

THIRTY-TWO GOLDBACH VARIATIONS

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ABSTRACT. We give thirty-two diverse proofs of a small mathematical gem—the fundamental Euler sum identity

$$\zeta(2, 1) = \zeta(3) = 8 \zeta(\bar{2}, 1).$$

We also discuss various generalizations for multiple harmonic (Euler) sums and some of their many connections, thereby illustrating both the wide variety of techniques fruitfully used to study such sums and the attraction of their study.

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1. INTRODUCTION

There are several ways to introduce and make attractive a new or unfamiliar subject. We choose to do so by emulating Glen Gould's passion for Bach's *Goldberg variations*. We shall illustrate most of the techniques used to study Euler sums by focusing almost entirely on the identities of (1.2) and (1.5)

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{m=1}^{n-1} \frac{1}{m} = \sum_{n=1}^{\infty} \frac{1}{n^3} = 8 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \sum_{m=1}^{n-1} \frac{1}{m}$$

and some of their many generalizations.

1.1. Euler, Goldbach and the birth of ζ . What follows is a transcription of correspondence between Euler and Goldbach [49] that led to the origin of the zeta-function and multi-zeta values, see also [8, 9, 42].

59. Goldbach an Euler, Moskau, 24. Dez. 1742.¹ [...] *Als ich neulich die vermeinten summas der beiden letzteren serierum in meinem vorigen Schreiben wieder betrachtet, habe ich alsofort wahrgenommen, daß selbige aus einem bloßem Schreibfehler entstanden, von welchem es aber in der Tat heißt: Si non errasset, fecerat ille minus.*²

This is the letter in which Goldbach precisely formulates the series which sparked Euler's further investigations into what would become the zeta-function. These investigations were apparently due to a serendipitous mistake. The above translates as follows:

*When I recently considered further the indicated sums of the last two series in my previous letter, I realized immediately that the same series arose due to a mere writing error, from which indeed the saying goes, "Had one not erred, one would have achieved less."*³

Goldbach continues

Ich halte dafür, daß es ein problema problematum ist, die summam huius:

$$\begin{aligned} & /54r/1 + \frac{1}{2^n} \left(1 + \frac{1}{2^m}\right) + \frac{1}{3^n} \left(1 + \frac{1}{2^m} + \frac{1}{3^m}\right) \\ & + \frac{1}{4^n} \left(1 + \frac{1}{2^m} + \frac{1}{3^m} + \frac{1}{4^m}\right) + \text{etc.} \end{aligned}$$

¹AAL: F.136, Op. 2, Nr.8, Blatt 54–55.

²Frei zitiert nach Marcus Valerius Martialis, I, 21,9.

³*Opera Omnia*, vol. IVA4, Birkhäuser Verlag.

in den casibus zu finden, wo m et n nicht numeri integri pares et sibi aequales sind, doch gibt es casus, da die summa angegeben werden kann, exempli gr[atia], si $m = 1$, $n = 3$, denn es ist

$$1 + \frac{1}{2^3} \left(1 + \frac{1}{2}\right) + \frac{1}{3^3} \left(1 + \frac{1}{2} + \frac{1}{3}\right) + \frac{1}{4^3} \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) + \text{etc.} = \frac{\pi^4}{72}.$$

1.2. The Modern Language of Euler Sums. For positive integers s_1, \dots, s_m and signs $\sigma_j = \pm 1$, consider [14] the m -fold Euler sum

$$\zeta(s_1, \dots, s_m; \sigma_1, \dots, \sigma_m) := \sum_{k_1 > \dots > k_m > 0} \prod_{j=1}^m \frac{\sigma_j^{k_j}}{k_j^{s_j}}.$$

As is now customary, we combine strings of exponents and signs by replacing s_j by \bar{s}_j in the argument list if and only if $\sigma_j = -1$, and denote n repetitions of a substring S by $\{S\}^n$. Thus, for example, $\zeta(\bar{1}) = -\log 2$, $\zeta(\{2\}^3) = \zeta(2, 2, 2) = \pi^6/7!$ and

$$\zeta(s_1, \dots, s_m) = \sum_{k_1 > \dots > k_m > 0} \prod_{j=1}^m k_j^{-s_j}. \quad (1.1)$$

The identity

$$\zeta(2, 1) = \zeta(3) \quad (1.2)$$

goes back to Euler [44] [45, p. 228] and has since been repeatedly rediscovered, (see, e.g., [29, 31, 46, 57]). In this language Goldbach had found

$$\zeta(3, 1) + \zeta(4) = \frac{\pi^4}{72}.$$

The more general formula

$$2\zeta(m, 1) = m\zeta(m+1) - \sum_{j=1}^{m-2} \zeta(j+1)\zeta(m-j), \quad 2 \leq m \in \mathbf{Z} \quad (1.3)$$

is also due to Euler [44] [45, p. 266]. Nielsen [62, p. 229] [63, p. 198] [64, pp. 47–49] developed a method for obtaining (1.3) and related results based on partial fractions. Formula (1.3) has also been rediscovered many times [76, 69, 67, 48, 22, 73]. Crandall and Buhler [40] deduced (1.3) from their general infinite series formula which expresses $\zeta(s, t)$ for real $s > 1$ and $t \geq 1$ in terms of Riemann zeta values.

Study of the the multiple zeta function (1.1) led to the discovery of a new generalization of (1.2), involving nested sums of arbitrary depth:

$$\zeta(\{2, 1\}^n) = \zeta(\{3\}^n), \quad n \in \mathbf{Z}^+. \quad (1.4)$$

Although numerous proofs of (1.2) and (1.3) are known (we give many in the sequel), the only proof of (1.4) of which we are aware involves making a simple change of variable in a multiple iterated integral (see [14, 15, 19] and (5.11) below).

An alternating version of (1.2) is

$$8\zeta(\bar{2}, 1) = \zeta(3), \tag{1.5}$$

which has also resurfaced from time to time [64, p. 50] [71, (2.12)] [35, p. 267] and hints at the generalization

$$8^n \zeta(\{\bar{2}, 1\}^n) \stackrel{?}{=} \zeta(\{3\}^n), \quad n \in \mathbf{Z}^+, \tag{1.6}$$

originally conjectured in [14], and which still remains open—despite abundant, even overwhelming, evidence [11].

1.3. Hilbert and Hardy Inequalities. Much of the early 20th century history—and philosophy—of the “*bright and amusing*” subject of inequalities charmingly discussed in G.H. Hardy’s retirement lecture as London Mathematical Society Secretary, [52]. He comments [52, p. 474] that *Harald Bohr is reported to have remarked “Most analysts spend half their time hunting through the literature for inequalities they want to use, but cannot prove.”*

Central to Hardy’s essay are:

Theorem 1. (Hilbert) *For non-negative sequences (a_n) and (b_n) , not both zero, and for $1 \leq p, q \leq \infty$ with $1/p + 1/q = 1$ one has*

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_n b_m}{n+m} < \pi \csc\left(\frac{\pi}{p}\right) \|a_n\|_p \|b_n\|_q. \tag{1.7}$$

Theorem 2. (Hardy) *For a non-negative sequence (a_n) and for $p > 1$*

$$\sum_{n=1}^{\infty} \left(\frac{a_1 + a_2 + \dots + a_n}{n}\right)^p \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p. \tag{1.8}$$

We return to these inequalities in Section 6.

Hardy [52, p. 485] remarks that his “*own theorem was discovered as a by-product of my own attempt to find a really simple and elementary proof of Hilbert’s.*” He reproduces Elliott’s proof of (1.8), writing “*it can hardly be possible to find a proof more concise or elegant*” and also “*I have given nine [proofs] in a lecture in Oxford, and more have been found since then.*” (See [52, p. 488].)

1.4. Our Motivation and Intentions. We wish to emulate Hardy and to present proofs that are either elementary, bright and amusing, concise or elegant—ideally all at the same time! In doing so we note that:

- (1) $\zeta(3)$, while provably irrational, is still quite mysterious, see [34, 13] and [9]. Hence, exposing more relationships and approaches can only help. We certainly hope one of them will lead to a proof of conjecture (1.6).

(2) Identities for $\zeta(3)$ are abundant and diverse. We give three each of which is the entry-point to a fascinating set:

- Our first favourite is a *binomial sum* [2] that played a role in Apéry's 1976 proof, see [13, 34] and [9, Chapter 3], of the irrationality of $\zeta(3)$:

$$\zeta(3) = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3 \binom{2k}{k}}. \quad (1.9)$$

- Our second is Broadhurst's binary *BBP formula* [30]:

$$\zeta(3) = \frac{48}{7} \mathcal{S}_1(1, -7, -1, 10, -1, -7, 1, 0) + \frac{32}{7} \mathcal{S}_3(1, 1, -1, -2, -1, 1, 1, 0),$$

where $\mathcal{S}_p(a_1, a_2, \dots, a_8) := \sum_{k=1}^{\infty} a_k 2^{-\lfloor p(k+1)/2 \rfloor} k^{-3}$, and the coefficients a_k repeat modulo 8. We refer to [8, Chapter 3] for the digit properties of such formulae. Explicitly,

$$\begin{aligned} \zeta(3) = & \frac{1}{672} \sum_{k=0}^{\infty} \frac{1}{2^{12k}} \left[\frac{2048}{(24k+1)^3} - \frac{11264}{(24k+2)^3} - \frac{1024}{(24k+3)^3} + \frac{11776}{(24k+4)^3} \right. \\ & - \frac{512}{(24k+5)^3} + \frac{4096}{(24k+6)^3} + \frac{256}{(24k+7)^3} + \frac{3456}{(24k+8)^3} + \frac{128}{(24k+9)^3} \\ & - \frac{704}{(24k+10)^3} - \frac{64}{(24k+11)^3} - \frac{128}{(24k+12)^3} - \frac{32}{(24k+13)^3} - \frac{176}{(24k+14)^3} \\ & + \frac{16}{(24k+15)^3} + \frac{216}{(24k+16)^3} + \frac{8}{(24k+17)^3} + \frac{64}{(24k+18)^3} - \frac{4}{(24k+19)^3} \\ & \left. + \frac{46}{(24k+20)^3} - \frac{2}{(24k+21)^3} - \frac{11}{(24k+22)^3} + \frac{1}{(24k+23)^3} \right]. \end{aligned}$$

It was this discovery that lead Bailey and Crandall to their striking recent work on normality of BBP constants [8, Chapter 4].

- Our third favourite due to Ramanujan [9, p. 138] is the *hyperbolic series* approximation

$$\zeta(3) = \frac{7\pi^3}{180} - 2 \sum_{k=1}^{\infty} \frac{1}{k^3 (e^{2\pi k} - 1)},$$

in which the 'error' is $\zeta(3) - 7\pi^3/180 \approx -0.003742745$, and which to our knowledge is the 'closest' one gets to writing $\zeta(3)$ as a rational multiple of π^3 .

- (3) Often results about $\zeta(3)$ are more precisely results about $\zeta(2, 1)$ or $\zeta(\bar{2}, 1)$, as we shall exhibit.
- (4) Double and multiple sums are still under-studied and under-appreciated. We should like to partially redress that.
- (5) One can now prove these seemingly analytic facts in an entirely finitary manner via words over alphabets, dispensing with notions of infinity and convergence;

- (6) Many subjects are touched upon—from computer algebra, integer relation methods, generating functions and techniques of integration to polylogarithms, hypergeometric and special functions, non-commutative rings, combinatorial algebras and Stirling numbers—so that most readers will find a proof worth showing in an undergraduate class;
- (7) For example, there has been an explosive recent interest in q -analogues, see §4.3, and in quantum field theory, algebraic K -theory and knot theory, see [8, 77].

For some of the broader issues relating to Euler sums, we refer the reader to the survey articles [8, 19, 36, 74, 75, 77, 78]. Computational issues are discussed in [9, 39] and to an extent in [15].

1.5. Further Notation.

Notation and Terminology. For positive integer N , denote the N th partial sum of the harmonic series by $H_N := \sum_{n=1}^N 1/n$. We also use $\psi = \Gamma'/\Gamma$ to denote the logarithmic derivative of the Euler gamma-function (also referred to as the *digamma* function), and recall the identity $\psi(N+1) + \gamma = H_N$, where $\gamma = 0.5772156649\dots$ is *Euler's constant*. Where convenient, we employ the Pochhammer symbol $(a)_n = a(a+1)\cdots(a+n-1)$ for complex a and non-negative integer n . As usual, the *Kronecker* $\delta_{m,n}$ is 1 if $m = n$ and 0 otherwise.

