v+L}$. In this section we compute the effect on $E_{k,v+L}$ of a Hecke operator that is supported away from N.

For any finitely generated A-submodule $L \subset F^r$ of rank r and any prime $\mathfrak{p} \subset A$ let $L_\mathfrak{p}$ denote the closure of L in $F_\mathfrak{p}^r$, which is a finitely generated $A_\mathfrak{p}$-submodule of rank r. Note that L can be recovered from the submodules $L_\mathfrak{p}$ for all $\mathfrak{p}$ as the intersection $F^r \cap \prod_\mathfrak{p} L_\mathfrak{p}$ within $(\mathbb{A}_F^\mathrm{f})^r$. Consider finitely generated projective A-submodules L, $L' \subset F^r$ of rank r, vectors v, $v' \in F^r$, and an element $\delta \in \mathrm{GL}_r(F)$, which together satisfy:

Assumption 14.1 *For every prime $\mathfrak{p} \subset A$ we have:*

(a) $v\delta + L_\mathfrak{p}\delta \subset v' + L'_\mathfrak{p}$,

(b) $v\delta + L_\mathfrak{p}\delta = v' + L'_\mathfrak{p}$ *whenever* $v \notin L_\mathfrak{p}$, *and*

(c) $L_\mathfrak{p}\delta \not\subset \mathfrak{p}L'_\mathfrak{p}$.

Here (a) is equivalent to $v\delta + L\delta \subset v' + L'$, which includes the fact that $L\delta \subset L'$. Given (a), condition (b) means that $E_{k,v+L}$ and $E_{k,v'+L'}$ are Eisenstein series of the same level N and that T_δ is supported only at primes not dividing N. Property (c) is equivalent to $L\delta \not\subset \mathfrak{p}L'$ for any prime $\mathfrak{p}$, which serves as normalisation. If $L = L' = A^r$, then (a) means that δ has coefficients in A and maps v into $v' + A^r$. Then, in addition, condition (b) means that the determinant of δ is relatively prime to N, and (c) means that δ is not congruent to the zero matrix modulo any prime of A. Assumption 14.1 will remain in force until Theorem 14.11 below.

To begin with we abbreviate

$$\begin{aligned} \Gamma' &:= \Gamma_{v'+L'}, \\ \Gamma &:= \delta^{-1}\Gamma_{v+L}\delta \cap \Gamma_{v'+L'} \;=\; \Gamma_{v\delta+L\delta} \cap \Gamma_{v'+L'} \;<\; \Gamma'. \end{aligned}$$

For any prime $\mathfrak{p} \subset A$ we consider the open compact subgroups

$$\begin{aligned} K'_\mathfrak{p} &:= \left\{ k \in \mathrm{GL}_r(F_\mathfrak{p}) \,\middle|\, v'k + L'_\mathfrak{p}k = v' + L'_\mathfrak{p} \right\}, \\ K_\mathfrak{p} &:= \left\{ k \in \mathrm{GL}_r(F_\mathfrak{p}) \,\middle|\, v'k + L'_\mathfrak{p}k = v' + L'_\mathfrak{p} \text{ and } v\delta k + L_\mathfrak{p}\delta k = v\delta + L_\mathfrak{p}\delta \right\} \;<\; K'_\mathfrak{p}. \end{aligned}$$

Since $L'/L\delta$ is finite, for any prime $\mathfrak{p}$ not dividing its annihilator we have $L_\mathfrak{p}\delta = L'_\mathfrak{p}$ and hence $v\delta + L_\mathfrak{p}\delta = v' + L'_\mathfrak{p}$. Thus for almost all $\mathfrak{p}$ we have $K_\mathfrak{p} = K'_\mathfrak{p}$. By Assumption 14.1 (b) this is so in particular if $v \notin L_\mathfrak{p}$. Also, the equalities $L' = F^r \cap \prod_\mathfrak{p} L'_\mathfrak{p}$ and $L = F^r \cap \prod_\mathfrak{p} L_\mathfrak{p}$ imply that $\Gamma' = \mathrm{GL}_r(F) \cap \prod_\mathfrak{p} K'_\mathfrak{p}$ and $\Gamma = \mathrm{GL}_r(F) \cap \prod_\mathfrak{p} K_\mathfrak{p}$.

Lemma 14.2 *For every* $\mathfrak{p}$ *we have* $\det(K_\mathfrak{p}) = \det(K'_\mathfrak{p})$.

Proof. If $v \notin L_\mathfrak{p}$, this follows from the fact that $K'_\mathfrak{p} = K_\mathfrak{p}$. Otherwise by assumption we have $L_\mathfrak{p}\delta = v\delta + L_\mathfrak{p}\delta \subset v' + L'_\mathfrak{p} = L'_\mathfrak{p}$ and both are free $A_\mathfrak{p}$-modules of rank r within $F_\mathfrak{p}^r$. To prove the desired statement we can conjugate everything by an arbitrary element of $\mathrm{GL}_r(F)$. By the elementary divisor theorem we may thus without loss of generality assume that $L'_\mathfrak{p} = A_\mathfrak{p}^r$ and that $L_\mathfrak{p}\delta = A_\mathfrak{p}^r h$ for some diagonal matrix $h \in \mathrm{GL}_r(F_\mathfrak{p})$. For any $a \in A_\mathfrak{p}^\times$ the diagonal matrix $\mathrm{diag}(1,\ldots,1,a)$ then lies in $K_\mathfrak{p}$ with determinant a; hence $A_\mathfrak{p}^\times < \det(K_\mathfrak{p})$. As $A_\mathfrak{p}^\times$ is the unique largest compact subgroup of $F_\mathfrak{p}^\times$, it follows that $\det(K_\mathfrak{p}) = \det(K'_\mathfrak{p}) = A_\mathfrak{p}^\times$, as desired. ∎

Lemma 14.3 *There is a natural bijection*

$$\begin{aligned} \Gamma\backslash\Gamma' &\longrightarrow \textstyle\prod_\mathfrak{p} K_\mathfrak{p}\backslash K'_\mathfrak{p}, \\ \Gamma\gamma &\longmapsto (K_\mathfrak{p}\gamma)_\mathfrak{p}. \end{aligned}$$

Proof. If two cosets $\Gamma\gamma_1$ and $\Gamma\gamma_2$ have the same image, we have $K_\mathfrak{p}\gamma_1 = K_\mathfrak{p}\gamma_2$ and hence $\gamma_1\gamma_2^{-1} \in K_\mathfrak{p}$ for all $\mathfrak{p}$. Thus $\gamma_1\gamma_2^{-1} \in \mathrm{GL}_r(F) \cap \prod_\mathfrak{p} K_\mathfrak{p} = \Gamma$, and so $\Gamma\gamma_1 = \Gamma\gamma_2$. The map is therefore injective. For the surjectivity consider any collection of cosets $K_\mathfrak{p}k_\mathfrak{p} \subset K'_\mathfrak{p}$. By Lemma 14.2 we may without loss of generality assume that $k_\mathfrak{p} \in \mathrm{SL}_r(F_\mathfrak{p}) \cap K'_\mathfrak{p}$. By strong approximation in the group SL_r there then exists an element $\gamma \in \mathrm{SL}_r(F) \cap \prod_\mathfrak{p} K_\mathfrak{p}k_\mathfrak{p}$. This element lies in $\mathrm{GL}_r(F) \cap \prod_\mathfrak{p} K'_\mathfrak{p} = \Gamma'$; hence the map is surjective. ◻

Next observe that for any $\gamma \in \Gamma'$ the subset $v\delta\gamma + L\delta\gamma \subset F^r$ depends only on the coset $\Gamma\gamma$. For any $x \in F^r$ we let $C(x)$ denote the number of such cosets for which $x \in v\delta\gamma + L\delta\gamma$. Similarly, for any $k \in K'_\mathfrak{p}$ the subset $v\delta k + L_\mathfrak{p}\delta k \subset F_\mathfrak{p}^r$ depends only on the coset $K_\mathfrak{p}k$. For any $x \in F_\mathfrak{p}^r$ we let $C_\mathfrak{p}(x)$ denote the number of such cosets for which $x \in v\delta k + L_\mathfrak{p}\delta k$. For any fixed $x \in F^r$ the module $(Ax + Av' + L')/L\delta$ is finite, so for any prime $\mathfrak{p}$ not dividing its annihilator we have $x \in v' + L'_\mathfrak{p} = v\delta + L_\mathfrak{p}\delta$ and $K'_\mathfrak{p} = K_\mathfrak{p}$ and hence $C_\mathfrak{p}(x) = 1$.

Lemma 14.4 *For any $x \in F^r$ we have $C(x) = \prod_\mathfrak{p} C_\mathfrak{p}(x)$.*

Proof. Since $v \in F^r$ and $L = F^r \cap \prod_\mathfrak{p} L_\mathfrak{p}$, for any $\gamma \in \Gamma'$ we have the equality $v\delta\gamma + L\delta\gamma = F^r \cap (v\delta\gamma + \prod_\mathfrak{p} L_\mathfrak{p}\delta\gamma)$ within $(\mathbb{A}_F^\mathrm{f})^r$. Since $x \in F^r$, it follows that $x \in v\delta\gamma + L\delta\gamma$ if and only if $x \in v\delta\gamma + L_\mathfrak{p}\delta\gamma$ for all $\mathfrak{p}$. But the latter condition depends only on the coset $K_\mathfrak{p}\gamma$, so the product formula follows from Lemma 14.3. ◻

Now let $q_\mathfrak{p}$ denote the order of the residue field $k(\mathfrak{p}) := A/\mathfrak{p}$. In principle one can give an explicit formula for $C_\mathfrak{p}(x)$ as a polynomial in $q_\mathfrak{p}$ with coefficients in $\mathbb{Z}$. But we are only interested in $C_\mathfrak{p}(x)$ modulo (p), so we restrict ourselves to determining this residue class. Let char_X denote the characteristic function of a subset $X \subset F_\mathfrak{p}^r$.

