AIMS-Stellenbosch Number Theory Conference 2019

14 – 18 January 2019

Room 1006, Mathematical Sciences and Industrial Psychology Building, Stellenbosch University

Monday 14 January	
09h30	Welcome & Coffee
10h00 - 11h00	René Schoof: *Equations for modular curves associated to normalizers of non-split Cartan subgroups*
11h20 - 12h20	Richard Griffon: *Elliptic curves with large Tate-Shafarevich groups over function fields*
12h30 - 13h00	Tuan Ngo Dac: *On special values of Anderson modules*
lunch	
14h30 - 15h30	Cécile Armana: *Sturm-type bounds for automorphic forms of Drinfeld type over function fields*
15h50 - 16h20	Alex Bamunoba: *A search for c-Wieferich primes*
16h30 - 17h00	Bruce Bartlett: *A geometric approach to the modular flow on the space of lattices*

Tuesday 15 January	
09h30 - 10h30	Emmanuel Peyre: *Unrandomness of rational points*
10h50 - 11h50	Ambrus Pàl: *Arithmetic and homotopy*
12h00 - 12h30	Augustine Munagi: *Combinatory classes of compositions with higher-order conjugation*
lunch	
14h30 - 15h30	Patrick Rabarison: *Arithmetic and topology of r-pyramids*
15h50 - 16h50	Florian Luca: *X-coordinates of Pell equations in various sequences*

Wednesday 16 January	
09h30	Bus to AIMS
10h30	*Tea*
11h00 - 12h00	Peter Stevenhagen: *On the solvability of the negative Pell equation*
12h05 - 12h35	Darlison Nyirenda: *A generalization of a partition theorem of M. V. Subbarao*
12h40 - 13h10	Taboka Prince Chalebgwa: *Algebraic values of the Weierstrass σ-function*
lunch	
15h30	Bus to Stellenbosch

Thursday 17 January	
09h30 - 10h30	Daniel Fiorilli: *Chebyshev's bias in Galois groups*
10h50 - 11h50	Sophie Marques: *When is the permutation ring Cohen Macaulay?*
12h00 - 12h30	Gareth Boxall: *Remarks on a question of Levin*
lunch	
14h30 - 15h30	Harry Schmidt: *Mahler functions and quadratic polynomials*
15h50 - 16h20	Marion le Gonidec: *About Hartmanis & Stearns problem: Algebraic irrational numbers and machines*
16h30 - 17h00	Liam Baker: *Computational results concerning the ring of Drinfeld modular forms for a full congruence subgroup*

Friday 18 January	
11h30	Bus to Winelands
12h00	Lunch at Tokara Deli
19h00	Return to Stellenbosch

Abstracts

Sturm-type bounds for automorphic forms of Drinfeld type over function fields

Cécile Armana

Besançon, France

In the function field setting, automorphic forms of Drinfeld type can be viewed as analogues of classical weight 2 modular forms. The Hecke algebra associated to these automorphic forms is related to various topics in function field arithmetic, such as elliptic curves over function fields. I will present Sturm-type bounds for the generators of the Hecke algebra (joint work with Fu-Tsun Wei).

Computational results concerning the ring of Drinfeld modular forms for a full congruence subgroup

Liam Baker

Stellenbosch, South Africa

Drinfeld modules were first defined by Drinfeld in order to prove a special case of the Langlands Conjecture for function fields. Later, other authors such as Goss, Gekeler and many others built on this setting by defining Drinfeld modular forms in analogy with the classical case and providing a theory for them, with the most studied case being that of rank 2. Cornelissen showed that the ring of modular forms for a congruence subgroup is generated by the forms of weight 1 and 2, but apart from some rather small cases it is not known whether the weight 1 forms suffice to generate the full ring. In this work, we show some computational results in this direction.

A search for c-Wieferich primes

Alex Bamunoba

Makere, Uganda

In this talk, I will define c-Wieferich primes in $\mathbb{F}_q[T]$ (where q is a prime power) and give a few examples for small values of q. I will then describe two ways of computing c-Wieferich primes in $\mathbb{F}_q[T]$ for $q > 2$ and modifying Thakur's arguments show how these primes are related to the T-adic Goss zeta value at 1.

A geometric approach to the modular flow on the space of lattices

Bruce Bartlett

Stellenbosch, South Africa

At the ICM in 2006 Etiénne Ghys gave a beautiful talk on "Knots and Dynamics". In particular he gave a fascinating topological interpretation of the variation in the phase of the Dedekind eta function as the "linking number with the trefoil" in the modular flow on the 3-dimensional space of lattices. I will give a geometric version of the same result, by exhibiting a canonical smooth 1-form on the 3-dimensional space of lattices, whose periods yields the Rademacher function.

