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On certain Drinfeld modular forms of higher rank

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Résumé. Nous donnons une introduction aux formes modulaires de Drinfeld pour des sous-groupes de congruence principaux de $\mathrm{GL}_r(\mathbb{F}_q[t])$, et puis nous construisons un analogue en rang r de la fonction h. Nous montrons que cette fonction est cuspidale de poids $(q^r-1)/(q-1)$ et de type 1 et qu'elle satisfait une formule de produit. Dans ce but, nous calculons le développement à l'infini des séries d'Eisenstein de poids 1 et de nivaux $N \in \mathbb{F}_q[t]$.

Abstract. We give an introduction to Drinfeld modular forms for principal congruence subgroups of $\mathrm{GL}_r(\mathbb{F}_q[t])$, and then construct a rank r analogue of the h-function. We show that this function is a cusp form of weight $(q^r - 1)/(q - 1)$ and type 1 which satisfies a product formula. Along the way, we compute the expansion at infinity of weight one Eisenstein series of level $N \in \mathbb{F}_q[t]$.

Dedicated to the memory of David Goss

1. Introduction and Outline

This paper is a companion to [2] and may serve as a sneak preview of joint work in progress [3] with Richard Pink, where the analytic theory of Drinfeld modular forms of higher rank is developed in greater generality.

The main protagonist of this note is the higher rank version of the function Gekeler denoted h. It is defined essentially as the $(q-1)$-st root of the discriminant function Δ. As $(q-1)$-st roots are only defined up to a unit, we take some care to make it explicit which $(q-1)$-st root is taken. As a corollary to the first author's product formula for Δ we then give a product formula for h (Theorem 5.3).

We also discuss other examples of higher rank Drinfeld modular forms, namely Eisenstein series for the full modular group, Eisenstein series for principal congruence subgroups, coefficient forms and exponential coefficient forms. In Section 3, we calculate the leading coefficients of these

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forms and in Section 6 we show that the weight one Eisenstein series for principal congruence subgroups also have a product expansion that generalizes one obtained by Gekeler [7] in the rank 2 case (Theorem 6.2).

Finally, we have sketched an outline of the history of Drinfeld modular forms of higher rank in Section 7, since we believe that it would be of interest to the community.

2. Basic Definitions

Let $A = \mathbb{F}_q[t]$ be the polynomial ring over the finite field of order q, let $F_\infty = \mathbb{F}_q((\frac{1}{t}))$ be the completion of $\mathbb{F}_q(t)$ at the place with uniformizer $\frac{1}{t}$, and denote by $\mathbb{C}_\infty = \hat{\bar{F}}_\infty$ the completion of an algebraic closure of F_∞.

Fix an integer $r \geq 2$, called the rank. Drinfeld's period domain is the rigid analytic space

$$\Omega^r := \mathbb{P}^r(\mathbb{C}_\infty) \smallsetminus \{F_\infty\text{-rational hyperplanes}\},$$

on which the group $\mathrm{GL}_r(F_\infty)$ acts from the left as usual. We represent elements of Ω^r as column vectors $\omega = (\omega_1, \omega_2, \ldots, \omega_r)^T$, normalized so that $\omega_r = \bar{\pi}$, where

$$\bar{\pi} = \sqrt[q-1]{t - t^q} \prod_{i=1}^{\infty} \left(1 - \frac{t^{q^i} - t}{t^{q^{i+1}} - t} \right)$$

is a period of the Carlitz module, defined up to a multiplicative constant in $\mathbb{F}_q^*$.

For $\gamma \in \mathrm{GL}_r(A)$ and $\omega \in \Omega^r$, we define

$$j(\gamma, \omega) := \bar{\pi}^{-1} \cdot (\text{last entry of } \gamma\omega),$$

where $\gamma\omega$ denotes the matrix product. Then

$$\gamma(\omega) := j(\gamma, \omega)^{-1} \gamma\omega$$

defines an action of $\mathrm{GL}_r(A)$ on Ω^r which preserves our choice of normalization.

Definition 2.1. Let $k \in \mathbb{Z}_{>0}$ and $m \in \mathbb{Z}/(q-1)\mathbb{Z}$. Let $\Gamma \subset \mathrm{GL}_r(A)$ be an arithmetic subgroup. A *weak modular form* of weight k and type m for Γ is a holomorphic (in the rigid analytic sense) function $f : \Omega^r \to \mathbb{C}_\infty$ satisfying

$$f\big(\gamma(\omega)\big) = (\det \gamma)^{-m} j(\gamma, \omega)^k f(\omega), \quad \text{for all } \gamma \in \Gamma. \tag{2.1}$$

If $m = 0$, or $\det \Gamma = \{1\}$, we suppress any mention of the type.

Suppose $\Gamma = \mathrm{GL}_r(A)$, then plugging scalars of the form $\gamma = \varepsilon I \in \mathrm{GL}_r(A)$, where $\varepsilon \in \mathbb{F}_q^*$, into (2.1), we find that if f is a non-zero weak modular form of weight k and type m, then

$$k \equiv rm \pmod{q-1}. \tag{2.2}$$

Let $0 \neq N \in A$, then we are particularly interested in the principal congruence group

$$\Gamma(N) := \ker\big(\mathrm{GL}_r(A) \longrightarrow \mathrm{GL}_r(A/NA)\big).$$

Elements of A^r or $(N^{-1}A)^r$ will be represented by row vectors, so when $a = (a_1, a_2, \dots, a_r) \in A^r$ and $\omega \in \Omega^r$, then

$$a\omega = \sum_{i=1}^{r} a_i\omega_i \in \mathbb{C}_\infty$$

will denote the matrix product. The set $A^r\omega = A\omega_1 + \cdots + A\omega_r \subset \mathbb{C}_\infty$ is a lattice of rank r, i.e. a projective A-submodule of rank r which has finite intersection with any ball of finite radius in $\mathbb{C}_\infty$. To a lattice $\Lambda \subset \mathbb{C}_\infty$ we associate its *exponential function*

$$e_\Lambda : \mathbb{C}_\infty \to \mathbb{C}_\infty; \qquad e_\Lambda(z) := z \prod_{\lambda\in\Lambda}{}' \left(1 - \frac{z}{\lambda}\right),$$

where as usual the prime denotes a product over non-zero indices. This function is entire, surjective, $\mathbb{F}_q$-linear and has simple zeros precisely at the elements of Λ. It satisfies

$$\frac{1}{e_\Lambda(z)} = \sum_{\lambda\in\Lambda} \frac{1}{z+\lambda} \quad \text{and} \quad e_{c\Lambda}(z) = ce_\Lambda(c^{-1}z) \quad \text{for all } c \in \mathbb{C}_\infty.$$

See [17, Chapter 4] for more details on lattices and Drinfeld modules.