Lemma 14.5 *For any prime $\mathfrak{p}$ consider the unique integers $\mu_{\mathfrak{p},1} \geqslant \ldots \geqslant \mu_{\mathfrak{p},r} \geqslant 0$ such that $L'_\mathfrak{p}/L_\mathfrak{p}\delta \cong \bigoplus_{j=1}^r A/\mathfrak{p}^{\mu_{\mathfrak{p},j}}$. Then for any $x \in F_\mathfrak{p}^r$ we have*

$$C_\mathfrak{p}(x) \;\equiv\; \left\{ \begin{array}{ll} \mathrm{char}_{v'+L'_\mathfrak{p}}(x) & \text{if } \mu_{\mathfrak{p},1} \leqslant 1 \\ \mathrm{char}_{L'_\mathfrak{p}\smallsetminus\mathfrak{p}L'_\mathfrak{p}}(x) & \text{if } 2 \leqslant \mu_{\mathfrak{p},1} \leqslant \mu_{\mathfrak{p},r-1}+1 \\ 0 & \text{if } \mu_{\mathfrak{p},1} \geqslant \mu_{\mathfrak{p},r-1}+2 \end{array} \right\} \bmod (q_\mathfrak{p}).$$

Proof. By Assumption 14.1 (a) we have $v\delta + L_\mathfrak{p}\delta \subset v' + L'_\mathfrak{p}$, so for any $k \in K'_\mathfrak{p}$ we also have $v\delta k + L_\mathfrak{p}\delta k \subset v' + L'_\mathfrak{p}$. Thus $C_\mathfrak{p}(x) = 0$ if $x \notin v' + L'_\mathfrak{p}$. So till the end of the proof we assume that $x \in v' + L'_\mathfrak{p}$. If in addition $v\delta + L_\mathfrak{p}\delta = v' + L'_\mathfrak{p}$, we have $K_\mathfrak{p} = K'_\mathfrak{p}$ and $v\delta k + L_\mathfrak{p}\delta k = v' + L'_\mathfrak{p}$ and hence $C_\mathfrak{p}(x) = 1$. Till the end of the proof we therefore assume that $v\delta + L_\mathfrak{p}\delta \neq v' + L'_\mathfrak{p}$. By Assumption 14.1 (b) this implies that $v \in L_\mathfrak{p}$ and hence $L_\mathfrak{p}\delta = v\delta + L_\mathfrak{p}\delta \subsetneqq v' + L'_\mathfrak{p} = L'_\mathfrak{p}$. For ease of notation we abbreviate the chosen exponents to $\mu_i := \mu_{\mathfrak{p},i}$. Then $\mu_1 \geqslant 1$, and Assumption 14.1 (c) requires that $\mu_r = 0$.

Both $L_\mathfrak{p}\delta \subset L'_\mathfrak{p}$ are free $A_\mathfrak{p}$-modules of rank r within $F_\mathfrak{p}^r$. To prove the desired statement we can conjugate everything by an arbitrary element of $\mathrm{GL}_r(F)$. By the elementary divisor theorem we may thus without loss of generality assume that $L'_\mathfrak{p} = A_\mathfrak{p}^r$ and $L_\mathfrak{p}\delta = \bigoplus_{j=1}^r \mathfrak{p}^{\mu_j}A_\mathfrak{p}$. Then $K'_\mathfrak{p} = \mathrm{GL}_r(A_\mathfrak{p})$ and

$$K_\mathfrak{p} \;=\; h^{-1}\,\mathrm{GL}_r(A_\mathfrak{p})h \cap \mathrm{GL}_r(A_\mathfrak{p}) \;=\; \big\{\, (a_{ij})_{ij} \in \mathrm{GL}_r(A_\mathfrak{p}) \,\big|\, \forall i \geqslant j\colon\ a_{ij} \in \mathfrak{p}^{\mu_j-\mu_i}A_\mathfrak{p} \,\big\}.$$

Next observe that $\mathfrak{p}^{\mu_1}L'_\mathfrak{p} \subset L_\mathfrak{p}\delta \subset L'_\mathfrak{p}$. Consider the factor module $\bar{L}' := L'_\mathfrak{p}/\mathfrak{p}^{\mu_1}L'_\mathfrak{p} = (A/\mathfrak{p}^{\mu_1})^r$ and its submodule $\bar{L} := L_\mathfrak{p}\delta/\mathfrak{p}^{\mu_1}L'_\mathfrak{p} = \bigoplus_{j=1}^r \mathfrak{p}^{\mu_j}A_\mathfrak{p}/\mathfrak{p}^{\mu_1}A_\mathfrak{p}$. Then $K'_\mathfrak{p}$ surjects to $\bar{K}' := \mathrm{GL}_r(A/\mathfrak{p}^{\mu_1})$, and the image $\bar{K} < \bar{K}'$ of $K_\mathfrak{p} < K'_\mathfrak{p}$ is the stabiliser of $\bar{L}$. In particular we have $[\bar{K}' : \bar{K}] = [K'_\mathfrak{p} : K_\mathfrak{p}]$. To compute this number note that the image of $K_\mathfrak{p}$ in $\mathrm{GL}_r(k(\mathfrak{p}))$ is the parabolic subgroup

$$P(k(\mathfrak{p})) \;:=\; \big\{\, (\bar{a}_{ij})_{ij} \in \mathrm{GL}_r(k(\mathfrak{p})) \,\big|\, \forall i \geqslant j\colon\ \mu_j > \mu_i \Rightarrow \bar{a}_{ij} = 0 \,\big\},$$

and a straightforward calculation shows that $[\mathrm{GL}_r(k(\mathfrak{p})) : P(k(\mathfrak{p}))] \equiv 1$ modulo $(q_\mathfrak{p})$. From this we deduce that

$$[K'_\mathfrak{p} : K_\mathfrak{p}] \;\in\; \prod_{i \geqslant j} q_\mathfrak{p}^{\max\{0,\mu_j-\mu_i-1\}} \cdot (1 + q_\mathfrak{p}\mathbb{Z}). \tag{14.6}$$

Also, let $\bar{x} \in \bar{L}'$ denote the image of $x \in v' + L'_\mathfrak{p} = L'_\mathfrak{p}$. Then

$$C_\mathfrak{p}(x) \;=\; \frac{|\{\bar{k} \in \bar{K}' \mid \bar{x} \in \bar{L}\bar{k}\}|}{|\bar{K}|} \;=\; [K'_\mathfrak{p} : K_\mathfrak{p}] \cdot \frac{|\{\bar{k} \in \bar{K}' \mid \bar{x}\bar{k}^{-1} \in \bar{L}\}|}{|\bar{K}'|}.$$

If $\bar{x} = 0$, we deduce that $C_\mathfrak{p}(x) = [K'_\mathfrak{p} : K_\mathfrak{p}]$. Otherwise $\bar{x}$ lies in the subset $\bar{S}_\nu := \mathfrak{p}^\nu\bar{L}' \smallsetminus \mathfrak{p}^{\nu+1}\bar{L}'$ for a unique exponent $0 \leqslant \nu < \mu_1$. Since $\bar{S}_\nu$ is an orbit under $\bar{K}'$, the last fraction is equal to the proportional size of $\bar{L} \cap \bar{S}_\nu$ versus $\bar{S}_\nu$; hence

$$C_\mathfrak{p}(x) \;=\; [K'_\mathfrak{p} : K_\mathfrak{p}] \cdot \frac{|\bar{L} \cap \bar{S}_\nu|}{|\bar{S}_\nu|}. \tag{14.7}$$

To compute these cardinalities observe that

$$\bar{L} \cap \mathfrak{p}^\nu\bar{L}' \;=\; \bigoplus_{j=1}^r (\mathfrak{p}^{\mu_j}A_\mathfrak{p} \cap \mathfrak{p}^\nu A_\mathfrak{p})/\mathfrak{p}^{\mu_1}A_\mathfrak{p} \;=\; \bigoplus_{j=1}^r \mathfrak{p}^{\max\{\mu_j,\nu\}}A_\mathfrak{p}/\mathfrak{p}^{\mu_j}A_\mathfrak{p}$$

and hence

$$|\bar{L} \cap \mathfrak{p}^\nu\bar{L}'| \;=\; \prod_{j=1}^r q_\mathfrak{p}^{\mu_1-\max\{\mu_j,\nu\}}.$$

The same calculation with $\nu + 1$ in place of ν shows that

$$|\bar{L} \cap \mathfrak{p}^{\nu+1}\bar{L}'| \;=\; \prod_{j=1}^r q_\mathfrak{p}^{\mu_1-\max\{\mu_j,\nu+1\}}.$$

Together this implies that

$$|\bar{L} \cap \bar{S}_\nu| \;=\; q_\mathfrak{p}^{\sum_{j=1}^r(\mu_1-\max\{\mu_j,\nu\})} - q_\mathfrak{p}^{\sum_{j=1}^r(\mu_1-\max\{\mu_j,\nu+1\})}.$$

Since $\mu_1 > \mu_r = 0$, we certainly have $\mu_1 - \max\{\mu_r, \nu\} > \mu_1 - \max\{\mu_r, \nu+1\}$, so the first exponent is greater than the second. Therefore

$$|\bar{L} \cap \bar{S}_\nu| \in q_\mathfrak{p}^{\sum_{j=1}^r (\mu_1 - \max\{\mu_j, \nu+1\})} \cdot (-1 + q_\mathfrak{p}\mathbb{Z}). \tag{14.8}$$

A similar, but simpler, computation shows that

$$|\bar{S}_\nu| \in q_\mathfrak{p}^{r(\mu_1-\nu-1)} \cdot (-1 + q_\mathfrak{p}\mathbb{Z}). \tag{14.9}$$

Combining the formulas (14.6) through (14.9) we deduce that

$$C_\mathfrak{p}(x) \in q_\mathfrak{p}^{c(\nu)} \cdot (1 + q_\mathfrak{p}\mathbb{Z})$$

for

$$c(\nu) := \sum_{i \geqslant j} \max\{0, \mu_j - \mu_i - 1\} + \sum_{j=1}^r (\mu_1 - \max\{\mu_j, \nu+1\}) - r(\mu_1 - \nu - 1).$$

By (14.6), the same formula is true in the case $\bar{x} = 0$ if we set $\nu := \mu_1$.

