THE GENERIC MONODROMY OF DRINFELD MODULAR VARIETIES IN SPECIAL CHARACTERISTIC

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Abstract. By combining theorems of Drinfeld and Strauch, we show that the monodromy representation on the special fiber of a Drinfeld modular variety, with level not divisible by the characteristic, is surjective. We illustrate this result in the special case of Drinfeld $\mathbb{F}_q[t]$-modules in level t, and apply this to show that the Kronecker factors of a Drinfeld modular polynomial in rank r are irreducible.

Dedicated to Gerhard Frey on the occasion of his 75^{th} *birthday.*

1. Statement of the main result

Throughout this paper we fix a global function field F of characteristic $p > 0$ with exact field of constants the finite field $\mathbb{F}_q$ of cardinality q. We fix a place ∞ of F, and let A denote the ring of elements of F which are regular away from ∞. This is a Dedekind domain with finite class group $\mathrm{Cl}(A)$ and unit group $A^\times = \mathbb{F}_q^\times$.

Let $I \subset A$ denote a proper non-zero ideal and n_I the order of I in the ideal class group $\mathrm{Cl}(A)$ of A. Let $g_I \in A$ be a generator of I^{n_I}; it is unique up to multiplication by $\mathbb{F}_q^\times$. Hence $A[1/I] := A[1/g_I]$ is independent of the choice of g_I. We also write $\mathrm{Cl}_I(A)$ for the I-class group of A, i.e., the group of fractional ideals of A of support prime to I modulo its subgroup of principal fractional ideals that possess a generator which is congruent to 1 modulo I. One has a short exact sequence $0 \longrightarrow (A/I)^\times/\mathbb{F}_q^\times \longrightarrow \mathrm{Cl}_I(A) \longrightarrow \mathrm{Cl}(A) \longrightarrow 0$.

Let $r \geq 1$ be a positive integer and consider the functor $\mathcal{M}_I^r$ from $A[1/I]$-schemes S to **Sets**, which to any such S assigns the set of isomorphism classes of tuples $(\mathcal{L}, \phi, \alpha)$, where $\mathcal{L}$ is a line bundle on S, where ϕ (together with $\mathcal{L}$) is a Drinfeld A-module $\phi\colon A \longrightarrow \mathrm{End}_{\mathbb{F}_q\text{-gp sch.}/S}(\mathcal{L})$ of rank r, and where α denotes a level I-structure on $(\mathcal{L}, \phi)$, subject to the condition that the characteristic $\partial\phi\colon A \longrightarrow \mathrm{End}_S(\mathrm{Lie}\,\mathcal{L}) = \mathcal{O}_S$ coincides with the structure morphism $S \longrightarrow \mathrm{Spec}\, A[1/I]$ composed with the open immersion $\mathrm{Spec}\, A[1/I] \longrightarrow \mathrm{Spec}\, A$. By [Dri74, Prop. 5.3 and Cor. to 5.4], the functor $\mathcal{M}_I^r$ is representable by a smooth finite type morphism $\mathfrak{M}_I^r \longrightarrow \mathrm{Spec}\, A[1/I]$ of relative dimension $r-1$. The universal Drinfeld module on $\mathfrak{M}_I^r$ we denote by

$$\phi_I^r\colon A \longrightarrow \mathrm{End}_{\mathbb{F}_q\text{-gp sch.}/\mathfrak{M}_I^r}(\mathcal{L}_I^r).$$

Let now $\mathfrak{p} \subset A$ denote a maximal ideal that is prime to I. We write $\kappa_\mathfrak{p}$ for its residue field, $A_\mathfrak{p}$ for the completion of A at $\mathfrak{p}$, and let $\overline{\kappa}_\mathfrak{p}$ be an algebraic closure of $\kappa_\mathfrak{p}$.

Definition 1.1. We call $\mathfrak{M}_{I,\mathfrak{p}}^r := \mathfrak{M}_I^r \times_{\mathrm{Spec}\, A[1/I]} \mathrm{Spec}\, \kappa_\mathfrak{p}$ the *special fiber of* $\mathfrak{M}_I^r$ *at* $\mathfrak{p}$.

Let $\mathfrak{M}^r_{I,\overline{\mathfrak{p}}}$ be the base change $\mathfrak{M}^r_{I,\mathfrak{p}} \times_{\kappa_\mathfrak{p}} \overline{\kappa}_\mathfrak{p}$ and let $\phi^r_{I,\overline{\mathfrak{p}}}$ be the corresponding universal Drinfeld module. The scheme $\mathfrak{M}^r_{I,\overline{\mathfrak{p}}}$ is regular. Its connected components can be naturally labelled by $\mathrm{Cl}_I(A)$: By [Pap06, proof of Cor. 4.6] the connected components of $\mathfrak{M}^r_{I,\overline{\mathfrak{p}}}$ are in bijection with those of $\mathfrak{M}^1_{I,\overline{\mathfrak{p}}} \cong \operatorname{Spec}\big(R_I \otimes_{A[1/I]} \overline{\kappa}_\mathfrak{p}\big)$, where by [Dri74, Thm. 1, §7] R_I is the integral closure of A in the class field of F associated to $\mathrm{Cl}_I(A)$. Now class field theory gives the desired labeling.

For each ideal class $\mathfrak{c}$ in $\mathrm{Cl}_I(A)$, denote by $\eta_\mathfrak{c} = \operatorname{Spec}\kappa_{\eta_\mathfrak{c}}$ the generic point of the corresponding component, and let $\overline{\eta}_\mathfrak{c} = \operatorname{Spec}\overline{\kappa}_{\eta_\mathfrak{c}}$ be a geometric point above $\kappa_{\eta_\mathfrak{c}}$. Observe that $\kappa_{\eta_\mathfrak{c}}$ contains $\overline{\kappa}_\mathfrak{p}$.

Let $\phi^r_{\eta_\mathfrak{c}}$ denote the pullback of $\phi^r_{I,\overline{\mathfrak{p}}}$ to $\eta_\mathfrak{c}$. By [Dri74, Prop. 5.5], it is a Drinfeld A-module of characteristic $\mathfrak{p}$ and height 1, i.e., $\phi^r_{\eta_\mathfrak{c}}$ is ordinary. This means that for any $n \geq 1$ the group of $\mathfrak{p}^n$-torsion points $\phi^r_{\eta_\mathfrak{c}}[\mathfrak{p}^n](\overline{\kappa}_{\eta_\mathfrak{c}})$ is a free $A/\mathfrak{p}^n$-module of rank $r-1$.

Let $g_\mathfrak{p}$ be a generator of the principal ideal $\mathfrak{p}^{n_\mathfrak{p}}$, so that $\phi^r_{\eta_\mathfrak{c}}[\mathfrak{p}^{nn_\mathfrak{p}}](\overline{\kappa}_{\eta_\mathfrak{c}})$ is the set of roots of $\phi^r_{\eta_\mathfrak{c},g^n_\mathfrak{p}}(X)$. We define the $\mathfrak{p}$-adic Tate module of $\phi^r_{\eta_\mathfrak{c}}$ as

$$\mathrm{Ta}_\mathfrak{p}\,\phi^r_{\eta_\mathfrak{c}} = \varprojlim_n \phi^r_{\eta_\mathfrak{c}}[\mathfrak{p}^{nn_\mathfrak{p}}](\overline{\kappa}_{\eta_\mathfrak{c}}),$$

where multiplication by $g_\mathfrak{p}$ defines the transition map in the inverse system. The limit is independent of the choice of $g_\mathfrak{p}$. By ordinariness of $\phi^r_{\eta_\mathfrak{c}}$ it is free of rank $r-1$ over $A_\mathfrak{p}$.