Next, we construct a suitable parameter to allow expansions at infinity of Drinfeld modular forms of rank $r \geq 2$. This parameter appears first in [8], in a more general setting. For more detail, see [1], although the action of $\mathrm{GL}_r(F_\infty)$ on Ω^r in that thesis differs from the action described here by the automorphism $\gamma \mapsto (\gamma^T)^{-1}$.

Definition 2.2. Let $0 \neq N \in A$. For $\omega = (\omega_1, \omega_2, \dots, \omega_r)^T \in \Omega^r$, we write $\omega = (\omega_1, \omega')^T$, where $\omega' = (\omega_2, \dots, \omega_r)^T \in \Omega^{r-1}$. Its associated lattice is $\Lambda' = A^{r-1}\omega' \subset \mathbb{C}_\infty$. We set

$$u_N := e_{\Lambda'}(N^{-1}\omega_1)^{-1}. \tag{2.3}$$

This is the *parameter at infinity* for $\Gamma(N)$. When $N = 1$, we write $u := u_1$ for the parameter at infinity for $\Gamma(1) = \mathrm{GL}_r(A)$.

This parameter also differs from the one constructed in [1] and [3] by a factor of N, but will allow slightly neater expressions.

Theorem 2.3 ([3])**.** *Every weak modular form $f : \Omega^r \to \mathbb{C}_\infty$ of weight k and type m for $\Gamma(N)$ admits a u_N-expansion*

$$f(\omega) = \sum_{n\in\mathbb{Z}} f_n(\omega')u_N^n, \tag{2.4}$$

where the $f_n : \Omega^{r-1} \to \mathbb{C}_\infty$ *are weak modular forms of weight* $k-n$ *and type* m *for* $\Gamma'(N) = \ker\big(\mathrm{GL}_{r-1}(A) \to \mathrm{GL}_{r-1}(A/NA)\big)$*, uniquely determined by* f*. The series (2.4) converges uniformly on suitable neighbourhoods of the boundary of* Ω^r*.*

If $N = 1$, then from (2.2) follows that the function f_n is identically zero unless

$$n \equiv k - (r-1)m \pmod{q-1}. \tag{2.5}$$

Using the above expansions, we can define the notion of a modular form for $\Gamma(N)$:

Definition 2.4 ([3])**.** A function $f : \Omega^r \to \mathbb{C}_\infty$ satisfying an expansion of the form (2.4) is *holomorphic at infinity* if the functions f_n in (2.4) are identically zero for all $n < 0$. We say f *vanishes at infinity* if f_0 is also identically zero.

A weak modular form f of weight k and type m for $\mathrm{GL}_r(A)$ is a *modular form* if it is holomorphic at infinity, and a *cusp form* if it vanishes at infinity.

A weak modular form f of weight k for $\Gamma(N)$ with $N \in A$ non-constant is a *modular form* if $j(\gamma,\omega)^{-k} f\big(\gamma(\omega)\big)$ is holomorphic at infinity for every $\gamma \in \mathrm{GL}_r(A)$. Furthermore, f is a *cusp form* if $j(\gamma,\omega)^{-k} f\big(\gamma(\omega)\big)$ vanishes at infinity for every $\gamma \in \mathrm{GL}_r(A)$.

3. First examples of Drinfeld modular forms for $\mathrm{GL}_r(\mathbb{F}_q[t])$

We describe here the standard examples of Drinfeld modular forms for $\mathrm{GL}_r(A)$; in essence, these constructions already appear in the early work of Drinfeld [6] and Goss [14]. For more details, see also [1, §§3.4 and 3.5], where these examples are studied for more general rings A.

Example 3.1 (Eisenstein series of level 1)**.** *Let* $k \in \mathbb{Z}_{>0}$ *and* $\omega \in \Omega^r$*. The Eisenstein series of weight* k *for* $\mathrm{GL}_r(A)$ *is*

$$E_k(\omega) := \sum_{a \in A^r}{}' \frac{1}{(a\omega)^k}.$$

For $\gamma \in \mathrm{GL}_r(A)$ we easily compute

$$E_k\big(\gamma(\omega)\big) = \sum_{a \in A^r}{}' \frac{1}{\big(aj(\gamma,\omega)^{-1}\gamma\omega\big)^k} = j(\gamma,\omega)^k \sum_{a \in A^r\gamma}{}' \frac{1}{(a\omega)^k} = j(\gamma,\omega)^k E_k(\omega),$$

so E_k is a weak modular form of weight k for $\mathrm{GL}_r(A)$. By (2.2), E_k is identically zero unless k is a multiple of $q-1$.

Example 3.2 (Exponential coefficient forms)**.** *Let* $\omega \in \Omega^r$*, then the exponential function associated to the lattice* $\Lambda = A^r\omega$ *is given by the* $\mathbb{F}_q$*-linear* *power series*

$$e_\Lambda(z) = \sum_{i=0}^{\infty} e_i(\omega) z^{q^i}, \qquad \text{with } e_0(\omega) = 1.$$

Let $\gamma \in \mathrm{GL}_r(A)$. Then the exponential function associated to the lattice $\gamma(\Lambda) := A^r\gamma(\omega) = j(\gamma,\omega)^{-1}A^r\omega = j(\gamma,\omega)^{-1}\Lambda$ satisfies

$$e_{\gamma(\Lambda)}(z) = e_{j(\gamma,\omega)^{-1}\Lambda}(z) = j(\gamma,\omega)^{-1} e_\Lambda\big(j(\gamma,\omega)z\big), \tag{3.1}$$

so for each $i \geq 0$

$$e_i\big(\gamma(\omega)\big) = j(\gamma,\omega)^{q^i-1} e_i(\omega),$$

and e_i is a weak modular form of weight $q^i - 1$ for $\mathrm{GL}_r(A)$.