It remains to find out when this exponent is greater than 0. Combining the terms for $i = r$ with the rest of the formula and using the fact that $\mu_r = 0$ yields

$$\begin{aligned} c(\nu) &= \textstyle\sum_{r>i\geqslant j} \max\{0, \mu_j - \mu_i - 1\} + \sum_{j=1}^r \big(\max\{0, \mu_j - 1\} - \max\{0, \mu_j - \nu - 1\}\big) \\ &= \textstyle\sum_{r>i\geqslant j} \max\{0, \mu_j - \mu_i - 1\} + \sum_{j=1}^r \max\{0, \min\{\mu_j - 1, \nu\}\}. \end{aligned}$$

Here all summands are $\geqslant 0$. Since $\mu_1 \geqslant \ldots \geqslant \mu_r$, the first sum contains a positive term if and only if $\mu_1 - \mu_{r-1} - 1 \geqslant 1$, and the second sum contains a positive term if and only if $\min\{\mu_1 - 1, \nu\} \geqslant 1$. Thus

$$\begin{aligned} &c(\nu) > 0 \quad \text{if } \mu_1 \geqslant \mu_{r-1} + 2 \text{ or } (\mu_1 \geqslant 2 \text{ and } \nu \geqslant 1), \\ &c(\nu) = 0 \quad \text{if } \mu_1 \leqslant \mu_{r-1} + 1 \text{ and } (\mu_1 \leqslant 1 \text{ or } \nu = 0). \end{aligned}$$

Combining all the cases we conclude that

$$\begin{array}{ll} C_\mathfrak{p}(x) = 0 & \text{if } x \notin v' + L'_\mathfrak{p}, \\ C_\mathfrak{p}(x) = 1 & \text{if } x \in v' + L'_\mathfrak{p} = v\delta + L_\mathfrak{p}\delta, \\ C_\mathfrak{p}(x) \equiv 0 \mod (q_\mathfrak{p}) & \text{if } x \in v' + L'_\mathfrak{p} \neq v\delta + L_\mathfrak{p}\delta \text{ and } (\mu_1 \geqslant \mu_{r-1} + 2 \text{ or } (\mu_1 \geqslant 2 \text{ and } x \in \mathfrak{p}L'_\mathfrak{p})), \\ C_\mathfrak{p}(x) \equiv 1 \mod (q_\mathfrak{p}) & \text{if } x \in v' + L'_\mathfrak{p} \neq v\delta + L_\mathfrak{p}\delta \text{ and } \mu_1 \leqslant \mu_{r-1} + 1 \text{ and } (\mu_1 \leqslant 1 \text{ or } x \notin \mathfrak{p}L'_\mathfrak{p}), \end{array}$$

Since $v' + L'_\mathfrak{p} = v\delta + L_\mathfrak{p}\delta$ if and only if $L'_\mathfrak{p} = L_\mathfrak{p}\delta$ if and only if $\mu_1 = 0$, the desired formula follows. □

Now recall from Definition 12.11 that the Hecke operator associated to the double coset $\Gamma_{v+L}\delta\Gamma_{v'+L'}$ is defined by

$$T_\delta : \mathcal{M}_k(\Gamma_{v+L}) \longrightarrow \mathcal{M}_k(\Gamma_{v'+L'}),\ f \longmapsto \textstyle\sum_\gamma f|_k \gamma, \tag{14.10}$$

where γ runs through a set of representatives of the quotient $\Gamma_{v+L}\backslash\Gamma_{v+L}\delta\Gamma_{v'+L'}$.

Theorem 14.11 *Under Assumption 14.1 consider the integers $\mu_{\mathfrak{p},i}$ from Lemma 14.5. If $\mu_{\mathfrak{p},1} \geqslant \mu_{\mathfrak{p},r-1} + 2$ for some $\mathfrak{p}$, we have*

$$T_\delta E_{k,v+L} \;=\; 0. \qquad .$$

Otherwise let S be the finite set of primes $\mathfrak{p}$ for which $2 \leqslant \mu_{\mathfrak{p},1} \leqslant \mu_{\mathfrak{p},r-1} + 1$. For any subset $I \subset S$ set $L'_I := \prod_{\mathfrak{p}\in I} \mathfrak{p} \cdot L'$. Then $v' + L' = v'' + L'$ for some element $v'' \in (v' + L') \cap \bigcap_{\mathfrak{p}\in S} \mathfrak{p}L'_{\mathfrak{p}}$ and

$$T_\delta E_{k,v+L} \;=\; \sum_{I\subset S} (-1)^{|I|} \cdot E_{k,v''+L'_I}.$$

Proof. By the construction of Γ and Γ' we have $T_\delta f = \sum_\gamma E_{k,v+L}|_k\, \delta\gamma$, where γ runs through a set of representatives $\mathcal{R}$ of $\Gamma\backslash\Gamma'$. Using the transformation rule from Proposition 13.3 (a) and the definition (13.1) of Eisenstein series we deduce that

$$(T_\delta E_{k,v+L})(\omega) \;=\; \sum_{\gamma\in\mathcal{R}} E_{k,v\delta\gamma+L\delta\gamma}(\omega) \;=\; \sum_{\gamma\in\mathcal{R}} \; \sum_{0\neq x\in v\delta\gamma+L\delta\gamma} (x\omega)^{-k} \;=\; \sum_{0\neq x\in F^r} C(x)\cdot(x\omega)^{-k}.$$

Here $C(x)$ is determined by Lemmas 14.4 and 14.5: If $\mu_{\mathfrak{p},1} \geqslant \mu_{\mathfrak{p},r-1} + 2$ for some $\mathfrak{p}$, we have $C(x) = 0$ for all $x \in F^r$. Otherwise, for any prime $\mathfrak{p}$ in the indicated set S, we have $v \in L_{\mathfrak{p}}$ and hence $v' \in L'_{\mathfrak{p}}$ by Assumption 14.1 (b). Thus $\mathfrak{p}$ does not divide the annihilator N of the coset $v' + L'/L'$. By the Chinese remainder theorem there therefore exists an element $a \in \bigcap_{\mathfrak{p}\in S} \mathfrak{p}$ with $a \equiv 1$ modulo N, and then $v'' := av'$ lies in $(v' + L') \cap \bigcap_{\mathfrak{p}\in S} \mathfrak{p}L'_{\mathfrak{p}}$. For any subset $I \subset S$ we then have

$$F^r \cap \prod_{\text{all } \mathfrak{p}} \left\{ \begin{array}{rl} v' + L'_{\mathfrak{p}} & \text{if } \mathfrak{p} \notin I, \\ \mathfrak{p}L'_{\mathfrak{p}} & \text{if } \mathfrak{p} \in I, \end{array} \right\} \;=\; v'' + \left(F^r \cap \prod_{\text{all } \mathfrak{p}} \left\{ \begin{array}{rl} L'_{\mathfrak{p}} & \text{if } \mathfrak{p} \notin I, \\ \mathfrak{p}L'_{\mathfrak{p}} & \text{if } \mathfrak{p} \in I, \end{array} \right\} \right) \;=\; v'' + L'_I.$$

Lemmas 14.4 and 14.5 then imply that

$$\begin{aligned} C(x) \;&\equiv\; \prod_{\mathfrak{p}\notin S} \mathrm{char}_{v'+L'_{\mathfrak{p}}}(x) \cdot \prod_{\mathfrak{p}\in S} \mathrm{char}_{L'_{\mathfrak{p}} \smallsetminus \mathfrak{p}L'_{\mathfrak{p}}}(x) \qquad\qquad \text{modulo } (q) \\ &=\; \prod_{\mathfrak{p}\notin S} \mathrm{char}_{v'+L'_{\mathfrak{p}}}(x) \cdot \prod_{\mathfrak{p}\in S} \left[\mathrm{char}_{L'_{\mathfrak{p}}}(x) - \mathrm{char}_{\mathfrak{p}L'_{\mathfrak{p}}}(x) \right] \\ &=\; \prod_{\mathfrak{p}\notin S} \mathrm{char}_{v'+L'_{\mathfrak{p}}}(x) \cdot \sum_{I\subset S} (-1)^{|I|} \cdot \prod_{\mathfrak{p}\in S\smallsetminus I} \mathrm{char}_{L'_{\mathfrak{p}}}(x) \cdot \prod_{\mathfrak{p}\in I} \mathrm{char}_{\mathfrak{p}L'_{\mathfrak{p}}}(x) \\ &=\; \sum_{I\subset S} (-1)^{|I|} \cdot \prod_{\mathfrak{p}\notin I} \mathrm{char}_{v'+L'_{\mathfrak{p}}}(x) \cdot \prod_{\mathfrak{p}\in I} \mathrm{char}_{\mathfrak{p}L'_{\mathfrak{p}}}(x) \\ &=\; \sum_{I\subset S} (-1)^{|I|} \cdot \mathrm{char}_{v''+L'_I}(x). \end{aligned}$$

The desired formula now follows from the definition (13.1) of Eisenstein series. □

Corollary 14.12 *Consider any $\delta \in \mathrm{GL}_r(F)$ such that for every prime $\mathfrak{p} \subset A$ we have:*

(a) $v\delta + L_{\mathfrak{p}}\delta \subset v' + L'_{\mathfrak{p}}$,

(b) $v\delta + L_\mathfrak{p}\delta = v' + L'_\mathfrak{p}$ *whenever* $v \notin L_\mathfrak{p}$, *and*

(c) $\mathfrak{p}L'_\mathfrak{p} \subsetneqq L_\mathfrak{p}\delta$.

Then the Hecke operator T_δ *associated to the double coset* $\Gamma_{v+L}\delta\Gamma_{v'+L'}$ *satisfies*

$$T_\delta E_{k,v+L} \;=\; E_{k,v'+L'}.$$

Proof. In that case Assumption 14.1 holds with $\mu_{\mathfrak{p},1} \leqslant 1$ for all $\mathfrak{p}$; hence we are in the second case of Theorem 14.11 with $S = \varnothing$. □

Proposition 14.13 *Consider any arithmetic subgroups* $\Gamma, \Gamma' < \mathrm{GL}_r(F)$, *any element* $\delta \in \mathrm{GL}_r(F)$, *and any scalar* $a \in F^\times$. *Then the Hecke operators* T_δ *and* $T_{a^{-1}\delta}$ *associated to the double cosets* $\Gamma\delta\Gamma'$ *and* $\Gamma a^{-1}\delta\Gamma'$ *satisfy*

$$T_{a^{-1}\delta} \;=\; a^k \cdot T_\delta.$$

Proof. As γ runs through a set of representatives of $\Gamma\backslash\Gamma\delta\Gamma'$, the element $a^{-1}\gamma$ runs through a set of representatives of $\Gamma\backslash\Gamma a^{-1}\delta\Gamma'$. Since $f|_k(a^{-1}\gamma) = f|_k(a^{-1}\cdot \mathrm{Id}_r)|_k\gamma = a^k \cdot f|_k\gamma$ by (1.6) and (1.7), the formula follows from the definition of Hecke operators 12.11. □

Remark 14.14 Using Proposition 14.13, one can express any Hecke operator in terms of another Hecke operator that is associated to a matrix with coefficients in A. If one prefers, one can also require that the inverse matrix has coefficients in A.

Remark 14.15 Combining Proposition 14.13 with Theorem 14.11 or Corollary 14.12, one obtains an explicit formula for $T_{a^{-1}\delta}E_{k,v+L}$ as well. In the special case $v' + L' = v + L$ one obtains many Hecke operators for which E_{v+L} is an eigenform with eigenvalue 1 or a^k.

Remark 14.16 In the case $r = 2$ Theorem 14.11 was proved by Gekeler [Ge86, VIII.1]. For instance, for $L = L' = A^2$, the Hecke operator in [Ge86] associated to a prime element $\pi \in A$ is T_δ for the matrix $\delta = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \pi^{-1}\end{smallmatrix}\right)$ and satisfies $T_\delta E_{k,L} = \pi^k \cdot E_{k,L}$.