Observe that $\phi^r_{\eta_\mathfrak{c},g^n_\mathfrak{p}}(X) = h_n \circ (X \mapsto X^{q^{n\deg\mathfrak{p}}})$ for some unique $\mathbb{F}_q$-linear polynomial $h_n \in \kappa_{\eta_\mathfrak{c}}[X]$ with non-vanishing linear term. The étale quotient $\phi^r_{\eta_\mathfrak{c}}[\mathfrak{p}^{nn_\mathfrak{p}}]^{\text{ét}}$ of the finite flat A-module scheme $\phi^r_{\eta_\mathfrak{c}}[\mathfrak{p}^{nn_\mathfrak{p}}]$ is $\operatorname{Spec}\kappa_{\eta_\mathfrak{c}}[X]/(h_n(X))$. The group schemes $\phi^r_{\eta_\mathfrak{c}}[\mathfrak{p}^{nn_\mathfrak{p}}]^{\text{ét}}$ also form an inverse system, and one has $\phi^r_{\eta_\mathfrak{c}}[\mathfrak{p}^{nn_\mathfrak{p}}](\overline{\kappa}_{\eta_\mathfrak{c}}) \cong \phi^r_{\eta_\mathfrak{c}}[\mathfrak{p}^{nn_\mathfrak{p}}]^{\text{ét}}(\kappa^{\text{sep}}_{\eta_\mathfrak{c}})$ as finite A-modules. Because the polynomials h_n are defined over $\kappa_{\eta_\mathfrak{c}}$, the absolute Galois group $G_{\kappa_{\eta_\mathfrak{c}}} = \mathrm{Gal}(\kappa^{\text{sep}}_{\eta_\mathfrak{c}}/\kappa_{\eta_\mathfrak{c}})$ acts on $\mathrm{Ta}_\mathfrak{p}\,\phi^r_{\eta_\mathfrak{c}}$ and by the very construction of $\mathrm{Ta}_\mathfrak{p}\,\phi^r_{\eta_\mathfrak{c}}$, the group $G_{\kappa_{\eta_\mathfrak{c}}}$ acts $A_\mathfrak{p}$-linearly. This yields a continuous group homomorphism

$$\rho_{\mathfrak{p},\eta_\mathfrak{c}}\colon G_{\kappa_{\eta_\mathfrak{c}}} \longrightarrow \mathrm{Aut}_{A_\mathfrak{p}}(\mathrm{Ta}_\mathfrak{p}\,\phi^r_{\eta_\mathfrak{c}}) \cong \mathrm{GL}_{r-1}(A_\mathfrak{p}).$$

Our main result is the following:

Theorem 1.2. *The map $\rho_{\mathfrak{p},\eta_\mathfrak{c}}$ is surjective.*

The proof is a simple consequence of the results [Dri74, Str10] by Drinfeld and Strauch, which seems not to have been recorded in the literature. In fact, combining the work of Drinfeld and Strauch, it even follows that the image under $\rho_{\mathfrak{p},\eta_\mathfrak{c}}$ of a decomposition group of $G_{\kappa_{\eta_\mathfrak{c}}}$ at a supersingular point of $\mathfrak{M}^r_{I,\overline{\mathfrak{p}}}$ in the component of $\eta_\mathfrak{c}$ is already surjective; cf. Remark 4.2.

One might wonder about refinements of Theorem 1.2. For any point x of $\mathfrak{M}^r_{I,\overline{\mathfrak{p}}}$ denote by $\rho_{\mathfrak{p},x}\colon G_x \longrightarrow \mathrm{Aut}_\mathfrak{p}(\mathrm{Ta}_\mathfrak{p}\,\phi^r_x)$ the action of the absolute Galois group $G_x = \mathrm{Gal}(\kappa^{\text{sep}}_x/\kappa_x)$ of the residue field at x on the corresponding Tate module $\mathrm{Ta}_\mathfrak{p}\,\phi^r_x$ of rank r_x with $0 \leq r_x \leq r-1$.

Question 1.3. Suppose that $\mathrm{End}_{\overline{\kappa}_x}(\phi^r_x) = A$. Is $\rho_{\mathfrak{p},x}(G_x)$ open in $\mathrm{Aut}_\mathfrak{p}(\mathrm{Ta}_\mathfrak{p}\,\phi^r_x) \cong \mathrm{GL}_{r_x}(A_\mathfrak{p})$?

This appears to be a natural analog of the results [DP12] of Devic and Pink on adelic openness for Drinfeld modules in special characteristic. They consider the Galois action of a Drinfeld A-module ϕ of rank r and characteristic $\mathfrak{p} \neq 0$ over a finitely generated field. If $\mathrm{End}_{\overline{\kappa}_x}(\phi_x) = A$, their results imply that the associated adelic Galois representation of G_x away from $\mathfrak{p}$ and ∞ has open image in $\mathrm{SL}_r(\prod_{v\neq\mathfrak{p}} A_v)$. They also give a complete answer with no condition on $\mathrm{End}_{\overline{\kappa}_x}(\phi_x)$. This leads to.

Question 1.4. Describe for any point x of $\mathfrak{M}^d_{I,\overline{\mathfrak{p}}}$ the Zariski closure $\mathcal{G}_x$ of $\rho_{\mathfrak{p},x}(G_x)$ in $\mathrm{Aut}_{\mathfrak{p}}(\mathrm{Ta}_{\mathfrak{p}}\,\phi^r_x) \cong \mathrm{GL}_{r_x}(A_{\mathfrak{p}})$. Is $\rho_{\mathfrak{p},x}(G_x)$ an open subgroup in $\mathcal{G}_x(A_{\mathfrak{p}})$?

We end this introduction with a quick survey of the content of the individual sections. Section 2 recalls the relevant work of Drinfeld on formal $\mathcal{O}$-modules and $\mathcal{O}$-divisible groups from [Dri74]. Section 3 recalls the main theorem of Strauch, so that in Section 4 we can combine the two and deduce the proof of Theorem 1.2. Section 5 illustrates the main result in the special case of $A = \mathbb{F}_q[t]$ and level t, where the moduli scheme can be described explicitly. In Section 6 we shall answer in Proposition 6.2 a question raised in [BR16] related to the reduction of modular polynomials of level $\mathfrak{p}$ in the case $A = \mathbb{F}_q[t]$. We shall prove that certain special polynomials which are the natural building blocks of the mod $\mathfrak{p}$ reduction of modular polynomials are irreducible as asked in [BR16, Question 4.5].

2. Formal $\mathcal{O}$-modules, $\mathcal{O}$-divisible groups and deformations of Drinfeld modules

Let K be a non-archimedean local field with ring of integers $\mathcal{O}$ and finite residue field k. The normalized valuation on K is v_K, its uniformizer ϖ_K and the cardinality of k will be q_K. Let $\breve{K}$ be the completion of the maximal unramified extension of K and write $\breve{\mathcal{O}}$ for its ring of integers. The residue field $\breve{k}$ of $\breve{\mathcal{O}}$ is an algebraic closure of k. Denote by $\mathrm{CNL}_{\breve{\mathcal{O}}}$ the category of complete noetherian local $\breve{\mathcal{O}}$-algebras C with residue field $\breve{k}$, and with morphisms being $\breve{\mathcal{O}}$-algebra homomorphisms $f\colon C \longrightarrow C'$ such that $f(\mathfrak{m}_C) \subset \mathfrak{m}_{C'}$, where for $C \in \mathrm{CNL}_{\breve{\mathcal{O}}}$ we denote by $\mathfrak{m}_C$ its maximal ideal.

Let B be a ring. The power series ring over B in indeterminates $x_1, . . , x_n$ will be $B[[x_1, . . , x_n]]$.

Definition 2.1 ([Dri74, § 1])**.** A *formal group*[1] over B is a series Φ in $B[[x, y]]$ such that $\Phi(x, y) = \Phi(y, x)$, $\Phi(x, 0) = x$ and $\Phi(\Phi(x, y), z) = \Phi(x, \Phi(y, z))$.

A *homomorphism* from a formal group Φ to a formal group Ψ over B is a series $\beta \in xB[[x]]$ such that $\Psi(\beta(x), \beta(y)) = \beta(\Phi(x, y))$. Composition of homomorphisms is composition of formal power series; it is well-defined because the series have zero constant term.

The endomorphism ring of a formal group Φ is denoted $\mathrm{End}(\Phi)$. It comes with a natural homomorphism $D\colon \mathrm{End}(\Phi) \longrightarrow B$, given by differentiation at zero $\beta \mapsto D\beta = \left(\frac{\mathrm{d}}{\mathrm{d}x}\beta\right)(0)$.