We organize our proofs by technique, although clearly this is somewhat arbitrary as many proofs fit well within more than one category. Broadly their sophistication increases as we move through the paper. In some of the later sections the proofs become more schematic. We invite readers to send additional selections for our collection, a collection which for us has all the beauty of Blake's grain of sand⁴:

*“To see a world in a grain of sand
And a heaven in a wild flower,
Hold infinity in the palm of your hand
And eternity in an hour.”*

2. TELESCOPING AND PARTIAL FRACTIONS

For a quick proof of (1.2), consider

$$S := \sum_{n,k>0} \frac{1}{nk(n+k)} = \sum_{n,k>0} \frac{1}{n^2} \left(\frac{1}{k} - \frac{1}{n+k} \right) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=1}^n \frac{1}{k} = \zeta(3) + \zeta(2, 1).$$

⁴William Blake from *Auguries of Innocence*.

On the other hand,

$$S = \sum_{n,k>0} \left(\frac{1}{n} + \frac{1}{k} \right) \frac{1}{(n+k)^2} = \sum_{n,k>0} \frac{1}{n(n+k)^2} + \sum_{n,k>0} \frac{1}{k(n+k)^2} = 2\zeta(2, 1),$$

by symmetry. □

The above argument goes back at least to Steinberg [57]. See also [58].

For (1.5), first consider

$$\begin{aligned} \zeta(\bar{2}, \bar{1}) + \zeta(3) &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \sum_{k=1}^n \frac{(-1)^k}{k} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \sum_{k=1}^{\infty} \left(\frac{(-1)^k}{k} - \frac{(-1)^{n+k}}{n+k} \right) \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \sum_{k=1}^{\infty} (-1)^k \left(\frac{n+k - (-1)^n k}{k(n+k)} \right) \\ &= \sum_{n,k>0} \frac{(-1)^{n+k}}{nk(n+k)} + \sum_{n,k>0} \frac{(-1)^{n+k}}{n^2(n+k)} - \sum_{n,k>0} \frac{(-1)^k}{n^2(n+k)} \\ &= \sum_{n,k>0} \left(\frac{1}{n} + \frac{1}{k} \right) \frac{(-1)^{n+k}}{(n+k)^2} + \zeta(\bar{1}, 2) - \sum_{n,k>0} \frac{(-1)^n (-1)^{n+k}}{n^2(n+k)} \\ &= \sum_{n,k>0} \frac{(-1)^{n+k}}{n(n+k)^2} + \sum_{n,k>0} \frac{(-1)^{n+k}}{k(n+k)^2} + \zeta(\bar{1}, 2) - \zeta(\bar{1}, \bar{2}) \\ &= 2\zeta(\bar{2}, 1) + \zeta(\bar{1}, 2) - \zeta(\bar{1}, \bar{2}). \end{aligned} \tag{2.1}$$

Similarly,

$$\begin{aligned} \zeta(2, \bar{1}) + \zeta(\bar{3}) &= \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=1}^n \frac{(-1)^k}{k} = \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=1}^{\infty} \left(\frac{(-1)^k}{k} - \frac{(-1)^{n+k}}{n+k} \right) \\ &= \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=1}^{\infty} (-1)^k \left(\frac{n+k - (-1)^n k}{k(n+k)} \right) \\ &= \sum_{n,k>0} \frac{(-1)^k}{nk(n+k)} + \sum_{n,k>0} \frac{(-1)^k}{n^2(n+k)} - \sum_{n,k>0} \frac{(-1)^{n+k}}{n^2(n+k)} \\ &= \sum_{n,k>0} \left(\frac{1}{n} + \frac{1}{k} \right) \frac{(-1)^k}{(n+k)^2} + \sum_{n,k>0} \frac{(-1)^n (-1)^{n+k}}{n^2(n+k)} - \zeta(\bar{1}, 2) \\ &= \sum_{n,k>0} \frac{(-1)^n (-1)^{n+k}}{n(n+k)^2} + \sum_{n,k>0} \frac{(-1)^k}{k(n+k)^2} + \zeta(\bar{1}, \bar{2}) - \zeta(\bar{1}, 2) \\ &= \zeta(\bar{2}, \bar{1}) + \zeta(2, \bar{1}) + \zeta(\bar{1}, \bar{2}) - \zeta(\bar{1}, 2). \end{aligned} \tag{2.2}$$

Adding equations (2.1) and (2.2) now gives

$$2\zeta(\bar{2}, 1) = \zeta(3) + \zeta(\bar{3}), \quad (2.3)$$

i.e.

$$8\zeta(\bar{2}, 1) = 4 \sum_{n=1}^{\infty} \frac{1 + (-1)^n}{n^3} = 4 \sum_{m=1}^{\infty} \frac{2}{(2m)^3} = \zeta(3),$$

which is (1.5). □

3. FINITE SERIES TRANSFORMATIONS

For any positive integer N , we have

$$\sum_{n=1}^N \frac{1}{n^3} - \sum_{n=1}^N \frac{1}{n^2} \sum_{k=1}^{n-1} \frac{1}{k} = \sum_{n=1}^N \frac{1}{n^2} \sum_{k=1}^n \frac{1}{N-k+1} \quad (3.1)$$

by induction. Alternatively, consider

$$T := \sum_{\substack{n,k=1 \\ k \neq n}}^N \frac{1}{nk(k-n)} = \sum_{\substack{n,k=1 \\ k \neq n}}^N \left(\frac{1}{n} - \frac{1}{k} \right) \frac{1}{(k-n)^2} = 0.$$

On the other hand,

$$\begin{aligned} T &= \sum_{\substack{n,k=1 \\ k \neq n}}^N \frac{1}{n^2} \left(\frac{1}{k-n} - \frac{1}{k} \right) \\ &= \sum_{n=1}^N \frac{1}{n^2} \left(\sum_{k=1}^{n-1} \frac{1}{k-n} + \sum_{k=n+1}^N \frac{1}{k-n} - \sum_{k=1}^N \frac{1}{k} + \frac{1}{n} \right) \\ &= \sum_{n=1}^N \frac{1}{n^3} - \sum_{n=1}^N \frac{1}{n^2} \sum_{k=1}^{n-1} \frac{1}{n-k} + \sum_{n=1}^N \frac{1}{n^2} \left(\sum_{k=n+1}^N \frac{1}{k-n} - \sum_{k=1}^N \frac{1}{k} \right). \end{aligned}$$

Since $T = 0$, this implies that

$$\sum_{n=1}^N \frac{1}{n^3} - \sum_{n=1}^N \frac{1}{n^2} \sum_{k=1}^{n-1} \frac{1}{k} = \sum_{n=1}^N \frac{1}{n^2} \left(\sum_{k=1}^N \frac{1}{k} - \sum_{k=1}^{N-n} \frac{1}{k} \right) = \sum_{n=1}^N \frac{1}{n^2} \sum_{k=1}^n \frac{1}{N-k+1},$$

which is (3.1). But the right hand side satisfies

$$\begin{aligned} \frac{H_N}{N} &= \sum_{n=1}^N \frac{1}{n^2} \cdot \frac{n}{N} \leq \sum_{n=1}^N \frac{1}{n^2} \sum_{k=1}^n \frac{1}{N-k+1} \\ &\leq \sum_{n=1}^N \frac{1}{n^2} \cdot \frac{n}{N-n+1} = \frac{1}{N+1} \sum_{n=1}^N \left(\frac{1}{n} + \frac{1}{N-n+1} \right) = \frac{2H_N}{N+1}. \end{aligned}$$

Letting N grow without bound now gives (1.2), since $\lim_{N \rightarrow \infty} \frac{H_N}{N} = 0$. □

4. GEOMETRIC SERIES

4.1. Convolution of Geometric Series. The following argument is suggested in [76]. A closely related derivation, in which our explicit consideration of the error term is suppressed by taking N infinite, appears in [22]. Let $2 \leq m \in \mathbf{Z}$, and consider

$$\begin{aligned} \sum_{j=1}^{m-2} \zeta(j+1)\zeta(m-j) &= \lim_{N \rightarrow \infty} \sum_{n=1}^N \sum_{k=1}^N \sum_{j=1}^{m-2} \frac{1}{n^{j+1}} \frac{1}{k^{m-j}} \\ &= \lim_{N \rightarrow \infty} \left\{ \sum_{\substack{n,k=1 \\ k \neq n}}^N \left(\frac{1}{n^{m-1}(k-n)k} - \frac{1}{n(k-n)k^{m-1}} \right) + \sum_{n=1}^N \frac{m-2}{n^{m+1}} \right\} \\ &= (m-2)\zeta(m+1) + 2 \lim_{N \rightarrow \infty} \sum_{\substack{n,k=1 \\ k \neq n}}^N \frac{1}{n^{m-1}k(k-n)}. \end{aligned}$$

Thus, we find that

$$\begin{aligned} &(m-2)\zeta(m+1) - \sum_{j=1}^{m-2} \zeta(j+1)\zeta(m-j) \\ &= 2 \lim_{N \rightarrow \infty} \sum_{n=1}^N \frac{1}{n^m} \sum_{\substack{k=1 \\ k \neq n}}^N \left(\frac{1}{k} - \frac{1}{k-n} \right) \\ &= 2 \lim_{N \rightarrow \infty} \sum_{n=1}^N \frac{1}{n^m} \left\{ \sum_{k=1}^{n-1} \frac{1}{k} - \frac{1}{n} + \sum_{k=1}^n \frac{1}{N-k+1} \right\} \\ &= 2\zeta(m, 1) - 2\zeta(m+1) + 2 \lim_{N \rightarrow \infty} \sum_{n=1}^N \frac{1}{n^m} \sum_{k=1}^n \frac{1}{N-k+1}, \end{aligned}$$

and hence

$$2\zeta(m, 1) = m\zeta(m+1) - \sum_{j=1}^{m-2} \zeta(j+1)\zeta(m-j) - 2 \lim_{N \rightarrow \infty} \sum_{n=1}^N \frac{1}{n^m} \sum_{k=1}^n \frac{1}{N-k+1}.$$

But, in light of

$$\sum_{n=1}^N \frac{1}{n^m} \sum_{k=1}^n \frac{1}{N-k+1} \leq \sum_{n=1}^N \frac{1}{n^m} \cdot \frac{n}{N-n+1} \leq \frac{1}{N+1} \sum_{n=1}^N \left(\frac{1}{N-n+1} + \frac{1}{n} \right) = \frac{2H_N}{N+1},$$

the identity (1.3) now follows. \square

4.2. A Sum Formula. Equation (1.2) is the case $n = 3$ of the following result. See [28].

Theorem 3. *If $3 \leq n \in \mathbf{Z}$ then*

$$\zeta(n) = \sum_{j=1}^{n-2} \zeta(n-j, j). \tag{4.1}$$

We discuss a generalization (10.4) of the sum formula (4.1) to arbitrary depth in §10.2.