15 Coefficient forms

As before we fix a finitely generated projective A-submodule $L \subset F^r$ of rank r. We will show that the coefficients of the exponential function $e_{L\omega}$ and of the associated Drinfeld A-module are modular forms for the group Γ_L; these are the *coefficient forms* in the title. We will also exhibit them as polynomials in Eisenstein series. The coefficients of $e_{L\omega}$ have been studied in a special case, for instance in [Ge86, II.2] and [Ge11].

For every $k \geqslant 0$ we write $e_{k,L}(\omega) := e_{L\omega,q^k}$, so that $e_{L\omega}(z) = \sum_{k=0}^{\infty} e_{k,L}(\omega)z^{q^k}$ with $e_{0,L} = 1$. Then by [BR09, (9)] we have

$$e_{k,L} \;=\; E_{q^k-1,L} + \sum_{j=1}^{k-1} e_{j,L} \cdot E_{q^{k-j}-1,L}^{q^j}. \tag{15.1}$$

By direct calculation [Ba14, Lemma 3.4.13] this is equivalent to the more suggestive fact that $z - \sum_{i\geqslant 1} E_{q^i-1,L}(\omega)z^{q^i}$ is the compositional inverse of $e_{L\omega}$, in other words, that for all $\omega \in \Omega^r$ and $z \in \mathbb{C}$ we have

$$e_{L\omega}\Big(z - \sum_{i\geqslant 1} E_{q^i-1,L}(\omega)z^{q^i}\Big) \;=\; z. \tag{15.2}$$

By induction on k the recursion formula (15.1) implies that $e_{k,L}$ is a universal polynomial with coefficients in $\mathbb{F}_p$ in the Eisenstein series $E_{q^i-1,L}$ for all $1 \leqslant i \leqslant k$.

Proposition 15.3 *For all $k \geqslant 0$ we have:*

(a) $e_{k,L}|_{q^k-1}\gamma = e_{k,L\gamma}$ for all $\gamma \in \mathrm{GL}_r(F)$.

(b) $e_{k,L}$ is a modular form of weight $q^k - 1$ for the group Γ_L.

(c) The u-expansion of $e_{k,L}$ has constant term $e_{k,L'}$ with L' as in (13.8). In particular $e_{k,L}$ is not a cusp form.

Proof. For any $\gamma \in \mathrm{GL}_r(F)$ the exponential function associated to the lattice $L\gamma(\omega) \subset \mathbb{C}_\infty$ satisfies

$$e_{L\gamma(\omega)} \;=\; e_{j(\gamma,\omega)^{-1}L\gamma\omega}(z) \;\overset{2.3}{=}\; j(\gamma,\omega)^{-1}e_{L\gamma\omega}\big(j(\gamma,\omega)z\big).$$

Comparing coefficients of z^{q^k} in the respective power series expansions yields

$$e_{L,q^k}\big(\gamma(\omega)\big) \;=\; j(\gamma,\omega)^{q^k-1}e_{L\gamma,q^k}(\omega),$$

proving (a). Part (b) follows from Theorem 13.16 and the formula (15.1) by induction on k. To prove (c), write $\omega = \binom{\omega_1}{\omega'}$ as before. For any fixed $\omega' \in \Omega^{r-1}$, if ω_1 goes to infinity, the defining formula (13.4) shows that $e_{L\omega}$ goes to $e_{L'\omega'}$ coefficientwise. Thus $e_{k,L}$ goes to $e_{k,L'}$, and since the latter is non-zero, it follows that $e_{k,L}$ is not a cusp form. □

Next let $(\mathbb{G}_{\mathrm{a},\Omega^r}, \psi^L)$ be the Drinfeld A-module of rank r over Ω^r that was associated to L in (7.3). Following (7.2) and (2.1) and Corollary 13.7, for any $a \in A \smallsetminus \{0\}$ and any $\omega \in \Omega^r$ we then have

$$\psi_a^{L\omega}(X) \;=\; a\cdot X\cdot \prod_{\substack{v\,\in\, a^{-1}L\smallsetminus L\\ \text{modulo } L}} \big(1 - E_{1,v+L}(\omega)\cdot X\big). \tag{15.4}$$

This is an $\mathbb{F}_q$-linear polynomial of degree $[a^{-1}L : L] = q^{r\deg(a)}$ in X. We expand it as

$$\psi_a^{L\omega}(X) \;=\; \sum_{i\geqslant 0} g_{a,i}^L(\omega)\cdot X^{q^i} \tag{15.5}$$

with holomorphic functions $g_{a,i}^L$ on Ω^r, which are non-zero for $i = 0$ and $i = r\deg(a)$ but zero whenever $i > r\deg(a)$. The formula (15.4) implies that each $g_{a,k}^L$ is a homogeneous symmetric polynomial of degree $q^k - 1$ in the functions $E_{1,v+L}$.

For an alternative description recall that $\psi_a^{L\omega}$ can be characterised as the unique $\mathbb{F}_q$-linear polynomial such that $\psi_a^{L\omega}(e_{L\omega}(z)) = e_{L\omega}(az)$. Plugging the expansions for $\psi_a^{L\omega}$ and $e_{L\omega}$ into this functional equation and using the fact that $e_{L,1} = 1$, we deduce that for all $k \geqslant 0$ we have

$$g_{a,k}^L + \sum_{i=0}^{k-1} g_{a,i}^L \cdot e_{k-i,L}^{q^i} \;=\; e_{k,L} \cdot a^{q^k}. \tag{15.6}$$

By induction on k this recursion relation implies that $g_{a,k}^L$ is a universal polynomial with coefficients in A in the functions $e_{j,L}$ for all $1 \leqslant j \leqslant k$, or again in the Eisenstein series $E_{q^i-1,L}$ for all $1 \leqslant i \leqslant k$.

More generally, consider any non-zero ideal $N \subset A$. Then some positive power of N is a principal ideal, say $N^n = (a)$ for $a \in A \smallsetminus \{0\}$, and we choose an element $N^* \in \mathbb{C}_\infty$ such that $(N^*)^n = a$. This element is well-defined up to multiplication by a root of unity, and for any principal ideal (a) the value $(a)^*$ is equal to a times a root of unity. We also set $\deg(N) := \dim_{\mathbb{F}_q}(A/N)$, so that $[N^{-1}L : L] = q^{r\deg(N)}$. In analogy with the definition (7.2) of $\psi_a^{L\omega}$ we define

$$\psi_N^{L\omega} \;:=\; N^* \cdot e_{e_{L\omega}(N^{-1}L\omega)}. \tag{15.7}$$

Note that for any principal ideal we have $g_{(a),i}^L = g_{a,i}^L$ times a root of unity; hence everything that follows about $g_{N,i}^L$ applies equally to $g_{a,i}^L$.

For general N, by (2.1) and Corollary 13.7 we have

$$\psi_N^{L\omega}(X) \;=\; N^* \cdot X \cdot \prod_{\substack{v \in N^{-1}L \smallsetminus L \\ \text{modulo } L}} \big(1 - E_{1,v+L}(\omega) \cdot X\big). \tag{15.8}$$

As in (15.5) we define holomorphic functions $g_{N,i}^L$ on Ω^r by expanding

$$\psi_N^{L\omega}(X) \;=\; \sum_{i\geqslant 0} g_{N,i}^L(\omega) \cdot X^{q^i}, \tag{15.9}$$

which are non-zero for $i = 0$ and $i = r\deg(N)$ but zero whenever $i > r\deg(N)$. The formula (15.8) implies that each $g_{N,k}^L$ is a homogeneous symmetric polynomial of degree $q^k - 1$ in the functions $E_{1,v+L}$.

For an alternative description observe that by the definition of $\psi_N^{L\omega}$ and Proposition 2.3 (a) we have

$$\psi_N^{L\omega}(e_{L\omega}(z)) \;=\; N^* \cdot e_{N^{-1}L\omega}(z). \tag{15.10}$$

Plugging the respective expansions into this functional equation and using the fact that $e_{L,1} = 1$, we deduce that for all $k \geqslant 0$ we have

$$g_{N,k}^L + \sum_{i=0}^{k-1} g_{N,i}^L \cdot e_{k-i,L}^{q^i} \;=\; N^* \cdot e_{k,N^{-1}L}. \tag{15.11}$$

By induction on k this recursion relation implies that $g^L_{N,k}$ is a polynomial with coefficients in $\mathbb{F}_p[N^*]$ in the functions $e_{j,L}$ and $e_{j,N^{-1}L}$ for all $1 \leqslant j \leqslant k$, or again in the Eisenstein series $E_{q^i-1,L}$ and $E_{q^i-1,N^{-1}L}$ for all $1 \leqslant i \leqslant k$.

Proposition 15.12 *For any non-zero ideal $N \subset A$ and any $k \geqslant 0$ we have:*

(a) $g^L_{N,k}|_{q^k-1}\gamma = g^{L\gamma}_{N,k}$ *for all* $\gamma \in \mathrm{GL}_r(F)$.

(b) $g^L_{N,k}$ *is a modular form of weight* $q^k - 1$ *for the group* Γ_L.

(c) The u-expansion of $g^L_{N,k}$ *has constant term* $g^{L'}_{N,k}$ *with* L' *as in (13.8). In particular* $g^L_{N,k}$ *is a cusp form whenever* $k > (r-1)\deg(N)$, *but not for* $k = (r-1)\deg(N)$.

Proof. By construction $g^L_{N,i}$ is a homogeneous symmetric polynomial of degree $q^i - 1$ in the functions $E_{1,v+L}$. Thus the transformation formula in Proposition 13.3 (a) directly implies (a). Part (b) follows from Theorem 13.16 and the formula (15.11) by induction on k. To prove (c), write $\omega = \binom{\omega_1}{\omega'}$ as before. For any fixed $\omega' \in \Omega^{r-1}$, if ω_1 goes to infinity, the defining formula (13.4) shows that $e_{L\omega}$ and $e_{N^{-1}L\omega}$ go to $e_{L'\omega'}$ and $e_{N^{-1}L'\omega'}$, respectively. The functional equation $\psi^{L\omega}_N(e_{L\omega}(z)) = N^* \cdot e_{N^{-1}L\omega}(z)$ and its counterpart for $L'\omega'$ in place of $L\omega$ thus imply that $\psi^{L\omega}_N$ goes to $\psi^{L'\omega'}_N$. Taking coefficients this shows that the u-expansion of each $g^L_{N,k}$ has constant term $g^{L'}_{N,k}$. Finally, that constant term is zero for $k > (r-1)\deg(N)$ and non-zero for $k = (r-1)\deg(N)$. □

16 Discriminant forms

Definition 16.1 *For any non-zero proper ideal* $N \subsetneq A$ *we call* $\Delta^L_N := g^L_{N,r\deg(N)}$ *the* discriminant form *associated to* N. *Likewise we set* $\Delta^L_a := g^L_{a,r\deg(a)}$.