Suppose that B is an $\mathcal{O}$-algebra via a map α. A *formal $\mathcal{O}$-module over B* is a pair $X = (\Phi, [\cdot]_X)$ where Φ is a formal group over B and $[\cdot]_X$ is a homomorphism $\mathcal{O} \longrightarrow \mathrm{End}(\Phi)$ such that $D \circ [\cdot]_X = \alpha$. Morphisms of formal $\mathcal{O}$-modules are defined in the obvious way.

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[1] More correctly we should add the attributes one-dimensional and commutative; but we shall not deal with any other kind of formal group; and so for the sake of brevity we suppress them.

If $\alpha(\varpi_K) = 0$, then $[\varpi_K]_X \in xB[[x]]$ can be written as the composition $\gamma \circ (x \mapsto x^{q_K^h})$ for some unique $\gamma \in xB[[x]]$ with linear term $D\gamma \neq 0$ and a unique $h \geq 1$. One calls h the *height* of the formal $\mathcal{O}$-module $X = (\Phi, [\cdot]_X)$.

Example 2.2 ([Dri74, Rem. after Prop. 2.2], [Ros03, § 4]). Let $s\colon A \longrightarrow \mathcal{O}$ be a ring homomorphism with $s(\mathfrak{p})\mathcal{O} = \varpi_K\mathcal{O}$, and let C be in $\mathrm{CNL}_{\breve{\mathcal{O}}}$. This induces an A-algebra structure on C, which we denote by γ. Let $\phi\colon A \longrightarrow C\{\tau\}, a \mapsto \phi_a$ be a Drinfeld A-module in standard form of rank r and characteristic γ; cf. [Dri74, Rem. after Prop. 5.2]. Let $\Phi(x, y) = x + y$ be the additive formal group law. Then $\mathrm{End}(\Phi) = C\{\{\tau\}\}$, the subring, under addition and composition, of $xC[[x]]$ of power series in the monomials x^{q^i}, $i \geq 0$, with coefficients in C. It can be shown that ϕ extends uniquely to a continuous ring homomorphism $\widehat{\phi}\colon A_{\mathfrak{p}} \longrightarrow \mathrm{End}(\Phi), a \mapsto \widehat{\phi}_a$. This uses that elements in $\mathfrak{p}$ map to topologically nilpotent elements in C under γ. This defines the structure of a formal $A_{\mathfrak{p}}$-module $\widehat{\phi}_{\mathfrak{p}} = (\Phi, \widehat{\phi})$ on Φ. Moreover the height of the formal $A_{\mathfrak{p}}$-module $\widehat{\phi}_{\mathfrak{p}} \pmod{\mathfrak{m}_C}$ agrees with the height of the Drinfeld A-module $\phi \pmod{\mathfrak{m}_C}$.

Let $\breve{k}$ be an $\mathcal{O}$-algebra via reduction, i.e., via the canonical maps $\mathcal{O} \longrightarrow \breve{\mathcal{O}} \longrightarrow \breve{k}$. Let $\overline{X}$ be a formal $\mathcal{O}$-module over $\breve{k}$ of finite height $h > 0$. A *deformation of* $\overline{X}$ *to* $C \in \mathrm{CNL}_{\breve{\mathcal{O}}}$ is a formal $\mathcal{O}$-module X_C over C whose reduction modulo $\mathfrak{m}_C$ is equal to $\overline{X}$. Two deformations X_C and X'_C to C are isomorphic if there exists an isomorphism of formal $\mathcal{O}$-modules over C that reduces to the identity modulo $\mathfrak{m}_C$. Since h is finite, by [Dri74, Prop. 4.1] there is at most one such isomorphism.

Theorem 2.3 ([Dri74, Prop. 4.2])**.** *The functor* $\mathrm{CNL}_{\breve{\mathcal{O}}} \longrightarrow$ **Sets** *that associates to* $C \in \mathrm{CNL}_{\breve{\mathcal{O}}}$ *the set of deformations of* $\overline{X}$ *to* C *up to isomorphism is representable by a ring* $R_{\overline{X}}$ *in* $\mathrm{CNL}_{\breve{\mathcal{O}}}$. *The universal ring* $R_{\overline{X}}$ *is a power series ring over* $\breve{\mathcal{O}}$ *in* $h - 1$ *indeterminates.*

Definition 2.4. The universal formal group over $R_{\overline{X}}$ is denoted by $X_{\overline{X}}$.

To recall the notion of $\mathcal{O}$-divisible module (again of dimension 1), we need some preparations. We fix a ring B in $\mathrm{CNL}_{\breve{\mathcal{O}}}$. Following [Tag93], for R any ring, we define an *R-module scheme* over B to be a pair $(\mathcal{G}, \phi)$, where $\mathcal{G}$ is a commutative group scheme over B and $\phi\colon R \longrightarrow \mathrm{End}(\mathcal{G})$ is a ring homomorphism. A *map* $(\mathcal{G}, \phi) \longrightarrow (\mathcal{G}', \phi')$ *of R-module schemes* is a map $\mathcal{G} \longrightarrow \mathcal{G}'$ of group schemes over B that is equivariant for the R-action.

If $\mathcal{G}$ is finite flat over B, then one can define the étale and connected parts $\mathcal{G}^{\text{ét}}$, $\mathcal{G}^{\text{loc}}$ of $\mathcal{G}$, and one has a short exact sequence $0 \longrightarrow \mathcal{G}^{\text{loc}} \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}^{\text{ét}} \longrightarrow 0$; see [Tat67, 1.4]. Because any endomorphism of $\mathcal{G}$ preserves $\mathcal{G}^{\text{loc}}$, if $\mathcal{G}$ carries an R-action, then the short exact sequence is one of R-module schemes. For the following, we assume that K has positive characteristic. Then the field k is canonically a subring of $\mathcal{O}$.

Definition 2.5 ([Dri74, § 4], [Tag93, § 1])**.** Let r be in $\mathbb{N}$. An *$\mathcal{O}$-divisible module of rank r over B* is an inductive system $\mathcal{F} = (\mathcal{F}_n, i_n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$ the following hold:

(a) $\mathcal{F}_n$ is a finite flat group scheme over B that carries an $\mathcal{O}$-module structure.
(b) There is a closed immersion $\mathcal{F}_n \hookrightarrow \mathbb{G}_{a,B}$ of k-module schemes.[2]

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[2] We restrict to $\mathcal{O}$-modules of dimension 1 and therefore suppress the dimension in the definition.

(c) The order of $\mathcal{F}_n$ over B is q_K^{rn},

(d) The following sequence of $\mathcal{O}$-module schemes over B is exact

$$0 \longrightarrow \mathcal{F}_n \xrightarrow{i_n} \mathcal{F}_{n+1} \xrightarrow{\varpi_K^n} \mathcal{F}_{n+1}.$$

A *morphism of $\mathcal{O}$-divisible modules over* B is a morphism of inductive systems of $\mathcal{O}$-module schemes.

Given an $\mathcal{O}$-divisible module $\mathcal{F} = (\mathcal{F}_n)_{n\geq 1}$, the connected and étale parts $\mathcal{F}^{\mathrm{loc}} = (\mathcal{F}_n^{\mathrm{loc}})_{n\geq 1}$ and $\mathcal{F}^{\text{ét}} = (\mathcal{F}_n^{\text{ét}})_{n\geq 1}$ form $\mathcal{O}$-divisible modules as well and one has a degree-wise short exact sequence of $\mathcal{O}$-divisible modules $0 \longrightarrow \mathcal{F}^{\mathrm{loc}} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}^{\text{ét}} \longrightarrow 0$. If $\mathcal{F}^{\text{ét}} = 0$, we call $\mathcal{F}$ *local.*

Concerning $\mathcal{F}^{\text{ét}}$ note that since $\breve{k}$ is algebraically closed, one has an isomorphism of $\mathcal{O}$-module schemes between $\mathcal{F}_n^{\text{ét}}$ mod $\mathfrak{m}_B$ and the constant $\mathcal{O}$-module scheme $\underline{\mathcal{O}^s/\varpi_K^n\mathcal{O}^s}$ for s the rank of $\mathcal{F}^{\text{ét}}$. Hensel lifting shows that the same isomorphism holds over B. Drinfeld writes $\mathcal{F}^{\text{ét}} = \underline{K^s/\mathcal{O}^s}$.