Example 3.3 (Drinfeld coefficient forms)**.** *Let φ^ω be the rank r Drinfeld module associated to the lattice $A^r\omega \subset \mathbb{C}_\infty$ and $a \in A$. Then*

$$\varphi_a^\omega(X) = \sum_{i=0}^{r\deg a} g_{i,a}(\omega) X^{q^i},$$

where $g_{0,a}(\omega) = a$ and $\Delta_a(\omega) := g_{r\deg a,a}(\omega) \neq 0$.

Let $\gamma \in \mathrm{GL}_r(A)$, then from

$$e_\Lambda(aX) = \varphi_a^\omega\big(e_\Lambda(X)\big) \tag{3.2}$$

and (3.1) we get

$$\varphi_a^{\gamma(\omega)}(X) = j(\gamma,\omega)^{-1}\varphi_a^\omega\big(j(\gamma,\omega)X\big),$$

so the coefficients $g_{i,a}$ are weak modular forms of weight $q^i - 1$ for $\mathrm{GL}_r(A)$.

These three constructions are interrelated by the following identities (see also [1, §3.4]):

(3.3)

$$e_i(\omega) = E_{q^i-1}(\omega) + \sum_{j=1}^{i-1} e_j(\omega) E_{q^{i-j}-1}(\omega)^{q^j}, \qquad \text{see e.g. [4, (9)]},$$

(3.4)

$$(a^{q^i} - a)e_i(\omega) = g_{i,a}(\omega) + \sum_{j=1}^{i-1} g_{j,a}(\omega) e_{i-j}(\omega)^{q^j}, \qquad \text{from (3.2)},$$

(3.5)

$$g_{i,a}(\omega) = (a^{q^i} - a)E_{q^i-1}(\omega) + \sum_{j=1}^{i-1} E_{q^j-1}(\omega) g_{i-j,a}(\omega)^{q^j}, \quad \text{see e.g. [9, (2.10)]}.$$

In (3.4) and (3.5) we adopt the convention that $g_{i,a} \equiv 0$ if $i > r\deg a$.

To show that these weak modular forms are holomorphic at infinity, it suffices to compute the u-expansion of the Eisenstein series E_k. In the next computation, we write $a=(a_1,a_2,\dots,a_r)\in A^r$ and $a'=(a_2,a_3,\dots,a_r)\in A^{r-1}$.

$$\begin{aligned}
E_k(\omega) &= \sum_{a\in A^r}{}' \frac{1}{(a\omega)^k}\\
&= \underbrace{\sum_{a'\in A^{r-1}}{}' \frac{1}{(a'\omega')^k}}_{a_1=0} \quad + \quad \sum_{a_1\in A}{}' \underbrace{\sum_{a'\in A^{r-1}} \frac{1}{(a_1\omega_1+a'\omega')^k}}_{a_1\neq 0}\\
&= E'_k(\omega') + \sum_{a_1\in A}{}' P_k\left(\sum_{a'\in A^{r-1}} \frac{1}{(a_1\omega_1+a'\omega')}\right)\\
&= E'_k(\omega') + \sum_{a_1\in A}{}' P_k\big(e_{\Lambda'}(a_1\omega_1)^{-1}\big)\\
&= E'_k(\omega') + \sum_{a_1\in A}{}' P_k\big(u(a_1\omega_1)\big)\\
&= E'_k(\omega') + \sum_{a_1\in A}{}' P_k\left(\frac{1}{\varphi^{\omega'}_{a_1}(u^{-1})}\right)\\
&= E'_k(\omega') + O(u^{q-1}). \qquad (3.6)
\end{aligned}$$

Here $P_k(X)$ is the degree k Goss polynomial associated to the lattice Λ', first introduced by Goss in [15], see also [9, §3]. We denote by $E'_k(\omega')$ the weight k Eisenstein series associated to the rank $r-1$ lattice $\Lambda'=A^{r-1}\omega'\subset\mathbb{C}_\infty$. The term $O(u^{q-1})$ denotes a sum of higher-degree terms in u, starting with $c_{q-1}(\omega')u^{q-1}$, whose coefficients are functions of $\omega'\in\Omega^{r-1}$. The exponent $q-1$ here comes from (2.5). The last equality follows because $P_k(0)=0$.

The above computation is due to Goss [14] in rank $r=2$, and appears in [4, §3.3] for arbitrary rank, see also [1, Prop. 3.5.3].

From (3.3), (3.5) and (3.6) we obtain, by induction on r and on i,

$$e_i(\omega) = e'_i(\omega') + O(u^{q-1}) \quad i=1,2,3,\dots, \quad \text{and} \qquad (3.7)$$

$$g_{i,a}(\omega) = g'_{i,a}(\omega') + O(u^{q-1}), \quad i=1,2,\dots,r\deg a, \qquad (3.8)$$

where e'_i and $g'_{i,a}$ denote the corresponding forms associated to the rank $r-1$ lattics $\Lambda'=A^{r-1}\omega'\subset\mathbb{C}_\infty$. These identities essentially appear in [4, §3.3], and are computed via a different method in [1, §3.5.3].

Note that if $i>(r-1)\deg a$ then $g'_{i,a}\equiv 0$, so that $g_{i,a}$ is a cusp form for $i>(r-1)\deg a$.

In particular, we have

Proposition 3.4. *The Eisenstein series E_k, exponential coefficient forms e_i and Drinfeld coefficient forms $g_{a,i}$ are Drinfeld modular forms of type 0 and weights $k, q^i - 1$ and $q^i - 1$, respectively, for $\Gamma = \mathrm{GL}_r(A)$.*

It is shown in [3] that the Drinfeld coefficient forms $g_{1,t}, g_{2,t}, \dots, g_{r,t}$ are algebraically independent and generate the algebra of Drinfeld modular forms of type 0 and arbitrary weight for $\mathrm{GL}_r(A)$.