Remarks on a question of Levin

Gareth Boxall

Stellenbosch, South Africa

I shall consider a question of Aaron Levin and a conjectured answer discussed by Umberto Zannier in his book on unlikely intersections [Z]. The question concerns algebraic curves $C_1, C_2 \subseteq \mathbb{G}_m^N$ and asks what can be said about the points $x \in C_1$ for which there is a positive integer n such that $x^n \in C_2$. A potential answer, which Zannier shows to be a consequence of Zilber's Conjecture on unlikely Intersections with Tori, is as follows (in which $[n]C_1 = \{x^n : x \in C_1\}$).

Conjecture 1. *Let $C_1, C_2 \subseteq \mathbb{G}_m^N$ be geometrically irreducible closed algebraic curves defined over a number field, with $N \geq 3$. Let $\mathcal{N} = \{n \in \mathbb{N} : [n]C_1 \subseteq C_2\}$. Suppose there does not exist an algebraic subgroup $H \leq \mathbb{G}_m^N$ of dimension 2 containing both C_1 and C_2. Then $\{x \in C_1 :$ there is some $n \in \mathbb{N} \setminus \mathcal{N}$ such that $x^n \in C_2\}$ is finite.*

I shall discuss partial results by Bays and Habegger [BH] towards this conjecture in the case where $C_1 = C_2$. Given a more recent result of Amoroso, Masser and Zannier [AMZ], it is possible to extend the work of Bays and Habegger to the two curve setting. I shall discuss how to do this and make some additional remarks in the direction of the conjecture.

[AMZ] Amoroso, F., Masser, D. and Zannier, U., Bounded height in pencils of finitely generated subgroups, *Duke Math. J.* 166(no. 13): 2599–2642, 2017.

[BH] Bays, M. and Habegger, P., A note on divisible points of curves, *Trans. Amer. Math. Soc.* 367: 1313–1328, 2015.

[Z] Zannier, U., *Some problems of unlikely intersections in arithmetic and geometry*, Annals of Mathematics Studies 181, Princeton University Press, Princeton, 2012.

Algebraic values of the Weierstrass σ-function

Taboka Prince Chalebgwa

Stellenbosch, South Africa

Let $\Lambda = \mathbb{Z}[i]$ be the lattice of Gaussian integers and $\sigma(z; \Lambda)$ the Weierstrass σ-function associated with Λ. Let $d \geq 1$ be an integer and $H > e$ a real number. Building on earlier work by Boxall-Jones and Besson, I proved a bound of the form $C(\log H)^7$ for the number of algebraic points of height at most H and degree at most d on the graph of $\sigma(z; \Lambda)$, excluding the zeroes. I will give a brief discussion of this result.

On special values of Anderson modules

Tuan Ngo Dac

Caen, France

Let K be a global function field over a finite field of characteristic p and let A be the ring of elements of K which are regular outside a fixed place of K. We provide a report on recent developments in the arithmetic of special L-values of Anderson A-modules.

Provided that p does not divide the class number of K, we prove an "analytic class number formula" for Anderson A-modules with the help of recent work of Debry. This is a joint work with B. Anglès and F. Tavares Ribeiro.

Chebyshev's bias in Galois groups

Daniel Fiorilli

Orsay, France

In an 1853 letter, Chebyshev noted that there seem to be more primes of the form $4n+3$ than of the form $4n+1$. Many generalizations of this phenomenon have been studied. In this talk we will discuss Chebyshev's bias in the context of the Chebotarev density theorem. We will focus on the generic case of S_n extensions, in which the question is strongly linked with the representation theory of this group and the ramification data of the extensions. We will see in detail how to take advantage of the rich representation theory of the symmetric group as well as bounds on characters due to Roichman, Féray, Sniady, Larsen and Shalev.

(joint with Florent Jouve)

About Hartmanis & Stearns problem: Algebraic irrational numbers and machines

Marion le Gonidec

la Réunion, France

The Hartmanis & Stearns problem is an "algorithmic version" of Borel's conjecture about normality of algebraic irrational numbers. The question asked by Hartmanis & Stearns is the following: Can the digits of an algebraic irrational number be computed in linear time by a deterministic Turing machine? The expected answer is no. Following the idea of Cobham to consider smaller classes of machines and ask whether the numbers generated by them could be algebraic irrational numbers or not, I will present the case of numbers generated by pushdown automata, morphism and a larger class of machines called one-way transducer-like machines, which (as expected) can not be algebraic irrational numbers.