To analyze $\mathcal{F}^{\mathrm{loc}}$, we present in the following paragraphs, up to and including Proposition 2.6, some results that are implicitly stated in [Dri74, § 4] and are straightforward to deduce from [Tag93, § 1]. Suppose that $\mathcal{F}^{\mathrm{loc}}$ is non-trivial. Then $\varinjlim \mathcal{F}_n^{\mathrm{loc}} \cong \operatorname{Spf} B[[x]]$, and this isomorphism is one of formal k-module schemes, if we identify $\operatorname{Spf} B[[x]]$ with the formal completion of $\mathbb{G}_{a,B}$ at the zero section. The action of $\mathcal{O}$ on $\mathcal{F}^{\mathrm{loc}}$ induces an $\mathcal{O}$-action on the formal additive group over B. The resulting formal $\mathcal{O}$-module will be denoted by $X_{\mathcal{F}}$.

Conversely, let $X = (\Phi, [\cdot]_X)$ be a formal $\mathcal{O}$-module over B whose reduction to $\breve{k}$ has finite height h. Then a local divisible $\mathcal{O}$-module is defined as follows: For $n \in \mathbb{N}$, write $[\varpi_K^n]_X = H_n u_n$ uniquely with $H_n \in B[x]$ monic of degree q_K^{hn} and $H_n \pmod{\mathfrak{m}_B} = x^{q_K^{hn}}$, and $u_n \in B[[x]]$ a unit. Then $X[\varpi_K^n] := \operatorname{Spec} B[x]/(H_n) = \operatorname{Spec} B[[x]]/([\varpi_K^n]_X)$ is a finite flat scheme over B and one can verify for all $n \geq 1$ that

(a) The formal $\mathcal{O}$-module structure $[\cdot]_X$ defines an $\mathcal{O}$-action on $X[\varpi_K^n]$, and in such a way that the closed immersion $X[\varpi_K^n] \hookrightarrow \mathbb{G}_{a,B}$ is one of k-module schemes.

(b) One has a short exact sequence $0 \longrightarrow X[\varpi_K^n] \longrightarrow X[\varpi_K^{n+1}] \overset{\varpi_K}{\longrightarrow} X[\varpi_K^{n+1}]$ of $\mathcal{O}$-module schemes.

The resulting $\mathcal{O}$-divisible local group is denoted by $\mathcal{F}_X$.

Proposition 2.6. *The constructions $X \mapsto \mathcal{F}_X$ and $\mathcal{F} \mapsto X_{\mathcal{F}}$ define mutual inverses between the set of local divisible $\mathcal{O}$-modules $\mathcal{F}$ of rank h and the set of formal $\mathcal{O}$-modules $X = (\Phi, [\cdot]_X)$ such that $x \pmod{\mathfrak{m}_B}$ has height h.*

Example 2.7 ([Dri74, before Prop. 5.4]). Let C, ϕ, $\mathcal{O}$, $\widehat{\phi}$ be as in Example 2.2, and let $g_{\mathfrak{p}}$ be a generator of the ideal $\mathfrak{p}^{n_{\mathfrak{p}}} \subset A$. Let r be the rank of ϕ and h its height. One verifies the following:

(a) For $n \geq 0$, the scheme $\phi[\mathfrak{p}^{nn_{\mathfrak{p}}}] := C[x]/(\phi_{g_{\mathfrak{p}}}^n(x))$ is finite flat over C and possess an $A_{\mathfrak{p}}$-module structure via ϕ.

(b) The sequence $(\phi[\mathfrak{p}^n])_n$ with $\phi[\mathfrak{p}^n] \hookrightarrow \phi[\mathfrak{p}^{n+1}]$ given by inclusion defines a divisible $A_{\mathfrak{p}}$-module $\phi[\mathfrak{p}^\infty]$ over C of height r.

(c) One has an isomorphism $\mathcal{F}^{\mathrm{loc}} \cong (\widehat{\phi}_{\mathfrak{p}}[\mathfrak{p}^n])_{n\geq 1}$.

(d) The rank h of $\mathcal{F}^{\mathrm{loc}}$ is the height of $\phi \pmod{\mathfrak{m}_C}$ over $\breve{k}$, and one has $\mathcal{F}^{\text{ét}} \cong (F_{\mathfrak{p}}/A_{\mathfrak{p}})^{r-h}$.

Let $\mathcal{F}_{\breve{k}}$ be an $\mathcal{O}$-divisible module of rank r over $\breve{k}$. There is an obvious notion of a deformation $\mathcal{F}_{\breve{k}}$ to $\mathcal{O}$-divisible modules over rings C in $\mathrm{CNL}_{\breve{\mathcal{O}}}$ and this defines a functor $\mathrm{CNL}_{\breve{\mathcal{O}}} \longrightarrow \mathbf{Sets}$.

Theorem 2.8 ([Dri74, Prop. 4.5])**.** *Suppose that $\mathcal{F}^{\mathrm{loc}}_{\breve{k}}$ has height $h > 0$. Then the functor $\mathrm{CNL}_{\breve{\mathcal{O}}} \longrightarrow \mathbf{Sets}$ that associates to $C \in \mathrm{CNL}_{\breve{\mathcal{O}}}$ the set of deformations of $\mathcal{F}_{\breve{k}}$ to C up to isomorphism is representable by some ring $R_{\mathcal{F}_{\breve{k}}}$ in $\mathrm{CNL}_{\breve{\mathcal{O}}}$. The universal ring $R_{\mathcal{F}_{\breve{k}}}$ is a power series ring over $\breve{\mathcal{O}}$ in $r-1$ indeterminates.*

From here on, we let $\mathcal{O} := A_{\mathfrak{p}}$ with $\mathfrak{p}$ a closed point of $\operatorname{Spec} A$ as in the introduction. We let $\phi_0 \colon A \longrightarrow \breve{k}\{\tau\}$ be a Drinfeld-module of rank r whose characteristic is given by $A \longrightarrow A_{\mathfrak{p}} = \mathcal{O} \longrightarrow \breve{\mathcal{O}} \longrightarrow \breve{k}$, for our chosen $\breve{\mathcal{O}}$. Let $I \subset A$ be a proper nonzero ideal with $I + \mathfrak{p} = A$. Choosing a level I-structure for ϕ_0, which can be done over $\breve{k}$, defines a point of $\mathfrak{M}^d_{I,\mathfrak{p}}(\breve{k})$ which we denote by x. Then x defines a Drinfeld A-module ϕ_x that is isomorphic to ϕ_0 (over $\breve{k}$) together with a level I-structure. A *deformation of ϕ_0 to $C \in \mathrm{CNL}_{\breve{\mathcal{O}}}$* is a Drinfeld A-module $\phi \colon A \longrightarrow C\{\tau\}$, in standard form, up to isomorphism, which reduces modulo $\mathfrak{m}_C$ to ϕ_0. By Hensel's Lemma, the level I-structure on ϕ_0 extends uniquely to a level I-structure of ϕ over C. Hence one can identify deformations of ϕ_0 with morphisms $\operatorname{Spec} C \longrightarrow \mathfrak{M}^r_I$ that when composed with $\operatorname{Spec} \breve{k} \longrightarrow \operatorname{Spec} C$ yield x. The following is Drinfeld's analog of the Serre-Tate theorem for Drinfeld A-modules.

