# I Prefer Pi: A Brief History and Anthology of Articles in the American Mathematical Monthly

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**Abstract.** In celebration of both a special 'big' $\pi$ Day (3/14/15) and the 2015 centennial of the Mathematical Association of America, we review the illustrious history of the constant $\pi$ in the pages of the American Mathematical Monthly.

**1. INTRODUCTION.** Once in a century Pi Day is accurate not just to three digits, but to five. The year the MAA was founded (1915) was such a year, and so is the MAA's centennial year (2015). To arrive at this auspicious conclusion, we consider the date to be given as month-day-two digit year.[1] This year Pi Day turns 26. For a more detailed discussion of Pi and its history, we refer to last year's article [46]. We do note that "I prefer pi" is a succinct palindrome.[2]

In honour of this happy coincidence, we have gone back and *selected* roughly seventy five representative papers relating to Pi (the constant not the symbol) published in this journal since its inception in 1894 (which predates that of the MAA itself). Those 75 papers listed in three periods (before 1945, 1945–1989, and 1990 on) form the core bibliography of this article. The first author and three undergraduate research students[3] ran a seminar in which they looked at the 75 papers. Here is what they discovered.

**Common themes** In each of the three periods one observes both the commonality of topics and the changing stye of presentation. We shall say more about this as we proceed.

- We see authors of varying notoriety. Many are top-tier research mathematicians whose names remain known. Others once famous are unknown. Articles come from small colleges, big ten universities, ivy league schools and everywhere else. In earlier days, articles came from people at big industrial labs, but nowadays, those labs no longer support research as they used to.
- These papers cover relatively few topics.
  - Every few years a 'simple proof' of the irrationality of $\pi$ is published. Such proofs can be found in [$^\star$58, 26, 29, 31, 39, 52, 59, 62, 76].
  - Many proofs of $\zeta(2) := \sum_{n\geq 1} 1/n^2 = \pi^2/6$ appear, each trying to be a bit more slick or elementary than the last. Of course, whether you prefer your proofs concise and high tech, or more leisurely and lower tech, is a matter of taste and context. See [$^\star$38, $^\star$58, 20, 28, 34, 42, 57, 68, 69].
  - Articles on mathematics outside the European tradition have appeared since the Monthly's earliest days. See the papers [3, 9, 11, 15].
- In the past thirty years, computer algebra begins to enter the discussions – sometimes in a fundamental way.

[1] For advocates of $\tau = 2\pi$, your big day 6/28/31 will come in 2031.

[2] Given by the Professor in Yōko Ozawa, *The Housekeeper and the Professor*, Picador Books, 2003. Kindle location 1095, as is "a nut for a jar of tuna?"

[3] The students are Elliot Catt from Newcastle, and Ghislain McKay & Corey Sinnamon from Waterloo.

- Of course the compositing style of the MONTHLY has changed several times.
- The process of constructing this selection highlights how much our scholarly life has changed over the past 30 years. Much more can be found and studied easily, but there is even more to find than in previous periods. The ease of finding papers in Google Scholar has the perverse consequence – like Gresham's law in economics – of making less easily accessible material even more likely to be ignored.

While our list is not completely exhaustive, almost every paper listed in the bibliography has been cited in the literature. In fact, several have been highly cited. Some highly used research, such as Ivan Niven's proof of the irrationality of $\pi$ [14], is rarely cited as it has been fully absorbed into the literature. Indeed, a quick look at the AMS's Mathematical Reviews reveals only 15 citations of Niven's paper.

We deem as pi-star (or $\pi^\star$) papers from our MONTHLY bibliography that have been cited in the literature more than 30 times. The existence of JSTOR means that most readers can access all these papers easily, but we have arranged for the $\pi^\star$s to be available free for the next year on our website (`http://www.shsu.edu/ bks006/Monthly.html`). Here are the $\pi^\star$s with citation numbers according to Google Scholar (as of 1/7/2015). These papers are marked with a $\star$ in the regular bibliography.

1. 133 citations: J. M. Borwein, P. B. Borwein, D. H. Bailey, Ramanujan, modular equations, and approximations to pi or how to compute one billion digits of pi, **96**(1989) 201–219.
2. 119 citations: G. Almkvist, B. Berndt, Gauss, Landen, Ramanujan, the arithmetic-geometric mean, ellipses, $\pi$, and the ladies diary, **95**(1988) 585–608.
3. 73 citations: A Kufner, L Maligrand, The prehistory of the Hardy inequality, **113**(2006) 715–732.
4. 63 citations: J.M. Borwein, P.B. Borwein, K. Dilcher, Pi, Euler numbers, and asymptotic expansions, **96**(1989) 681–687.
5. 56 citations: N.D. Baruah, B.C. Berndt, H.H. Chan, Ramanujan's series for $1/\pi$: a survey, **116**(2009) 567–587.
6. 40 citations: J. Sondow, Double integrals for Euler's constant and $\ln \pi/4$ and an analog of Hadjicostas's formula, **112**(2005) 61–65.
7. 39 citations: D. H. Lehmer, On arccotangent relations for $\pi$, **45**(1938) 657–664.
8. 39 citations: I. Papadimitriou, A simple proof of the formula $\sum_{k=1}^{\infty} 1/k^2 = \pi^2/6$, **80**(1973) 424–425.
9. 36 citations: V. Adamchik, S. Wagon, A simple formula for $\pi$, **104**(1997) 852–855.
10. 35 citations: D. Huylebrouck, Similarities in irrationality proofs for $\pi$, $\ln 2$, $\zeta(2)$, and $\xi(3)$, **108**(2001) 222–231.
11. 35 citations: L. J. Lange, An elegant continued fraction for $\pi$, **106**(1999) 456–458.
12. 33 citations: S. Rabinowitz, S. Wagon, A spigot algorithm for the digits of $\pi$, **102**(1995) 195–203.
13. 32 citations: W. S. Brown, Rational exponential expressions and a conjecture concerning $\pi$ and $e$, **76**(1969) 28–34.

**The remainder of this article.** We begin with a very brief history of Pi, both mathematical and algorithmic, which can be followed in more detail in [80] and [46]. We then turn to our three periods, and make a very few extra comments about some of the articles. For the most part the title of each article is a pretty good abstract. We then make a few summatory remarks and list a handful of references from outside the MONTHLY, such as David Blattner's *Joy of Pi* [79] and Arndt and Haenel's *Pi Unleashed* [78].

**2. PI: A BRIEF HISTORY.** Pi is arguably the most resilient of mathematical objects. It has been studied seriously over many millennia and by every major culture, remaining as intensely examined today as in the Syracuse of Archimedes' time. Its role in popular culture was described in last year's Pi Day article [46]. We also recall the recent movies *Life of Pi* ((2012, PG) directed by Ang Lee) and *Pi* ((1998, R) directed by Darren Aronofsky)[4].