Proof. Summing the geometric series on the right hand side gives

$$\begin{aligned} \sum_{j=1}^{n-2} \sum_{h=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{h^j(h+m)^{n-j}} &= \sum_{h,m=1}^{\infty} \left[\frac{1}{h^{n-2}m(h+m)} - \frac{1}{m(h+m)^{n-1}} \right] \\ &= \sum_{h=1}^{\infty} \frac{1}{h^{n-1}} \sum_{m=1}^{\infty} \left(\frac{1}{m} - \frac{1}{h+m} \right) - \zeta(n-1, 1) \\ &= \sum_{h=1}^{\infty} \frac{1}{h^{n-1}} \sum_{k=1}^h \frac{1}{k} - \zeta(n-1, 1) \\ &= \sum_{h=1}^{\infty} \frac{1}{h^n} + \sum_{h=1}^{\infty} \frac{1}{h^{n-1}} \sum_{k=1}^{n-1} \frac{1}{k} - \zeta(n-1, 1) \\ &= \zeta(n). \end{aligned}$$

\square

4.3. A q -Analogue. The following argument is based on an idea of Zudilin [78]. We begin with the finite geometric series identity

$$\frac{uv}{(1-u)(1-uv)^s} + \frac{uv^2}{(1-v)(1-uv)^s} = \frac{uv}{(1-u)(1-v)^s} - \sum_{j=1}^{s-1} \frac{uv^2}{(1-v)^{j+1}(1-uv)^{s-j}},$$

valid for all positive integers s and real u, v with $u \neq 1, uv \neq 1$. We now assume $s > 1$, q is real and $0 < q < 1$. Put $u = q^m, v = q^n$ and sum over all positive integers m and n . Thus,

$$\begin{aligned}
& \sum_{m,n>0} \frac{q^{m+n}}{(1-q^m)(1-q^{m+n})^s} + \sum_{m,n>0} \frac{q^{m+2n}}{(1-q^n)(1-q^{m+n})^s} \\
&= \sum_{m,n>0} \frac{q^{m+n}}{(1-q^m)(1-q^n)^s} - \sum_{m,n>0} \frac{q^{m+2n}}{(1-q^n)^s(1-q^{m+n})} \\
&\quad - \sum_{j=1}^{s-2} \sum_{m,n>0} \frac{q^{m+2n}}{(1-q^n)^{j+1}(1-q^{m+n})^{s-j}} \\
&= \sum_{m,n>0} \frac{q^n}{(1-q^n)^s} \left[\frac{q^m}{1-q^m} - \frac{q^{m+n}}{1-q^{m+n}} \right] - \sum_{j=1}^{s-2} \sum_{m,n>0} \frac{q^{m+2n}}{(1-q^n)^{j+1}(1-q^{m+n})^{s-j}} \\
&= \sum_{n>0} \frac{q^n}{(1-q^n)^s} \sum_{m=1}^n \frac{q^m}{1-q^m} - \sum_{j=1}^{s-2} \sum_{m,n>0} \frac{q^{m+2n}}{(1-q^n)^{j+1}(1-q^{m+n})^{s-j}} \\
&= \sum_{n>0} \frac{q^{2n}}{(1-q^n)^{s+1}} + \sum_{n>m>0} \frac{q^{n+m}}{(1-q^n)^s(1-q^m)} - \sum_{j=1}^{s-2} \sum_{m,n>0} \frac{q^{m+2n}}{(1-q^n)^{j+1}(1-q^{m+n})^{s-j}}
\end{aligned}$$

Cancelling the second double sum on the left with the corresponding double sum on the right and replacing $m+n$ by k in the remaining sums now yields

$$\sum_{k>m>0} \frac{q^k}{(1-q^k)^s(1-q^m)} = \sum_{n>0} \frac{q^{2n}}{(1-q^n)^{s+1}} - \sum_{j=1}^{s-2} \sum_{k>m>0} \frac{q^{k+m}}{(1-q^m)^{j+1}(1-q^k)^{s-j}},$$

or equivalently, that

$$\sum_{k>0} \frac{q^{2k}}{(1-q^k)^{s+1}} = \sum_{k>m>0} \frac{q^k}{(1-q^k)^s(1-q^m)} + \sum_{j=1}^{s-2} \sum_{k>m>0} \frac{q^{k+m}}{(1-q^k)^{s-j}(1-q^m)^{j+1}}. \quad (4.2)$$

Multiplying (4.2) through by $(1-q)^{s+1}$ and letting $q \rightarrow 1$ gives

$$\zeta(s+1) = \zeta(s, 1) + \sum_{j=1}^{s-2} \zeta(s-j, j+1),$$

which is just a restatement of (4.1). Taking $s = 2$ gives (1.2) again. \square

As in [24], define the q -analog of a non-negative integer n by

$$[n]_q := \sum_{k=0}^{n-1} q^k = \frac{1 - q^n}{1 - q},$$

and the multiple q -zeta function

$$\zeta[s_1, \dots, s_m] := \sum_{k_1 > \dots > k_m > 0} \prod_{j=1}^m \frac{q^{(s_j-1)k_j}}{[k_j]_q^{s_j}}, \quad (4.3)$$

where s_1, s_2, \dots, s_m are real numbers with $s_1 > 1$ and $s_j \geq 1$ for $2 \leq j \leq m$. Then multiplying (4.2) by $(1 - q)^{s+1}$ and then setting $s = 2$ gives $\zeta[2, 1] = \zeta[3]$, which is a q -analog of (1.2). That is, the latter may be obtained from the former by letting $q \rightarrow 1-$. On the other hand, $s = 3$ in (4.2) gives

$$\zeta[4] + (1 - q)\zeta[3] = \zeta[3, 1] + (1 - q)\zeta[2, 1] + \zeta[2, 2],$$

which, in light of $\zeta[2, 1] = \zeta[3]$ implies $\zeta[3, 1] = \zeta[4] - \zeta[2, 2]$. By Theorem 1 of [24], we know that $\zeta[2, 2]$ reduces to depth 1 multiple q -zeta values. Indeed, by the q -stuffle multiplication rule [24], $\zeta[2]\zeta[2] = 2\zeta[2, 2] + \zeta[4] + (1 - q)\zeta[3]$. Thus,

$$\zeta[3, 1] = \zeta[4] - \zeta[2, 2] = \frac{3}{2}\zeta[4] - \frac{1}{2}(\zeta[2])^2 + \frac{1}{2}(1 - q)\zeta[3],$$

which is a q -analog of the evaluation [14, 15, 16, 18, 19, 20, 21]

$$\zeta(3, 1) = \frac{\pi^4}{360}.$$

Additional material concerning q -analogs of multiple harmonic sums and multiple zeta values can be found in [24, 25, 26, 27].

5. INTEGRAL REPRESENTATIONS

5.1. Single Integrals I. We use the fact that

$$\int_0^1 u^{k-1}(-\log u) du = \frac{1}{k^2}, \quad k > 0. \quad (5.1)$$

Thus

$$\begin{aligned} \sum_{k>n>1} \frac{1}{k^2 n} &= \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k>n} \int_0^1 u^{k-1}(-\log u) du \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 (-\log u) \sum_{k>n} u^{k-1} du \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 (-\log u) \frac{u^n}{1-u} du \end{aligned}$$

$$\begin{aligned}
&= - \int_0^1 \frac{\log u}{1-u} \sum_{n=1}^{\infty} \frac{u^n}{n} du \\
&= \int_0^1 (-\log u)(1-u)^{-1} \log(1-u)^{-1} du.
\end{aligned} \tag{5.2}$$

The interchanges of summation and integration are in each case justified by Lebesgue's monotone convergence theorem. After making the change of variable $t = 1 - u$, we obtain

$$\sum_{k>n>1} \frac{1}{k^2 n} = \int_0^1 \log(1-t)^{-1} (-\log t) \frac{dt}{t} = \int_0^1 (-\log t) \sum_{n=1}^{\infty} \frac{t^{n-1}}{n} dt. \tag{5.3}$$

Again, since all terms of the series are positive, Lebesgue's monotone convergence theorem permits us to interchange the order of summation and integration. Thus, invoking (5.1) again, we obtain

$$\sum_{k>n>1} \frac{1}{k^2 n} = \sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 (-\log t) t^{n-1} dt = \sum_{n=1}^{\infty} \frac{1}{n^3},$$

which is (1.2). □

5.2. Single Integrals II. The Laplace transform

$$\int_0^1 x^{r-1} (-\log x)^\sigma dx = \int_0^\infty e^{-ru} u^\sigma du = \frac{\Gamma(\sigma+1)}{r^{\sigma+1}}, \quad r > 0, \quad \sigma > -1, \tag{5.4}$$

generalizes (5.1) and yields the representation

$$\zeta(m+1) = \frac{1}{m!} \sum_{r=1}^{\infty} \frac{\Gamma(m+1)}{r^{m+1}} = \frac{1}{m!} \sum_{r=1}^{\infty} \int_0^1 x^{r-1} (-\log x)^m dx = \frac{(-1)^m}{m!} \int_0^1 \frac{\log^m x}{1-x} dx.$$

The interchange of summation and integration is valid if $m > 0$. The change of variable $x \mapsto 1 - x$ now yields

$$\zeta(m+1) = \frac{(-1)^m}{m!} \int_0^1 \log^m(1-x) \frac{dx}{x}, \quad 1 \leq m \in \mathbf{Z}. \tag{5.5}$$

In [46], equation (5.4) in conjunction with clever use of change of variable and integration by parts, is used to prove the identity

$$k! \zeta(k+2) = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \cdots \sum_{n_k=1}^{\infty} \frac{1}{n_1 n_2 \cdots n_k} \sum_{p=1+n_1+n_2+\cdots+n_k} \frac{1}{p^2}, \quad 0 \leq k \in \mathbf{Z}. \tag{5.6}$$

The case $k = 1$ of (5.6) is precisely (1.2). We give here a slightly simpler proof of (5.6), dispensing with the integration by parts.

From (5.4),

$$\begin{aligned}
 k!\zeta(k+2) &= \sum_{r=1}^{\infty} \frac{1}{r} \cdot \frac{\Gamma(k+1)}{r^{k+1}} = \sum_{r=1}^{\infty} \frac{1}{r} \int_0^1 x^{r-1} (-\log x)^k dx \\
 &= \int_0^1 (-\log x)^k \log(1-x)^{-1} \frac{dx}{x} \\
 &= \int_0^1 \log^k(1-x)^{-1} (-\log x) \frac{dx}{1-x} \\
 &= \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \cdots \sum_{n_k=1}^{\infty} \frac{1}{n_1 n_2 \cdots n_k} \int_0^1 \frac{x^{n_1+n_2+\cdots+n_k}}{1-x} (-\log x) dx \\
 &= \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \cdots \sum_{n_k=1}^{\infty} \frac{1}{n_1 n_2 \cdots n_k} \sum_{p>n_1+n_2+\cdots+n_k} \int_0^1 x^{p-1} (-\log x) dx \\
 &= \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \cdots \sum_{n_k=1}^{\infty} \frac{1}{n_1 n_2 \cdots n_k} \sum_{p>n_1+n_2+\cdots+n_k} \frac{1}{p^2}.
 \end{aligned}$$

□

5.3. Double Integrals I. Write

$$\begin{aligned}
 \zeta(2, 1) &= \sum_{k,m>0} \frac{1}{k(m+k)^2} = \int_0^1 \int_0^1 \sum_{k>0} \frac{(xy)^k}{k} \sum_{m>0} (xy)^{m-1} dx dy \\
 &= - \int_0^1 \int_0^1 \frac{\log(1-xy)}{1-xy} dx dy.
 \end{aligned}$$

Now make the change of variable $u = xy$, $v = x/y$ with Jacobian $1/(2v)$, obtaining

$$\zeta(2, 1) = -\frac{1}{2} \int_0^1 \frac{\log(1-u)}{1-u} \int_u^{1/u} \frac{dv}{v} du = \int_0^1 \frac{(\log u) \log(1-u)}{1-u} du,$$

which is (5.2). Now continue as in §5.1.

□

5.4. Double Integrals II. The following is reconstructed from a phone conversation with Krishna Alladi. See also [6]. Let $\varepsilon > 0$. By expanding the integrand as a geometric series, one sees that

$$\sum_{n=1}^{\infty} \frac{1}{(n+\varepsilon)^2} = \int_0^1 \int_0^1 \frac{(xy)^\varepsilon}{1-xy} dx dy.$$

Differentiating with respect to ε and then letting $\varepsilon = 0$ gives

$$\zeta(3) = -\frac{1}{2} \int_0^1 \int_0^1 \frac{\log(xy)}{1-xy} dx dy = -\frac{1}{2} \int_0^1 \int_0^1 \frac{\log x + \log y}{1-xy} dx dy = - \int_0^1 \int_0^1 \frac{\log x}{1-xy} dx$$

by symmetry. Now integrate with respect to y to get

$$\zeta(3) = \int_0^1 (\log x) \log(1-x) \frac{dx}{x}. \quad (5.7)$$

Comparing (5.7) with (5.3) completes the proof of (1.2). \square

5.5. Integration by Parts. Start with (5.7) and integrate by parts, obtaining

$$2\zeta(3) = \int_0^1 \frac{\log^2 x}{1-x} dx = \int_0^1 \log^2(1-x) \frac{dx}{x} = \sum_{n,k>0} \int_0^1 \frac{x^{n+k-1}}{nk} dx = \sum_{n,k>0} \frac{1}{nk(n+k)}.$$

Now see §2. \square

5.6. Triple Integrals I. This time, instead of (5.1) we use the elementary identity

$$\frac{1}{k^2 n} = \int_0^1 y_1^{-1} \int_0^{y_1} y_2^{k-n-1} \int_0^{y_2} y_3^{n-1} dy_3 dy_2 dy_1, \quad k > n > 0.$$

This yields

$$\sum_{k>n>0} \frac{1}{k^2 n} = \int_0^1 y_1^{-1} \int_0^{y_1} (1-y_2)^{-1} \int_0^{y_2} (1-y_3)^{-1} dy_3 dy_2 dy_1. \quad (5.8)$$

