Experimental Mathematics:

And Its Implications

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Elsewhere Kronecker said "In mathematics, I recognize true scientific value only in concrete mathematical truths, or to put it more pointedly, only in mathematical formulas." ... I would rather say "computations" than "formulas", but my view is essentially the same. (Harold M. Edwards, 2004)

D Drive www.cs.dal.ca/ddrive mathagate

Two Scientific Quotations

Kurt Gödel overturned the mathematical apple cart entirely deductively, but he held quite different ideas about legitimate forms of mathematical reasoning:

> *If mathematics describes an objective world just like physics, there is no reason why inductive methods should not be applied in mathematics just the same as in physics.**

and *Christof Koch* accurately captures scientific distaste for philosophizing:

> *Whether we scientists are inspired, bored, or infuriated by philosophy, all our theorizing and experimentation depends on particular philosophical background assumptions. This hidden influence is an acute embarrassment to many researchers, and it is therefore not often acknowledged.* (Christof Koch†, 2004)

*Taken from an until then unpublished 1951 manuscript in his *Collected Works*, Volume III.

†In "Thinking About the Conscious Mind," a review of John R. Searle's *Mind. A Brief Introduction*, OUP 2004.

Three Mathematical Definitions

mathematics, n. *a group of related subjects, including algebra, geometry, trigonometry and calculus , concerned with the study of number, quantity, shape, and space, and their inter-relationships, applications, generalizations and abstractions.*

This definition taken from the *Collins Dictionary* makes no immediate mention of proof, nor of the means of reasoning to be allowed. *Webster's Dictionary* contrasts:

> **induction, n.** *any form of reasoning in which the conclusion, though supported by the premises, does not follow from them necessarily*; and

> **deduction, n.** *a process of reasoning in which a conclusion follows necessarily from the premises presented, so that the conclusion cannot be false if the premises are true.*

I, like Gödel, and as I shall show many others, suggest that both should be openly entertained in mathematical discourse.

My Intentions in these Lectures

I aim to discuss Experimental M*a*thodology, its *philosophy*, *history*, *current practice* and *proximate future*, and using concrete accessible—entertaining I hope—examples, to explore implications for mathematics and for mathematical philosophy.

> *Thereby, to persuade you both of the power of mathematical experiment and that the traditional accounting of mathematical learning and research is largely an ahistorical caricature.*

The four lectures are largely independent

The tour mirrors that from the recent books:

> Jonathan M. Borwein and David H. Bailey, *Mathematics by Experiment: Plausible Reasoning in the 21st Century*; and with Roland Girgensohn, *Experimentation in Mathematics: Computational Paths to Discovery*, A.K. Peters, Natick, MA, 2004.

1. Plausible Reasoning in the 21st Century, I.

This first lecture will be a general introduction to

Experimental Mathematics, its Practice and its Philosophy.

It will reprise the sort of 'Experimental methodology' that David Bailey and I—among many others—have come to practice over the past two decades.*

Dalhousie-DRIVE

*All resources are available at `www.experimentalmath.info`.

2. Plausible Reasoning in the 21st Century, II.

The second lecture will focus on the differences between

Determining Truths or Proving Theorems.

We shall explore various of the tools available for deciding what to believe in mathematics, and—using accessible examples—illustrate the rich experimental tool-box mathematicians can now have access to.

Dalhousie-DRIVE

3. Ten Computational Challenge Problems.

This lecture will make a more advanced analysis of the themes developed in Lectures 1 and 2. It will look at 'lists and challenges' and discuss *Ten Computational Mathematics Problems* including

$$\int_0^\infty \cos(2x) \prod_{n=1}^\infty \cos\left(\frac{x}{n}\right) dx \stackrel{?}{=} \frac{\pi}{8}.$$

This problem set was stimulated by Nick Trefethen's recent more numerical *SIAM 100 Digit, 100 Dollar Challenge*.*

... ...

Die ganze Zahl schuf der liebe Gott, alles Ubrige ist Menschenwerk. God made the integers, all else is the work of man. (Leopold Kronecker, 1823-1891)

*The talk is based on an article to appear in the May 2005 *Notices of the AMS*, and related resources such as `www.cs.dal.ca/~jborwein/digits.pdf`.