4. Eisenstein series for $\Gamma(N)$

Now let $N \in A$ be non-constant and recall that $\Gamma(N) = \ker\big(\mathrm{GL}_r(A) \longrightarrow \mathrm{GL}_r(A/NA)\big)$.

Example 4.1 (Eisenstein series of level N)**.** *Set $V_N := (N^{-1}A/A)^r$. Let $k \in \mathbb{Z}_{k>0}$. To each $v \in V_N \smallsetminus \{0\}$ we associate the weight k Eisenstein series*

$$E_{k,v}(\omega) := \sum_{a \in A^r} \frac{1}{\big((a+v)\omega\big)^k}.$$

By abuse of notation, the v on the right hand side above actually denotes a representative in $(N^{-1}A)^r$ for the class in $(N^{-1}A/A)^r$.

When $k = 1$ we drop the k from the notation, and we have, additionally,

$$E_v(\omega) := E_{1,v}(\omega) = \frac{1}{e_\Lambda(v\omega)}. \tag{4.1}$$

It is again easy to show that $E_{k,v}$ is a weak modular form of weight k for $\Gamma(N)$. A computation similar to (3.6), but more complicated, again shows that $E_{k,v}$ is holomorphic at infinity. This was essentially carried out by Kapranov in [20]. To show that $E_{k,v}$ is actually a modular form, we need to show that $j(\gamma,\omega)^{-k}E_{k,v}\big(\gamma(\omega)\big)$ is holomorphic at infinity for all $\gamma \in \mathrm{GL}_r(A)$, which will follow from the next result.

Proposition 4.2. *Let $N \in A \smallsetminus \mathbb{F}_q^*$, $v \in V_N \smallsetminus \{0\}$, $\varepsilon \in \mathbb{F}_q^*$, $k \geq 1$ and $\gamma \in \mathrm{GL}_r(A)$. Then*

(1) $E_{k,\varepsilon v}(\omega) = \varepsilon^{-k} E_{k,v}(\omega)$.
(2) $j(\gamma,\omega)^{-k} E_{k,v}\big(\gamma(\omega)\big) = E_{k,v\gamma}(\omega)$.

In particular, $E_{k,v}$ is a modular form of weight k for $\Gamma(N)$.

Proof. For (1), we have

$$E_{k,\varepsilon v}(\omega) = \sum_{a \in A^r} \frac{1}{\big((a+\varepsilon v)\omega\big)^k} = \varepsilon^{-k} \sum_{a \in \varepsilon^{-1}A^r} \frac{1}{\big((a+v)\omega\big)^k} = \varepsilon^{-k} E_{k,v}(\omega).$$

For (2), we compute

$$\begin{aligned}E_{k,v}\big(\gamma(\omega)\big) &= \sum_{a\in A^r} \frac{1}{\big((a+v)j(\gamma,\omega)^{-1}\gamma\omega\big)^k} \\ &= j(\gamma,\omega)^k \sum_{a\in A^r\gamma} \frac{1}{\big((a+v\gamma)\omega\big)^k} = j(\gamma,\omega)^k E_{k,v\gamma}(\omega).\end{aligned}$$

□

In this paper, we will obtain an alternative expression, due to Gekeler [7] in the rank $r=2$ case, for the u_N-expansion of $E_v = E_{1,v}$.

5. The cusp forms Δ and h

The cusp form $g_{r,t}$ is non-vanishing on Ω^r and is called the Drinfeld discriminant form:

$$\Delta(\omega) := g_{r,t}(\omega).$$

Let $V := V_t = (t^{-1}A/A)^r \cong \mathbb{F}_q^r$. The weight 1 Eisenstein series of level $N=t$ are the reciprocals of the non-zero t-torsion points of φ^ω, by (4.1), i.e.

$$\varphi_t^\omega(X) = tX \prod_{v\in V}{}' \big(1 - XE_v(\omega)\big).$$

In particular,

$$\Delta(\omega) = t \prod_{v\in V}{}' E_v(\omega). \tag{5.1}$$

So far, all of our examples have had type $m=0$. The first example of a (weak) modular form in rank $r=2$ of non-zero type was a Poincaré series in [13, Chapter X]. Gekeler showed in [9] that this is indeed a modular form, now called the h-function, which has many interesting properties.

Our goal is to define a rank r generalization of the h-function and relate it to Δ.

Definition 5.1. Let

$$S = t^{-1}\{(1,*,\ldots,*),(0,1,*,\ldots,*),\ldots,(0,\ldots,0,1)\} \subset V = (t^{-1}A/A)^r$$

be a set of representatives of the quotient space $(V \smallsetminus \{0\})/\mathbb{F}_q^* \cong \mathbb{P}^r(\mathbb{F}_q)$.

Let $\lambda_t := e_{A\omega_r}(t^{-1}\omega_r) = \bar{\pi}^{-1}e_A(t^{-1})$. Note that λ_t generates the t-torsion of the Carlitz module, and $\lambda_t^{q-1} = -t$.

Then we set

$$h(\omega) := -\lambda_t \prod_{v\in S} E_v(\omega). \tag{5.2}$$

For $a \in A$, we define

$$d(a) := \begin{cases} q^{(r-1)\deg a} & \text{if } a \neq 0, \\ 0 & \text{if } a = 0. \end{cases}$$

Let $\omega' \in \Omega^{r-1}$ and denote by $\varphi^{\omega'}$ the rank $r-1$ Drinfeld module associated to the lattice $\Lambda' = A^{r-1}\omega'$. If $a \neq 0$ we let $\Delta'_a(\omega')$ denote the leading coefficient of $\varphi_a^{\omega'}(X)$, so that

$$\varphi_a^{\omega'}(X) = aX + \cdots + \Delta'_a(\omega')X^{d(a)}.$$

To $0 \neq a \in A$ and $\omega' \in \Omega^{r-1}$ we associate the polynomial $f_a(X)$, defined by

$$f_a(X) := X^{d(a)}\Delta'_a(\omega')^{-1}\varphi_a^{\omega'}(X^{-1}) - 1. \tag{5.3}$$

We have

$$\varphi_a^{\omega'}(X^{-1}) = \Delta'_a(\omega')X^{-d(a)}\big(1 + f_a(X)\big). \tag{5.4}$$

The polynomial $f_a(X)$ has degree $d(a) - 1$ in X and is divisible by $X^{(q-1)q^{(r-1)\deg a - 1}} = X^{(1-1/q)d(a)}$.