Now make the change of variable $y_i = 1 - x_i$ for $i = 1, 2, 3$ to obtain

$$\begin{aligned} \sum_{k>n>0} \frac{1}{k^2 n} &= \int_0^1 (1-x_1)^{-1} \int_{x_1}^1 x_2^{-1} \int_{x_2}^1 x_3^{-1} dx_3 dx_2 dx_1 \\ &= \int_0^1 x_3^{-1} \int_0^{x_3} x_2^{-1} \int_0^{x_2} (1-x_1)^{-1} dx_1 dx_2 dx_3. \end{aligned}$$

After expanding $(1-x_1)^{-1}$ into a geometric series and interchanging the order of summation and integration, one arrives at

$$\sum_{k>n>0} \frac{1}{k^2 n} = \sum_{n=1}^{\infty} \int_0^1 x_3^{-1} \int_0^{x_3} x_2^{-1} \int_0^{x_2} x_1^{n-1} dx_1 dx_2 dx_3 = \sum_{n=1}^{\infty} \frac{1}{n^3},$$

as required. \square

More generally [14, 15, 16, 19, 20, 55],

$$\zeta(s_1, \dots, s_k) = \sum_{n_1 > \dots > n_k > 0} \prod_{j=1}^k n_j^{-s_j} = \int \prod_{j=1}^k \left(\prod_{r=1}^{s_j-1} \frac{dt_r^{(j)}}{t_r^{(j)}} \right) \frac{dt_{s_j}^{(j)}}{1-t_{s_j}^{(j)}}, \quad (5.9)$$

where the integral is over the simplex

$$1 > t_1^{(1)} > \dots > t_{s_1}^{(1)} > \dots > t_1^{(k)} > \dots > t_{s_k}^{(k)} > 0,$$

and is abbreviated by

$$\int_0^1 \prod_{j=1}^k a^{s_j-1} b, \quad a = \frac{dt}{t}, \quad b = \frac{dt}{1-t}. \quad (5.10)$$

The change of variable $t \mapsto 1-t$ at each level of integration switches the differential forms a and b , thus yielding the duality formula [14] [55, p. 483] (conjectured in [53])

$$\zeta(s_1 + 2, \{1\}^{r_1}, \dots, s_n + 2, \{1\}^{r_n}) = \zeta(r_n + 2, \{1\}^{s_n}, \dots, r_1 + 2, \{1\}^{s_1}), \quad (5.11)$$

which is valid for all nonnegative integers $s_1, r_1, \dots, s_n, r_n$. The case $s_1 = 0, r_1 = 1$ of (5.11) is (1.2). More generally, (1.4) can be restated as

$$\int_0^1 (ab^2)^n = \int_0^1 (a^2b)^n$$

and thus (1.4) is recovered by taking each $s_j = 0$ and each $r_j = 1$ in (5.11). For further generalizations and extensions of duality, see [15, 24, 25].

For alternations, we require in addition the differential form $c := -dt/(1+t)$ with which we may form the generating function

$$\sum_{n=1}^{\infty} z^{3n} \zeta(\{\bar{2}, 1\}^n) = \sum_{n=0}^{\infty} \left\{ z^{6n+3} \int_0^1 (ac^2ab^2)^n ac^2 + z^{6n+6} \int_0^1 (ac^2ab^2)^{6n+6} \right\}.$$

A lengthy calculation verifies that the only changes of variable that preserve the unit interval and send the non-commutative polynomial ring $\mathbf{Q}\langle a, b \rangle$ into $\mathbf{Q}\langle a, b, c \rangle$ are

$$S(a, b) = S(a, b), \quad t \mapsto t, \quad (5.12)$$

$$S(a, b) = R(b, a), \quad t \mapsto 1-t, \quad (5.13)$$

$$S(a, b) = S(2a, b+c), \quad t \mapsto t^2, \quad (5.14)$$

$$S(a, b) = S(a+c, b-c), \quad t \mapsto \frac{2t}{1+t}, \quad (5.15)$$

$$S(a, b) = S(a+2c, 2b-2c), \quad t \mapsto \frac{4t}{(1+t)^2}, \quad (5.16)$$

and compositions thereof, such as $t \mapsto 1-2t/(1+t) = (1-t)/(1+t)$, etc. In (5.12)–(5.16), $S(a, b)$ denotes a non-commutative word on the alphabet $\{a, b\}$ and $R(b, a)$ denotes the word formed by switching a and b and then reversing the order of the letters.

Now view a, b and c as indeterminates. In light of the polynomial *identity*

$$ab^2 - 8ac^2 = 2[ab^2 - 2a(b+c)^2] + 8[ab^2 - (a+c)(b-c)^2] + [(a+2c)(2b-2c)^2 - ab^2]$$

in the non-commutative ring $\mathbf{Z}\langle a, b, c \rangle$ and the transformations (5.14), (5.15) and (5.16) above, each bracketed term vanishes when we make the identifications $a = dt/t, b =$

$dt/(1-t)$, $c = -dt/(1+t)$ and perform the requisite iterated integrations. Thus,

$$\zeta(2, 1) - 8\zeta(\bar{2}, 1) = \int_0^1 ab^2 - 8 \int_0^1 ac^2 = 0,$$

which in light of (1.2) proves (1.5). \square

5.7. Triple Integrals II. First, note that by expanding the integrands in geometric series and integrating term by term,

$$\zeta(2, 1) = 8 \int_0^1 \frac{dx}{x} \int_0^x \frac{y dy}{1-y^2} \int_0^y \frac{z dz}{1-z^2}.$$

Now make the change of variable

$$\frac{x dx}{1-x^2} = \frac{du}{1+u}, \quad \frac{y dy}{1-y^2} = \frac{dv}{1+v}, \quad \frac{z dz}{1-z^2} = \frac{dw}{1+w}$$

to obtain the equivalent integral

$$\zeta(2, 1) = 8 \int_0^\infty \left(\frac{du}{2u} + \frac{du}{2(2+u)} - \frac{du}{1+u} \right) \int_0^u \frac{dv}{1+v} \int_0^v \frac{dw}{1+w}.$$

The two inner integrals can be directly performed, leading to

$$\zeta(2, 1) = 4 \int_0^\infty \frac{\log^2(u+1)}{u(u+1)(u+2)} du.$$

Finally, make the substitution $u+1 = 1/\sqrt{1-x}$ to obtain

$$\zeta(2, 1) = \frac{1}{2} \int_0^1 \frac{\log^2(1-x)}{x} dx = \zeta(3),$$

by (5.5).

5.8. Complex Line Integrals I. Here we apply the Mellin inversion formula [3, p. 243], [72, pp. 130–132]

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} y^z \frac{dz}{z} = \begin{cases} 1, & y > 1 \\ 0, & y < 1 \\ \frac{1}{2}, & y = 1 \end{cases}$$

which is valid for fixed $c > 0$. It follows that if $c > 0$ and $s-1 > c > 1-t$ then the Perron-type formula

$$\begin{aligned} \zeta(s, t) + \frac{1}{2}\zeta(s+t) &= \sum_{n=1}^{\infty} n^{-s} \sum_{k=1}^{\infty} k^{-t} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(\frac{n}{k}\right)^z \frac{dz}{z} \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \zeta(s-z)\zeta(t+z) \frac{dz}{z} \end{aligned} \quad (5.17)$$

is valid. (Interchanging the order of summation and integration is permissible by absolute convergence.) Although we have not yet found a way to exploit (5.17) in proving identities such as (1.2), we note that by integrating around the rectangular contour with corners $(\pm c \pm iM)$ and then letting $M \rightarrow +\infty$, one can readily establish the stuffle [15, 19, 23, 24] formula in the form

$$\zeta(s, t) + \frac{1}{2}\zeta(s+t) + \zeta(t, s) + \frac{1}{2}\zeta(t+s) = \zeta(s)\zeta(t), \quad s, t > 1 + c.$$

The right hand side arises as the residue contribution of the integrand at $z = 0$. One can also use (5.17) to establish

$$\sum_{s=2}^{\infty} [\zeta(s, 1) + \frac{1}{2}\zeta(s+1)] x^{s-1} = \sum_{n>m>0} \frac{x}{mn(n-x)} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{x}{n(n-x)},$$

but this is easy to prove directly.

5.9. Complex Line Integrals II. We let $\lambda(s) := \sum_{n>0} \lambda_n n^{-s}$ represent a formal Dirichlet series, with real coefficients λ_n and we set $s := \sigma + i\tau$ with $\sigma = \Re(s) > 0$, and consider the following integral:

$$\iota_{\lambda}(\sigma) := \int_0^{\infty} \left| \frac{\lambda(s)}{s} \right|^2 d\tau = \frac{1}{2} \int_{-\infty}^{\infty} \left| \frac{\lambda(s)}{s} \right|^2 d\tau, \quad (5.18)$$

as a function of λ . We begin with a useful variant of the Mellin inversion formula, namely

$$\int_{-\infty}^{\infty} \frac{\cos(at)}{t^2 + u^2} dt = \frac{\pi}{u} e^{-au}, \quad (5.19)$$

for $u, a > 0$, as follows by contour integration, from a computer algebra system, or otherwise. This leads to

Theorem 4. (Theorem 1 of [7]). *For $\lambda(s) = \sum_{n=1}^{\infty} \lambda_n n^{-s}$ and $s = \sigma + i\tau$ with fixed $\sigma = \Re(s) > 0$ such that the Dirichlet series is absolutely convergent it is true that*

$$\iota_{\lambda}(\sigma) = \int_0^{\infty} \left| \frac{\lambda(s)}{s} \right|^2 d\tau = \frac{\pi}{2\sigma} \sum_{n=1}^{\infty} \frac{\Lambda_n^2 - \Lambda_{n-1}^2}{n^{2\sigma}}, \quad (5.20)$$

where $\Lambda_n := \sum_{k=1}^n \lambda_k$ and $\Lambda_0 := 0$.

More generally, for given absolutely convergent Dirichlet series $\alpha(s) := \sum_{n=1}^{\infty} \alpha_n n^{-s}$ and $\beta(s) := \sum_{n=1}^{\infty} \beta_n n^{-s}$

$$\frac{1}{2} \int_{-\infty}^{\infty} \frac{\alpha(s) \overline{\beta(s)}}{\sigma^2 + \tau^2} d\tau = \frac{\pi}{2\sigma} \sum_{n=1}^{\infty} \frac{A_n \overline{B_n} - A_{n-1} \overline{B_{n-1}}}{n^{2\sigma}}, \quad (5.21)$$

in which $A_n = \sum_{k=1}^n \alpha_k$ and $B_n = \sum_{k=1}^n \beta_k$.

Note that the righthand side of (5.20) is always a generalized Euler sum. \square

For the Riemann zeta function, and for $\sigma > 1$, Theorem 4 applies and yields

$$\frac{\sigma}{\pi} \iota_{\zeta}(\sigma) = \zeta(2\sigma - 1) - \frac{1}{2} \zeta(2\sigma),$$

as $\lambda_n = 1$ and $\Lambda_n = n - 1/2$. By contrast it is known that on the critical line

$$\frac{1/2}{\pi} \iota_{\zeta} \left(\frac{1}{2} \right) = \log(\sqrt{2\pi}) - \frac{1}{2} \gamma.$$

There are similar formulae for $s \mapsto \zeta(s - k)$ with k integral. For instance, applying the result in (5.20) with $\zeta_1 := t \mapsto \zeta(t + 1)$ yields

$$\frac{1}{\pi} \int_0^{\infty} \frac{|\zeta(3/2 + i\tau)|^2}{1/4 + \tau^2} d\tau = \frac{1}{\pi} \iota_{\zeta_1} \left(\frac{1}{2} \right) = 2\zeta(2, 1) + \zeta(3) = 3\zeta(3),$$

on using (1.2). For the *alternating zeta function*, $\alpha := s \mapsto (1 - 2^{1-s})\zeta(s)$, the same approach via (5.21) produces

$$\frac{1}{\pi} \int_0^{\infty} \frac{\alpha(3/2 + i\tau) \overline{\alpha(3/2 + i\tau)}}{1/4 + \tau^2} d\tau = 2\zeta(\bar{2}, \bar{1}) + \zeta(3) = 3\zeta(2) \log(2) - \frac{9}{4}\zeta(3),$$

and

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\alpha(3/2 + i\tau) \overline{\zeta(3/2 + i\tau)}}{1/4 + \tau^2} d\tau = \zeta(\bar{2}, 1) + \zeta(2, \bar{1}) + \alpha(3) = \frac{9}{8}\zeta(2) \log(2) - \frac{3}{4}\zeta(3),$$

since as we have seen repeatedly $\zeta(\bar{2}, 1) = \zeta(3)/8$; while $\zeta(2, \bar{1}) = \zeta(3) - 3/2\zeta(2)\log(2)$ and $\zeta(\bar{2}, \bar{1}) = 3/2\zeta(2)\log(2) - 13/8\zeta(3)$, (e.g., [17]).