Theorem 2.9 ([Dri74, 5.C, in part. Prop. 5.4])**.** *The following holds*

(a) *The functor $\mathrm{CNL}_{\breve{\mathcal{O}}} \longrightarrow \mathbf{Sets}$ which associates to $C \in \mathrm{CNL}_{\breve{\mathcal{O}}}$ the set of deformations of ϕ_0 to C is representable by the completion of the stalk of $\mathcal{O}_{\mathfrak{M}^r_I \otimes_{A[1/I]} \breve{\mathcal{O}}}$ at x; in particular, this completion is independent of the choice of I.*

(b) *The natural transformation from deformations of ϕ_0 to deformations of the $\mathcal{O}$-divisible group $\phi_0[\mathfrak{p}^\infty]$ defined in Example 2.7, is an isomorphism. I.e., there is a natural isomorphism of $\breve{\mathcal{O}}$-algebras*

$$R_{\phi_0[\mathfrak{p}^\infty]} \longrightarrow \widehat{\mathcal{O}}_{\mathfrak{M}^r_I \otimes_{A[1/I]} \breve{\mathcal{O}},x}.$$

3. The result of Strauch

Let K, $\mathcal{O}$, k, $\breve{K}$, $\breve{\mathcal{O}}$, $\breve{k}$ and $\mathrm{CNL}_{\breve{\mathcal{O}}}$ be as in the previous section. Let $\overline{X}$ be a formal group over k of height h and let $R_{\overline{X}}$ and $X_{\overline{X}}$ be as in Theorem 2.3. The following is from [Str10, § 1,2]. First one may choose an identification $R_{\overline{X}} \cong \breve{\mathcal{O}}[[u_1, \ldots, u_{h-1}]]$ such that the multiplication by ϖ_K on $X_{\overline{X}}$ is given by a power series $[\varpi_K]_{X_{\overline{X}}} \in R_{\overline{X}}[[x]]$ with the property that for all $i = 0, \ldots, h$ one has

$$[\varpi_K]_{X_{\overline{X}}} \equiv u_i x^{q_K^i} \pmod{(u_0, \ldots, u_{i-1}, x^{q_K^i+1})}, \tag{1}$$

with the conventions $u_0 = \varpi_K$ and $u_h = 1$.

For $m \in \{0, \ldots, h-1\}$ put

$$R_m := \breve{\mathcal{O}}[[u_1, \ldots, u_{h-1}]]/(u_0, \ldots, u_m)$$

with the convention that $R_0 = R_{X_{\overline{X}}}$. Then the closed reduced subscheme of $\operatorname{Spec} R_0$ where the height of the connected component of $X_{\overline{X}}[\varpi_K^\infty]$ is at least m is equal to $\operatorname{Spec} R_m$, and the open part of $\operatorname{Spec} R_m$ where the height of the connected component is equal to m is

$$U_m := \operatorname{Spec} R_m \setminus V(u_m).$$

Let κ_m be the field of fractions of R_m and put $\eta_m = \operatorname{Spec} \kappa_m$. Let $\overline{\kappa}_m$ be an algebraic closure of κ_m and put $\overline{\eta}_m = \operatorname{Spec} \overline{\kappa}_m$. Fix a positive integer n. Denote by

$$\mathrm{Ta}_{X_{\overline{X}},\eta_m} := \varprojlim_n X_{\overline{X}}[\varpi_K^n]_{\eta_m}(\overline{\kappa}_m)$$

the Tate-module of $X_{\overline{X}}$ at η_m. It is a free $\mathcal{O}$-module of rank $h-m$. The absolute Galois group $\pi_1(\eta_m, \overline{\eta}_m)$ of κ_m acts $\mathcal{O}$-linearly on it. We denote the resulting representation by

$$\rho_{X_{\overline{X}},m} \colon \pi_1(\eta_m, \overline{\eta}_m) \longrightarrow \mathrm{Aut}_{\mathcal{O}}(\mathrm{Ta}_{X_{\overline{X}},\eta_m}) \cong \mathrm{GL}_{h-m}(\mathcal{O}).$$

It clearly factors via $\pi_1(U_m, \overline{\eta}_m)$. Then [Str10, Thm. 2.1], asserts:

Theorem 3.1. *For any $m \in \{0, \ldots, h-1\}$ the homomorphism $\rho_{X_{\overline{X}},m}$ is surjective.*

4. Proof of Theorem 1.2

Let the notation be as in Section 1. Set in addition $\mathcal{O} = A_{\mathfrak{p}}$, $K = \operatorname{Frac} \mathcal{O}$, $k = A/\mathfrak{p}$ and take $\breve{K}, \breve{\mathcal{O}}, \breve{k}$ as in Section 2. Let $\xi_{\mathfrak{c}} \in \mathfrak{M}^r_{I,\mathfrak{p}}(\breve{k})$ be a supersingular point in the component of $\mathfrak{M}^r_I$ labelled by $\mathfrak{c}$.[3] Consider the following canonical morphisms of schemes

$$\begin{tikzcd}
\operatorname{Spec} \widehat{\mathcal{O}}_{\mathfrak{M}^r_{I,\overline{\mathfrak{p}}},\xi_{\mathfrak{c}}} \arrow[r] \arrow[rr, bend left, "\widehat{\iota}"] & \operatorname{Spec} \mathcal{O}_{\mathfrak{M}^r_{I,\overline{\mathfrak{p}}},\xi_{\mathfrak{c}}} \arrow[r, "\iota"] & \mathfrak{M}^r_{I,\overline{\mathfrak{p}}},
\end{tikzcd}$$

with $\widehat{\mathcal{O}}_{\mathfrak{M}^r_{I,\overline{\mathfrak{p}}},\xi_{\mathfrak{c}}}$ the completion of the local ring $\mathcal{O}_{\mathfrak{M}^r_{I,\overline{\mathfrak{p}}},\xi_{\mathfrak{c}}}$. Denote by $\widehat{\eta}_{\mathfrak{c}}$ the generic point of $\operatorname{Spec} \widehat{\mathcal{O}}_{\mathfrak{M}^r_{I,\overline{\mathfrak{p}}},\xi_{\mathfrak{c}}}$ and choose a minimal geometric point $\overline{\widehat{\eta}}_{\mathfrak{c}}$ over $\widehat{\eta}_{\mathfrak{c}}$ together with a map $\overline{\widehat{\eta}}_{\mathfrak{c}} \longrightarrow \overline{\eta}_{\mathfrak{c}}$. Let $\mathfrak{M}^{r,\mathrm{ord}}_{I,\overline{\mathfrak{p}}} \subset \mathfrak{M}^r_{I,\overline{\mathfrak{p}}}$ be the locus of ordinary Drinfeld A-modules. It is an open subscheme since its complement is defined by the vanishing of the coefficient of $(\phi^r_{I,\overline{\mathfrak{p}}})_{g_{\mathfrak{p}}} \in M^r_{I,\overline{\mathfrak{p}}}[\tau]$ in degree $\deg g_{\mathfrak{p}}$, where $M^r_{I,\overline{\mathfrak{p}}}$ is the coordinate ring of the affine scheme $\mathfrak{M}^r_{I,\overline{\mathfrak{p}}}$, and hence its complement is closed in $\mathfrak{M}^r_{I,\overline{\mathfrak{p}}}$. We obtain a corresponding diagram of fundamental groups with continuous group homomorphisms

(2)
$$\begin{tikzcd}
\pi_1\big(\widehat{\iota}^{-1}(\mathfrak{M}^{r,\mathrm{ord}}_{I,\overline{\mathfrak{p}}}), \overline{\widehat{\eta}}_{\mathfrak{c}}\big) \arrow[r] & \pi_1\big(\iota^{-1}(\mathfrak{M}^{r,\mathrm{ord}}_{I,\overline{\mathfrak{p}}}), \overline{\eta}_{\mathfrak{c}}\big) \arrow[r] & \pi_1(\mathfrak{M}^{r,\mathrm{ord}}_{I,\overline{\mathfrak{p}}}, \overline{\eta}_{\mathfrak{c}}) \\
\pi_1(\widehat{\eta}_{\mathfrak{c}}, \overline{\widehat{\eta}}_{\mathfrak{c}}) \arrow[u] \arrow[r] & \pi_1(\eta_{\mathfrak{c}}, \overline{\eta}_{\mathfrak{c}}). \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Over $\mathfrak{M}^{r,\mathrm{ord}}_{I,\overline{\mathfrak{p}}}$ the Tate-module

$$\mathrm{Ta}_{\mathfrak{p}}\, \phi^r_{\eta_{\mathfrak{c}}} = \varprojlim_n \phi^r_{\eta_{\mathfrak{c}}}[\mathfrak{p}^{n n_{\mathfrak{p}}}](\overline{\kappa}_{\eta_{\mathfrak{c}}}),$$

is free over $\mathcal{O}$ of rank $r-1$, and we have continuous homomorphisms.