Elliptic curves with large Tate-Shafarevich groups over function fields

Richard Griffon

Basel, Switzerland

Tate-Shafarevich groups of elliptic curves remain mysterious objects. Given an elliptic curve E, $\text{Ш}(E)$ is conjectured to be finite and, assuming finiteness, some upper bounds on its size $\#\text{Ш}(E)$ are known in terms of the conductor or the height of E. These are proven in an analytic way (i.e. by using the L-function of E) via the BSD conjecture.

In this talk I will report on a recent work with Guus de Wit, where we construct an explicit family of elliptic curves over $\mathbb{F}_q(t)$ with 'large' Tate-Shafarevich groups, indeed essentially as large as they can possibly be according to the above mentioned bounds. Our result is completely unconditional and actually provides additional information about the structure of the Tate-Shafarevich groups under study. We use various techniques, including the computation of the relevant L-functions, a detailed study of the distribution of their zeros, and the proof of the BSD conjecture for these curves.

X-coordinates of Pell equations in various sequences

Florian Luca

Witwatersrand, South Africa

Let $d > 1$ be a squarefree integer and (X_n, Y_n) be the nth solution of the Pell equation $X^2 - dY^2 = \pm 1$. Given your favourite set of positive integers U, one can ask what can we say about those d such that $X_n \in U$ for some n?

Formulated in this way, the question has many solutions d since one can always pick $u \in U$ and write $u^2 \pm 1 = dv^2$ with integers d and v such that d is squarefree obtaining in this way that (u, v) is a solution of the Pell equation corresponding to d.

What about if we ask that $X_n \in U$ for at least two different n's? Then the answer is very different. For example, if U is the set of squares, then it is a classical result of Ljunggren that the only such d is 1785 for which both X_1 and X_2 are squares.

In my talk, I will survey recent results about this problem when U is the set of Fibonacci numbers, Tribonacci numbers, k-Generalized Fibonacci numbers, sums of two Fibonacci numbers, rep-digits (in base 10 or any integer base $b \geq 2$), and factorials. The proofs use linear forms in logarithms and computations. These results have been obtained in joint work with various colleagues such as J. J. Bravo, C. A. Gómez, A. Montejano, L. Szalay and A. Togbé and recent Ph.D. students M. Ddamulira, B. Faye and M. Sias.

When is the permutation ring Cohen Macaulay?

Sophie Marques

Stellenbosch, South Africa

Understanding when the ring of invariants of a Cohen-Macaulay ring is Cohen-Macaulay has been a question which has interested mathematicians for some time. In particular the question of "When is the ring of polynomial invariants of a permutation group Cohen-Macaulay?" has been largely studied: Ellingsrud and Skjelbred, 1980, Smith, 1996, Campbell et. al. and Kemper 1999... One reason is that this question is closely related to the inverse Galois problem and a very interesting result of Noether (1913), that permits to describe general field extensions through their transitive Galois groups using the pure transcendence of permutation fields.

In our work, we proved that the polynomial invariants of a permutation group are Cohen-Macaulay for any choice of coefficient field if and only if the group is generated by transpositions, double transpositions, and 3-cycles. This generalizes several previously known results, bringing together combinatorics, topology and algebra. We will explain how the "if" direction of the argument uses Stanley-Reisner theory and a recent result of Christian Lange in orbifold theory. The "only-if" direction uses a local-global result, which is based on a theorem of Raynaud of reduction of the problem to an analysis of inertia groups, and on a combinatorial argument of identification of the inertia groups that obstruct Cohen-Macaulayness. We will explain how the only-if direction interestingly results from the "if" direction and involves a purely algebraic argument.

We will present, in proving this direction, a helpful result which is true under very general assumptions: Cohen-Macaulayness of the ring of invariants only depends on actions of the inertia groups at any prime ideal on a nice neighborhood of this prime. We will conclude presenting questions remained open related with this work.

(Joint work with Ben Blum Smith)

Combinatory classes of compositions with higher-order conjugation

Augustine Munagi

Witwatersrand, South Africa

The classical development of the theory of integer compositions by P. A. MacMahon (1854–1929) has recently been extended to compositions with conjugates of higher orders. This discussion will be based on certain classes of compositions possessing conjugates of a prescribed order - their enumeration and some identities they satisfy.