As in the previous subsection we have not been able to directly obtain (1.5) or even (1.2), but we have connected them to quite difficult line integrals.

5.10. Contour Integrals and Residues. Following [69], let \mathcal{C}_n ($n \in \mathbf{Z}^+$) be the square contour with vertices $(\pm 1 \pm i)(n + 1/2)$. Using the asymptotic expansion

$$\psi(z) \sim \log z - \frac{1}{2z} - \sum_{r=1}^{\infty} \frac{B_{2r}}{2r z^{2r}}, \quad |\arg z| < \pi$$

in terms of the Bernoulli numbers

$$\frac{t}{1 - e^{-t}} = 1 + \frac{t}{2} + \sum_{r=1}^{\infty} \frac{B_{2r}}{(2r)!} t^{2r}, \quad |t| < 2\pi$$

and the identity

$$\psi(z) = \psi(-z) - \frac{1}{z} - \pi \cot \pi z,$$

we can show that for each integer $k \geq 2$,

$$\lim_{n \rightarrow \infty} \int_{\mathcal{C}_n} z^{-k} \psi^2(-z) dz = 0.$$

Then by the residue theorem, we obtain

Theorem 5 (Theorem 3 of [69]). *For every integer $k \geq 2$,*

$$2 \sum_{n=1}^{\infty} n^{-k} \psi(n) = k\zeta(k+1) - 2\gamma\zeta(k) - \sum_{j=1}^{k-1} \zeta(j)\zeta(k-j+1),$$

where $\gamma = 0.577215664\dots$ is Euler's constant.

In light of the identity

$$\psi(n) + \gamma = H_{n-1} = \sum_{k=1}^{n-1} \frac{1}{k}, \quad n \in \mathbf{Z}^+,$$

Theorem 5 is equivalent to (1.3). The case $k = 2$ thus gives (1.2). \square

Flajolet and Salvy [47] developed the residue approach more systematically, and applied it to a number of other Euler sum identities in addition to (1.3).

6. WITTEN ZETA-FUNCTIONS

We recall that for $r, s > 1/2$:

$$\mathcal{W}(r, s, t) := \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{n^r m^s (n+m)^t}$$

is a *Witten ζ -function*, [77, 43, 40]. We refer to [77] for a description of the uses of more general Witten ζ -functions. Ours are also called *Tornheim double sums*, [43]. There is a simple algebraic relation

$$\mathcal{W}(r, s, t) = \mathcal{W}(r-1, s, t+1) + \mathcal{W}(r, s-1, t+1). \quad (6.1)$$

This is based on writing

$$\frac{m+n}{(m+n)^{t+1}} = \frac{m}{(m+n)^{t+1}} + \frac{n}{(m+n)^{t+1}}.$$

Also

$$\mathcal{W}(r, s, t) = \mathcal{W}(s, r, t), \quad (6.2)$$

and

$$\mathcal{W}(r, s, 0) = \zeta(r)\zeta(s) \quad \text{while} \quad \mathcal{W}(r, 0, t) = \zeta(t, r). \quad (6.3)$$

Hence, $\mathcal{W}(s, s, t) = 2\mathcal{W}(s, s-1, t+1)$ and so

$$\mathcal{W}(1, 1, 1) = 2\mathcal{W}(1, 0, 2) = 2\zeta(2, 1) = 2\zeta(3).$$

Note the analogue to (6.1), viz. $\zeta(s, t) + \zeta(t, s) = \zeta(s)\zeta(t) - \zeta(s+t)$, shows $\mathcal{W}(s, 0, s) = 2\zeta(s, s) = \zeta^2(s) - \zeta(2s)$. Thus, $\mathcal{W}(2, 0, 2) = 2\zeta(2, 2) = \pi^4/36 - \pi^4/90 = \pi^4/72$.

More generally, recursive use of (6.1) and (6.2), along with initial conditions (6.3) shows that *all integer $\mathcal{W}(s, r, t)$ values are expressible in terms of double (and single) Euler sums*. If we start with $\Gamma(s)/(m+n)^t = \int_0^1 (-\log \sigma)^{t-1} \sigma^{m+n-1} d\sigma$ we obtain

$$\mathcal{W}(r, s, t) = \frac{1}{\Gamma(t)} \int_0^1 \text{Li}_r(\sigma) \text{Li}_s(\sigma) \frac{(-\log \sigma)^{t-1}}{\sigma} d\sigma. \quad (6.4)$$

For example, we recover an analytic proof of

$$2\zeta(2, 1) = \mathcal{W}(1, 1, 1) = \int_0^1 \frac{\ln^2(1-\sigma)}{\sigma} d\sigma = 2\zeta(3), \quad (6.5)$$

Indeed S in the proof of §2 is precisely $\mathcal{W}(1, 1, 1)$.

We may now discover analytic as opposed to algebraic relations. Integration by parts yields

$$\mathcal{W}(r, s+1, 1) + \mathcal{W}(r+1, s, 1) = \text{Li}_{r+1}(1) \text{Li}_{s+1}(1) = \zeta(r+1)\zeta(s+1), \quad (6.6)$$

So, in particular, $\mathcal{W}(s+1, s, 1) = \zeta^2(s+1)/2$.

Symbolically, *Maple* immediately evaluates $\mathcal{W}(2, 1, 1) = \pi^4/72$, and while it fails directly with $\mathcal{W}(1, 1, 2)$, we know it must be a multiple of π^4 or equivalently $\zeta(4)$; and numerically obtain $\mathcal{W}(1, 1, 2)/\zeta(4) = .499999999999999998\dots$

6.1. The Hilbert Matrix. Letting $a_n := 1/n^r$ and $b_n := 1/n^s$, inequality (1.7) of Section 1.3 yields

$$\mathcal{W}(r, s, 1) \leq \pi \csc\left(\frac{\pi}{p}\right) \sqrt[p]{\zeta(pr)} \sqrt[q]{\zeta(qs)}. \quad (6.7)$$

Indeed, the constant in (1.7) is best possible [52, 70]. We consider

$$\mathcal{R}_p(s) := \frac{\mathcal{W}((p-1)s, s, 1)}{\pi \zeta(ps)},$$

and observe that with $\sigma_n^p(s) := \sum_{m=1}^{\infty} (n/m)^{-(p-1)s} / (n+m) \rightarrow \pi \csc\left(\frac{\pi}{q}\right)$, we have

$$\begin{aligned} \mathcal{L}_p &:= \lim_{s \rightarrow 1/p} (ps-1) \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{n^{-s} m^{-(p-1)s}}{n+m} = \lim_{s \rightarrow 1/p} (ps-1) \sum_{n=1}^{\infty} \frac{1}{n^{ps}} \sigma_n^p(s) \\ &= \lim_{s \rightarrow 1/p} (ps-1) \sum_{n=1}^{\infty} \frac{\{\sigma_n^p(s) - \pi \csc(\pi/q)\}}{n^{ps}} + \lim_{s \rightarrow 1/p} (2s-1)\zeta(ps) \pi \csc\left(\frac{\pi}{q}\right) \end{aligned}$$

$$= 0 + \pi \operatorname{csc} \left(\frac{\pi}{q} \right).$$

Setting $r := (p - 1)s$, $s \rightarrow 1/p^+$ we check that $\zeta(ps)^{1/p} \zeta(qr)^{1/q} = \zeta(ps)$ and hence the best constant in (6.7) is the one given.

To recapitulate in terms of the celebrated infinite *Hilbert matrix*, $\mathcal{H}_0 := \{1/(m + n)\}_{m,n=1}^\infty$, [9, pp. 250–252], we have actually proven:

Theorem 6. *Let $1 < p, q < \infty$ be given with $1/p + 1/q = 1$. The Hilbert matrix \mathcal{H}_0 determines a bounded linear mappings from the sequence space ℓ^p to itself such that*

$$\|\mathcal{H}_0\|_{p,p} = \lim_{s \rightarrow 1/p} \frac{\mathcal{W}(s, (p - 1)s, 1)}{\zeta(ps)} = \pi \operatorname{csc} \left(\frac{\pi}{p} \right).$$

Proof. Appealing to the isometry between $(\ell^p)^*$ and ℓ^q , and given the evaluation \mathcal{L}_p above, we directly compute the operator norm of \mathcal{H}_0 as

$$\|\mathcal{H}_0\|_{p,p} = \sup_{\|x\|_p=1} \|\mathcal{H}_0 x\|_p = \sup_{\|y\|_q=1} \sup_{\|x\|_p=1} \langle \mathcal{H}_0 x, y \rangle = \pi \operatorname{csc} \left(\frac{\pi}{p} \right).$$

□

A delightful operator-theoretic introduction to the Hilbert matrix \mathcal{H}_0 is given by Choi in his Chauvenet prize winning article [37].

One may also study the corresponding behaviour of Hardy’s inequality (1.8). For example, setting $a_n := 1/n$ in (1.8) and denoting $H_n := \sum_{k=1}^n 1/k$ yields

$$\sum_{n=1}^\infty \left(\frac{H_n}{n} \right)^p \leq \left(\frac{p}{p-1} \right)^p \zeta(p).$$

Application of the integral test and the evaluation

$$\int_1^\infty \left(\frac{\log x}{x} \right)^p dx = \frac{\Gamma(1+p)}{(p-1)^{p+1}},$$

for $p > 1$ easily shows the constant is again best possible.

7. A STIRLING NUMBER GENERATING FUNCTION

Following [35], we begin with the integral representation (5.5) of §5.2. In light of the expansion

$$\frac{(-1)^m}{m!} \log^m(1-x) = \sum_{n=0}^\infty u(n, m) \frac{x^n}{n!}, \quad 0 \leq m \in \mathbf{Z},$$

in terms of the unsigned Stirling numbers of the first kind (also referred to as the Stirling cycle numbers in [50]), we have

$$\zeta(m+1) = \int_0^1 \left\{ \sum_{n=1}^{\infty} u(n, m) \frac{x^n}{n!} \right\} \frac{dx}{x} = \sum_{n=1}^{\infty} \frac{u(n, m)}{n! n}, \quad 1 \leq m \in \mathbf{Z}.$$

Telescoping the known recurrence

$$u(n, m) = u(n-1, m-1) + (n-1)u(n-1, m), \quad 1 \leq m \leq n, \quad (7.1)$$

yields

$$u(n, m) = (n-1)! \left\{ \delta_{m,1} + \sum_{j=1}^{n-1} \frac{u(j, m-1)}{j!} \right\}. \quad (7.2)$$

Iterating this gives the representation

$$\zeta(m+1) = \zeta(2, \{1\}^{m-1}), \quad 1 \leq m \in \mathbf{Z},$$

the $m=2$ case of which is (1.2). See also $n=0$ in (10.3) below. \square

For the alternating case, we begin by writing the recurrence (7.1) in the form

$$u(n+1, k) + (j-n)u(n, k) = u(n, k-1) + j u(n, k).$$

Following [35], multiply both sides by $(-1)^{n+k+1} j^{k-m-1} / (j-n)_n$, where $1 \leq n \leq j-1$ and $k, m \in \mathbf{Z}^+$, yielding

$$\begin{aligned} (-1)^k \left\{ \frac{(-1)^{n+1} u(n+1, k)}{(j-n)_n} - \frac{(-1)^n u(n, k)}{(j-n+1)_{n-1}} \right\} j^{k-m-1} \\ = \frac{(-1)^n}{(j-n)_n} \left\{ (-1)^{k-1} u(n, k-1) j^{k-m-1} - (-1)^k u(n, k) j^{k-m} \right\}. \end{aligned}$$

Now sum on $1 \leq k \leq m$ and $1 \leq n \leq j-1$, obtaining

$$\sum_{k=1}^m \frac{(-1)^{k+j} u(j, k)}{j! j^{m-k}} - \frac{1}{j^m} = \frac{(-1)^{m+1}}{(j-1)!} \sum_{n=m}^{j-1} (-1)^n (j-n-1)! u(n, m).$$

Finally, sum on $j \in \mathbf{Z}^+$ to obtain

$$\zeta(m) = \sum_{k=1}^m \sum_{j=k}^{\infty} \frac{(-1)^{k+j} u(j, k)}{j! j^{m-k}} + \sum_{n=m}^{\infty} (-1)^{n+m} u(n, m) \sum_{j=n+1}^{\infty} \frac{(j-1-n)!}{(j-1)!}.$$