From both an analytic and computational viewpoint, it makes sense to begin with Archimedes. Around 250 BCE, Archimedes of Syracuse (287–212 BCE) is thought to have been the first (in *Measurement of the Circle*) to show that the "two possible Pi's" are the same. For a circle of radius $r$ and diameter $d$, *Area*$= \pi_1 r^2$ while *Perimeter* $= \pi_2 d$, but that $\pi_1 = \pi_2$ is not obvious, and is often overlooked; see [55].

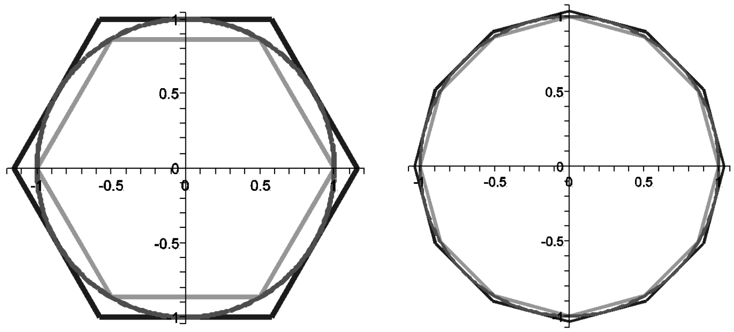


**Figure 1.** Archimedes' method of computing $\pi$ with 6- and 12-gons

**Archimedes' Method** The first rigorous mathematical calculation of $\pi$ was also due to Archimedes, who used a brilliant scheme based on *doubling inscribed and circumscribed polygons*,

$$6 \mapsto 12 \mapsto 24 \mapsto 48 \mapsto 96$$

and computing the perimeters to obtain the bounds $3\frac{10}{71} < \pi < 3\frac{10}{70} = \ldots$.[5] The case of 6-gons and 12-gons is shown in Figure 1; for $n = 48$ one already 'sees' near-circles. No computational mathematics approached this level of rigour again until the 19th century. Phillips in [41] or [80, pp. 15-19] calls Archimedes the 'first numerical analyst'.

Archimedes' scheme constitutes the first true algorithm for $\pi$, in that it can produce an arbitrarily accurate value for $\pi$. It also represents the birth of numerical and error analysis – all without positional notation or modern trigonometry. As discovered in the 19th century, this scheme can be stated as a simple, numerically stable, recursion, as follows [82].

**Archimedean Mean Iteration (Pfaff-Borchardt-Schwab).** Set $a_0 = 2\sqrt{3}$ and $b_0 = 3$, which are the values for circumscribed and inscribed 6-gons. If

$$a_{n+1} = \frac{2a_n b_n}{a_n + b_n} \qquad (H) \qquad \text{and} \qquad b_{n+1} = \sqrt{a_{n+1} b_n} \qquad (G), \tag{1}$$

[4] Imagine, an R–rated movie involving Pi!

[5] All rules are meant to be broken. Writing $10/70$ without cancellation makes it easier to see that $1/7$ is larger than $10/71$.

then $a_n$ and $b_n$ converge to $\pi$, with the error decreasing by a factor of four with each iteration. In this case the error is easy to estimate – look at $a_{n+1}^2 - b_{n+1}^2$ – and the limit is somewhat less accessible, but still reasonably easy to determine [82].

Variations of Archimedes' geometrical scheme were the basis for all high-accuracy calculations of $\pi$ over the next 1800 years – far after its 'best before' date. For example, in fifth century China, Tsu Chung-Chih used a variant of this method to obtain $\pi$ correct to seven digits. A millennium later, al-Kāshī in Samarkand "*who could calculate as eagles can fly*" obtained $2\pi$ in *sexadecimal*:

$$2\pi \approx 6 + \frac{16}{60^1} + \frac{59}{60^2} + \frac{28}{60^3} + \frac{01}{60^4} + \frac{34}{60^5} + \frac{51}{60^6} + \frac{46}{60^7} + \frac{14}{60^8} + \frac{50}{60^9},$$

good to 16 decimal places (using $3 \cdot 2^{28}$-gons). This is a personal favourite; reentering it in a computer centuries later and getting the predicted answer gives the authors horripilation ('goose-bumps').

Pi's centrality is emphasised by the many ways it turns up early in new subjects from irrationality theory to probability and harmonic analysis. For instance, Francois Viéta's (1540–1603) formula

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2}\frac{\sqrt{2+\sqrt{2}}}{2}\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots \tag{2}$$

and John Wallis' (1616–1703) infinite product [67, 74, 75]

$$\frac{\pi}{2} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9}\cdots \tag{3}$$

are accounted among the first infinitary objects in mathematics. The latter leads to the Gamma function, Stirling's formula, and much more [64] including the *first infinite continued fraction*[6] for $2/\pi$ by Lord Brouncker (1620–1684), first President of the Royal Society of London:

$$\frac{2}{\pi} = \frac{1}{1+}\,\frac{9}{2+}\,\frac{25}{2+}\,\frac{49}{2\cdots}. \tag{4}$$

Here we use the modern concise notation for a continued fraction.

**Arctangents and Machin formulas** With the development of calculus, it became possible to extend calculations of $\pi$ dramatically as shown in Figure 4. Almost all calculations between 1700 and 1980 reduce to exploiting the series for the arctangent (or another inverse trig function) and using identities to require computation only near the centre of the interval of convergence. Thus, one starts with

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad \text{for } -1 \le x \le 1 \tag{5}$$

and $\arctan(1) = \pi/4$. Substituting $x = 1$ proves the *Gregory-Leibniz formula* (1671–74)

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots. \tag{6}$$

[6] This was discovered without proof as was (3) .

James Gregory (1638–75) was the greatest of a large Scottish mathematical family. The point $x = 1$, however, is on the boundary of the interval of convergence of the series. Justifying substitution requires a careful error estimate for the remainder or Lebesgue's monotone convergence theorem, but most introductory calculus texts ignore the issue. The $\arctan$ integral and series were known centuries earlier to the Kerala school, which was identified with Madhava (c. 1350 – c. 1425) of Sangamagrama near Kerala, India. Madhava may well have computed 13 digits of $\pi$.

To make (5) computationally feasible, we can use one of many formulas such as:

$$\arctan(1) = 2\arctan\left(\frac{1}{3}\right) + \arctan\left(\frac{1}{7}\right) \quad \text{(Hutton)} \tag{7}$$

$$\arctan(1) = \arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{5}\right) + \arctan\left(\frac{1}{8}\right) \quad \text{(Euler)} \tag{8}$$

$$\arctan(1) = 4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right) \quad \text{(Machin).} \tag{9}$$

All of this, including the efficiency of different *Machin formulas* as they are now called, is lucidly described by the early and distinguished computational number theorist D.H. Lehmer [⋆13]. See also [2,5,49] and [19] by Wrench, who in 1961 with Dan Shanks performed extended computer computation of $\pi$ using these formulas; see Figure 5.