Since $[N^{-1}L : L]$ is a power of q, we have $(-1)^{[N^{-1}L:L]-1} = 1$ in $\mathbb{F}_q$; hence by (15.8) and (15.9) the above definition means that

$$\Delta^L_N(\omega) \;=\; N^* \cdot \prod_{\substack{v \in N^{-1}L \smallsetminus L \\ \text{modulo } L}} E_{1,v+L}(\omega). \tag{16.2}$$

Proposition 16.3 *(a)* $\Delta^L_N(\omega) \neq 0$ *for all* $\omega \in \Omega^r$.

(b) Δ^L_N *is a cusp form of weight* $q^{r\deg(N)} - 1$ *for the group* Γ_L.

(c) $\Delta^{aL}_N = a^{1-q^{r\deg(N)}} \cdot \Delta^L_N$ *for any* $a \in F$.

Proof. (a) follows from (16.2) and Corollary 13.7, and (b) is a special case of Proposition 15.12. Assertion (c) results from applying Proposition 15.12 (a) to $\gamma = a \cdot \mathrm{Id}_r$. □

Next recall that for any $a \in A \smallsetminus \{0\}$ the degree $\deg(a)$ is a multiple of the degree $\deg(\infty)$ of the residue field at ∞ over $\mathbb{F}_q$. Therefore $q^{r\deg(a)} - 1$ is a multiple of $q^{r\deg(\infty)} - 1$.

Proposition 16.4 *There exists a non-zero cusp form Δ^L (cf. [Ge86, VI.(5.14) & 5.15]) of weight $q^{r\deg(\infty)}-1$ for the group Γ_L, such that for every $a \in A \smallsetminus \{0\}$ we have*

$$\Delta^L_a \;=\; (\Delta^L)^{\frac{q^{r\deg(a)}-1}{q^{r\deg(\infty)}-1}} \cdot (\textit{some root of unity}).$$

Moreover this Δ^L is unique up to multiplication by some root of unity.

Proof. Since ψ^L is a Drinfeld module, for all $a, b \in A \smallsetminus \{0\}$ we have $\psi^L_{ab}(X) = \psi^L_a(\psi^L_b(X))$. Substituting the expansions from (15.5) for ψ^L_{ab} and ψ^L_a and ψ^L_b and taking highest coefficients implies that $\Delta^L_{ab} = \Delta^L_a \cdot (\Delta^L_b)^{q^{r\deg(a)}}$. As the ring A is commutative, interchanging a and b yields the same value; hence

$$\Delta^L_b \cdot (\Delta^L_a)^{q^{r\deg(b)}} \;=\; \Delta^L_a \cdot (\Delta^L_b)^{q^{r\deg(a)}}.$$

By Proposition 16.3 we may divide by $\Delta^L_a \Delta^L_b$, obtaining the equality

$$(\Delta^L_a)^{q^{r\deg(b)}-1} \;=\; (\Delta^L_b)^{q^{r\deg(a)}-1}. \tag{16.5}$$

To exploit this fact, recall that by the Riemann-Roch theorem, every sufficiently large multiple of $\deg(\infty)$ arises as $\deg(a)$ for some element $a \in A \smallsetminus \{0\}$. In particular we can find non-constant elements b, $c \in A$ such that $\deg(b) = \deg(c) + \deg(\infty)$. Then by Proposition 16.3 the quotient

$$\Delta^L \;:=\Delta^L_b \big/ (\Delta^L_c)^{q^{r\deg(\infty)}} \tag{16.6}$$

is a well-defined holomorphic function on Ω^r. The fact that Δ^L_b and Δ^L_c are modular forms of respective weights $q^{r\deg(b)}-1$ and $q^{r\deg(c)}-1$ for Γ_L implies that Δ^L is a weak modular form of weight

$$\big(q^{r\deg(b)}-1\big) - \big(q^{r\deg(c)}-1\big) \cdot q^{r\deg(\infty)} \;=\; q^{r\deg(\infty)}-1$$

for Γ_L. Also, by direct calculation the formula (16.5) in the case $a = c$ implies that

$$(\Delta^L)^{q^{r\deg(b)}-1} \;=\; (\Delta^L_b)^{q^{r\deg(\infty)}-1}.$$

Combining this with the formula (16.5) for arbitrary a we deduce that

$$(\Delta^L_a)^{(q^{r\deg(\infty)}-1)(q^{r\deg(b)}-1)} \;=\; (\Delta^L)^{(q^{r\deg(a)}-1)(q^{r\deg(b)}-1)}.$$

Thus $\Delta^L_a/(\Delta^L)^{\frac{q^{r\deg(a)}-1}{q^{r\deg(\infty)}-1}}$ is a holomorphic function on Ω^r whose $(q^{r\deg(\infty)}-1)(q^{r\deg(b)}-1)$-th power is identically 1. As the rigid analytic space Ω^r is connected, this function is therefore constant and a root of unity. The last formula also shows that a positive power of Δ^L is holomorphic at every boundary component; hence the same holds for Δ^L. Thus Δ^L has all the desired properties. Finally, the uniqueness is clear from the stated condition. $\square$

Proposition 16.7 *For every non-zero proper ideal $N \subsetneq A$ we have*

$$\Delta^{N^{-1}L} \cdot (\Delta^L_N)^{q^{r\deg(\infty)}-1} \;=\; (\Delta^L)^{q^{r\deg(N)}-1} \cdot (\textit{some constant}).$$

Proof. The formulas (15.7) and (15.10) imply that $\psi^{L\omega}_N = N^* \cdot h^L_N$, where h^L_N is an isogeny of Drinfeld modules $(\mathbb{G}_{\mathrm{a},\Omega^r}, \psi^L) \to (\mathbb{G}_{\mathrm{a},\Omega^r}, \psi^{N^{-1}L})$. For any $a \in A$ we then have $\psi^{N^{-1}L}_a \circ h^L_N = h^L_N \circ \psi^L_a$. Taking highest coefficients implies that

$$\Delta^{N^{-1}L}_a \cdot (\Delta^L_N)^{q^{r\deg(a)}} \;=\; \Delta^L_N \cdot (\Delta^L_a)^{q^{r\deg(N)}} \cdot (\text{some constant}).$$

Dividing by Δ^L_N and substituting the formulas for $\Delta^{N^{-1}L}_a$ and Δ^L_a from Proposition 16.4 we obtain

$$(\Delta^{N^{-1}L})^{\frac{q^{r\deg(a)}-1}{q^{r\deg(\infty)}-1}} \cdot (\Delta^L_N)^{q^{r\deg(a)}-1} \;=\; (\Delta^L)^{q^{r\deg(N)}\cdot\frac{q^{r\deg(a)}-1}{q^{r\deg(\infty)}-1}} \cdot (\text{some constant}).$$

Varying a or extracting roots as in Proposition 16.4 yields the desired formula. □

Remark 16.8 If the class group $\mathrm{Cl}(A)$ of A is trivial, the above relations show that Δ^L is the unique fundamental discriminant form for Γ_L.

In general, for any non-zero proper ideal $M \subsetneq A$ we have $\Gamma_{M^{-1}L} = \Gamma_L$. The discriminant forms $\Delta^{M^{-1}L}_a$ and $\Delta^{M^{-1}L}$ and $\Delta^{M^{-1}L}_N$ are therefore cusp forms for the same group Γ_L. Let $\mathcal{H}$ denote the multiplicative group generated by all of them, modulo constants, which thus consists of nowhere vanishing holomorphic functions on Ω^r. Then the formulas in Propositions 16.3 (c) and 16.4 and 16.7 imply that as N runs through a set of representatives of the ideal class group $\mathrm{Cl}(A)$, the functions $\Delta^{N^{-1}L}$ generate a subgroup of finite index, say $\mathcal{H}'$.

On the other hand each discriminant form corresponds to a section of a certain invertible sheaf on the Satake compactification of $\Gamma_L\backslash\Omega$. As such, its divisor is a formal $\mathbb{Z}$-linear combination of the irreducible components of codimension 1 of the boundary of the Satake compactification. These irreducible components are in bijection with $\mathrm{Cl}(A)$, so the group $\mathcal{D}$ of divisors supported on the boundary of the compactification is a free abelian group of rank $\mathrm{Cl}(A)$. Taking divisors maps the above group $\mathcal{H}$ injectively into $\mathcal{D}$.

One can expect that the image of $\mathcal{H}$ has finite index in $\mathcal{D}$. In fact, precisely such a statement is proved for an arbitrary congruence subgroup in the case $r = 2$ by Gekeler [Ge86, VII Thm. 5.11] and [Ge97, Thm. 4.1], and by Kapranov [Ka87, top of page 546] for arbitrary r in the case $A = \mathbb{F}_q[t]$.

Note that, since $\mathcal{H}'$ is generated by $|\mathrm{Cl}(A)|$ elements and has finite index in $\mathcal{H}$, the expectation is equivalent to saying that $\mathcal{H}$ is a free abelian group of rank $|\mathrm{Cl}(A)|$. This in turn means that, up to taking roots, the formulas in Propositions 16.3 (c) and 16.4 and 16.7 generate all multiplicative relations up to constant factors between the discriminant forms.

Example 16.9 Suppose that $\operatorname{Spec} A$ is a rational curve and ∞ is a point of degree 2 over $\mathbb{F}_q$. Let $P \subset A$ be the prime ideal associated to a point of degree 1 over $\mathbb{F}_q$. Then the ideal class group of A has order 2 and is generated by the class of P. Write $P^2 = (a)$ for an element $a \in A$ of degree 2. Then by Proposition 16.4 we have $\Delta_a^L \sim \Delta^L$, where "$\sim$" denotes equality up to a constant. Also, in the notation of the proof of Proposition 16.7 we have $a \cdot h_P^{P^{-1}L} \circ h_P^L = \psi_a^L$. Taking highest coefficients implies that $\Delta_P^{P^{-1}L} \cdot (\Delta_P^L)^{q^r} \sim \Delta_a^L \sim \Delta^L$. Together with the same relation for $P^{-1}L$ in place of L and with the fact that $\Delta_P^{P^{-2}L} = \Delta_P^{a^{-1}L} \sim \Delta_P^L$ by Proposition 16.3 (c), we conclude that

$$\begin{aligned} \Delta_P^{P^{-1}L} \cdot (\Delta_P^L)^{q^r} &\sim \Delta^L \quad \text{and} \\ \Delta_P^L \cdot (\Delta_P^{P^{-1}L})^{q^r} &\sim \Delta^{P^{-1}L}. \end{aligned}$$

In this case we can therefore view Δ_P^L and $\Delta_P^{P^{-1}L}$ as the two fundamental discriminant forms for Γ_P, and by Remark 16.8 they should be multiplicatively independent.