Noting that

$$\sum_{j=n+1}^{\infty} \frac{(j-1-n)!}{(j-1)!} = \sum_{k=0}^{\infty} \frac{k!}{(k+n)!} = \frac{1}{n!} {}_2F_1(1, 1; n+1; 1) = \frac{1}{(n-1)!(n-1)},$$

we find that

$$\zeta(m) = \sum_{k=1}^m \sum_{j=k}^{\infty} \frac{(-1)^{j+k} u(j, k)}{j! j^{m-k}} + \sum_{n=m}^{\infty} \frac{(-1)^{n+m} u(n, m)}{(n-1)!(n-1)}.$$

Now employ the recurrence (7.1) again to get

$$\begin{aligned} \zeta(m) &= \sum_{k=1}^{m-2} \sum_{j=k}^{\infty} \frac{(-1)^{j+k} u(j, k)}{j! j^{m-k}} + \sum_{j=m-1}^{\infty} \frac{(-1)^{j+m-1} u(j, m-1)}{j! j} + \sum_{j=m}^{\infty} \frac{(-1)^{j+m} u(j, m)}{j!} \\ &\quad + \sum_{n=m}^{\infty} \frac{(-1)^{n+m} u(n-1, m)}{(n-1)!} + \sum_{n=m}^{\infty} \frac{(-1)^{n+m} u(n-1, m-1)}{(n-1)!(n-1)} \\ &= \sum_{k=1}^{m-2} \sum_{j=k}^{\infty} \frac{(-1)^{j+k} u(j, k)}{j! j^{m-k}} + 2 \sum_{j=m-1}^{\infty} \frac{(-1)^{j+m-1} u(j, m-1)}{j! j}. \end{aligned} \tag{7.3}$$

Using (7.2) again, we find that the case $m = 3$ of (7.3) gives

$$\begin{aligned} \zeta(3) &= \sum_{j=1}^{\infty} \frac{(-1)^j u(j, 1)}{j! j^2} + 2 \sum_{j=2}^{\infty} \frac{(-1)^j u(j, 2)}{j! j} \\ &= \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j^3} + 2 \sum_{j=2}^{\infty} \frac{(-1)^j}{j! j} (j-1)! \sum_{k=1}^{j-1} \frac{u(k, 1)}{k!} \\ &= \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j^3} + 2 \sum_{j=2}^{\infty} \frac{(-1)^j}{j^2} \sum_{k=1}^{j-1} \frac{1}{k} \\ &= 2\zeta(\bar{2}, 1) - \zeta(\bar{3}), \end{aligned}$$

which easily rearranges to give (2.3), shown in §2 to be trivially equivalent to (1.5). \square

8. POLYLOGARITHM IDENTITIES

8.1. Dilogarithm and Trilogarithm. Consider the power series

$$J(x) := \zeta_x(2, 1) = \sum_{n>k>0} \frac{x^n}{n^2 k}, \quad 0 \leq x \leq 1.$$

In light of (12.3), we have

$$J(x) = \int_0^x \frac{dt}{t} \int_0^t \frac{du}{1-u} \int_0^v \frac{dv}{1-v} = \int_0^x \frac{\log^2(1-t)}{2t} dt.$$

The computer algebra package MAPLE readily evaluates

$$\int_0^x \frac{\log^2(1-t)}{2t} dt = \zeta(3) + \frac{1}{2} \log^2(1-x) \log(x) + \log(1-x) \text{Li}_2(1-x) - \text{Li}_3(1-x) \tag{8.1}$$

where

$$\text{Li}_s(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^s}$$

is the classical polylogarithm [59, 60]. (One can also readily verify the identity (8.1) by differentiating both sides by hand, and then checking (8.1) trivially holds as $x \rightarrow 0+$. See also [5, p. 251, Entry 9].) Thus,

$$J(x) = \zeta(3) + \frac{1}{2} \log^2(1-x) \log(x) + \log(1-x) \text{Li}_2(1-x) - \text{Li}_3(1-x).$$

Letting $x \rightarrow 1-$ gives (1.2) again. \square

In [5, p. 251, Entry 9], we also find that

$$J(-z) + J(-1/z) = -\frac{1}{6} \log^3 z - \text{Li}_2(-z) \log z + \text{Li}_3(-z) + \zeta(3) \quad (8.2)$$

and

$$J(1-z) = \frac{1}{2} \log^2 z \log(z-1) - \frac{1}{3} \log^3 z - \text{Li}_2(1/z) \log z - \text{Li}_3(1/z) + \zeta(3). \quad (8.3)$$

Putting $z = 1$ in (8.2) and employing the well-known dilogarithm evaluation [59, p. 4]

$$\text{Li}_2(-1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

gives (1.5). Putting $z = 2$ in (8.3) and employing the dilogarithm evaluation [59, p. 6]

$$\text{Li}_2\left(\frac{1}{2}\right) = \sum_{n=1}^{\infty} \frac{1}{n^2 2^n} = \frac{\pi^2}{12} - \frac{1}{2} \log^2 2$$

and the trilogarithm evaluation [59, p. 155]

$$\text{Li}_3\left(\frac{1}{2}\right) = \sum_{n=1}^{\infty} \frac{1}{n^3 2^n} = \frac{7}{8} \zeta(3) - \frac{\pi^2}{12} \log 2 + \frac{1}{6} \log^3 2$$

gives (1.5) again. \square

Finally, as in [15, Lemma 10.1], differentiation shows that

$$J(-x) = -J(x) + \frac{1}{4} J(x^2) + J\left(\frac{2x}{x+1}\right) - \frac{1}{8} J\left(\frac{4x}{(x+1)^2}\right). \quad (8.4)$$

Putting [15, Theorem 10.3] $x = 1$ gives $8J(-1) = J(1)$ immediately, i.e. (1.5). \square

In [15], it is noted that once the component functions in (8.4) are known, the coefficients can be deduced by computing each term to high precision with a common transcendental value of x and then employing a linear relations finding algorithm. We note here a somewhat more satisfactory method for arriving at (8.4).

First, as in §5.6 one must determine the fundamental transformations (5.12)–(5.16). While this is not especially difficult, as the calculations are somewhat lengthy, we do not

include them here. By performing these transformations on the function $J(x)$, one finds that

$$\begin{aligned} J(x) &= \int_0^x ab^2, & J\left(\frac{2x}{1+x}\right) &= \int_0^x (a+c)(b-c)^2, \\ J(-x) &= \int_0^x ac^2, \\ J(x^2) &= \int_0^x 2a(b+c)^2, & J\left(\frac{4x}{(1+x)^2}\right) &= \int_0^x (a+2c)4(b-c)^2. \end{aligned}$$

It now stands to reason that we should seek rational numbers r_1, r_2, r_3 and r_4 such that

$$ac^2 = r_1ab^2 + 2r_2a(b+c)^2 + r_3(a+c)(b-c)^2 + r_4(a+2c)4(b-c)^2$$

is an identity in the non-commutative polynomial ring $\mathbf{Q}\langle a, b, c \rangle$. The problem of finding such rational numbers reduces to solving a finite set of linear equations. For example, comparing coefficients of the monomial ab^2 tells us that $r_1 + 2r_2 + r_3 + 4r_4 = 0$. Coefficients of other monomials give us additional equations, and we readily find that $r_1 = -1$, $r_2 = 1/4$, $r_3 = 1$ and $r_4 = -1/8$, thus proving (8.4) as expected.

8.2. Convolution of Polylogarithms. Motivated by [32, 33], for real $0 < x < 1$ and integers s and t , consider

$$\begin{aligned} T_{s,t}(x) &:= \sum_{\substack{m,n=1 \\ m \neq n}}^{\infty} \frac{x^{n+m}}{n^s m^t (m-n)} = \sum_{\substack{m,n=1 \\ m \neq n}}^{\infty} \frac{x^{n+m}(m-n+n)}{n^s m^{t+1}(m-n)} \\ &= \sum_{\substack{m,n=1 \\ m \neq n}}^{\infty} \frac{x^{n+m}}{n^s m^{t+1}} + \sum_{\substack{m,n=1 \\ m \neq n}}^{\infty} \frac{x^{n+m}}{n^{s-1} m^{t+1}(m-n)} \\ &= \sum_{n=1}^{\infty} \frac{x^n}{n^s} \sum_{m=1}^{\infty} \left(\frac{x^m}{m^{t+1}} - \frac{x^n}{n^{t+1}} \right) + T_{s-1,t+1}(x) \\ &= \text{Li}_s(x) \text{Li}_{t+1}(x) - \text{Li}_{s+t+1}(x^2) + T_{s-1,t+1}(x). \end{aligned}$$

Telescoping this gives

$$T_{s,t}(x) = T_{0,s+t}(x) - s \text{Li}_{s+t+1}(x^2) + \sum_{j=1}^s \text{Li}_j(x) \text{Li}_{s+t+1-j}(x), \quad 0 \leq s \in \mathbf{Z}.$$

With $t = 0$, this becomes

$$T_{s,0}(x) = T_{0,s}(x) - s \text{Li}_{s+1}(x^2) + \sum_{j=1}^s \text{Li}_j(x) \text{Li}_{s+1-j}(x), \quad 0 \leq s \in \mathbf{Z}.$$

But for any integers s and t , there holds

$$T_{s,t}(x) = \sum_{\substack{m,n=1 \\ m \neq n}}^{\infty} \frac{x^{n+m}}{n^t m^s (m-n)} = - \sum_{\substack{m,n=1 \\ m \neq n}}^{\infty} \frac{x^{n+m}}{m^s n^t (n-m)} = -T_{s,t}(x).$$

Therefore,

$$T_{s,0}(x) = \frac{1}{2} \sum_{j=1}^s \text{Li}_j(x) \text{Li}_{s+1-j}(x) - \frac{s}{2} \text{Li}_{s+1}(x^2), \quad 0 \leq s \in \mathbf{Z}. \quad (8.5)$$

On the other hand,

$$\begin{aligned} T_{s,0}(x) &= \sum_{n=1}^{\infty} \frac{x^n}{n^s} \sum_{\substack{m=1 \\ m \neq n}}^{\infty} \frac{x^m}{m-n} = \sum_{n=1}^{\infty} \frac{x^{2n}}{n^s} \sum_{m=n+1}^{\infty} \frac{x^{m-n}}{m-n} - \sum_{n=1}^{\infty} \frac{x^n}{n^s} \sum_{m=1}^{n-1} \frac{x^m}{n-m} \\ &= \text{Li}_s(x^2) \text{Li}_1(x) - \sum_{n=1}^{\infty} \frac{x^n}{n^s} \sum_{j=1}^{n-1} \frac{x^{n-j}}{j}. \end{aligned}$$

Comparing this with (8.5) gives

$$\sum_{n=1}^{\infty} \frac{x^n}{n^s} \sum_{j=1}^{n-1} \frac{x^{n-j}}{j} = \frac{s}{2} \text{Li}_{s+1}(x^2) - [\text{Li}_s(x) - \text{Li}_s(x^2)] \text{Li}_1(x) - \frac{1}{2} \sum_{j=2}^{s-1} \text{Li}_j(x) \text{Li}_{s+1-j}(x), \quad (8.6)$$

where in (8.6) and what follows, we now require $2 \leq s \in \mathbf{Z}$ because the terms $j = 1$ and $j = s$ in the sum (8.5) were separated, and assumed to be distinct.