4. Apéry-Like Identities for $\zeta(n)$.

The final lecture comprises a research level case study of generating functions for zeta functions. This lecture is based on past research with David Bradley and current research with David Bailey.

One example is:

$$\begin{aligned} \mathcal{Z}(x) &:= 3\sum_{k=1}^{\infty} \frac{1}{\binom{2k}{k}(k^2-x^2)} \prod_{n=1}^{k-1} \frac{4x^2-n^2}{x^2-n^2} \\ &= \sum_{n=1}^{\infty} \frac{1}{n^2-x^2} \qquad (1) \\ &\left[= \sum_{k=0}^{\infty} \zeta(2k+2)\, x^{2\,k} = \frac{1-\pi x \cot(\pi x)}{2x^2}\right]. \end{aligned}$$

Note that with $x=0$ this recovers

$$3\sum_{k=1}^{\infty} \frac{1}{\binom{2k}{k}k^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2).$$

Experiments and Implications

I shall talk broadly about experimental and heuristic mathematics, giving accessible, primarily visual and symbolic, examples. The typographic to digital culture shift is vexing in math, viz:

- There is still no truly satisfactory way of displaying mathematics on the web

- We respect authority* but value authorship deeply

- And we care more about the reliability of our literature than does any other science

While the traditional central role of proof in mathematics is arguably under siege, the opportunities are enormous.

- Via examples, I intend to ask:

*Judith Grabiner, "Newton, Maclaurin, and the Authority of Mathematics," MAA, December 2004

MY QUESTIONS

★ What constitutes secure mathematical knowledge?

★ When is computation convincing? Are humans less fallible?

- What tools are available? What methodologies?

- What of the 'law of the small numbers'?

- Who cares for certainty? What is the role of proof?

★ How is mathematics actually done? How should it be?

DEWEY on HABITS

Old ideas give way slowly; for they are more than abstract logical forms and categories. They are habits, predispositions, deeply engrained attitudes of aversion and preference. ... Old questions are solved by disappearing, evaporating, while new questions corresponding to the changed attitude of endeavor and preference take their place. Doubtless the greatest dissolvent in contemporary thought of old questions, the greatest precipitant of new methods, new intentions, new problems, is the one effected by the scientific revolution that found its climax in the "Origin of Species." * (John Dewey)

* *The Influence of Darwin on Philosophy*, 1910. Dewey knew 'Comrade Van' in Mexico.

and MY ANSWERS

⊨ "Why I am a computer assisted fallibilist/social constructivist"

★ Rigour (proof) follows Reason (discovery)

★ Excessive focus on rigour drove us away from our wellsprings

• Many ideas are false. Not all truths are provable. Not all provable truths are worth proving ...

★ Near certainly is often as good as it gets— intellectual context (community) matters

• Complex human proofs are fraught with error (FLT, simple groups, $\cdots$)

★ Modern computational tools dramatically change the nature of available evidence

- Many of my more sophisticated examples originate in the boundary between mathematical physics and number theory and involve the ζ-function, $\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}$, and its relatives.

They often rely on the sophisticated use of *Integer Relations Algorithms* — recently ranked among the 'top ten' algorithms of the century. Integer Relation methods were first discovered by our colleague **Helaman Ferguson** the mathematical sculptor.

In 2000, Sullivan and Dongarra wrote "Great algorithms are the poetry of computation," when they compiled a list of the 10 algorithms having "the greatest influence on the development and practice of science and engineering in the 20th century".*

- Newton's method was apparently ruled ineligible for consideration.

*From "Random Samples", *Science* page 799, February 4, 2000. The full article appeared in the January/February 2000 issue of *Computing in Science & Engineering*. Dave Bailey wrote the description of 'PSLQ'.

The 20th century's Top Ten

#1. 1946: **The Metropolis Algorithm for Monte Carlo**. Through the use of random processes, this algorithm offers an efficient way to stumble toward answers to problems that are too complicated to solve exactly.

#2. 1947: **Simplex Method for Linear Programming**. An elegant solution to a common problem in planning and decision-making.