Remark 16.10 In the case $A = \mathbb{F}_q[t]$ one can take $\Delta^L = \Delta_t^L$ in Proposition 16.4. In [Ba16] this function is shown to satisfy a product formula which generalises the Jacobi product formula in the rank 2 case of Gekeler [Ge85]. Another product formula, involving $r-1$ separate parameters with constant coefficients, rather than u-expansions treated in the present monograph, was obtained by Hamahata [Ha02].

Remark 16.11 For any $v \in F^r \smallsetminus L$, the Eisenstein series $E_{1,v+L}$ is a non-zero modular form of weight 1 for the group Γ_{v+L} by Corollary 13.7 and Theorem 13.16. Using Proposition 16.4 it follows that for any integer $k \geqslant 0$, the product $\Delta^L \cdot E_{1,v+L}^k$ is a non-zero cusp form of weight $q^{r\deg(\infty)} - 1 + k$ for Γ_{v+L}. In this way we can explicitly produce non-zero cusp forms for Γ_{v+L} of any sufficiently large weight, giving more substance to the abstract result of Proposition 11.2.

To finish this section we construct Drinfeld modular forms of non-zero type by extracting roots from discriminant forms. This rests on the observation that for every $\alpha \in \mathbb{F}_q^\times$, applying Proposition 13.3 (a) to $\gamma = \alpha \cdot \mathrm{Id}_r$ implies that

$$E_{1,\alpha v+L} \;=\; E_{1,v+L}|_1\, \alpha \cdot \mathrm{Id}_r \;\overset{(1.7)}{=}\; \alpha^{-1} \cdot E_{1,v+L}. \tag{16.12}$$

Plugging this into (16.2), we can write each discriminant form as a $(q-1)$-st power of another holomorphic function on Ω^r.

Specifically, choose a set of representatives $\mathcal{R}_N^L$ of $N^{-1}L \smallsetminus L$ modulo addition by L and multiplication by $\mathbb{F}_q^\times$. Choose an element $\lambda_N \in \mathbb{C}_\infty$ satisfying $\lambda_N^{q-1} = -N^*$. Consider the function

$$\delta_N^L(\omega) \;:=\; \lambda_N \cdot \prod_{v \in \mathcal{R}_N^L} E_{1,v+L}(\omega). \tag{16.13}$$

Proposition 16.14 *(a) We have $(\delta_N^L)^{q-1} = \Delta_N^L$. In particular, another choice of representatives or of λ_N changes δ_N^L only by a factor in $\mathbb{F}_q^\times$.*

(b) The function δ_N^L is a cusp form of weight $\frac{q^{r\deg(N)}-1}{q-1}$ and type $\deg(N)$ *for the group* Γ_L.

Proof. Abbreviate $k := |\mathcal{R}_N^L| = \frac{q^{r\deg(N)}-1}{q-1}$ and note that $(\prod_{\alpha\in\mathbb{F}_q^\times}\alpha)^k = (-1)^k = -1$. Using this, the definitions of δ_L^N and Δ_L^N and (16.12) imply that

$$(\delta_N^L)^{q-1} = -N^*\cdot\prod_{v\in\mathcal{R}_N^L} E_{1,v+L}^{q-1} = N^*\cdot\prod_{v\in\mathcal{R}_N^L}\prod_{\alpha\in\mathbb{F}_q^\times}\alpha^{-1}E_{1,v+L} = N^*\cdot\prod_{v\in\mathcal{R}_N^L}\prod_{\alpha\in\mathbb{F}_q^\times}E_{1,\alpha v+L} = \Delta_N^L(\omega),$$

proving (a). The proof of (b) rests on properties of the Moore determinant, assembled in [Go96, Chapter 1.3]: For any elements $x_1,\ldots,x_n$ of an $\mathbb{F}_q$-algebra the Moore determinant is defined as

$$M(x_1,x_2,\ldots,x_n) := \begin{vmatrix} x_1 & \cdots & x_n \\ x_1^q & & x_n^q \\ \vdots & & \vdots \\ x_1^{q^{n-1}} & \cdots & x_n^{q^{n-1}} \end{vmatrix}. \tag{16.15}$$

Its most important property is [Go96, Cor. 1.3.7]

$$M(x_1,x_2,\ldots,x_n) = \prod_{(\alpha_1,\ldots,\alpha_n)}\Big(\sum_{i=1}^n \alpha_i x_i\Big), \tag{16.16}$$

where the product extends over all tuples in $\mathbb{F}_q^n\smallsetminus\{(0,\ldots,0)\}$ whose first non-zero entry is 1. Also, for any matrix $B = (\beta_{ij})_{i,j=1,\ldots,n}$ with coefficients in $\mathbb{F}_q$ we have $\beta_{ij}^q = \beta_{ij}$; hence the multiplicativity of the determinant implies that

$$M\Big(\sum_{i=1}^n \beta_{i1}x_i,\ldots,\sum_{i=1}^n \beta_{in}x_i\Big) = \det(B)\cdot M(x_1,x_2,\ldots,x_n). \tag{16.17}$$

To apply this, choose elements $v_1,\ldots,v_n\in N^{-1}L\smallsetminus L$ whose residue classes form a basis of the $\mathbb{F}_q$-vector space $N^{-1}L/L$. Then the set $\mathcal{R}_N^L$ of all elements of the form $\sum_{i=1}^n\alpha_i v_i$, for tuples $(\alpha_1,\ldots,\alpha_n)\in\mathbb{F}_q^n\smallsetminus\{(0,\ldots,0)\}$ whose first non-zero entry is 1, is a set of representatives of $N^{-1}L\smallsetminus L$ modulo addition by L and multiplication by $\mathbb{F}_q^\times$. The formula (16.16) and the additivity of the exponential function then imply that

$$M\big(e_{L\omega}(v_1\omega),\ldots,e_{L\omega}(v_n\omega)\big) = \prod_{(\alpha_1,\ldots,\alpha_n)}\Big(\sum_{i=1}^n\alpha_i e_{L\omega}(v_i\omega)\Big) = \prod_{v\in\mathcal{R}_N^L} e_{L\omega}(v\omega).$$

Take an arbitrary element $\gamma\in\Gamma_L$. Then the same calculation with the basis $v_1\gamma,\ldots,v_n\gamma$ yields

$$M\big(e_{L\omega}(v_1\gamma\omega),\ldots,e_{L\omega}(v_n\gamma\omega)\big) = \prod_{v\in\mathcal{R}_N^L} e_{L\omega}(v\gamma\omega).$$

For each j choose $\beta_{ij}\in\mathbb{F}_q$ such that $v_j\gamma\equiv\sum_{i=1}^n\beta_{ij}v_i$ modulo L. Then by the $\mathbb{F}_q$-linearity of the exponential function we have $e_{L\omega}(v_j\gamma\omega) = \sum_{i=1}^n\beta_{ij}e_{L\omega}(v_i\gamma\omega)$; hence with $B := (\beta_{ij})_{i,j=1,\ldots,n}$ the formula (16.17) implies that

$$M\big(e_{L\omega}(v_1\gamma\omega),\ldots,e_{L\omega}(v_n\gamma\omega)\big) = \det(B)\cdot M\big(e_{L\omega}(v_1\omega),\ldots,e_{L\omega}(v_n\omega)\big).$$

Combining these computations we deduce that

$$\prod_{v\in\mathcal{R}_N^L} e_{L\omega}(v\gamma\omega) \;=\; \det(B)\cdot \prod_{v\in\mathcal{R}_N^L} e_{L\omega}(v\omega).$$

Using Proposition 13.3 (a) and Corollary 13.7 we find that

$$\begin{aligned}(\delta_N^L|_k\gamma)(\omega) \;&=\; \lambda_N\cdot \prod_{v\in\mathcal{R}_N^L}(E_{1,v+L}|_1\gamma)(\omega) \;=\; \lambda_N\cdot\prod_{v\in\mathcal{R}_N^L} E_{1,v\gamma+L}(\omega)\\ &=\; \lambda_N\cdot\det(B)^{-1}\cdot \prod_{v\in\mathcal{R}_N^L} E_{1,v+L}(\omega) \;=\; \det(B)^{-1}\cdot\delta_N^L(\omega).\end{aligned}$$

To determine $\det(B)$ note that since L is a projective module of rank r over A, the module $N^{-1}L/L$ is a free module of rank r over A/N. Without loss of generality we may therefore assume that the $\mathbb{F}_q$-basis $v_1,\ldots,v_n$ is formed by multiplying an A/N-basis of $N^{-1}L/L$ with an $\mathbb{F}_q$-basis of A/N. For a suitable order of this basis, the matrix A is then simply a block diagonal matrix with $m \coloneqq \dim_{\mathbb{F}_q}(A/N) = \deg(N)$ copies of γ on the diagonal. Therefore $\det(B) = \det(\gamma)^m$. In view of (1.5) the above calculation thus implies that

$$\delta_N^L|_{k,m}\gamma \;=\; \det(\gamma)^m\cdot\delta_N^L|_k\gamma \;=\; \delta_N^L.$$

In other words δ_N^L is a weak modular form of weight k and type m for the group Γ_L. But by Theorem 13.16 and construction it is already a modular form for the congruence subgroup $\Gamma_L(N)$. It is therefore a modular form for Γ_L. Finally, since Δ_N^L is a cusp form, assertion (a) implies that δ_N^L is a cusp form as well. This finishes the proof of (b). □

Remark 16.18 In the case $A = \mathbb{F}_q[t]$ and $L = A^r$, the cusp form δ_t^L was first constructed by Gekeler in the 1980's, and is called $h(\omega)$ in the literature. The $r = 2$ case appears in [Ge88a] while the $r > 2$ case was unpublished until [Ge17]. In the meantime, it made an appearance as a weak modular form in [Ge89] and was shown to be holomorphic at infinity by Perkins [Pe14]. In [BB17, Thm. 5.3] it is shown to satisfy a product formula derived from the product formula of Δ_t^L.

17 The special case $A = \mathbb{F}_q[t]$

Throughout this section we set $A \coloneqq \mathbb{F}_q[t]$ and $L \coloneqq A^r$. Then $\Gamma_L = \mathrm{GL}_r(A)$, and $\Gamma(t) \coloneqq \Gamma_L((t))$ is the subgroup of matrices in $\mathrm{GL}_r(A)$ which are congruent to the identity matrix modulo (t). Recall from (6.11) that the graded ring of modular forms of all weights for an arithmetic group Γ is defined as

$$\mathcal{M}_*(\Gamma) \;\coloneqq\; \bigoplus_{k\geqslant 0}\mathcal{M}_k(\Gamma).$$

For $\Gamma = \Gamma(t)$ this ring can be described very explicitly, and for a subgroup containing $\Gamma(t)$ a description can be deduced by taking invariants. In the case $r = 2$ the ring was determined by Cornelissen [Co96] for $\Gamma(t)$, by Goss [Go80a] for $\mathrm{GL}_2(A)$, and by Gekeler [Ge88a] for $\mathrm{SL}_2(A)$.