(3)
$$\pi_1(\widehat{\eta}_{\mathfrak{c}}, \overline{\widehat{\eta}}_{\mathfrak{c}}) \longrightarrow \pi_1(\eta_{\mathfrak{c}}, \overline{\eta}_{\mathfrak{c}}) \longrightarrow \pi_1(\mathfrak{M}^{r,\mathrm{ord}}_{I,\overline{\mathfrak{p}}}, \overline{\eta}_{\mathfrak{c}}) \longrightarrow \mathrm{Aut}_{\mathcal{O}}(\mathrm{Ta}_{\mathfrak{p}}\, \phi^r_{\eta_{\mathfrak{c}}}).$$

---

[3] Supersingular points exist; and via the action of Hecke correspondences, which preserves the supersingular locus, they can be seen to lie in every component.

By Theorem 2.9, the ring $\operatorname{Spec}\widehat{\mathcal{O}}_{\mathfrak{M}^r_{I,\overline{\mathfrak{p}}},\xi_{\mathfrak{c}}}$ is naturally identified with the special fiber of the universal deformation ring of the $\mathcal{O}$-divisible module $\phi^r_{\xi_{\mathfrak{c}}}[\mathfrak{p}^\infty]$. Because $\xi_{\mathfrak{c}}$ is supersingular, it is a local $\mathcal{O}$-divisible module of rank r, and thus by Proposition 2.6 it arises from a formal $\mathcal{O}$-module of height r. Now by Theorem 3.1 of Strauch, the composition of the maps in (3) is surjective.

We have thus proved the following result.

Theorem 4.1. *The monodromy representation* $\pi_1(\widehat{\eta}_{\mathfrak{c}}, \overline{\widehat{\eta}}_{\mathfrak{c}}) \longrightarrow \operatorname{Aut}_{\mathcal{O}}(\operatorname{Ta}_{\mathfrak{p}} \phi^r_{\eta_{\mathfrak{c}}})$ *is surjective.*

Hence the map $\pi_1(\eta_{\mathfrak{c}}, \overline{\eta}_{\mathfrak{c}}) \longrightarrow \operatorname{Aut}_{\mathcal{O}}(\operatorname{Ta}_{\mathfrak{p}} \phi^r_{\eta_{\mathfrak{c}}})$ is surjective, as well. This completes the proof of Theorem 1.2.

Remark 4.2. One can think of the image of $\pi_1(\widehat{\eta}_{\mathfrak{c}}, \overline{\widehat{\eta}}_{\mathfrak{c}})$ in $\pi_1(\mathfrak{M}^{r,\mathrm{ord}}_{I,\overline{\mathfrak{p}}}, \overline{\eta}_{\mathfrak{c}})$ as the decomposition group at $\xi_{\mathfrak{c}}$. From this viewpoint, Theorem 3.1 says that already the image of this decomposition group surjects onto $\operatorname{Aut}_{\mathcal{O}}(\operatorname{Ta}_{\mathfrak{p}} \phi^r_{\eta_{\mathfrak{c}}}) \cong \mathrm{GL}_{r-1}(\mathcal{O})$. According to the same theorem, the decomposition groups at points of height $m < r$ map onto a natural subgroup of $\operatorname{Aut}_{\mathcal{O}}(\operatorname{Ta}_{\mathfrak{p}} \phi^r_{\eta_{\mathfrak{c}}})$ isomorphic to $\mathrm{GL}_{m-1}(\mathcal{O})$.

5. An example

For the remainder of this article we specialize to the case $A = \mathbb{F}_q[t]$ and level $I = tA$. We also set $B = A[\frac{1}{t}] = \mathbb{F}_q[t, \frac{1}{t}]$, we let $\mathfrak{p} \in A$ be a non-zero prime (monic irreducible polynomial) and suppose $\mathfrak{p} \neq t$ (otherwise, just replace t by $t+1$), and we write $|\mathfrak{p}| = q^{\deg(\mathfrak{p})}$. Let $\kappa_{\mathfrak{p}} = A/\mathfrak{p}$ with algebraic closure $\bar{\kappa}_{\mathfrak{p}}$. As a preparation for Section 6, in the present section we will work out an explicit example of the main result.

We start by recalling Pink's explicit description of $\mathfrak{M}^r_t$ [PS14, Pin13], see also [Bre16, Theorem 2] for more details: Let V be an $\mathbb{F}_q$-vector space of dimension $r \geq 1$ and write $V' = V \smallsetminus \{0\}$. Denote by $S_V = \operatorname{Sym}_B(V)$ the symmetric algebra of V over B and by K_V the fraction field of S_V. Denote by $RS_{V,0} = B[\frac{v}{v'} \mid v, v \in V']$ the subalgebra of K_V generated over B by quotients of non-zero elements of V. Then the base-change of $\mathfrak{M}^r_t$ to $\operatorname{Spec} B$ is given by

$$\mathfrak{M}^r_{t,B} = \operatorname{Spec} RS_{V,0},$$

which has geometrically irreducible fibres. Furthermore, for any fixed $v_1 \in V'$, the universal Drinfeld module $\phi = \phi^r_\eta$ on $\mathfrak{M}^r_{t,B}$ is determined by

$$t \mapsto \phi_t(X) = tX \prod_{v \in V'} \Big(1 - \frac{v_1}{v} X\Big) \in RS_{V,0}[X]$$

with level structure

$$V \xrightarrow{\sim} \phi[t]; \quad v \mapsto \frac{v}{v_1}.$$

The base change of the moduli scheme $\mathfrak{M}^r_{t,B}$ to $\bar{\kappa}_{\mathfrak{p}}$ is $\mathfrak{M}^r_{t,\overline{\mathfrak{p}}} = \operatorname{Spec}\big(RS_{V,0} \otimes_B \bar{\kappa}_{\mathfrak{p}}[X]\big)$, with universal Drinfeld module the reduction of ϕ modulo $\mathfrak{p}$, which we denote $\overline{\phi}$. We have

$$\overline{\phi}_{\mathfrak{p}^n}(X) = \overline{\phi}^{\text{ét}}_{\mathfrak{p}^n}(X^{|\mathfrak{p}|^n}),$$

where $\overline{\phi}^{\text{ét}}_{\mathfrak{p}^n}(X) \in RS_{V,0} \otimes_B \bar{\kappa}_{\mathfrak{p}}[X]$ is a separable $\mathbb{F}_q$-linear polynomial of degree $|\mathfrak{p}|^{n(r-1)}$. The outer terms in the local-étale decomposition

$$0 \longrightarrow \overline{\phi}[\mathfrak{p}^n]^{\mathrm{loc}} \longrightarrow \overline{\phi}[\mathfrak{p}^n] \longrightarrow \overline{\phi}[\mathfrak{p}^n]^{\text{ét}} \longrightarrow 0$$

are given by

$$\overline{\phi}[\mathfrak{p}^n]^{\mathrm{loc}} = \operatorname{Spec}\big(RS_{V,0} \otimes_B \overline{\kappa}_{\mathfrak{p}}[X]/\langle X^{|\mathfrak{p}|^n}\rangle\big)$$

and

$$\overline{\phi}[\mathfrak{p}^n]^{\text{ét}} = \operatorname{Spec}\big(RS_{V,0} \otimes_B \overline{\kappa}_{\mathfrak{p}}[X]/\langle \overline{\phi}^{\text{ét}}_{\mathfrak{p}^n}(X)\rangle\big).$$

Denote by $\overline{\kappa}_\eta$ the fraction field of $RS_{V,0} \otimes_B \overline{\kappa}_{\mathfrak{p}}$, which is the function field of $\mathfrak{M}^r_{t,\overline{\mathfrak{p}}}$ over $\overline{\kappa}_{\mathfrak{p}}$, and by $\overline{\kappa}_\eta(\overline{\phi}[\mathfrak{p}^n]^{\text{ét}})$ the splitting field of $\overline{\phi}^{\text{ét}}_{\mathfrak{p}^n}(X)$ over $\overline{\kappa}_\eta$.