In [⋆13] Lehmer gives what he considered to be a best possible self-checking pair of arctan relations for computing $\pi$. The pair was:

$$\arctan(1) = 8\arctan\left(\frac{1}{10}\right) - \arctan\left(\frac{1}{239}\right) - 4\arctan\left(\frac{1}{515}\right) \tag{10}$$

$$\arctan(1) = 12\arctan\left(\frac{1}{18}\right) + 8\arctan\left(\frac{1}{57}\right) - 5\arctan\left(\frac{1}{239}\right). \tag{11}$$

In [2], Ballantine shows that this pair makes a good choice since the series for $\arctan(1/18)$ and $\arctan(1/57)$ have terms that differ by a constant factor of '0,' a decimal shift. This observation was implemented in both the 1961 and 1973 computations listed in Figure 4.

**Mathematical landmarks in the life of Pi.** The irrationality of $\pi$ was first shown by Lambert in 1761 using continued fractions [⋆63]. This is a good idea since a number $\alpha$ has an eventually repeating non-terminating simple continued fraction if and only if $\alpha^2$ is rational, as made rigorous in 1794 by Legendre. Legendre conjectured that $\pi$ is non algebraic[7], that is, that $\pi$ is *transcendental*. Unfortunately all the pretty continued fractions for $\pi$ are not simple [⋆63, 70, 83]. In [⋆63] Lange examines various proofs of

$$\pi = 3 + \frac{1^2}{2\,+}\,\frac{3^2}{2\,+}\,\frac{5^2}{2\,+}\,\frac{7^2}{2\cdots}. \tag{12}$$

[7] It can be argued that he was anticipated by Maimonides (the Rambam, 1135–1204) [81].

Legendre was validated when in 1882 Lindemann proved $\pi$ transcendental. He did this by extending Hermite's 1873 proof of the transcendence of $e$. There followed a spate of simplifications by Weierstrass in 1885, Hilbert in 1893, and many others. Oswald Veblen's article [18], written only ten years later, is a lucid description of the topic by one of the leaders of the early 20th century American mathematical community.[8] A 1939 proof of the transcendence of $\pi$ by Ivan Niven [14] is reproduced exactly in Appendix A since it remains entirely appropriate for a class today.

We next reproduce our personal favorite MONTHLY proof of the irrationality of $\pi$. All such proofs eventually arrive at a putative integer that must lie strictly between zero and one.

**Theorem 1 (Breusch [26]).** *$\pi$ is irrational.*

*Proof.* Assume $\pi = a/b$ with $a$ and $b$ integers. Then, with $N = 2a$, $\sin N = 0$, $\cos N = 1$, and $\cos(N/2) = \pm 1$. If $m$ is zero or a positive integer, then

$$A_m(x) \equiv \sum_{k=0}^{\infty}(-1)^k(2k+1)^m \frac{x^{2k+1}}{(2k+1)!} = P_m(x)\,\cos x + Q_m(x)\,\sin x$$

where $P_m(x)$ and $Q_m(x)$ are polynomials in $x$ with integral coefficients. (The proof follows by induction on $m : A_{m+1} = x dA_m/dx$, and $A_0 = \sin x$.) Thus $A_m(N)$ is an integer for every positive integer $m$.

If $t$ is any positive integer, then

$$\begin{aligned} B_t(N) &\equiv \sum_{k=0}^{\infty}(-1)^k \frac{(2k+1-t-1)(2k+1-t-2)\cdots(2k+1-2t)}{(2k+1)!} N^{2k+1} \\ &= \sum_{k=0}^{\infty}(-1)^k \frac{(2k+1)^t - b_1(2k+1)^{t-1} + \cdots \pm b_t}{(2k+1)!} N^{2k+1} \\ &= A_t(N) - b_1 A_{t-1}(N) + \cdots \pm b_t A_0(N). \end{aligned}$$

Since all the $b_i$ are integers, $B_t(N)$ must be an integer too. Break the sum for $B_t(N)$ into the three pieces

$$\sum_{k=0}^{[(t-1)/2]}, \quad \sum_{k=[(t+1)/2]}^{t-1}, \text{ and } \sum_{k=t}^{\infty}.$$

In the first sum, the numerator of each fraction is a product of $t$ consecutive integers, therefore it is divisible by $t!$, and hence by $(2k+1)!$ since $2k+1 \le t$. Thus each term of the first sum is an integer. Each term of the second sum is zero. Thus the third sum must be an integer, for every positive integer $t$.

This third sum is

$$\sum_{k=t}^{\infty}(-1)^k \frac{(2k-t)!}{(2k+1)!(2k-2t)!} N^{2k+1}$$

[8]He was also nephew of Thorstein Veblen, one of the founders of sociology and originator of the term 'conspicuous consumption.'

$$= (-1)^t \frac{t!}{(2t+1)!} N^{2k+1} \left(1 - \frac{(t+1)(t+2)}{(2t+2)(2t+3)} \frac{N^2}{2!} + \frac{(t+1)(t+2)(t+3)(t+4)}{(2t+2)(2t+3)(2t+4)(2t+5)} \frac{N^4}{4!} - \cdots \right).$$

Let $S(t)$ stand for the sum in the parenthesis. Certainly

$$|S(t)| < 1 + N + \frac{N^2}{2!} + \cdots = e^N.$$

Thus the whole expression is absolutely less than

$$\frac{t!}{(2t+1)!} N^{2t+1} e^N < \frac{N^{2t+1}}{t^{t+1}} e^N < (N^2/t)^{t+1} e^N,$$

which is less than 1 for $t > t_0$.

Therefore $S(t) = 0$ for every integer $t > t_0$. But this is impossible, because

$$\lim_{t\to\infty} S(t) = 1 - \frac{1}{2^2} \cdot \frac{N^2}{2!} + \frac{1}{2^4} \cdot \frac{N^4}{4!} - \cdots = \cos(N/2) = \pm 1. \qquad \blacksquare$$

A similar argument shows that the natural logarithm of a rational number must be irrational. From $\log(a/b) = c/d$ would follow that $e^c = a^d/b^d = A/B$. Then

$$B \cdot \sum_{k=0}^{\infty} \frac{(k-t-1)(k-t-2)\cdots(k-2t)}{k!} c^k$$

would have to be an integer for every positive integer $t$, which leads to a contradiction.