Theorem 17.1 *The ring $\mathcal{M}_*(\Gamma(t))$ is generated over $\mathbb{C}_\infty$ by the Eisenstein series $E_{1,v+L}$ of weight 1 for all $v \in t^{-1}L \smallsetminus L$, and all polynomial equations between them are induced by the relations*

$$\begin{array}{rcll} E_{1,\alpha v+L} & = & \alpha^{-1} \cdot E_{1,v+L} & \text{for all } v \in t^{-1}L \smallsetminus L \text{ and } \alpha \in \mathbb{F}_q^\times, \text{ and} \\ E_{1,v+L} \cdot E_{1,v'+L} & = & E_{1,v+v'+L} \cdot (E_{1,v+L} + E_{1,v'+L}) & \text{for all } v,\ v' \in t^{-1}L \smallsetminus L \text{ with } v+v' \notin L. \end{array}$$

Proof. Let $K(t) < \mathrm{GL}_r(\hat{A})$ denote the subgroup of matrices that are congruent to the identity matrix modulo (t). By construction it is open compact and fine in the sense of [Pi13, Def. 1.4]. Let $M^r_{A,K(t)}$ be the associated fine moduli space of Drinfeld A-modules of rank r with a full level (t) structure. Then $\mathrm{GL}_r(\mathbb{A}^{\mathrm{f}}_F) = \mathrm{GL}_r(F) \cdot K(t)$, and so (8.4) with $g = 1$ provides an isomorphism $\pi_1 : \Gamma(t)\backslash\Omega^r \xrightarrow{\sim} M^r_{A,K(t)}(\mathbb{C}_\infty)$. The Satake compactification $\overline{M}^r_{A,K(t)}$ was described explicitly in [PS14] and [Pi13], as follows.

Abbreviate $\bar{V} := t^{-1}L/L$, and let $A_{\bar{V}}$ denote the graded polynomial ring over $\mathbb{F}_q$ in independent variables $Y_{\bar{v}}$ of degree 1 for all $\bar{v} \in \bar{V} \smallsetminus \{0\}$. Let $\mathfrak{a}_{\bar{V}} \subset A_{\bar{V}}$ be the homogeneous ideal that is generated by the elements of the form

$$\begin{array}{rl} Y_{\alpha\bar{v}} - \alpha^{-1} Y_{\bar{v}} & \text{for all } \bar{v} \in \bar{V} \smallsetminus \{0\} \text{ and } \alpha \in \mathbb{F}_q^\times, \text{ and} \\ Y_{\bar{v}} Y_{\bar{v}'} - Y_{\bar{v}+\bar{v}'} \cdot (Y_{\bar{v}} + Y_{\bar{v}'}) & \text{for all } \bar{v},\ \bar{v}' \in \bar{V} \smallsetminus \{0\} \text{ with } \bar{v}+\bar{v}' \neq 0. \end{array}$$

Let $R_{\bar{V}} := A_{\bar{V}}/\mathfrak{a}_{\bar{V}}$ denote the graded factor ring. Then by [Pi13, Thm. 7.4] there is a natural isomorphism

$$\overline{M}^r_{A,K(t)} \;\cong\; \mathrm{Proj}(R_{\bar{V}} \otimes_{\mathbb{F}_q} F), \tag{17.2}$$

which also identifies the invertible sheaf $\mathcal{L}$ from Section 10 with the ample sheaf $\mathcal{O}(1)$ on $\mathrm{Proj}(R_{\bar{V}} \otimes_{\mathbb{F}_q} F)$. Combined with Theorem 10.9 we thus obtain an isomorphism of graded $\mathbb{C}_\infty$-algebras

$$\mathcal{M}_*(\Gamma(t)) \;\cong\; R_{\bar{V}} \otimes_{\mathbb{F}_q} \mathbb{C}_\infty. \tag{17.3}$$

By the proof of [Pi13, Thm. 7.4], the isomorphism (17.2) also realises the universal generalised Drinfeld A-module over $\overline{M}^r_{A,K(t)}$ as the pair $(\bar{E},\bar{\varphi})$ consisting of the line bundle whose sheaf of sections is the invertible sheaf dual to $\mathcal{O}(1)$ and the generalised Drinfeld A-module with

$$\bar{\varphi}_t(X) \;=\; t \cdot X \cdot \prod_{\bar{v} \in \bar{V} \smallsetminus \{0\}} (1 - \bar{Y}_{\bar{v}} \cdot X),$$

where $\bar{Y}_{\bar{v}} \in R_V$ denotes the residue class of $Y_{\bar{v}}$. On the other hand from (8.9) we have a natural isomorphism

$$\pi_g^*(\bar{E},\bar{\varphi}) \;\cong\; (\mathbb{G}_{\mathrm{a},\Omega^r}, \psi^L),$$

and by equation (15.4) we have

$$\psi_t^{L\omega}(X) \;=\; t \cdot X \cdot \prod_{\substack{v \in t^{-1}L \smallsetminus L \\ \text{modulo } L}} \big(1 - E_{1,v+L}(\omega) \cdot X\big).$$

Furthermore, the respective level structures send a non-zero residue class $\bar{v} = v + L$ to the element $\bar{Y}_{\bar{v}}^{-1}$ in one case and to the function $E_{1,v+L}(\omega)^{-1} = e_{L\omega}(v\omega)$ in the other. Under the isomorphism (17.3) the element $\bar{Y}_{\bar{v}}$ therefore corresponds precisely to the Eisenstein series $E_{1,v+L}$. By the construction of $R_{\bar{V}}$ these Eisenstein series therefore generate $\mathcal{M}_*(\Gamma(t))$ and satisfy precisely the stated algebraic relations. □

Corollary 17.4 *The quotient field of $\mathcal{M}_*(\Gamma(t))$ is a rational function field over $\mathbb{C}_\infty$ that is generated by the algebraically independent elements E_{1,v_i+L} as $v_i + L$ runs through any $\mathbb{F}_q$-basis of $t^{-1}L/L$.*

Proof. By [PS14] the ring R_V is an integral domain and its quotient field is a rational function field over $\mathbb{F}_q$ that is generated by the algebraically independent elements $\bar{Y}_{\bar{v}_i}$ for any basis $\bar{v}_1, \ldots, \bar{v}_r$ of $\bar{V}$. The corollary thus follows from the isomorphism (17.3). □

Theorem 17.5 *(a) The ring $\mathcal{M}_*(\mathrm{GL}_r(A))$ is generated over $\mathbb{C}_\infty$ by the coefficient forms $g_{t,i}^L$ of weight $q^i - 1$ for all $1 \leqslant i \leqslant r$, which are algebraically independent over $\mathbb{C}_\infty$. The same statement holds with the coefficient forms $e_{i,L}$ or the Eisenstein series $E_{q^i-1,L}$ in place of $g_{t,i}^L$.*

(b) The ring $\mathcal{M}_*(\mathrm{SL}_r(A))$ is generated over $\mathbb{C}_\infty$ by the coefficient forms $g_{t,i}^L$ of weight $q^i - 1$ for all $1 \leqslant i \leqslant r-1$ and the determinant form δ_t^L of weight $\frac{q^r-1}{q-1}$, which are algebraically independent over $\mathbb{C}_\infty$. The same statement holds with the coefficient forms $e_{i,L}$ or the Eisenstein series $E_{q^i-1,L}$ in place of $g_{t,i}^L$.

(c) Let $\Gamma_1(t)$ denote the subgroup of matrices in $\mathrm{GL}_r(A)$ which are congruent modulo (t) to an upper triangular matrix with diagonal entries 1. The ring $\mathcal{M}_*(\Gamma_1(t))$ is generated over $\mathbb{C}_\infty$ by the modular forms

$$\sum_{\alpha_{i+1},\ldots,\alpha_r \in \mathbb{F}_q} E_{1,t^{-1}(0,\ldots,0,1,\alpha_{i+1},\ldots,\alpha_r)+L}$$

of weight 1 for all $1 \leqslant i \leqslant r$, which are algebraically independent over $\mathbb{C}_\infty$.

Proof. For any subgroup $\Gamma < \mathrm{GL}_r(A)$ containing $\Gamma(t)$, the formula (6.7) shows that $\mathcal{M}_*(\Gamma)$ is the subring of Γ-invariants in $\mathcal{M}_*(\Gamma(t))$ for the natural action by $f \mapsto f|_k\gamma$ on each $\mathcal{M}_k(\Gamma(t))$. By Proposition 13.3 (a) the action is given on the generators of $\mathcal{M}_*(\Gamma(t))$ by $E_{1,v+L}|_1\gamma = E_{1,v\gamma+L}$. This action factors through the factor group $\Gamma/\Gamma(t)$, which is $\mathrm{GL}_r(\mathbb{F}_q)$ in the case (a), respectively $\mathrm{SL}_r(\mathbb{F}_q)$ in the case (b), respectively the subgroup of upper triangular matrices with diagonal entries 1 in the case (c). Using a theorem of Dickson, the respective ring of invariants was shown in [PS14, Theorem 3.1] to have the set of generators that is first named in each case. The recursion relations (15.6) and (15.1) imply that by induction on i, each generator $g_{a,i}^L$ can be replaced by $e_{i,L}$ or again by $E_{q^i-1,L}$.

Since we are taking invariants under a finite group, the ring $\mathcal{M}_*(\Gamma(t))$ is an integral extension of $\mathcal{M}_*(\Gamma(t))^\Gamma$. The respective quotient fields therefore have the same transcendence degree over $\mathbb{C}_\infty$. For the former this transcendence degree is r by Corollary 17.4. In each case the r given generators of the subring $\mathcal{M}_*(\Gamma(t))^\Gamma$ must therefore be algebraically independent over $\mathbb{C}_\infty$. □

Theorem 17.6 *For any integer k we have*

$$\mathcal{M}_k(\mathrm{SL}_r(A)) = \bigoplus_{0\leqslant m<q-1} \mathcal{M}_{k,m}(\mathrm{GL}_r(A)).$$

In addition, for any integer $0 \leqslant m < q-1$ we have

$$\mathcal{M}_{k,m}(\mathrm{GL}_r(A)) = (\delta_t^L)^m \cdot \mathcal{M}_{k-m\frac{q^r-1}{q-1}}(\mathrm{GL}_r(A)).$$

In particular, every modular form for $\mathrm{GL}_r(A)$ of type $\not\equiv 0$ modulo $(q-1)$ is a cusp form.