Now Theorem 1.2 says the following: For every positive integer n, we have

$$\operatorname{Gal}\big(\overline{\kappa}_\eta(\overline{\phi}[\mathfrak{p}^n]^{\text{ét}})/\overline{\kappa}_\eta\big) \cong \mathrm{GL}_{r-1}(A/\mathfrak{p}^n). \tag{4}$$

6. An application

In this last section, we consider a variant of the above example and answer a question posed in [BR16].

Suppose $g_1, g_2, \ldots, g_{r-1}$ are algebraically independent over $\mathbb{F}_q(t)$ and set $L = \mathbb{F}_q(t, g_1, \ldots, g_{r-1})$, a rational function field of transcendence degree r over $\mathbb{F}_q$.

We define the Drinfeld module $\psi : A \longrightarrow L\{\tau\}$ by

$$t \mapsto \psi_t(X) = tX + g_1X^q + \cdots + g_{r-1}X^{q^{r-1}} + X^{q^r} \in L[X].$$

It is shown in [Bre16, Thm. 6] that, for every non-zero proper ideal $\mathfrak{n} \subset A$,

$$\operatorname{Gal}(L(\psi[\mathfrak{n}])/L) \cong \mathrm{GL}_r(A/\mathfrak{n}).$$

Our goal is to prove a similar result in special characteristic.

Denote by $L_t = L(\psi[t])$ the splitting field of $\psi_t(X)$ over L, and set

$$RS_t = B\big[v, \frac{1}{v} \mid 0 \neq v \in \psi[t]\big] \subset L_t,$$

the subalgebra of L_t generated over B by v and $\frac{1}{v}$ for $0 \neq v \in \psi[t]$. It is a graded ring if we set $\deg(v) = 1$ for all $0 \neq v \in \psi[t]$. We have $\psi_t(X) \in RS_t[X]$.

Fix a non-zero t-torsion point $0 \neq v_1 \in \psi[t]$, and consider the isomorphic Drinfeld module $\phi = v_1^{-1}\psi v_1$ over L_t. We denote by $L_{t,0} = L(\phi[t]) \subset L_t$ the splitting field of $\phi_t(X)$ over L, and set

$$RS_{t,0} = B\big[\frac{v}{v'} \mid v, v' \in \psi[t],\ v' \neq 0\big] \subset L_{t,0}.$$

This is the degree zero component of RS_t.

We have $\phi_t(X) = tX\prod_{0\neq v\in\psi[t]}\big(1 - \frac{v_1}{v}X\big) \in RS_{t,0}[X]$.

By [Bre16, Thm. 5] and its proof, there is an isomorphism

$$\theta : \operatorname{Spec}(RS_{t,0}) \xrightarrow{\sim} \mathfrak{M}^r_{t,B}, \tag{5}$$

and ϕ is the pullback via θ of the universal Drinfeld module described in Section 5.

Now consider the reduced Drinfeld modules $\overline{\psi}$ and $\overline{\phi}$ over $RS_t \otimes_B \kappa_{\mathfrak{p}}$ and $RS_{t,0} \otimes_B \kappa_{\mathfrak{p}}$, respectively.

Again, for every positive integer n, we have $\overline{\psi}_{\mathfrak{p}^n}(X) = \overline{\psi}^{\text{ét}}_{\mathfrak{p}^n}(X^{|\mathfrak{p}|^n})$, where $\overline{\psi}^{\text{ét}}_{\mathfrak{p}^n}(X) \in RS_t \otimes_B \overline{\kappa}_{\mathfrak{p}}[X]$ is a separable $\mathbb{F}_q$-linear polynomial of degree $|\mathfrak{p}|^{n(r-1)}$, and

$$\overline{\psi}[\mathfrak{p}^n]^{\text{ét}} = \operatorname{Spec}\big(RS_t \otimes_B \overline{\kappa}_{\mathfrak{p}}[X]/\langle \overline{\psi}^{\text{ét}}_{\mathfrak{p}^n}(X)\rangle\big).$$

Analoguous statements hold for $\overline{\phi}$.

We define the A-field $\ell = \overline{\kappa}_{\mathfrak{p}}(g_1, g_2, \ldots, g_{r-1})$, equipped with the homomorphism $A \longrightarrow A/\mathfrak{p} \subset \overline{\kappa}_{\mathfrak{p}} \subset \ell$. The Drinfeld module $\overline{\psi}$ is defined over ℓ. Denote by $\ell(\overline{\psi}[\mathfrak{p}^n]^{\text{ét}})$ the splitting field of $\overline{\psi}^{\text{ét}}_{\mathfrak{p}^n}(X)$ over ℓ. We have

Theorem 6.1. $\mathrm{Gal}\left(\ell(\overline{\psi}[\mathfrak{p}^n]^{\text{ét}})/\ell\right) \cong \mathrm{GL}_{r-1}(A/\mathfrak{p}^n)$.

Proof. We define the following fields.

$$\begin{aligned} \ell_{t,0} &= \overline{\kappa}_{\mathfrak{p}}(\overline{\phi}[t]) = \mathrm{Frac}\left(RS_{t,0} \otimes_B \overline{\kappa}_{\mathfrak{p}}\right), \\ \ell_t &= \ell(\overline{\psi}[t]) = \ell_{t,0}(v_1) = \mathrm{Frac}\left(RS_t \otimes_B \overline{\kappa}_{\mathfrak{p}}\right), \end{aligned}$$

and consider the following field extensions.

$$\begin{tikzcd}
 & \ell_t(\overline{\psi}[\mathfrak{p}^n]^{\text{ét}}) \arrow[dl, dash] \arrow[dd, dash] \arrow[dr, dash] & \\
\ell_{t,0}(\overline{\phi}[\mathfrak{p}^n]^{\text{ét}}) \arrow[dd, dash, "\mathrm{GL}_{r-1}(A/\mathfrak{p}^n)"'] & & \ell(\overline{\psi}[\mathfrak{p}^n]^{\text{ét}}) \arrow[dd, dash] \\
 & \ell_t \arrow[dl, dash] \arrow[dr, dash] & \\
\ell_{t,0} & & \ell
\end{tikzcd}$$

The field $\ell_{t,0}$ is (isomorphic via θ to) the function field of $\mathfrak{M}^r_{t,\overline{\mathfrak{p}}} = \mathfrak{M}^r_{t,B} \times_{\mathrm{Spec}\, B} \mathrm{Spec}\, \overline{\kappa}_{\mathfrak{p}}$ over $\overline{\kappa}_{\mathfrak{p}}$. It follows from (4) that $\mathrm{Gal}\left(\ell_{t,0}(\overline{\phi}[\mathfrak{p}^n]^{\text{ét}})/\ell_{t,0}\right) \cong \mathrm{GL}_{r-1}(A/\mathfrak{p}^n)$, and our goal is to show that the other two vertical extensions have this same Galois group.

We write

$$\overline{\phi}_t(X) = \bar{t}X + c_1X^q + \cdots + c_rX^{q^r} \in RS_{t,0} \otimes_B \overline{\kappa}_{\mathfrak{p}}[X],$$

where $\bar{t}$ denotes the image of t in $A \longrightarrow A/\mathfrak{p} \subset \overline{\kappa}_{\mathfrak{p}}$, $c_i = v_1^{q^i-1}g_i$ for $i = 1, \ldots, r-1$ and $c_r = v_1^{q^r-1}$. Because 1 is a t-torsion point, we have the algebraic relation $0 = \overline{\phi}_t(1) = \bar{t} + c_1 + c_2 + \cdots + c_r$. Observe also that $\overline{\kappa}_{\mathfrak{p}} \subset \ell_{t,0}$ contains roots of unity of all orders prime to p. It follows that $\ell_t = \ell_{t,0}(v_1) = \ell_{t,0}(\sqrt[q^r-1]{c_r})$ is a Kummer extension of $\ell_{t,0}$.