Irrationality measures, denoted $\mu(\alpha)$, as described in [83] seem not to have seen much attention in the MONTHLY. The *irrationality measure* of a real number is the infimum over $\mu > 0$ such that the inequality

$$\left|\alpha - \frac{p}{q}\right| \le \frac{1}{q^\mu}$$

has at most finitely many solutions in $p \in Z$ and $q \in N$. Currently, the best irrationality measure known for $\pi$ is $7.6063$. For $\pi^2$ it is $5.095412$, and for $\log 2$ it is $3.57455391$. For every rational number the irrationality measure is $1$ and the Thue-Siegel-Roth theorem states that if $\alpha$ is a real algebraic irrational then $\mu(\alpha) = 2$. Indeed, almost all real numbers have an irrationality measure of 2 and transcendental numbers have irrationality measure 2 or greater. For example, the transcendental number $e$ has $\mu(e) = 2$ while *Liouville numbers* such as $\sum_{n\ge 0} 1/10^{n!}$ are precisely those numbers having infinite irrationality measure. The fact that $\mu(\pi) < \infty$ (equivalently $\pi$ is not a Louisville number) was first proved by Mahler [85] in 1953.[9] This fact does figure in the solution of many MONTHLY problems over the years; for instance, it lets one estimate how far $\sin(n)$ is from zero.

[9] He showed $\mu(\pi) \le 42$. Douglas Adams would be pleased. The entire Mahler Archive is on line at `http://carma.newcastle.edu.au/mahler/`.

The *Riemann zeta* function[10] is defined for $s > 1$ by $\zeta(s) = \sum_{n\geq 1} 1/n^s$. The *Basel problem*, first posed by Pietro Mengoli in 1644, which asked for the evaluation of $\zeta(2) = \sum_{n\geq 1} 1/n^2$, was popularized by the Bernoullis, who came from Basel in Switzerland, and hence the name. In 1735, all even values of $\zeta$ were evaluated by Euler. He argued that $\sin(\pi x)$ could be thought of as an infinite polynomial and so

$$\frac{\sin(\pi x)}{x} = \pi \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right), \tag{13}$$

since both sides have the same zeros and value at zero. Comparing the coefficients of the Taylor series of both sides of (13) establishes that $\zeta(2) = \pi^2/6$ and then one recursively can determine a closed form (involving Bernoulli polynomials). In particular, $\zeta(4) = \pi^4/90, \zeta(6) = \pi^6/945$, and $\zeta(8) = \pi^8/9450$ and so on. By contrast, $\zeta(3)$ was only proven irrational in the late 1970s and the status of $\zeta(5)$ is unsettled – although every one who has thought about this *knows* it is irrational. It is a nice exercise to confirm the values of $\zeta(4), \zeta(6)$ from (13). A large number of the papers in this collection centre on the Basel problem and its extensions; see [$^\star$58, $^\star$73, 50, 72]. An especially nice accounting is in [43]. As is discussed in [$^\star$24, 46], it is striking how little more is known about the number–theoretic structure of $\pi$.

**Algorithmic high spots in the life of Pi.** In the large, only three methods have been used to make significant computations of $\pi$: before 1700 by Archimedes' method, between 1700 and 1980 using calculus methods (usually based on the arctangent's Maclaurin series and Machin formulas), and since 1980 using spectacular series or iterations both based on elliptic integrals and the arithmetic-geometric mean. The progress of this multi-century project is shown in Figures 2, 4, and 5. If plotted on a log linear scale, the records line up well, especially in Figure 5, which neatly tracks Moore's law.

| Name | Year | Digits |
|---|---|---|
| Babylonians | 2000? BCE | 1 |
| Egyptians | 2000? BCE | 1 |
| Hebrews (1 Kings 7:23) | 550? BCE | 1 |
| Archimedes | 250? BCE | 3 |
| Ptolemy | 150 | 3 |
| Liu Hui | 263 | 5 |
| Tsu Ch'ung Chi | 480? | 7 |
| Al-Kashi | 1429 | 14 |
| Romanus | 1593 | 15 |
| van Ceulen (**Ludolph's number**) | 1615 | 35 |

**Figure 2.** Pre-calculus $\pi$ calculations

The 'post-calculus' era was made possible by the simultaneous discovery by Eugene Salamin and Richard Brent in 1976 of identities – actually known to Gauss but not recognised for their value [$^\star$24, 37, 82] – that lead to the following two illustrative reduced complexity algorithms.

[10]As expressed in Stigler's law of eponymy, discoveries are often named after later researchers, but in Euler's case he needs no more glory.

**Quadratic Algorithm (Salamin-Brent).** Set $a_0 = 1, b_0 = 1/\sqrt{2}$, and $s_0 = 1/2$. Calculate

$$a_k = \frac{a_{k-1} + b_{k-1}}{2} \qquad \text{(Arithmetic)}, \qquad b_k = \sqrt{a_{k-1} b_{k-1}} \qquad \text{(Geometric)}, \tag{14}$$

$$c_k = a_k^2 - b_k^2, \qquad s_k = s_{k-1} - 2^k c_k \quad \text{and compute} \quad p_k = \frac{2a_k^2}{s_k}. \tag{15}$$

Then $p_k$ converges *quadratically* to $\pi$. Note the similarity between the arithmetic-geometric mean iteration (14) (which for general initial values converges quickly to a non-elementary limit), and the out-of-kilter harmonic-geometric mean iteration (1) (which in general converges slowly to an elementary limit), and which is an arithmetic-geometric iteration in the reciprocals (see [82]).

Each iteration of the Brent-Salamin algorithm *doubles* the correct digits. Successive iterations produce $1, 4, 9, 20, 42, 85, 173, 347$, and $697$ good decimal digits of $\pi$, and take $\log N$ operations to compute $N$ digits. Twenty-five iterations compute $\pi$ to over 45 million decimal digit accuracy. A disadvantage is that each of these iterations must be performed to the precision of the final result. Likewise, we have the following.

**Quartic Algorithm (The Borweins).** Set $a_0 = 6 - 4\sqrt{2}$ and $y_0 = \sqrt{2} - 1$. Iterate

$$y_{k+1} = \frac{1 - (1 - y_k^4)^{1/4}}{1 + (1 - y_k^4)^{1/4}} \qquad \text{and} \qquad a_{k+1} = a_k(1 + y_{k+1})^4 - 2^{2k+3} y_{k+1}(1 + y_{k+1} + y_{k+1}^2).$$

Then $1/a_k$ *converges quartically*[11] to $\pi$. Note that only the power of 2 used in $a_k$ depends on $k$. Twenty five iterations yield an algebraic number that agrees with $\pi$ to in excess of a quadrillion digits. This iteration is nicely derived in [56].