In [2], the first author proves the following product formula for Δ. Denote by $A_+ \subset A$ the subset of monic elements.

Theorem 5.2 ([2])**.** *The discriminant Δ satisfies*

$$\Delta(\omega) = -\Delta'(\omega')^q u^{q-1} \prod_{a \in A_+} \big(1 + f_a(u)\big)^{(q^r-1)(q-1)}. \tag{5.5}$$

Here $\Delta'(\omega')$ is the rank $r-1$ Drinfeld discriminant function (defined to be $\Delta' := 1$ if $r = 2$), and the product converges in suitable neighbourhoods of the boundary of Ω^r.

Our main goal will be to the prove the following.

Theorem 5.3.

(1) *h is a cusp form of weight $k = (q^r - 1)/(q - 1)$ and type $m = 1$ for $\mathrm{GL}_r(A)$.*

(2) *$\Delta(\omega) = (-1)^{r-1}h(\omega)^{q-1}$*

(3) *$h(\omega) = -\lambda_t M\big(E_{w_r}(\omega)^{-1}, \ldots, E_{w_1}(\omega)^{-1}\big)^{-1}$, where $M(x_1, \ldots, x_r) = \det\big(x_i^{q^j}\big)$ denotes the Moore determinant of $x_1, x_2, \ldots, x_r \in \mathbb{C}_\infty$ and the $w_i = t^{-1}(0, \ldots, 0, 1, 0 \ldots, 0)$ (with the 1 in position i) form a standard basis for V.*

(4) *h satisfies the product formula*

$$h(\omega) = (-1)^r h'(\omega')^q u \prod_{a \in A_+} \big(1 + f_a(u)\big)^{q^r-1}, \tag{5.6}$$

where $h'(\omega')$ denotes the rank $r-1$ h-function, which is defined to be -1 when $r = 2$.

This generalizes known results for the rank 2 case, see [5].

Remark 5.4. It has come to our attention that the construction of h in higher rank has been known to Ernst-Ulrich Gekeler since the mid 1980's (unpublished); it appears in [11].

Remark 5.5. Assuming that there are no non-constant modular forms of weight 0 for $\mathrm{GL}_r(A)$ (which follows from the results in [3]), one shows in the usual way that there are no modular forms of negative weight, and thus h (vanishing with order one at infinity) is a cusp form for $\mathrm{GL}_r(A)$ of minimal weight, and furthermore generates the space of modular forms of weight $k = (q^r - 1)/(q - 1)$. In particular, h is proportional to the reciprocal of the Legendre determinant form constructed by Gekeler in [10] and developed further by Perkins [21].

A natural approach to proving (5.6) is to repeat the first author's proof of (5.5), starting with (5.2) instead of (5.1). Then Theorem 5.3.2 will automatically yield (5.5). Instead, we start with (5.5) and Theorem 5.3.2 to deduce that (5.6) must hold up to multiplication by a unit in $\mathbb{F}_q^*$, and then compute the first term of the u-expansion of h to show that this unit is 1. This will be an instructive computation.

We first prove part (2) of Theorem 5.3. Notice that $V \smallsetminus \{0\} = \bigcup_{\varepsilon \in \mathbb{F}_q^*} \varepsilon S$.

Then

$$\begin{aligned}
\Delta(\omega) &= t \prod_{v \in V}{}' E_v(\omega) = -\lambda_t^{q-1} \prod_{\varepsilon \in \mathbb{F}_q^*} \prod_{v \in S} E_{\varepsilon v}(\omega) \\
&= -\lambda_t^{q-1} \prod_{\varepsilon \in \mathbb{F}_q^*} \prod_{v \in S} \varepsilon^{-1} E_v(\omega) \\
&= -\lambda_t^{q-1} \left(\prod_{\varepsilon \in \mathbb{F}_q^*} \varepsilon^{-1} \right)^k \left(\prod_{v \in S} E_v(\omega) \right)^{q-1} \\
&= -(-1)^k \left(-\lambda_t \prod_{v \in S} E_v(\omega) \right)^{q-1} \\
&= (-1)^{r-1} h(\omega)^{q-1}
\end{aligned}$$

since $\#S = k \equiv r \pmod 2$ when q is odd and $\prod_{\varepsilon \in \mathbb{F}_q^*} \varepsilon^{-1} = -1$.

We next prove part (3) using the Moore Determinant Formula [17, Cor. 1.3.7]:

$$\begin{aligned} h(\omega) &= -\lambda_t \left(\prod_{v \in S} e_\Lambda(v\omega) \right)^{-1} \\ &= -\lambda_t M\big(e_\Lambda(w_r\omega), e_\Lambda(w_{r-1}\omega), \ldots, e_\Lambda(w_1\omega)\big)^{-1}. \end{aligned}$$