Next, note that if n is a positive integer and $0 < x < 1$, then

$$1 - x^n = (1-x) \sum_{j=0}^{n-1} x^j < (1-x)n.$$

Thus, if $2 \leq s \in \mathbf{Z}$ and $0 < x < 1$, then

$$\begin{aligned} 0 < [\text{Li}_s(x) - \text{Li}_s(x^2)] \text{Li}_1(x) &= \text{Li}_1(x) \sum_{n=1}^{\infty} \frac{x^n(1-x^n)}{n^s} < (1-x) \text{Li}_1(x) \sum_{n=1}^{\infty} \frac{x^n}{n^{s-1}} \\ &< (1-x) \log^2(1-x). \end{aligned}$$

Since the latter expression tends to zero in the limit as $x \rightarrow 1-$, taking the limit in (8.6) gives

$$\zeta(s, 1) = \frac{1}{2} s \zeta(s+1) - \frac{1}{2} \sum_{j=1}^{s-2} \zeta(j+1) \zeta(s-j), \quad 2 \leq s \in \mathbf{Z},$$

which is (1.3). □

9. FOURIER SERIES

The Fourier expansions

$$\sum_{n=1}^{\infty} \frac{\sin(nt)}{n} = \frac{\pi - t}{2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\cos(nt)}{n} = -\log |2 \sin(t/2)|$$

are both valid in the open interval $0 < t < 2\pi$. Multiplying these together, simplifying, and doing a partial fraction decomposition gives

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} \sum_{k=1}^{n-1} \frac{1}{k} &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} \sum_{k=1}^{n-1} \left(\frac{1}{k} + \frac{1}{n-k} \right) = \frac{1}{2} \sum_{n>k>0} \frac{\sin(nt)}{k(n-k)} \\ &= \frac{1}{2} \sum_{m,n=1}^{\infty} \frac{\sin(m+n)t}{mn} = \sum_{m,n=1}^{\infty} \frac{\sin(mt) \cos(nt)}{mn} \\ &= -\frac{\pi - t}{2} \log |2 \sin(t/2)|, \end{aligned} \tag{9.1}$$

again for $0 < t < 2\pi$. Integrating (9.1) term by term yields

$$\sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n^2} \sum_{k=1}^{n-1} \frac{1}{k} = \zeta(2, 1) + \frac{1}{2} \int_0^\theta (\pi - t) \log |2 \sin(t/2)| dt, \tag{9.2}$$

valid for $0 \leq \theta \leq 2\pi$. Likewise for $0 \leq \theta \leq 2\pi$,

$$\sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n^3} = \zeta(3) + \int_0^\theta (\theta - t) \log |2 \sin(t/2)| dt. \tag{9.3}$$

Setting $\theta = \pi$ in (9.2) and (9.3) produces

$$\zeta(2, 1) - \zeta(\bar{2}, 1) = -\frac{1}{2} \int_0^\pi (\pi - t) \log |2 \sin(t/2)| dt = \frac{\zeta(3) - \zeta(\bar{3})}{2}.$$

In light of (1.2), this implies

$$\zeta(\bar{2}, 1) = \frac{\zeta(3) + \zeta(\bar{3})}{2} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1 + (-1)^n}{n^3} = \sum_{m=1}^{\infty} \frac{1}{(2m)^3} = \frac{1}{8} \zeta(3),$$

which is (1.5). □

Applying Parseval's equation to (9.1) gives (via [10, 12, 47]) the integral evaluation

$$\frac{1}{4\pi} \int_0^{2\pi} (\pi - t)^2 \log^2(2 \sin(t/2)) dt = \sum_{n=1}^{\infty} \frac{H_n^2}{(n+1)^2} = \frac{11}{4} \zeta(4).$$

A reason for valuing such integral representations is that they are frequently easier to use numerically.

10. FURTHER GENERATING FUNCTIONS

10.1. Hypergeometric Functions. Note that in the notation of (1.1), $\zeta(2, 1)$ is the coefficient of xy^2 in

$$G(x, y) := \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x^{m+1} y^{n+1} \zeta(m+2, \{1\}^n) = y \sum_{m=0}^{\infty} x^{m+1} \sum_{k=1}^{\infty} \frac{1}{k^{m+2}} \prod_{j=1}^{k-1} \left(1 + \frac{y}{j}\right). \quad (10.1)$$

Now recall the notation $(y)_k := y(y+1)\cdots(y+k-1)$ for the rising factorial with k factors. Thus,

$$\frac{y}{k} \prod_{j=1}^{k-1} \left(1 + \frac{y}{j}\right) = \frac{(y)_k}{k!}.$$

Substituting this into (10.1), interchanging order of summation, and summing the resulting geometric series yields the hypergeometric series

$$G(x, y) = \sum_{k=1}^{\infty} \frac{(y)_k}{k!} \left(\frac{x}{k-x}\right) = - \sum_{k=1}^{\infty} \frac{(y)_k (-x)_k}{k! (1-x)_k} = 1 - {}_2F_1 \left(\begin{matrix} y, -x \\ 1-x \end{matrix} \middle| 1 \right).$$

But, Gauss's summation theorem for the hypergeometric function [1, p. 557] [4, p. 2] and the power series expansion for the logarithmic derivative of the gamma function [1, p. 259] imply that

$${}_2F_1 \left(\begin{matrix} y, -x \\ 1-x \end{matrix} \middle| 1 \right) = \frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)} = \exp \left\{ \sum_{k=2}^{\infty} (x^k + y^k - (x+y)^k) \frac{\zeta(k)}{k} \right\}.$$

Thus, we have derived the generating function equality [14] (see [24] for a q -analog)

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x^{m+1} y^{n+1} \zeta(m+2, \{1\}^n) = 1 - \exp \left\{ \sum_{k=2}^{\infty} (x^k + y^k - (x+y)^k) \frac{\zeta(k)}{k} \right\}. \quad (10.2)$$

Extracting coefficients of xy^2 from both sides of (10.2) yields (1.2). \square

The generalization (1.3) can be similarly derived: extract the coefficient of $x^{m-1}y^2$ from both sides of (10.2). In fact, it is easy to see that (10.2) provides a formula for $\zeta(m+2, \{1\}^n)$ for all nonnegative integers m and n in terms of sums of products of values of the Riemann zeta function at the positive integers. In particular, Markett's formula [61] (cf. also [12]) for $\zeta(m, 1, 1)$ for positive integers $m > 1$ is most easily obtained in this way. Noting symmetry between x and y in (10.2) gives Drinfeld's duality formula [41]

$$\zeta(m+2, \{1\}^n) = \zeta(n+2, \{1\}^m) \quad (10.3)$$

for non-negative integers m and n , a special case of the more general duality formula (5.11). Note that (1.2) is just the case $m = n = 0$.

Similarly [38, 2.1b] equating coefficients of xy^2 in Kummer's summation theorem [56, p. 53] [4, p. 9]

$${}_2F_1 \left(\begin{matrix} x, y \\ 1 + x - y \end{matrix} \middle| -1 \right) = \frac{\Gamma(1 + x/2)\Gamma(1 + x - y)}{\Gamma(1 + x)\Gamma(1 + x/2 - y)}$$

yields (1.5).

10.2. A Generating Function for Sums. The identity (1.2) can also be recovered by setting $x = 0$ in the following result:

Theorem 7 (Theorem 1 of [11]). *If x is any complex number not equal to a positive integer, then*

$$\sum_{n=1}^{\infty} \frac{1}{n(n-x)} \sum_{m=1}^{n-1} \frac{1}{m-x} = \sum_{n=1}^{\infty} \frac{1}{n^2(n-x)}.$$

Proof. Fix $x \in \mathbf{C} \setminus \mathbf{Z}^+$. Let S denote the left hand side. By partial fractions,

$$\begin{aligned} S &= \sum_{n=1}^{\infty} \sum_{m=1}^{n-1} \left(\frac{1}{n(n-m)(m-x)} - \frac{1}{n(n-m)(n-x)} \right) \\ &= \sum_{m=1}^{\infty} \frac{1}{m-x} \sum_{n=m+1}^{\infty} \frac{1}{n(n-m)} - \sum_{n=1}^{\infty} \frac{1}{n(n-x)} \sum_{m=1}^{n-1} \frac{1}{n-m} \\ &= \sum_{m=1}^{\infty} \frac{1}{m(m-x)} \sum_{n=m+1}^{\infty} \left(\frac{1}{n-m} - \frac{1}{n} \right) - \sum_{n=1}^{\infty} \frac{1}{n(n-x)} \sum_{m=1}^{n-1} \frac{1}{m}. \end{aligned}$$

Now for fixed $m \in \mathbf{Z}^+$,

$$\begin{aligned} \sum_{n=m+1}^{\infty} \left(\frac{1}{n-m} - \frac{1}{n} \right) &= \lim_{N \rightarrow \infty} \sum_{n=m+1}^N \left(\frac{1}{n-m} - \frac{1}{n} \right) = \sum_{n=1}^m \frac{1}{n} - \lim_{N \rightarrow \infty} \sum_{n=1}^m \frac{1}{N-n+1} \\ &= \sum_{n=1}^m \frac{1}{n}, \end{aligned}$$

since m is fixed. Therefore, we have

$$\begin{aligned} S &= \sum_{m=1}^{\infty} \frac{1}{m(m-x)} \sum_{n=1}^m \frac{1}{n} - \sum_{n=1}^{\infty} \frac{1}{n(n-x)} \sum_{m=1}^{n-1} \frac{1}{m} = \sum_{n=1}^{\infty} \frac{1}{n(n-x)} \left(\sum_{m=1}^n \frac{1}{m} - \sum_{m=1}^{n-1} \frac{1}{m} \right) \\ &= \sum_{n=1}^{\infty} \frac{1}{n^2(n-x)}. \end{aligned}$$

□

Theorem 7 is in fact equivalent to the sum formula [51, 65]

$$\sum_{\substack{\sum a_i = s \\ a_i \geq 0}} \zeta(a_1 + 2, a_2 + 1, \dots, a_r + 1) = \zeta(r + s + 1), \quad (10.4)$$

valid for all integers $s \geq 0$, $r \geq 1$, and which generalizes Theorem 3 (4.1) to arbitrary depth. The identity (1.2) is simply the case $r = 2$, $s = 0$. A q -analog of the sum formula (10.4) is derived as a special case of more general results in [24]. See also [26].

10.3. An Alternating Generating Function. An alternating counterpart to Theorem 7 is given below.

Theorem 8. (Theorem 3 of [11]). *For all non-integer x*

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 - x^2} \left\{ H_n + \sum_{n=1}^{\infty} \frac{x^2}{n(n^2 - x^2)} \right\} &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 - x^2} \left\{ \psi(n) - \psi(x) - \frac{\pi}{2} \cot(\pi x) - \frac{1}{2x} \right\} \\ &= \sum_{o>0 \text{ odd}}^{\infty} \frac{1}{o(o^2 - x^2)} + \sum_{n=1}^{\infty} \frac{(-1)^n n}{(n^2 - x^2)^2} \\ &= \sum_{e>0 \text{ even}}^{\infty} \frac{e}{(x^2 - e^2)^2} - x^2 \sum_{o>0 \text{ odd}}^{\infty} \frac{1}{o(x^2 - o^2)^2}. \end{aligned}$$

Setting $x = 0$ reproduces (1.5) in the form $\zeta(\bar{2}, 1) = \sum_{n>0}^{\infty} (2n)^{-3}$. We record that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 - x^2} = \frac{1}{2x^2} - \frac{\pi}{2x \sin(\pi x)},$$

while

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 - x^2} \left\{ \psi(n) - \psi(x) - \frac{\pi}{2} \cot(\pi x) - \frac{1}{2x} \right\} &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 - x^2} \left\{ H_n + \sum_{n=1}^{\infty} \frac{x^2}{n(n^2 - x^2)} \right\} \\ &= \sum_{n=1}^{\infty} \frac{1}{(2n-1)((2n-1)^2 - x^2)} + \sum_{n=1}^{\infty} \frac{n(-1)^n}{(n^2 - x^2)^2}. \end{aligned}$$

10.4. The Digamma Function. Define an auxiliary function Λ by

$$x\Lambda(x) := \frac{1}{2}\psi'(1-x) - \frac{1}{2}(\psi(1-x) + \gamma)^2 - \frac{1}{2}\zeta(2).$$

We note, but do not use, that

$$x\Lambda(x) = \frac{1}{2} \int_0^{\infty} \frac{t(e^{-t} + e^{-t(1-x)})}{1 - e^{-t}} dt - \frac{1}{2} \left(\int_0^{\infty} \frac{e^{-t} - e^{-t(1-x)}}{1 - e^{-t}} dt \right)^2 - \zeta(2).$$

It is easy to verify that

$$\begin{aligned}\psi(1-x) + \gamma &= \sum_{n=1}^{\infty} \frac{x}{n(x-n)}, \\ \psi'(1-x) - \zeta(2) &= \sum_{n=1}^{\infty} \left(\frac{1}{(x-n)^2} - \frac{1}{n^2} \right) = \sum_{n=1}^{\infty} \frac{2nx - x^2}{n^2(n-x)^2},\end{aligned}\tag{10.5}$$

and

$$\sum_{n=0}^{\infty} \zeta(n+2, 1)x^n = \sum_{n=1}^{\infty} \frac{1}{n(n-x)} \sum_{m=1}^{n-1} \frac{1}{m}.$$