As charmingly detailed in [*21], see also [*47, 82], Ramanujan discovered that

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!\,(1103 + 26390k)}{(k!)^4 396^{4k}}. \tag{16}$$

Each term of this series produces an additional *eight* correct digits in the result. When Gosper used this formula to compute 17 million digits of $\pi$ in 1985, and it agreed to many millions of places with the prior estimates, *this concluded the first proof* of (16). As described in [*24], this computation can be shown to be exact enough to constitute a bona fide proof! Actually, Gosper first computed the simple continued fraction for $\pi$, hoping to discover some new things in its expansion, but found none. At the time of this writing, 500 million terms of the continued fraction for $\pi$ have been computed by Neil Bickford (then a teenager) without shedding light on whether the sequence is unbounded (see [77]).

G.N. Watson, on looking at various of Ramanujan's formulas such as (16), reports the following sensations [86]:

> ...a thrill which is indistinguishable from the thrill I feel when I enter the Sagrestia Nuovo of the Capella Medici and see before me the austere beauty of the four statues representing 'Day', 'Night', 'Evening', and 'Dawn' which Michelangelo has set over the tomb of Guiliano de'Medici and Lorenzo de'Medici. – G. N. Watson, 1886–1965.

[11] A fourth-order iteration might be a compound of two second-order ones; this one cannot be so decomposed.

Soon after Gosper did his computation, David and Gregory Chudnovsky found the following even more rapidly convergent variation of Ramanujan's formula. It is a consequence of the fact that $\sqrt{-163}$ corresponds to an imaginary quadratic field with class number one:

$$\frac{1}{\pi} = 12 \sum_{k=0}^{\infty} \frac{(-1)^k\,(6k)!\,(13591409 + 545140134k)}{(3k)!\,(k!)^3\,640320^{3k+3/2}}. \tag{17}$$

Each term of this series produces an extraordinary additional 14 correct digits. Note that in both (16) and (17) one computes a rational series and has a single multiplication by a surd to compute at the end.

**Some less familiar themes** While most of the articles in our collection fit into one of the big themes (irrationality [57] , transcendence, arctangent formulas, Euler's product for $\sin x$, evaluation of $\zeta(2)$, $\pi$ in other cultures) there are of course some lovely sporadic examples. These include the following.

- ***Spigot Algorithms*, which drip off one more digit at a time for $\pi$ and use only integer arithmetic** [*71, 54]. As described in [*44], the first spigot algorithm was discovered for $e$. While the ideas are simple, the specifics for $\pi$ need some care; we refer to Rabinowitz and Wagon [*71] for the carefully explained details.
- **Series for $\pi \cdot e$ and $\pi/e$** [35]. Melzack, then at Bell Labs, proved[12] that

$$\frac{\pi}{2e} = \lim_{N\to\infty} \prod_{n=1}^{2N} \left(1 + \frac{2}{n}\right)^{(-1)^{n+1} n} \tag{18}$$

$$\frac{6}{\pi e} = \lim_{N\to\infty} \prod_{n=2}^{2N+1} \left(1 + \frac{2}{n}\right)^{(-1)^{n} n}. \tag{19}$$

Melzak begins by showing that $\lim_{n\to\infty} V(C_n)/V(S_n) = \sqrt{2/(\pi e)}$, where $S_n$ is the $n$-sphere and $C_n$ is the inscribed $n$-dimensional cylinder of greatest volume. He then proves (18) and (19), saying that it closely follows the derivation of Wallis' formula, and he *conjectures* that (18) can be used to prove that $e/\pi$ is irrational. We remind the reader that the transcendentality of $e^\pi$ follows from the *Gelfond-Schneider* theorem (1934) [82] since $e^{\pi/2} = i^{-i}$, but the statuses of $e+\pi, e/\pi, e\cdot\pi$, and $\pi^e$ are unsettled.

Both (18) and (19) are very slowly convergent. To check (19), one may take logs and expand the series for log, then exchange the order of summation to arrive at the more rapidly convergent 'zeta'-series

$$\sum_{n=2}^{\infty} \frac{(-2)^n}{n} \left(\alpha\,(n-1) - 1\right) = \log\left(\frac{\pi\, e}{6}\right)$$

where $\alpha(s) := \sum_{k\ge 0} (-1)^k/(k+1)^s$ is the alternating zeta function, which is well defined for $\operatorname{Re} s > 0$.

If we consider the partial products for (18), then we obtain

$$\left(\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{8}{7}\cdot\frac{8}{9}\cdots\frac{2N}{2N+1}\right)\cdot\left(\frac{2N+1}{2N+2}\right)^{2N}.$$

[12] We correct errors in Melzak's original formulas.

As $N \to \infty$ the left factor yields Wallis's product for $\pi/2$ and the right factor tends to $1/e$, which confirms (18). A similar partial product can be obtained from (19).

- **A curious predictability in the error in the Gregory-Liebnitz series** (6) **for** $\pi/4$ [$^\star$25, 45]. In 1988, it was observed that the series

$$\pi = 4\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} \; = \; 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots\right), \tag{20}$$

when truncated to 5,000,000 terms, differs strangely from the true value of $\pi$:

```
3.141592453589793238464643383279502784197169399387305820974941822307816 40...
3.141592653589793238462643383279502884197169399375105820974944592307816 40...
        2             -2             10           -122           2770.
```

Values differ as expected from truncating an alternating series: in the seventh place a "4" that should be a "6." But the next 13 digits are correct, and after another blip, for 12 digits. Of the first 46 digits, only four differ from the corresponding digits of $\pi$. Further, the "error" digits seemingly occur with a period of 14. Such anomalous behavior begs for explanation. A great place to start is by using Neil Sloane's internet-based integer sequence recognition tool, available at `www.oeis.org`. This tool has no difficulty recognizing the sequence of errors as twice the *Euler numbers*. Even Euler numbers are generated by $\sec x = \sum_{k=0}^{\infty}(-1)^k E_{2k}x^{2k}/(2k)!$. The first few are $1, -1, 5, -61, 1385, -50521, 2702765$. This discovery led to the following *asymptotic expansion*:

$$\frac{\pi}{2} - 2\sum_{k=1}^{N/2} \frac{(-1)^{k+1}}{2k-1} \approx \sum_{m=0}^{\infty} \frac{E_{2m}}{N^{2m+1}}. \tag{21}$$

Now the genesis of the anomaly is clear: by chance the series had been truncated at 5,000,000 terms – exactly one-half of a fairly large power of ten. Indeed, setting $N = 10,000,000$ in equation (21) shows that the first hundred or so digits of the truncated series value are small perturbations of the correct decimal expansion for $\pi$.