A generalization of a partition theorem of M. V. Subbarao

Darlison Nyirenda

Witwatersrand, South Africa

It is known that the number of partitions of a positive integer n into parts with multiplicities 2, 3 or 5 is equal to the number of partitions of n into parts congruent to $\pm 2, \pm 3, 6 \pmod{12}$. We give a generalisation of this result and further present parity formulas for some partition functions in the style of Subbarao.

Arithmetic and homotopy

Ambrus Pàl

Imperial College, London, UK

Methods of modern homotopy theory found several applications recently in number theory. I plan to talk about some of my work using motivic cohomology and motivic homotopy with arithmetic applications. These include the solvability of certain classes of embedding problems, the arithmetic of K3 surfaces and even some links to modular forms. I will try to be gentle while introducing the motivic homotopy machinery.

Unrandomness of rational points

Emmanuel Peyre

Grenoble, France

One of the most interesting developments in arithmetic geometry during the last 30 years comes from the discovery of the complexity of the distribution of rational solutions of polynomial equations. We will first explain what it means to be equidistributed for such solutions, then we will describe several well known examples for which "obvious" solutions of the equations are far more numerous than they ought to be, thus preventing equidistribution. The last part of the talk will relate this unrandomness to geometric invariants.

Arithmetic and topology of r-pyramids

PATRICK RABARISON

Antananarivo, Madagascar

Kronecker became famous for his discovery that all Abelian extensions of $\mathbb{Q}$ are subfields of fields generated by root of unity. Many generalizations of his result are now known in modern langage as class field theory. The importance of the cyclotomic polynomial and the unit circle in many domains of Mathematics is crucial. r-Pyramids can be viewed as a collection of numbers with each level interpreted as polynomials. In number theory, the work of Kummer and Iwasawa relate the Riemann Zeta Function with the arithmetic of cyclotomic fields. I conjecture the existence of a similar situation for families of fields generated by polynomials having their roots on some real plane algebraic curves. In the Galois theory setting, for some given groups A and B, I will show how to realise some groups satisfying the short exact sequence:

$$1 \longrightarrow A \longrightarrow G \longrightarrow B \longrightarrow 1$$

as Galois groups over $\mathbb{Q}$. In the analytic settings, the case of the roses and rings curves will be discussed. For instance, the series

$$Z_{C_m}(s) = \sum_{n=1}^{\infty} \frac{1}{(n(2^{m-1})^{n-1})^s} \quad , \quad Z_{R_2} = \sum_{n=1}^{\infty} \Big(\frac{2}{3^n - 1}\Big)^s \quad \text{and} \quad Z_{R_3} = \sum_{n=1}^{\infty} \Big(\frac{6}{7^n - 1}\Big)^s$$

are attached respectively to the m-circle, to the lemniscate and to the trefoil in the same way that the Riemann Zeta function is attached to the unit circle. Some bizarre "curvoids" from the algebra generated by pyramids will also appear in this talk and similar situation also occur for fractals which lead to a definition of (branched) fractal coverings. Situation for high dimensional case will also be discussed, for instance the case of the unbounded 3-real manifold (torus bundle):

$$M_3 : (x^2 + y^2)(z^2 + t^2) = 1$$

will be treated. I must mention that the constructions that I use are all explicit and many phenomena from experimentation remain inexplicable.

Mahler functions and quadratic polynomials

HARRY SCHMIDT

Manchester, UK

In this talk I will report on work in progress on Mahler functions associated with quadratic polynomials. My main focus lies on functional transcendence and arithmetic applications.

On the solvability of the negative Pell equation

PETER STEVENHAGEN

Leiden, Netherlands

Unlike the "ordinary" Pell equation $x^2 - dy^2 = 1$, which is solvable in positive integers x, y for all non-square $d > 0$, the negative Pell equation $x^2 - dy^2 = -1$ only has non-trivial integral solutions for a thin, but infinite set of d's. I will explain a conjecture of mine, which predicts the (irrational!) fraction of the d's satisfying the obvious local conditions for which we have solvability. After 25 years, it is still open, but substantial progress has been obtained by Fouvry, Klüners and others.

Equations for modular curves associated to normalizers of non-split Cartan subgroups

René Schoof

Rome, Italy

Recent work by Bilu, Parent and Rebolledo have reduced Serre's Uniformity Conjecture to the problem of determining the rational points of the modular curves associated to the normalizers of non-split Cartan subgroups of $\mathrm{GL}_2(\mathbb{F}_p)$.

In a joint work with Pietro Mercuri, we describe a method to compute for a prime p, equations over $\mathbb{Q}$ of these modular curves. For $p = 17, 19$ and 23 we present explicit equations.