Proof. The determinant induces an isomorphism $\mathrm{GL}_r(\mathbb{F}_q)/\mathrm{SL}_r(\mathbb{F}_q) \xrightarrow{\sim} \mathbb{F}_q^\times$; hence the action $f \mapsto f|_k\gamma$ of $\mathrm{GL}_r(\mathbb{F}_q)$ on $\mathcal{M}_k(\mathrm{SL}_r(A))$ factors through an action of $\mathbb{F}_q^\times$. As any linear action of $\mathbb{F}_q^\times$ on an $\mathbb{F}_q$-vector space is diagonalisable, it follows that $\mathcal{M}_k(\mathrm{SL}_r(A))$ is a direct sum of eigenspaces. By Definition 1.9 and (1.5) these eigenspaces are just the spaces $\mathcal{M}_{k,m}(\mathrm{GL}_r(A))$, proving the first equality.

The descriptions from Theorem 17.5 (a) and (b) imply that $\mathcal{M}_*(\mathrm{SL}_r(A))$ is a free module with basis $1, \delta_t^L, \ldots, (\delta_t^L)^{q-2}$ over the subring $\mathcal{M}_*(\mathrm{GL}_r(A))$. Since $(\delta_t^L)^m$ is a modular form of weight $m\frac{q^r-1}{q-1}$, this results in the second assertion. The last one now follows from the fact that δ_t^L is a cusp form. □

Remark 17.7 The last statement of Theorem 17.6 was already established independently in Corollary 6.4 (b) using the u-expansion. Combined with Proposition 17.8 below and the fact that δ_t^L is a modular form of weight $\frac{q^r-1}{q-1}$ and type 1 it directly implies the second statement of Theorem 17.6 by induction on m.

Proposition 17.8 *The Satake compactification $\overline{M}^r_{A,\mathrm{GL}_r(A)}$ has only one boundary component of codimension 1, and the cusp form δ_t^L has vanishing order 1 there.*[1]

Proof. The first statement can be deduced from the fact from Proposition 6.3 (a) that $\mathrm{GL}_r(\mathbb{A}_F^{\mathrm{f}}) = \mathrm{GL}_r(A)\cdot P(F)$ with the parabolic subgroup $P < \mathrm{GL}_r$ from (5.6).

[1]Thus $\Delta_t^L = (\delta_t^L)^{q-1}$ has vanishing order $q-1$ here! This is at odds with the intuition that Δ_t^L should have algebraic order of vanishing 1 at the cusp.

We point out that the identification of analytic modular forms with sections of a line bundle (Theorem 10.9) is only established when the moduli scheme is fine, which $M^r_{A,\mathrm{GL}_r(A)}$ is not. Indeed, since $\mathrm{GL}_r(A)\cap U(F) = \mathrm{SL}_r(A)\cap U(F)$, the u-parameter does not even distinguish between $\mathrm{GL}_r(A)\backslash\Omega^r$ and $\mathrm{SL}_r(A)\backslash\Omega^r$.

We can rescue our intuition as follows. By Corollary 5.10, the coefficient forms $g_{t,k}^L$ (and thus, by Theorem 17.5, all modular forms for $\mathrm{GL}_r(A)$) have u-expansions in which the only non-zero terms have exponent divisible by $q-1$. Therefore, these forms have expansions in the parameter $u' = u^{q-1}$, and with respect to this parameter, Δ_t^L has order of vanishing 1 at the cusp, as expected.

We thank Mihran Papikian for raising this issue.

For the second statement note first that under the isomorphism ι of (4.3) the subgroups $\Gamma(t) \cap U(F) < \mathrm{GL}_r(A) \cap U(F)$ correspond to the subgroups $(At)^{r-1} \subset A^{r-1}$ of F^{r-1}, which have index q^{r-1} in each other. Now consider any element $v \in t^{-1}L \smallsetminus L$. By the proof of Proposition 13.15 the subgroup $\Gamma_{v+L} \cap U(F)$ corresponds to the subgroup $(At)^{r-1}$ if $v \notin L + (\{0\} \times F^{r-1})$. By Proposition 13.15 we thus have

$$\mathrm{ord}_{\Gamma(t)\cap U(F)}(E_{1,v+L}) \;=\; \mathrm{ord}_{\Gamma_{v+L}\cap U(F)}(E_{1,v+L}) \;=\; \begin{cases} 0 & \text{if } v \in L + (\{0\} \times F^{r-1}), \\ 1 & \text{otherwise.} \end{cases}$$

Taking the product over a set of representatives as in (16.13), where the second case occurs $\frac{q^r - q^{r-1}}{q-1} = q^{r-1}$ times, we deduce that

$$\mathrm{ord}_{\Gamma(t)\cap U(F)}(\delta_t^L) \;=\; q^{r-1}.$$

Since $[\mathrm{GL}_r(A) \cap U(F) : \Gamma(t) \cap U(F)] = q^{r-1}$, it follows that $\mathrm{ord}_{\mathrm{GL}_r(A)\cap U(F)}(\delta_t^L) = 1$, as desired. □

Corollary 17.9 *The cusp forms of all weights and type* 0 *for* $\mathrm{GL}_r(A)$ *form the principal ideal of* $\mathcal{M}_*(\mathrm{GL}_r(A))$ *that is generated by* Δ_t^L. *In other words, for every integer k we have*

$$\mathcal{S}_k(\mathrm{GL}_r(A)) \;=\; \Delta_t^L \cdot \mathcal{M}_{k-q^r+1}(\mathrm{GL}_r(A)).$$

Proof. The cusp form δ_t^L is non-zero everywhere by Propositions 16.3 (a) and 16.14 (a). Thus for every cusp form $f \in \mathcal{S}_{k,0}(\Gamma)$, the quotient f/δ_t^L is again a weak modular form, and by Proposition 17.8 it is holomorphic at infinity; hence $f/\delta_t^L \in \mathcal{M}_{k-\frac{q^r-1}{q-1},-1}(\mathrm{GL}_r(A))$. By Theorem 17.6 with $m = q-2$ this in turn implies that $f \in (\delta_t^L)^{q-1}\mathcal{M}_{k-q^r+1,0}(\mathrm{GL}_r(A))$, as desired. □

Corollary 17.10 *The space of cusp forms* $\mathcal{S}_k(\mathrm{GL}_r(A))$ *is zero for* $k < q^r - 1$ *and one-dimensional with basis* Δ_t^L *for* $k = q^r - 1$. *In particular* Δ_t^L *is an eigenform for the Hecke operator associated to any double coset* $\mathrm{GL}_r(A)\delta\,\mathrm{GL}_r(A) \subset \mathrm{GL}_r(F)$.

Proof. By Theorem 17.5 (a) we have $\mathcal{M}_k(\mathrm{GL}_r(A)) = 0$ for $k < 0$ and $= \mathbb{C}_\infty$ for $k = 1$. By Corollary 17.9 this implies the first statement, which in turn implies the second. □

Theorem 17.11 *We have the following dimension formulas for all $k \geqslant 0$ and m:*

(a) $\dim_{\mathbb{C}_\infty} \mathcal{M}_k(\Gamma(t)) = \displaystyle\sum_{i_1,\ldots,i_{r-1}\in\{0,1\}} q^{\sum_\nu \nu\cdot i_\nu} \cdot \binom{k}{\sum_\nu i_\nu}.$

(b) Denote by $P_S(k)$ the number of partitions of k with parts in $S = \{q-1, q^2-1, \ldots, q^r-1\}$. Then

$$\dim_{\mathbb{C}_\infty} \mathcal{M}_k(\mathrm{GL}_r(A)) = P_S(k) = \begin{cases} 0 & \text{if } (q-1) \nmid k, \\ \dfrac{1}{\prod_{i=2}^r (q^i - 1)} \cdot \dfrac{k^{r-1}}{(r-1)!} + O(k^{r-2}) & \text{if } (q-1) | k. \end{cases}$$

(c) $\dim_{\mathbb{C}_\infty} \mathcal{M}_{k,m}(\mathrm{GL}_r(A)) = \begin{cases} P_S\big(k - m\frac{q^r-1}{q-1}\big) & \text{if } k \geqslant m\frac{q^r-1}{q-1}, \\ 0 & \text{otherwise.} \end{cases}$

(d) $\dim_{\mathbb{C}_\infty} \mathcal{M}_k(\Gamma_1(t)) = \binom{k+r-1}{r-1}$.

Proof. Assertion (a) follows from Theorem 17.1 together with [PS14, Thm. 1.10]. The first equality in (b) results from Theorem 17.5 (a). Clearly $P_S(k)$ is the number of partitions of $\frac{k}{q-1}$ with parts in $\{\frac{q-1}{q-1}, \frac{q^2-1}{q-1}, \ldots, \frac{q^r-1}{q-1}\}$, which by [Na00, Thm. 15.2] has the asymptotic behaviour given in (b). Assertion (c) is a direct consequence of Theorem 17.6. Finally, by Theorem 17.5 (c) the dimension in (d) is just the dimension of the space of homogeneous polynomials of degree k in the polynomial ring $\mathbb{C}_\infty[X_1, \ldots, X_r]$, which is well-known to be $\binom{k+r-1}{r-1}$. □

Remark 17.12 Taking invariants one may obtain similar dimension formulas for arbitrary arithmetic subgroups Γ containing $\Gamma(t)$. In particular [Pi13, Thm. 8.4] gives an explicit formula when $\Gamma(t) < \Gamma < \Gamma_1(t)$. It seems an interesting problem to find a dimension formula in general.

Remark 17.13 Combining Theorem 17.1 and [PS14, Thm. 1.7] shows that $\mathcal{M}_*(\Gamma(t))$ is a Cohen-Macaulay normal integral domain. By taking invariants, the argument in [PS14, §2] shows the same for $\mathcal{M}_*(\Gamma)$ whenever $\Gamma(t) < \Gamma < \Gamma_1(t)$. For $\Gamma = \mathrm{GL}_r(A)$ and $\mathrm{SL}_r(A)$ the same follows from the explicit description in Theorem 17.5. One may ask: Is this only a rare event for small level, or is it a general phenomenon?

Remark 17.14 *In the case of classical modular forms and also in the case of rank 2 Drinfeld modular forms, there are two approaches to dimension formulas. The one is algebro-geometric, similar to the approach in 17.11. The other uses valence formulas and vector space homomorphisms from $\mathcal{M}_k(\Gamma) \to \mathbb{C}_\infty$, mapping a modular form f to the constant coefficient f_0 in its u-expansion. One may wonder whether Gekeler's recent valence formula [Ge18] could be used in the same way.*

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