#3. 1950: **Krylov Subspace Iteration Method**. A technique for rapidly solving the linear equations that abound in scientific computation.

#4. 1951: **The Decompositional Approach to Matrix Computations**. A suite of techniques for numerical linear algebra.

#5. 1957: **The Fortran Optimizing Compiler**. Turns high-level code into efficient computer-readable code.

#6. 1959: QR Algorithm for Computing Eigenvalues. Another crucial matrix operation made swift and practical.

#7. 1962: **Quicksort Algorithms for Sorting**. For the efficient handling of large databases.

#8. 1965: **Fast Fourier Transform**. Perhaps the most ubiquitous algorithm in use today, it breaks down waveforms (like sound) into periodic components.

#9. 1977: **Integer Relation Detection**. A fast method for spotting simple equations satisfied by collections of seemingly unrelated numbers.

#10. 1987: **Fast Multipole Method**. A breakthrough in dealing with the complexity of n-body calculations, applied in problems ranging from celestial mechanics to protein folding.

Eight of these appeared in the first two decades of serious computing. Most are multiply embedded in every major mathematical computing package.

FOUR FORMS of EXPERIMENTS

We should discuss what Experiments are!

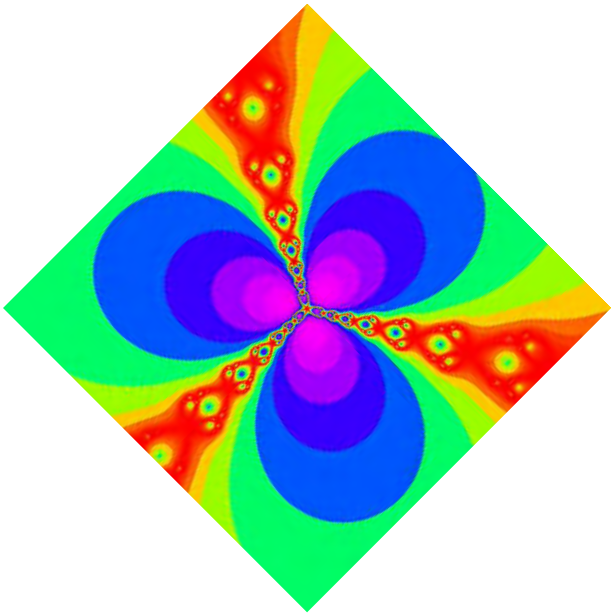
♣ **Kantian** examples: generating "the classical non-Euclidean geometries (hyperbolic, elliptic) by replacing Euclid's axiom of parallels (or something equivalent to it) with alternative forms."

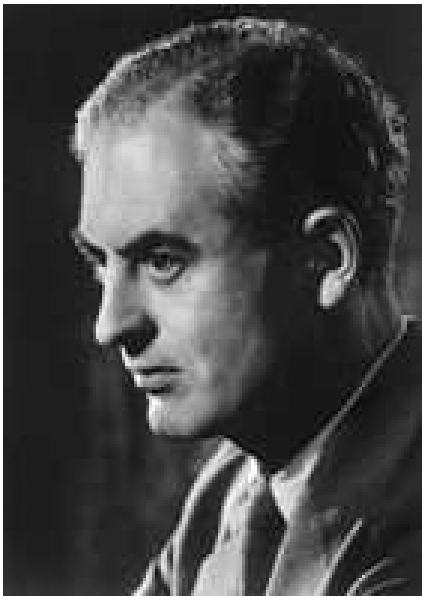
♢ The **Baconian** experiment is a contrived as opposed to a natural happening, it "is the consequence of 'trying things out' or even of merely messing about."

♡ **Aristotelian** demonstrations: "apply electrodes to a frog's sciatic nerve, and lo, the leg kicks; always precede the presentation of the dog's dinner with the ringing of a bell, and lo, the bell alone will soon make the dog dribble."

♠ The most important is **Galilean**: "a critical experiment – one that discriminates between possibilities and, in doing so, either gives us confidence in the view we are taking or makes us think it in need of correction."

- The only form which will make Experimental Mathematics a serious enterprise.