Now we can show that h is a weak modular form of weight k and type 1. Let $\gamma \in \mathrm{GL}_r(A)$. Then

$$\begin{aligned} h\big(\gamma(\omega)\big) &= -\lambda_t M\Big(E_{w_r}\big(\gamma(\omega)\big)^{-1}, E_{w_{r-1}}\big(\gamma(\omega)\big)^{-1}, \ldots, E_{w_1}\big(\gamma(\omega)\big)^{-1}\Big)^{-1} \\ &= -\lambda_t M\Big(j(\gamma,\omega)^{-1} E_{w_r\gamma}(\omega)^{-1}, j(\gamma,\omega)^{-1} E_{w_{r-1}\gamma}(\omega)^{-1}, \\ &\qquad \ldots, j(\gamma,\omega)^{-1} E_{w_1\gamma}(\omega)^{-1}\Big)^{-1} \\ &= -\lambda_t j(\gamma,\omega)^k M\big(e_\Lambda(w_r\gamma\omega), e_\Lambda(w_{r-1}\gamma\omega), \ldots, e_\Lambda(w_1\gamma\omega)\big)^{-1} \\ &= -\lambda_t j(\gamma,\omega)^k (\det\gamma)^{-1} M\big(e_\Lambda(w_r\omega), e_\Lambda(w_{r-1}\omega), \ldots, e_\Lambda(w_1\omega)\big)^{-1} \\ &= (\det\gamma)^{-1} j(\gamma,\omega)^k h(\omega). \end{aligned}$$

Alternatively, one can prove the above functional equation directly from the product definition (5.2) of $h(\omega)$ using the following amusing lemma.

Lemma 5.6. *Let $S = \{s_1, s_2, \ldots, s_k\} \subset V$ be any set of representatives of $(V \smallsetminus \{0\})/\mathbb{F}_q^* \cong \mathbb{P}^{r-1}(\mathbb{F}_q)$. Let $\gamma \in \mathrm{GL}_r(\mathbb{F}_q)$. Then $S\gamma$ is again a set of representatives, so we have (in some order) $S\gamma = \{c_1s_1, c_2s_2, \ldots, c_ks_k\}$ for factors $c_i \in \mathbb{F}_q^*$. Then $\prod_{i=1}^k c_i = \det\gamma$.*

We leave the proof as an exercise for the reader (hint: decompose γ as a product of elementary matrices and $\mathrm{diag}(1, \ldots, 1, \det(\gamma))$).

6. Expansions of weight-one Eisenstein series

Our main goal in this section is a product formula for the u_N-expansion of the weight 1 Eisenstein series E_v. Combined with the following result, this will give us the first term in the u-expansion of h, thus completing the proof of Theorem 5.3.

Proposition 6.1. *Let $N \in A$ be non-constant. For $\omega' \in \Omega^{r-1}$, denote by $\Delta'_N(\omega')$ the leading coefficient of $\varphi_N^{\omega'}(X)$. Then*

$$\begin{aligned} u &= \Delta'_N(\omega')^{-1} u_N^{d(N)} \big(1 + f_N(u_N)\big)^{-1} \\ &= \Delta'_N(\omega')^{-1} u_N^{d(N)} \Big(1 + O(u_N^{(1-1/q)d(N)})\Big). \end{aligned} \tag{6.1}$$

Proof. This is a straightforward calculation:

$$\begin{aligned} u &= \frac{1}{e_{\Lambda'}(\omega_1)} = \frac{1}{\varphi_N^{\omega'}\big(e_{\Lambda'}(N^{-1}\omega_1)\big)} = \varphi_N^{\omega'}(u_N^{-1})^{-1} \\ &\overset{(5.4)}{=} \Delta'_N(\omega')^{-1} u_N^{d(N)} \big(1 + f_N(u_N)\big)^{-1} \\ &= \Delta'_N(\omega')^{-1} u_N^{d(N)} \Big(1 + O(u_N^{(1-1/q)d(N)})\Big). \end{aligned}$$

Here the last equality comes from expanding $\big(1+f_N(u_N)\big)^{-1}$ as a geometric series. ☐

Theorem 6.2. *Let $N \in A$ be non-constant. Let $v = (v_1, \dots, v_r) \in V_N \smallsetminus \{0\}$ and write $v_1 = N^{-1}\alpha$ with $\alpha \in A$ satisfying $\deg \alpha < \deg N$ and $v' = (v_2, \dots, v_r)$. Then*

$$(6.2) \quad E_v(\omega) = \frac{u_N^{d(\alpha)}}{\Delta'_\alpha(\omega')\big(1 + f_\alpha(u_N)\big) + e_{\Lambda'}(v'\omega')u_N^{d(\alpha)}} \times$$
$$\prod_{a \in A_+} \frac{\big(1 + f_{aN}(u_N)\big)^{q-1}}{\left[\begin{array}{l} \big(1 + f_{aN}(u_N)\big)^{q-1} - \Delta'_{aN}(\omega')^{1-q}\, u_N^{(q-1)(d(aN)-d(\alpha))} \times \\ \qquad\qquad \big[\Delta'_\alpha(\omega')\big(1 + f_\alpha(u_N)\big) + e_\Lambda(v'\omega')u_N^{d(\alpha)}\big]^{q-1} \end{array}\right]},$$

where we recall that $d(a) = q^{(r-1)\deg(a)}$, and $d(0) := 0$.

In particular,

$$(6.3) \qquad E_v(\omega) = \begin{cases} E'_{v'}(\omega') + O(u_N) & \text{if } \alpha = 0 \\ \Delta'_\alpha(\omega')^{-1} u_N^{d(\alpha)} \Big(1 + O(u_N^{d(\alpha)(1-1/q)})\Big) & \text{if } \alpha \neq 0, \end{cases}$$

where $E'_{v'}(\omega') = e_{\Lambda'}(v'\omega')^{-1}$ is the corresponding rank $r-1$ Eisenstein series.

The weight 1 Eisenstein series E_v are thus modular forms of weight 1 for $\Gamma(N)$.

The expansion (6.3) appears also in [1, Prop. 3.5.6], but the normalization $\omega_r = \bar{\pi}$ and definition of u_N leads to a slightly simpler expression.