Let $\xi \in \mathfrak{M}^r_{t,\overline{\mathfrak{p}}}(\overline{\kappa}_{\mathfrak{p}})$ correspond to a supersingular Drinfeld module

$$\phi^{\xi}_t(X) = \bar{t}X + s_1X^q + \cdots + s_rX^{q^r} \in \overline{\kappa}_{\mathfrak{p}}[X].$$

Then the completion $\widehat{\ell_{t,0}}$ of $\ell_{t,0}$ at ξ contains the ring of formal power series

$$\overline{\kappa}_{\mathfrak{p}}[[c_2 - s_2, \cdots, c_r - s_r]].$$

This ring, in turn, contains

$$v_1 = \sqrt[q^r-1]{c_r} = \sqrt[q^r-1]{(c_r - s_r) + s_r} = \sqrt[q^r-1]{s_r} \sum_{i=0}^{\infty} \binom{\frac{1}{q^r-1}}{i} \left(\frac{c_r - s_r}{s_r}\right)^i$$

since $s_r \in \overline{\kappa}^{\times}_{\mathfrak{p}}$ and $q^r - 1$ is not divisible by the characteristic p.

This implies that $\overline{\psi}$ and $\overline{\phi}$ are isomorphic over $\widehat{\ell_{t,0}}$. Also, $\widehat{\ell_{t,0}}$ contains ℓ_t and $\widehat{\ell_{t,0}}(\overline{\phi}[\mathfrak{p}^n]^{\text{ét}}) = \widehat{\ell_{t,0}}(\overline{\psi}[\mathfrak{p}^n]^{\text{ét}})$.

Theorem 4.1 implies that the Galois representation

$$\mathrm{Gal}\left(\widehat{\ell_{t,0}}^{\mathrm{sep}}/\widehat{\ell_{t,0}}\right) \longrightarrow \mathrm{Ta}_{\mathfrak{p}}\,\overline{\phi}$$

is surjective. Since $\overline{\phi}$ and $\overline{\psi}$ are isomorphic over $\widehat{\ell_{t,0}}$, the same holds for $\mathrm{Ta}_{\mathfrak{p}}\,\overline{\psi}$, and in particular the Galois representation

$$\mathrm{Gal}\left(\ell_t^{\mathrm{sep}}/\ell_t\right) \longrightarrow \mathrm{Ta}_{\mathfrak{p}}\,\overline{\psi}$$

is surjective. This implies that

$$\mathrm{Gal}\left(\ell_t(\overline{\psi}[\mathfrak{p}^n]^{\text{ét}})/\ell_t\right) \cong \mathrm{GL}_{r-1}(A/\mathfrak{p}^n).$$

Finally, we have

$$[\ell(\overline{\psi}[\mathfrak{p}^n]^{\text{ét}}) : \ell] \geq [\ell_t(\overline{\psi}[\mathfrak{p}^n]^{\text{ét}}) : \ell_t] = \#\,\mathrm{GL}_{r-1}(A/\mathfrak{p}^n).$$

Since $\mathrm{Gal}\left(\ell(\overline{\psi}[\mathfrak{p}^n]^{\text{ét}})/\ell\right)$ is isomorphic to a subgroup of $\mathrm{GL}_{r-1}(A/\mathfrak{p}^n)$, it must be isomorphic to the whole group. This completes the proof of Theorem 6.1. □

Finally, we address [BR16, Question 4.5]. For this, we must first recall the construction of Drinfeld modular polynomials from [BR16].

Denote by C the subring of $A[g_1, \ldots, g_{r-1}] \subset L$ generated by monomials of the form $ag_1^{e_1} \cdots g_{r-1}^{e_{r-1}}$ satisfying $a \in A$ and $\sum_{k=1}^{r-1} e_k(q^k - 1) \equiv 0 \bmod q^r - 1$. Then the elements of C are the isomorphism invariants of rank r Drinfeld A-modules, i.e. $\mathrm{Spec}\, C$ is the coarse moduli scheme of Drinfeld modules of rank r and no level structure, see [BR16, Prop. 1.1].

Let $1 \leq s \leq r-1$. An isogeny $f : \psi \longrightarrow \psi^{(f)}$ is said to have *type* $(A/\mathfrak{p})^s$ if $\ker f(\overline{L}) \cong (A/\mathfrak{p})^s$, and such an isogeny is called *special* if $\ker f$ contains $U_0 := \ker\left(\psi[\mathfrak{p}](\overline{L}) \longrightarrow \overline{\psi}[\mathfrak{p}](\overline{\ell})\right)$. Because ψ has ordinary reduction $\bar{\psi}$, and so $U_0 \cong A/\mathfrak{p}$, f is special if and only if its reduction is inseparable.

To each invariant $J \in C$ we associate the *Drinfeld modular polynomial of type* $(A/\mathfrak{p})^s$, defined by

$$\Phi_{J,(A/\mathfrak{p})^s}(X) = \prod_{f\,:\,\psi \longrightarrow \psi^{(f)} \text{ of type } (A/\mathfrak{p})^s} \left(X - J(\psi^{(f)})\right) \in C[X].$$

This is irreducible over L if its roots in $\overline{L}$ are distinct (there always exist $J \in C$ for which the roots are distinct).

Modulo $\mathfrak{p}$, we have the Kronecker congruence relation [BR16, Thm. 4.4]:

$$\Phi_{J,(A/\mathfrak{p})^s}(X) \equiv \Phi^{\mathrm{spec}}_{J,(A/\mathfrak{p})^s}(X) \cdot \left(\Phi^{\mathrm{spec}}_{J,(A/\mathfrak{p})^{s+1}}(X^{|\mathfrak{p}|})\right)^{|\mathfrak{p}|^{s-1}} \bmod \mathfrak{p}, \tag{6}$$

where

$$\Phi^{\mathrm{spec}}_{J,(A/\mathfrak{p})^s}(X) := \prod_{f\,:\,\psi \longrightarrow \psi^{(f)} \text{ special of type } (A/\mathfrak{p})^s} \left(X - (J(\psi^{(f)}) \bmod \mathfrak{p})\right) \in \ell[X].$$

We answer [BR16, Question 4.5] in the affirmative, as follows.

Proposition 6.2. *Suppose $J \in C$ is such that the roots of $\Phi^{\mathrm{spec}}_{J,(A/\mathfrak{p})^s}(X)$ in $\overline{\ell}$ are distinct. Then $\Phi^{\mathrm{spec}}_{J,(A/\mathfrak{p})^s}(X) \in \ell[X]$ is irreducible.*

Proof. If $s = 1$, then $\Phi^{\mathrm{spec}}_{J,(A/\mathfrak{p})}(X) = X - J^{|\mathfrak{p}|}$ by [BR16, Example 5.1], and we are done.

Now suppose that $s > 1$. Let R be the integral closure of $A[g_1, g_2, \ldots, g_{r-1}]$ in $L(\psi[\mathfrak{p}])$, then $\psi[\mathfrak{p}] \subset R$. Let $f : \phi \longrightarrow \phi^{(f)}$ be a special isogeny of type $(A/\mathfrak{p})^s$.

Then $f(X) \in R[X]$ is an $\mathbb{F}_q$-linear polynomial, and $f(X) \equiv f^{\text{ét}}(X^{|\mathfrak{p}|}) \bmod \mathfrak{p}$, where $f^{\text{ét}}(X) \in R \otimes_A \kappa_{\mathfrak{p}}[X]$ is separable and $\ker f^{\text{ét}}$ is an A-submodule of $\overline{\psi}[\mathfrak{p}]^{\text{ét}}$ isomorphic to $(A/\mathfrak{p})^{s-1}$.

By Theorem 6.1 $\mathrm{Gal}(\ell^{\text{sep}}/\ell)$ acts transitively on the set of such submodules of $\overline{\psi}[\mathfrak{p}]^{\text{ét}}$, and thus also on the set of Drinfeld modules $\overline{\psi}^{(f)}$. Because J maps different $\overline{\psi}^{(f)}$ to different elements of $\overline{\ell}$, the group $\mathrm{Gal}(\ell^{\text{sep}}/\ell)$, in turn, acts transitively on the roots of $\Phi^{\text{spec}}_{J,(A/\mathfrak{p})^s}(X)$. $\square$

Remark 6.3. Equation (6) thus describes the decomposition of the Hecke correspondence associated to $(A/\mathfrak{p})^s$-isogenies on $\mathfrak{M}^r_{I,\overline{\mathfrak{p}}}$ into irreducible components with multiplicities.

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