A Julia set

From Peter Medawar (1915–87) *Advice to a Young Scientist* (1979)

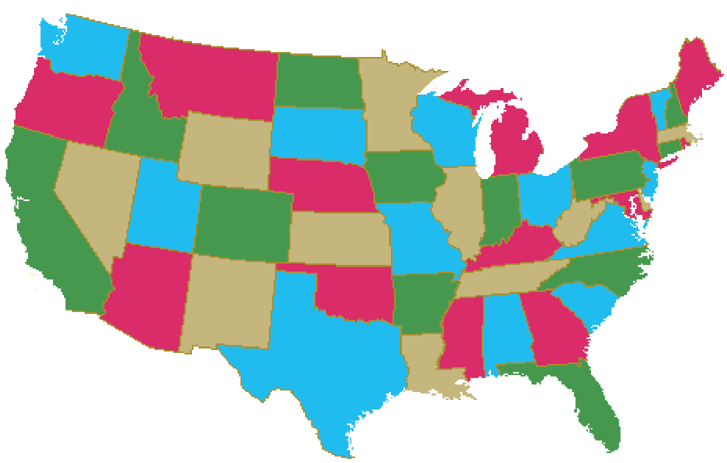
A PARAPHRASE of HERSH

In any event mathematics is and will remain a uniquely human undertaking. Indeed Reuben Hersh's arguments for a humanist philosophy of mathematics, as paraphrased below, become more convincing in our computational setting:

1. *Mathematics is human*. It is part of and fits into human culture. It does not match Frege's concept of an abstract, timeless, tenseless, objective reality.

2. *Mathematical knowledge is fallible*. As in science, mathematics can advance by making mistakes and then correcting or even re-correcting them. The "**fallibilism**" of mathematics is brilliantly argued in Lakatos' *Proofs and Refutations*.

3. There are different versions of proof or rigor. Standards of rigor can vary depending on time, place, and other things. The use of computers in formal proofs, exemplified by the computer-assisted proof of the four color theorem in 1977 (1997), is just one example of an emerging nontraditional standard of rigor.

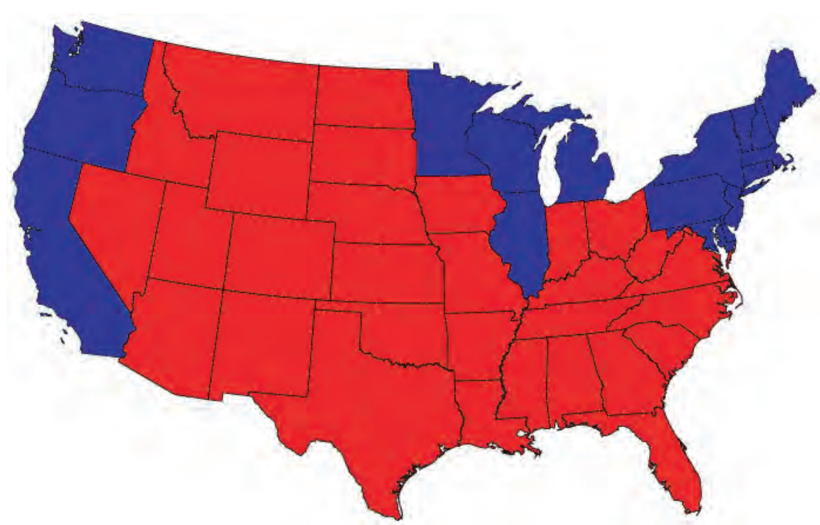


A 4-coloring

4. Empirical evidence, numerical experimentation and probabilistic proof all can help us decide what to believe in mathematics. Aristotelian logic isn't necessarily always the best way of deciding.

5. *Mathematical objects are a special variety of a social-cultural-historical object*. Contrary to the assertions of certain post-modern detractors, mathematics cannot be dismissed as merely a new form of literature or religion. Nevertheless, many mathematical objects can be seen as shared ideas, like Moby Dick in literature, or the Immaculate Conception in religion.

- “Fresh Breezes in the Philosophy of Mathematics”, *MAA Monthly*, Aug 1995, 589–594.



A 2-coloring?