On a hexadecimal computer with $N = 16^7$ the corresponding strings and hex errors are:

```
3.243F6A8885A308D313198A2E03707344A4093822299F31D0082EFA98EC4E6C89452821E...
3.243F6A6885A308D31319AA2E03707344A3693822299F31D7A82EFA98EC4DBF69452821E...
        2             -2             A            -7A            2AD2
```

with the first being the correct value of $\pi$. (In hexadecimal or *hex* one uses 'A,B, . . ., F' to write 10 through 15 as single 'hex-digits'.) Similar phenomena occur for other constants; see [80]. Also, knowing the errors means we can correct them and use (21) to make Gregory's formula computationally tractable.

- **Hilbert's inequality** [$^\star$61, 48] In its simplest incarnation, Hilbert's inequality is

$$\sum_{m,n=1}^{\infty} \frac{a_n\, b_m}{n+m} \leq \pi\sqrt{\sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} b_n^2} \qquad (\text{for } a_n, b_m \in \mathbb{R},\ a_n, b_m > 0) \tag{22}$$

with the assertion that the constant $\pi$ is best possible. Actually $2\pi$ was the best constant that Hilbert could obtain. Hardy's inequality, which originated in his successful attempt to prove (22) early in the development of the modern theory of inequalities, is well described in [*61]. One could write a nice book on the places in which $\pi$ or $\zeta(2)$ arise as the best possible constant in an inequality.

- **The distribution of the digits of** $\pi$ [46]. Single digit distribution of the first trillion digits base ten and sixteen is shown in Figure 3. All the counts in these figures are consistent with $\pi$ being random.

| Decimal Digit | Occurrences |
|---|---|
| 0 | 99999485134 |
| 1 | 99999945664 |
| 2 | 100000480057 |
| 3 | 99999787805 |
| 4 | 100000357857 |
| 5 | 99999671008 |
| 6 | 99999807503 |
| 7 | 99999818723 |
| 8 | 100000791469 |
| 9 | 99999854780 |
| Total | **1000000000000** |

| Hex Digit | Occurrences |
|---|---|
| 0 | 62499881108 |
| 1 | 62500212206 |
| 2 | 62499924780 |
| 3 | 62500188844 |
| 4 | 62499807368 |
| 5 | 62500007205 |
| 6 | 62499925426 |
| 7 | 62499878794 |
| 8 | 62500216752 |
| 9 | 62500120671 |
| A | 62500266095 |
| B | 62499955595 |
| C | 62500188610 |
| D | 62499613666 |
| E | 62499875079 |
| F | 62499937801 |
| Total | **1000000000000** |

**Figure 3.** Seemingly random behaviour of single digits of $\pi$ in base 10 and 16

| Name | Year | Correct Digits |
|---|---|---|
| Sharp (and Halley) | 1699 | 71 |
| Machin | 1706 | 100 |
| Strassnitzky and Dase | 1844 | 200 |
| Rutherford | 1853 | 440 |
| Shanks | 1874 | (707) 527 |
| Ferguson (**Calculator**) | 1947 | 808 |
| Reitwiesner et al. (**ENIAC**) | 1949 | 2,037 |
| Genuys | 1958 | 10,000 |
| Shanks and Wrench | 1961 | 100,265 |
| Guilloud and Bouyer | 1973 | 1,001,250 |

**Figure 4.** Calculus $\pi$ calculations

**3. PI IN THIS MONTHLY: 1894-1944.** This period yielded twenty papers for our selection. The July 1894 issue of this MONTHLY contained the most embarrassing article on Pi [10] ever to grace the pages of the MONTHLY. Flagged only by "published by the request of the author", who indicated it was copyrighted in 1889, it is the origin of the famous usually garbled story of the attempt by Indiana in 1897 to legislate the value of $\pi$; see [81] and [80, D. Singmaster, The legal values of Pi]. It contains a nonsensical geometric construction of $\pi$. So $\pi$ and the MONTHLY got off on a bad footing.

Luckily the future was brighter. While most early articles would meet today's criteria for publication, this is not true of all. For example, [20] offers a carefully organised list of 68 consequences of Euler's product for $\sin$ given in (13) with almost no English. By contrast, [6] is perhaps the first discussion of the efficiency of calculation in the MONTHLY.

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**4. PI IN THIS MONTHLY: 1945-1989.** This second period collects 22 papers. It saw the birth and evolution of the digital computer with many consequences for the computation of $\pi$. Even old topics are new when new ideas and tools arise. A charming example is as follows.

**Why $\pi$ is not $22/7$.** Did you know that

$$0 < \int_0^1 \frac{(1-x)^4 x^4}{1+x^2}\, dx \;=\; \frac{22}{7} - \pi? \tag{23}$$

| Name | Year | Correct Digits |
|---|---|---|
| Miyoshi and Kanada | 1981 | 2,000,036 |
| Kanada-Yoshino-Tamura | 1982 | 16,777,206 |
| Gosper | 1985 | 17,526,200 |
| Bailey | Jan. 1986 | 29,360,111 |
| Kanada and Tamura | Sep. 1986 | 33,554,414 |
| Kanada and Tamura | Oct. 1986 | 67,108,839 |
| Kanada et. al | Jan. 1987 | 134,217,700 |
| Kanada and Tamura | Jan. 1988 | 201,326,551 |
| Chudnovskys | May 1989 | 480,000,000 |
| Kanada and Tamura | Jul. 1989 | 536,870,898 |
| Kanada and Tamura | Nov. 1989 | 1,073,741,799 |
| Chudnovskys | Aug. 1991 | 2,260,000,000 |
| Chudnovskys | May 1994 | 4,044,000,000 |
| Kanada and Takahashi | Oct. 1995 | 6,442,450,938 |
| Kanada and Takahashi | Jul. 1997 | 51,539,600,000 |
| Kanada and Takahashi | Sep. 1999 | 206,158,430,000 |
| Kanada-Ushiro-Kuroda | Dec. 2002 | 1,241,100,000,000 |
| Takahashi | Jan. 2009 | 1,649,000,000,000 |
| Takahashi | April. 2009 | 2,576,980,377,524 |
| Bellard | Dec. 2009 | 2,699,999,990,000 |
| Kondo and Yee | Aug. 2010 | 5,000,000,000,000 |
| Kondo and Yee | Oct. 2011 | 10,000,000,000,000 |
| Kondo and Yee | Dec. 2013 | 12,200,000,000,000 |

**Figure 5.** Post-calculus $\pi$ calculations

The integrand is strictly positive on $(0, 1)$, so the integral in (23) is strictly positive – despite claims that $\pi$ is $22/7$ which rage over the millennia.[13] Why is this identity true? We have

$$\int_0^t \frac{x^4\,(1-x)^4}{1+x^2}\,dx = \frac{1}{7}\,t^7 - \frac{2}{3}\,t^6 + t^5 - \frac{4}{3}\,t^3 + 4\,t - 4\,\arctan(t)\,,$$

as differentiation easily confirms, and so the Newtonian Fundamental Theorem of Calculus proves (23).