Hence,

$$\Lambda(x) = \sum_{n=1}^{\infty} \frac{1}{n^2(n-x)} - x \sum_{n=1}^{\infty} \frac{1}{n(n-x)} \sum_{m=1}^{n-1} \frac{1}{m(m-x)}.$$

Now,

$$\sum_{n=1}^{\infty} \frac{1}{n^2(n-x)} - x \sum_{n=1}^{\infty} \frac{1}{n(n-x)} \sum_{m=1}^{n-1} \frac{1}{m(m-x)} = \sum_{n=1}^{\infty} \frac{1}{n(n-x)} \sum_{m=1}^{n-1} \frac{1}{m}$$

is directly equivalent to Theorem 7 of §10.2—see [11, Section 3]—and we have proven

$$\Lambda(x) = \sum_{n=0}^{\infty} \zeta(n+2, 1)x^n,$$

so that comparing coefficients yields yet another proof of Euler’s reduction (1.3). In particular, setting $x = 0$ again produces (1.2). \square

10.5. The Beta Function. Recall that the beta function is defined for positive real x and y by

$$B(x, y) := \int_0^1 t^{x-1}(1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

We begin with the following easily obtained generating function:

$$\sum_{n=1}^{\infty} t^n H_n = -\frac{\log(1-t)}{1-t}.$$

For $m \geq 2$ the Laplace integral (5.4) now gives

$$\begin{aligned}\zeta(m, 1) &= \frac{(-1)^m}{(m-1)!} \int_0^1 \frac{\log^{m-1}(t) \log(1-t)}{1-t} dt \\ &= \frac{(-1)^m}{2(m-1)!} \int_0^1 (m-1) \log^{m-2}(t) \log^2(1-t) \frac{dt}{t} \\ &= \frac{(-1)^m}{2(m-2)!} b_1^{(m-2)}(0),\end{aligned}\tag{10.6}$$

where

$$b_1(x) := \left. \frac{\partial^2}{\partial y^2} B(x, y) \right|_{y=1} = 2\Lambda(-x)$$

(cf. §10.4). Since

$$\frac{\partial^2}{\partial y^2} B(x, y) = B(x, y) [(\psi(y) - \psi(x+y))^2 + \psi'(y) - \psi'(x+y)],$$

we derive

$$b_1(x) = \frac{(\psi(1) - \psi(x+1))^2 + \psi'(1) - \psi'(x+1)}{x}.$$

Now observe that from (10.6),

$$\begin{aligned}\zeta(2, 1) &= \frac{1}{2} b_1(0) = \lim_{x \downarrow 0} \frac{(-\gamma - \psi(x+1))^2}{2x} - \lim_{x \downarrow 0} \frac{\psi'(x+1) - \psi'(1)}{2x} = -\frac{1}{2} \psi''(1) \\ &= \zeta(3).\end{aligned}$$

□

Continuing, from the following two identities, cognate to (10.5),

$$\begin{aligned}(-\gamma - \psi(x+1))^2 &= \left(\sum_{m=1}^{\infty} (-1)^m \zeta(m+1) x^m \right)^2 \\ &= \sum_{m=1}^{\infty} (-1)^m \sum_{k=1}^{m-1} \zeta(k+1) \zeta(m-k+1) x^m, \\ \zeta(2) - \psi'(x+1) &= \sum_{m=1}^{\infty} (-1)^{m+1} (m+1) \zeta(m+2) x^m,\end{aligned}$$

we get

$$\begin{aligned}2 \sum_{m=2}^{\infty} (-1)^m \zeta(m, 1) x^{m-2} &= \sum_{m=2}^{\infty} \frac{b_1^{(m-2)}(0)}{(m-2)!} x^{m-2} = b_1(x) \\ &= \sum_{m=1}^{\infty} (-1)^{m-1} \left((m+1) \zeta(m+2) - \sum_{k=1}^{m-1} \zeta(k+1) \zeta(m-k+1) \right) x^{m-1},\end{aligned}$$

from which Euler’s reduction (1.3) follows—indeed this is close to Euler’s original path.

Observe that (10.6) is especially suited to symbolic computation. We also note the pleasing identity

$$\psi'(x) = \frac{\Gamma''(x)}{\Gamma(x)} - \psi^2(x). \quad (10.7)$$

In some informal sense (10.7) generates (1.3), but we have been unable to make this sense precise.

11. A DECOMPOSITION FORMULA OF EULER

For positive integers s and t and distinct non-zero real numbers α and x , the partial fraction expansion

$$\frac{1}{x^s(x-\alpha)^t} = (-1)^t \sum_{r=0}^{s-1} \binom{t+r-1}{t-1} \frac{1}{x^{s-r}\alpha^{t+r}} + \sum_{r=0}^{t-1} \binom{s+r-1}{s-1} \frac{(-1)^r}{\alpha^{s+r}(x-\alpha)^{t-r}} \quad (11.1)$$

implies [64, p. 48] [61] Euler’s decomposition formula

$$\begin{aligned} \zeta(s, t) = & (-1)^t \sum_{r=0}^{s-2} \binom{t+r-1}{t-1} \zeta(s-r, t+r) + \sum_{r=0}^{t-2} (-1)^r \binom{s+r-1}{s-1} \zeta(t-r) \zeta(s+r) \\ & - (-1)^t \binom{s+t-2}{s-1} \{ \zeta(s+t) + \zeta(s+t-1, 1) \}. \end{aligned} \quad (11.2)$$

The depth-2 sum formula (4.1) is obtained by setting $t = 1$ in (11.2). If we also set $s = 2$, the identity (1.2) results. To derive (11.2) from (11.1) we follow [61], separating the last term of each sum on the right hand side of (11.1), obtaining

$$\begin{aligned} \frac{1}{x^s(x-\alpha)^t} = & (-1)^t \sum_{r=0}^{s-2} \binom{t+r-1}{t-1} \frac{1}{x^{s-r}\alpha^{t+r}} + \sum_{r=0}^{t-2} \binom{s+r-1}{s-1} \frac{(-1)^r}{\alpha^{s+r}(x-\alpha)^{t-r}} \\ & - (-1)^t \binom{s+t-2}{s-1} \frac{1}{\alpha^{s+t-1}} \left(\frac{1}{x-\alpha} - \frac{1}{x} \right). \end{aligned}$$

Now sum over all integers $0 < \alpha < x < \infty$. □

Nielsen states (11.1) without proof [64, p. 48, eq. (9)]. Markett proves (11.1) by induction [61, Lemma 3.1], which is the proof technique suggested for the $\alpha = 1$ case of (11.1) in [12, Lemma 1]. However, it is easy to prove (11.1) directly by expanding the left hand side into partial fractions with the aid of the residue calculus. Alternatively, as in [27] note that (11.1) is an immediate consequence of applying the partial derivative operator

$$\frac{1}{(s-1)!} \left(-\frac{\partial}{\partial x} \right)^{s-1} \frac{1}{(t-1)!} \left(-\frac{\partial}{\partial y} \right)^{t-1}$$

to the identity

$$\frac{1}{xy} = \frac{1}{(x+y)x} + \frac{1}{(x+y)y},$$

and then setting $y = \alpha - x$. This latter observation is extended in [27] to establish a q -analog of another of Euler's decomposition formulas for $\zeta(s, t)$.

12. EQUATING SHUFFLES AND STUFFLES

We begin with an informal argument. By the *stuffle multiplication* rule [15, 19, 23, 24]

$$\zeta(2)\zeta(1) = \zeta(2, 1) + \zeta(1, 2) + \zeta(3). \quad (12.1)$$

On the other hand, the shuffle multiplication rule [15, 16, 19, 20, 21] gives $ab \sqcup b = 2abb + bab$, whence

$$\zeta(2)\zeta(1) = 2\zeta(2, 1) + \zeta(1, 2). \quad (12.2)$$

The identity (1.2) now follows immediately on subtracting (12.1) from (12.2). \square

Of course, this argument needs justification, because it involves cancelling divergent series. To make the argument rigorous, we introduce the multiple polylogarithm [15, 18, 19]. For real $0 \leq x \leq 1$ and positive integers s_1, \dots, s_k with $x = s_1 = 1$ excluded for convergence, define

$$\zeta_x(s_1, \dots, s_k) := \sum_{n_1 > \dots > n_k > 0} x^{n_1} \prod_{j=1}^k n_j^{-s_j} = \int \prod_{j=1}^k \left(\prod_{r=1}^{s_j-1} \frac{dt_r^{(j)}}{t_r^{(j)}} \right) \frac{dt_{s_j}^{(j)}}{1 - t_{s_j}^{(j)}}, \quad (12.3)$$

where the integral is over the simplex

$$x > t_1^{(1)} > \dots > t_{s_1}^{(1)} > \dots > t_1^{(k)} > \dots > t_{s_k}^{(k)} > 0,$$

and is abbreviated by

$$\int_0^x \prod_{j=1}^k a^{s_j-1} b, \quad a = \frac{dt}{t}, \quad b = \frac{dt}{1-t}. \quad (12.4)$$

Then

$$\zeta(2)\zeta_x(1) = \sum_{n>0} \frac{1}{n^2} \sum_{k>0} \frac{x^k}{k} = \sum_{n>k>0} \frac{x^k}{n^2 k} + \sum_{k>n>0} \frac{x^k}{k n^2} + \sum_{k>0} \frac{x^k}{k^3},$$

and

$$\zeta_x(2)\zeta_x(1) = \int_0^x ab \int_0^x b = \int_0^x (2abb + bab) = 2\zeta_x(2, 1) + \zeta_x(1, 2).$$

Subtracting the two equations gives

$$[\zeta(2) - \zeta_x(2)]\zeta_x(1) = \zeta_x(3) - \zeta_x(2, 1) + \sum_{n>k>0} \frac{x^k - x^n}{n^2 k}.$$

We now take the limit as $x \rightarrow 1 -$. Uniform convergence implies the right hand side tends to $\zeta(3) - \zeta(2, 1)$. That the left hand side tends to zero follows immediately from the inequalities

$$\begin{aligned} 0 \leq x[\zeta(2) - \zeta_x(2)]\zeta_x(1) &= x \int_x^1 \log(1-t) \log(1-x) \frac{dt}{t} \\ &\leq \int_x^1 \log^2(1-t) dt \\ &= (1-x) \{1 + (1 - \log(1-x))^2\}. \end{aligned}$$

□

The alternating case (1.5) is actually easier using this approach, since the role of the divergent sum $\zeta(1)$ is taken over by the conditionally convergent sum $\zeta(\bar{1}) = -\log 2$. By the stuffle multiplication rule,

$$\zeta(\bar{2})\zeta(\bar{1}) = \zeta(\bar{2}, \bar{1}) + \zeta(\bar{1}, \bar{2}) + \zeta(3), \tag{12.5}$$

$$\zeta(2)\zeta(\bar{1}) = \zeta(2, \bar{1}) + \zeta(\bar{1}, 2) + \zeta(\bar{3}). \tag{12.6}$$

On the other hand, the shuffle multiplication rule gives $ac \sqcup c = 2ac^2 + cac$ and $ab \sqcup c = abc + acb + cab$, whence

$$\zeta(\bar{2})\zeta(\bar{1}) = 2\zeta(\bar{2}, 1) + \zeta(\bar{1}, 2), \tag{12.7}$$

$$\zeta(2)\zeta(\bar{1}) = \zeta(2, \bar{1}) + \zeta(\bar{2}, \bar{1}) + \zeta(\bar{1}, \bar{2}). \tag{12.8}$$

Comparing (12.5) with (12.7) and (12.6) with (12.8) yields the two equations

$$\zeta(\bar{2}, \bar{1}) = \zeta(\bar{1}, 2) + 2\zeta(\bar{2}, 1) - \zeta(\bar{1}, \bar{2}) - \zeta(3),$$

$$\zeta(\bar{2}, \bar{1}) = \zeta(\bar{1}, 2) - \zeta(\bar{1}, \bar{2}) + \zeta(\bar{3}).$$

Subtracting the latter two equations yields $2\zeta(\bar{2}, 1) = \zeta(3) + \zeta(\bar{3})$, i.e. (2.3), which was shown to be trivially equivalent to (1.5) in §2. □

13. CONCLUSION

There are doubtless other roads to Rome, and as indicated in the introduction we should like to learn of them. We finish with the three open questions we are most desirous of answers to.

- A truly combinatorial proof, perhaps of the form considered in [16].
- A direct proof that the appropriate line integrals in sections 5.8 and 5.9 evaluate to the appropriate multiple of $\zeta(3)$.
- A proof of (1.6), or at least some additional cases of it.

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