Corollary 6.3. *$h(\omega) = (-1)^r h'(\omega')^q u + O(u^q)$. In particular, the product formula (5.6) holds and h is a cusp form.*

Proof. By Theorem 5.2 and Theorem 5.3.2 the product formula (5.6) holds up to a constant factor in $\mathbb{F}_q^*$. In particular, $h(\omega)$ has a u-expansion; we now compute its u_t-expansion. In this computation, we let $V' := (t^{-1}A/A)^{r-1}$

and $S' \subset V'$ the corresponding set of representatives for $(V' \smallsetminus \{0\})/\mathbb{F}_q^* \cong \mathbb{P}^{r-1}(\mathbb{F}_q)$.

$$\begin{aligned} h(\omega) &= -\lambda_t \prod_{v \in S} E_v(\omega) = -\lambda_t \Big(\prod_{\substack{v \in S \\ \alpha = 0}} E_v(\omega) \Big) \Big(\prod_{\substack{v \in S \\ \alpha = 1}} E_v(\omega) \Big) \\ &= -\lambda_t \Big(\prod_{v' \in S'} \big(E'_{v'}(\omega') + O(u_t) \big) \Big) \cdot \Big({\prod_{v' \in V'}}' \big(u_t + O(u_t^2) \big) \Big) \\ &= h'(\omega') u_t^{q^{r-1}} + O(u_t^{q^{r-1}+1}) \\ &\overset{(6.1)}{=} h'(\omega') \Delta'(\omega') u + O(u^q) \\ &= (-1)^r h'(\omega')^q u + O(u^q), \quad \text{since } \Delta'(\omega') = (-1)^r h'(\omega')^{q-1}. \end{aligned}$$

Again, the exponent in $O(u^q)$ comes from (2.5). $\square$

Proof of Theorem 6.2. We generalize the argument leading up to [7, (2.1)].

We break up the product over $a = (a_1, \dots, a_r) \in A^r$ for $e_\Lambda(v\omega)$ into parts where $a_1 = 0$ and $a_1 \neq 0$, respectively:

$$\begin{aligned} e_\Lambda(v\omega) &= v\omega {\prod_{a \in A^r}}' \Big(1 - \frac{v\omega}{a\omega}\Big) \\ &= \underbrace{\Big(v\omega {\prod_{a' \in A^{r-1}}}' \Big(1 - \frac{v\omega}{a'\omega'}\Big) \Big)}_{a_1 = 0} \cdot \Big({\prod_{a_1 \in A}}' \underbrace{\prod_{a' \in A^{r-1}} \Big(1 - \frac{v\omega}{a_1\omega_1 + a'\omega'}\Big) \Big)}_{a_1 \neq 0}. \end{aligned} \tag{6.4}$$

The $a_1 = 0$ part of (6.4) simplifies to

$$\begin{aligned} & v\omega {\prod_{a' \in A^{r-1}}}' \Big(1 - \frac{v\omega}{a'\omega'}\Big) = e_{\Lambda'}(v\omega) \\ &= e_{\Lambda'}(v_1\omega_1 + v'\omega') = e_{\Lambda'}(\alpha N^{-1}\omega_1 + v'\omega') = e_{\Lambda'}(\alpha N^{-1}\omega_1) + e_{\Lambda'}(v'\omega') \\ &= \varphi_\alpha^{\omega'}\big(e_{\Lambda'}(N^{-1}\omega_1)\big) + e_{\Lambda'}(v'\omega') = \varphi_\alpha^{\omega'}(u_N^{-1}) + e_{\Lambda'}(v'\omega') \\ &\overset{(5.4)}{=} \frac{\Delta'_\alpha(\omega')\big(1 + f_\alpha(u_N)\big) + e_{\Lambda'}(v'\omega') u_N^{d(\alpha)}}{u_N^{d(\alpha)}}. \end{aligned} \tag{6.5}$$

Now suppose $a_1 \neq 0$. The function

$$z \mapsto \prod_{a' \in A^{r-1}} \Big(1 - \frac{z}{a_1\omega_1 + a'\omega'}\Big)$$

has a (simple) zero at z precisely when $z - a_1\omega_1 \in \Lambda'$, thus it is proportional to $e_{\Lambda'}(z - a_1\omega_1)$. The constant of proportionality is found by setting $z = 0$, so

$$\prod_{a'\in A^{r-1}}\left(1-\frac{z}{a_1\omega_1+a'\omega'}\right)=\frac{e_{\Lambda'}(z-a_1\omega_1)}{e_{\Lambda'}(-a_1\omega_1)}.$$

Set $a_1=\varepsilon a$ with $a\in A_+$ and $\varepsilon\in\mathbb{F}_q^*$. Then we obtain the following expression for the $a_1\neq 0$ part of (6.4):

$$\begin{aligned}
&\prod_{a'\in A^{r-1}}\left(1-\frac{v\omega}{a_1\omega_1+a'\omega'}\right)=\frac{e_{\Lambda'}(v\omega-a_1\omega_1)}{e_{\Lambda'}(-a_1\omega_1)}=\frac{e_{\Lambda'}(a_1\omega_1-v\omega)}{e_{\Lambda'}(a_1\omega_1)}\\
&=\frac{e_{\Lambda'}\big((a_1-v_1)\omega_1\big)-e_{\Lambda'}(v'\omega')}{e_{\Lambda'}(a_1\omega_1)}=\frac{e_{\Lambda'}\big((a_1N-\alpha)N^{-1}\omega_1\big)-e_{\Lambda'}(v'\omega')}{e_{\Lambda'}(a_1N\cdot N^{-1}\omega_1)}\\
&=\frac{\varphi^{\omega'}_{a_1N-\alpha}\big(e_{\Lambda'}(N^{-1}\omega_1)\big)-e_{\Lambda'}(v'\omega')}{\varphi^{\omega'}_{a_1N}\big(e_{\Lambda'}(N^{-1}\omega_1)\big)}=\frac{\varphi^{\omega'}_{a_1N-\alpha}(u_N^{-1})-e_{\Lambda'}(v'\omega')}{\varphi^{\omega'}_{a_1N}(u_N^{-1})}\\
&=\frac{\varepsilon\varphi^{\omega'}_{aN}(u_N^{-1})-\varphi^{\omega'}_{\alpha}(u_N^{-1})-e_{\Lambda'}(v'\omega')}{\varepsilon\varphi^{\omega'}_{aN}(u_N^{-1})}\\
&=\frac{\varphi^{\omega'}_{aN}(u_N^{-1})-\varepsilon^{-1}\left[\varphi^{\omega'}_{\alpha}(u_N^{-1})+e_{\Lambda'}(v'\omega')\right]}{\varphi^{\omega'}_{aN}(u_N^{-1})}\\
&\overset{(5.4)}{=}\frac{\left[\begin{array}{l}\big(1+f_{aN}(u_N)\big)-\varepsilon^{-1}\Delta'_{aN}(\omega')^{-1}\times\\ \qquad\left[\Delta'_{\alpha}(\omega')u_N^{-d(\alpha)}\big(1+f_{\alpha}(u_N)\big)+e_{\Lambda'}(v'\omega')\right]u_N^{d(aN)}\end{array}\right]}{\big(1+f_{aN}(u_N)\big)}\\
&=\frac{\left[\begin{array}{l}\big(1+f_{aN}(u_N)\big)-\varepsilon^{-1}\Delta'_{aN}(\omega')^{-1}\ u_N^{d(aN)-d(\alpha)}\times\\ \qquad\left[\Delta'_{\alpha}(\omega')\big(1+f_{\alpha}(u_N)\big)+e_{\Lambda'}(v'\omega')u_N^{d(\alpha)}\right]\end{array}\right]}{\big(1+f_{aN}(u_N)\big)}.
\end{aligned}$$