One can take the idea in (23) a bit further. Note that

$$\int_0^1 x^4\,(1-x)^4\,dx = \frac{1}{630}, \tag{24}$$

and we observe that

$$\frac{1}{2}\int_0^1 x^4\,(1-x)^4\,dx < \int_0^1 \frac{(1-x)^4 x^4}{1+x^2}\,dx < \int_0^1 x^4\,(1-x)^4\,dx. \tag{25}$$

[13] One may still find adverts in newspapers offering such proofs for sale. A recent and otherwise very nice children's book "Sir Cumference and the the Dragon of Pi (A Math Adventure)" published in (1999) repeats the error, and email often arrives in our in-boxes offering to show why things like this are true.

Combine this with (23) and (24) to derive

$$\frac{223}{71} < \frac{22}{7} - \frac{1}{630} < \pi < \frac{22}{7} - \frac{1}{1260} < \frac{22}{7},$$

and so we re-obtain Archimedes' famous computation

$$3\frac{10}{71} < \pi < 3\frac{10}{70}. \tag{26}$$

This derivation was popularized in *Eureka*, a Cambridge University student journal, in 1971.[14] A recent study of related approximations is made by Lucas [65]. It seems largely happenstance that $22/7$ is an early continued fraction approximate to $\pi$.

Another less standard offering is in [33] where Y. V. Matiyasevich shows that

$$\pi = \lim_{m\to\infty} \sqrt{\frac{6 \log \operatorname{fcm}(F_1, \ldots, F_m)}{\log \operatorname{lcm}(u_1, \ldots, u_m)}}. \tag{27}$$

Here 'lcm' is the least common multiple, 'fcm 'is the formal common multiple (the product), and $F_n$ is the $n$-th Fibonacci number with $F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}, n \geq 2$ (without the square root we obtain a formula for $\zeta(2)$).

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**5. PI IN THIS MONTHLY: 1990-2015.** In the final period we have collected 32 papers, and see no sign that interest in $\pi$ is lessening. A new topic [$^\star$44, 46, 51, 81] is that of *BBP formulas*, which can compute individual digits of certain constants such as $\pi$ in base 2 or $\pi^2$ in bases 2 and 3 without using the earlier digits. The phenomenon is based on the formula

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i}\left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6}\right). \tag{28}$$

On August 27, 2012, Ed Karrel used (28) to extract 25 hex digits of $\pi$ starting after the $10^{15}$ position. They are `353CB3F7F0C9ACCFA9AA215F2`.[15] In 1990 a billion digits had not yet been computed, see [80], and even now it is inconceivable to compute the full first quadrillion digits in any base.

Over this period the use of the computer has become more routine even in pure mathematics, and concrete mathematics is back in fashion. In this spirit, we record the following evaluation of $\zeta(2)$, which to our knowledge first appeared as an exercise in [82].

**Theorem 2 (Sophomore's Dream).** *One may square term-wise to obtain*

$$\left(\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{2n+1}\right)^2 = \sum_{n=-\infty}^{\infty} \frac{1}{(2n+1)^2}. \tag{29}$$

*In particular* $\zeta(2) = \pi^2/6$.

*Proof.* Let

$$\delta_N := \sum_{n=-N}^{N}\sum_{m=-N}^{N} \frac{(-1)^{m+n}}{(2m+1)(2n+1)} - \sum_{k=-N}^{N} \frac{1}{(2k+1)^2},$$

and note that $\delta_N = \sum_{n=-N}^{N} \frac{(-1)^n}{(2n+1)} \sum_{n\neq m=-N}^{N} \frac{(-1)^m}{m-n}$. We leave it to the reader to show that for large $N$ the inner sum $\epsilon_N(n)$ is of order $1/(N-n+1)$, which goes to zero.

The proof is finished by evaluating the left side of (29) to $\pi^2/4$ using Gregory's formula (6) and then noting that this means $\sum_{n=0}^{\infty} 1/(2n+1)^2 = \pi^2/8$. ∎

Another potent and concrete way to establish an identity is to obtain an appropriate differential equation. For example, consider

$$f(x) := \left(\int_0^x e^{-s^2}\,ds\right)^2 \quad \text{and} \quad g(x) := \int_0^1 \frac{\exp(-x^2(1+t^2))}{1+t^2}\,dt.$$

The derivative of $f+g$ is zero: in *Maple*,

[15] All processing was done on four NVIDIA GTX 690 graphics cards (GPUs) installed in CUDA; the computation took 37 days. CUDA is a parallel computing platform and programming mode developed by NVIDIA for use in their graphics processing units (GPUs).

```
f:=x->Int(exp(-s^2),s=0..x)^2;
g:=x->Int(exp(-x^2*(1+t^2))/(1+t^2),t=0..1);
with(student):d:=changevar(s=x*t,diff(f(x),x),t)+diff(g(x),x);
d:=expand(d);
```

shows this. Hence, $f(x) + g(x)$ is constant for $0 \le x \le \infty$ and so, after justifying taking the limit at $\infty$,

$$\left(\int_0^\infty \exp(-t^2)\, dt\right)^2 = f(\infty) \ = \ g(0) \ = \ \arctan(1) \ = \ \frac{\pi}{4}.$$

Thus, we have evaluated the Gaussian integral using only elementary calculus and Gregory's formula (6). The change of variables $t^2 = x$ shows that this evaluation of the normal distribution agrees with $\Gamma(1/2) = \sqrt{\pi}$.

In similar fashion, we may evaluate

$$F(y) := \int_0^\infty \exp(-x^2)\cos(2xy)\, dx$$

by checking that it satisfies the differential equation $F'(y) + 2y\,F(y) = 0$. We obtain

$$F(y) = \frac{\sqrt{\pi}}{2}\exp(-y^2),$$

since we have just evaluated $F(0) = \sqrt{\pi}/2$.

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## 6. CONCLUDING REMARKS.

It's generally the way with progress that it looks much greater than it really is. – Ludwig Wittgenstein[16]

It is a great strength of mathematics that 'old' and 'inferior' are not synonyms. As we have seen in this selection, many seeming novelties are actually rediscoveries. That is not at all a bad thing, but it does behoove authors to write "I have not seen this before" or "this is to my knowledge new" rather than unnecessarily claiming ontological or epistemological primacy.

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## A. APPENDIX: I. NIVEN - THE TRANSCENDENCE OF $\pi$ [14]

Among the proofs of the transcendence of $e$, which are in general variations and simplifications of the original proof of Hermite, perhaps the simplest is that of A. Hurwitz.[17] His solution of the problem contains an ingenious device, which we now employ to prove the transcendence of $\pi$.