Taking the product of this last expression as ε ranges over $\mathbb{F}_q^*$, we obtain

$$(6.6)\quad \prod_{\varepsilon\in\mathbb{F}_q^*}\prod_{a'\in A^{r-1}}\left(1-\frac{v\omega}{v_1a_1+a'\omega'}\right)=$$

$$\frac{\left[\begin{array}{l}\big(1+f_{aN}(u_N)\big)^{q-1}-\Delta'_{aN}(\omega')^{1-q}\ u_N^{(q-1)(d(aN)-d(\alpha))}\times\\ \qquad\left[\Delta'_{\alpha}(\omega')\big(1+f_{\alpha}(u_N)\big)+e_{\Lambda'}(v'\omega')u_N^{d(\alpha)}\right]^{q-1}\end{array}\right]}{\big(1+f_{aN}(u_N)\big)^{q-1}}.$$

The result follows after plugging (6.5) and (6.6) into (6.4).

Since E_v is holomorphic at infinity for every $v\in V_N\smallsetminus\{0\}$, it follows from Proposition 4.2 that the E_v's are modular forms for $\Gamma(N)$. $\square$

7. A brief history of Drinfeld modular forms in higher rank

Drinfeld modular forms were first defined in David Goss's 1977 Harvard thesis (partially published in [14]), both algebraically in arbitrary rank using Katz's formalism, and analytically in the rank 2 case, where a compactification of the moduli curve was available and a parameter at infinity could be defined. He wrote [18]:

> Barry Mazur saw Deligne lecture on Drinfeld's work [6] in Paris and came back to Harvard and suggested to me circa '75 that the cusps should generate subgroups of finite order in the Jacobian of Drinfeld's modular curves based on the analogy with elliptic modular forms. But Barry stated that he did not know how to define Eisenstein series. I looked at it and the definition just worked (though it took me a while to truly prove that these series were rigid analytic on Ω^2). Then the definitions of modular forms basically wrote themselves. The main thing was to show that they had analytic expansions at the cusps and that took a bit of work.

The rank 2 theory has since developed rapidly, especially in the hands of Goss, Ernst-Ulrich Gekeler and others. In higher rank, the next step came when Mikhail Kapranov [20] constructed a compactification of the moduli variety of Drinfeld $\mathbb{F}_q[t]$-modules with level-N structure, and studied the behaviour of Eisenstein series at the boundary components; in the process he also computed the u_N-expansions of Eisenstein series of level N. This allowed him to sketch a proof of finite-dimensionality of spaces of Drinfeld modular forms of higher rank, as related by Goss in [16, §6.4].

Around the same time, Gekeler investigated Drinfeld modular forms of higher rank, including the form h, but most of this work remained unpublished. In [8] he outlined a compactification of Drinfeld modular varieties in greater generality, although the details were never worked out. That paper contains a parameter at infinity in full generality, and states a result on the order of vanishing of a generalization of Δ at the cusps. The form h appears as a weak modular form in [10, §4] as a Legendre determinant involving Anderson generating functions.

Around the turn of the millenium, Yoshinori Hamahata [19] gave a product formula for the rank r Drinfeld discriminant function in terms of $r-1$ separate parameters.

In [4] expansions are computed for Eisenstein series, coefficient forms and modular functions, using a parameter at infinity for $\mathrm{GL}_r(\mathbb{F}_q[t])$, equivalent to u above, but normalized so that $\Delta'(\omega') = 1$ (this parameter was constructed by Hans-Georg Rück, unaware of [8]). Inspired by this, the second author and Richard Pink initiated a project in 2009, later joined by the first author, to develop the theory of Drinfeld modular forms of arbitrary rank [3]. In [22], Pink constructed a Satake compactification of Drinfeld moduli schemes in great generality (a construction which differs from that outlined in [8]), and defined algebraic modular forms in terms of these. In the case $A = \mathbb{F}_q[t]$ and level-t, Pink's construction is very explicit and leads to a complete description of algebraic modular forms for Γ, where $\Gamma(t) \subset \Gamma \subset \mathrm{GL}_r(\mathbb{F}_q[t])$.

In [3] Drinfeld modular forms are defined analytically in terms of u-expansions (for general A and Γ), and it is shown that such modular forms correspond to Pink's algebraic modular forms. In his thesis [1], the first author also studied Hecke operators on modular forms and the coefficients of u-expansions, obtaining amongst other results a new product formula for Δ [2].

Rudolph Perkins [21] has used the u-expansions from [1] to show that Gekeler's rank r Legendre determinant from [10, §4] is indeed a modular form, and furthermore starts to develop the notion of deformations of vectorial Drinfeld modular forms of higher rank.

Most recently, in [11, 12] Gekeler has written up a construction of Drinfeld modular forms for $\mathrm{GL}_r(\mathbb{F}_q[t])$ and studied in detail the growth order and vanishing of the forms Δ, h and E_v.

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