We assume that $\pi$ is an algebraic number, and show that this leads to a contradiction. Since the product of two algebraic numbers is an algebraic number, the quantity $i\pi$ is a root of an algebraic equation with integral coefficients

$$\theta_1(x) = 0, \tag{30}$$

whose roots are $\alpha_1 = i\pi,\ \alpha_2,\ \alpha_3, \ldots,\ \alpha_n$. Using Euler's relation $e^{i\pi} + 1 = 0$, we have

$$(e^{\alpha_1} + 1)\,(e^{\alpha_2} + 1) \cdots (e^{\alpha_n} + 1) = 0. \tag{31}$$

We now construct an algebraic equation with integral coefficients whose roots are the exponents in the expansion of (2). First consider the exponents

$$\alpha_1 + \alpha_2,\ \alpha_1 + \alpha_3,\ \alpha_2 + \alpha_3,\ \ldots,\ \alpha_{n-1} + \alpha_n. \tag{32}$$

[17] A. Hurwitz, Beweis der Transendenz der Zahl $e$, Mathematische Annalen, vol. 43, 1893, pp. 220-221 (also in his Mathematische Werke, vol. 2, pp. 134-135).

By equation (30), the elementary symmetric functions of $\alpha_1,\,\alpha_2,\,\ldots,\,\alpha_n$ are rational numbers. Hence the elementary symmetric functions of the quantities (32) are rational numbers. It follows that the quantities (32) are roots of

$$\theta_2(x) = 0, \tag{33}$$

an algebraic equation with integral coefficients. Similarily, the sums of the $\alpha$'s taken three at a time are the ${}_nC_3$ roots of

$$\theta_3(x) = 0. \tag{34}$$

Proceeding in the same way, we obtain

$$\theta_4(x) = 0,\, \theta_5(x) = 0,\, \ldots,\, \theta_n(x) = 0, \tag{35}$$

algebraic equations with integral coefficients, whose roots are the sums of the $\alpha$'s taken $4,\,5,\,\cdots,\,n$ at a time respectively. The product equation

$$\theta_1(x)\theta_2(x)\cdots\theta_n(x) = 0, \tag{36}$$

has roots that are precisely the exponents in the expansion of (31).

The deletion of zero roots (if any) from equation (36) gives

$$\theta(x) = cx^r + c_1x^{r-1} + \cdots + c_r = 0, \tag{37}$$

whose roots $\beta_1,\,\beta_2,\,\ldots,\,\beta_r$ are the non-vanishing exponents in the expansion of (31), and whose coefficients are integers. Hence (31) may be written in the form

$$e^{\beta_1} + e^{\beta_2} + \cdots + e^{\beta_r} + k = 0, \tag{38}$$

where $k$ is a positive integer.

We define

$$f(x) = \frac{c^s x^{p-1}\left\{\theta(x)\right\}^p}{(p-1)!}, \tag{39}$$

where $s = rp - 1$, and $p$ is a prime to be specified. Also, we define

$$F(x) = f(x) + f^{(1)}(x) + f^{(2)}(x) + \cdots + f^{(s+p+1)}(x), \tag{40}$$

noting, with thanks to Hurwitz, that the derivative of $e^{-x}F(x)$ is $-e^{-x}f(x)$. Hence we may write

$$e^{-x}F(x) - e^0F(0) = \int_0^x -e^{-\xi}f(\xi)d\xi.$$

The substitution $\xi = \tau x$ produces

$$F(x) - e^x F(0) = -x \int_0^1 e^{(1-\tau)x} f(\tau x) d\tau.$$

Let $x$ range over the values $\beta_1,\ \beta_2,\ \ldots,\ \beta_r$ and add the resulting equations. Using (38) we obtain

$$\sum_{j=1}^{r} F(\beta_j) + kF(0) = -\sum_{j=1}^{r} \beta_j \int_0^1 e^{(1-\tau)\beta_j} f(\tau\beta_j) d\tau. \tag{41}$$

This result gives the contradiction we desire. For we shall choose the prime $p$ to make the left side a non-zero integer, and make the right side as small as we please.

By (39), we have

$$\sum_{j=1}^{r} f^{(t)} = 0, \quad \text{for} \quad 0 \le t < p.$$

Also by (39) the polynomial obtained by multiplying $f(x)$ by $(p-1)!$ has integral coefficients. Since the product of $p$ consecutive positive integers is divisible by $p!$, the $p$th and higher derivatives of $(p-1)!f(x)$ are polynomials in $x$ with integral coefficients divisible by $p!$. Hence the $p$th and higher derivatives of $f(x)$ are polynomials with integral coefficients, each of which is divisible by $p$. That each of these coefficients is also divisible by $c^s$ is obvious from the definition (39). Thus we have shown that, for $t \ge p$, the quantity $f^{(t)}(\beta_j)$ is a polynomial in $\beta_j$ of degree at most $s$, each of whose coefficients is divisible by $pc^s$. By (37), a symmetric function of $\beta_1,\ \beta_2,\ \ldots,\ \beta_r$ with integral coefficients and of degree at most $s$ is an integer, *provided that* each coefficient is divisible by $c^s$ (by the fundamental theorem on symmetric functions). Hence

$$\sum_{j=1}^{r} f^{(1)}(\beta_j) = pk_t, \qquad (t = p,\ p+1,\ \cdots,\ p+s)$$

where the $k_t$ are integers. It follows that

$$\sum_{j=1}^{r} F(\beta_j) = p \sum_{t=p}^{n+s} k_t.$$

In order to complete the proof that the left side of (41) is a non-zero integer, we now show that $kF(0)$ is an integer that is prime to $p$. From (39) it is clear that

$$\begin{aligned} f^{(t)}(0) &= 0, && (t = 0,\ 1,\ \cdots,\ p-2) \\ f^{(p-1)}(0) &= c^s c_r^p, && \\ f^{(t)}(0) &= pK_t, && (t = p,\ p+1,\ \cdots,\ p+s) \end{aligned}$$

where the $K_t$ are integers. If $p$ is chosen greater than each of $k, c, c_r$ (possible since the number of primes is infinite), the desired result follows from (40).

Finally, the right side of (41) equals

$$-\sum_{j=1}^{r} \frac{1}{c} \int_0^1 \frac{\{c^r \beta_j \theta(\tau\beta_j)\}^p}{(p-1)!} e^{(1-r)\beta_j} \, d\tau.$$

This is a finite sum, each term of which may be made as small as we wish by choosing $p$ very large, because

$$\lim_{p\to\infty} \frac{\{c^r \beta_j \theta(\tau\beta_j)\}^p}{(p-1)!} = 0. \qquad \blacksquare$$