A PARAPHRASE of ERNEST

The idea that what is accepted as mathematical knowledge is, *to some degree*, dependent upon a community's methods of knowledge acceptance is central to the *social constructivist* school of mathematical philosophy.

> *The social constructivist thesis is that mathematics is a social construction, a cultural product, fallible like any other branch of knowledge.* (Paul Ernest)

Associated most notably with the writings of Paul Ernest* social constructivism seeks to define mathematical knowledge and epistemology through the social structure and interactions of the mathematical community and society as a whole.

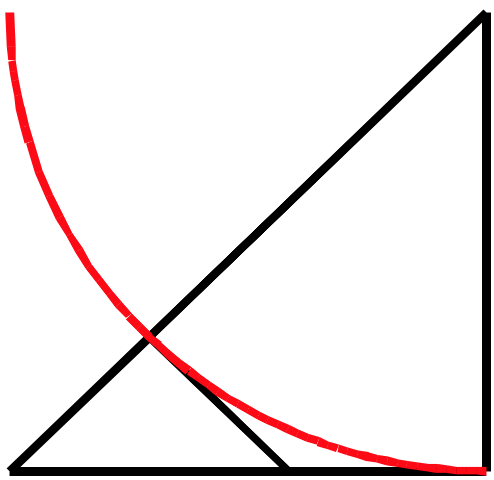
® **DISCLAIMER**: Social Constructivism is not Cultural Relativism

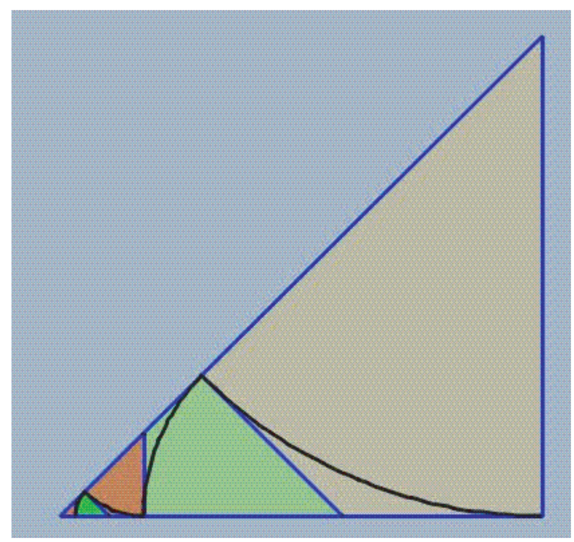
*In Social Constructivism As a Philosophy of Mathematics, Ernest, an English Mathematician and Professor in the Philosophy of Mathematics Education, carefully traces the intellectual pedigree for his thesis, a pedigree that encompasses the writings of Wittgenstein, Lakatos, Davis, and Hersh among others.

A NEW PROOF $\sqrt{2}$ is IRRATIONAL

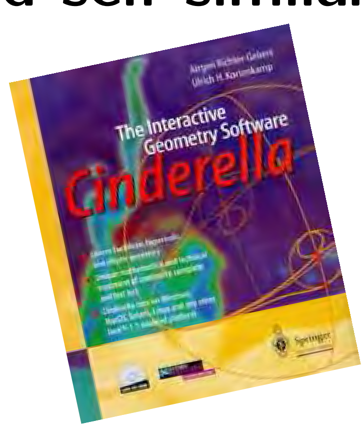
One can find new insights in the oldest areas:

- Here is Tom Apostol's lovely new graphical proof* of the irrationality of $\sqrt{2}$. I like very much that this was published in the present millennium.





Root two is irrational
(static and self-similar pictures)



MAA Monthly, November 2000, 241–242.

PROOF. To say $\sqrt{2}$ is rational is to draw a right-angled isoceles triangle with integer sides. Consider the *smallest* right-angled isoceles *triangle* with integer sides—that is with shortest hypotenuse.

Circumscribe a circle of radius one side and construct the tangent on the hypotenuse [See picture].

Repeating the process once yields a yet smaller such triangle in the same orientation as the initial one.

The *smaller* triangle again has integer sides . . . **QED**

Note the philosophical transitions.

- *Reductio ad absurdum* $\Rightarrow$ *minimal configuration*

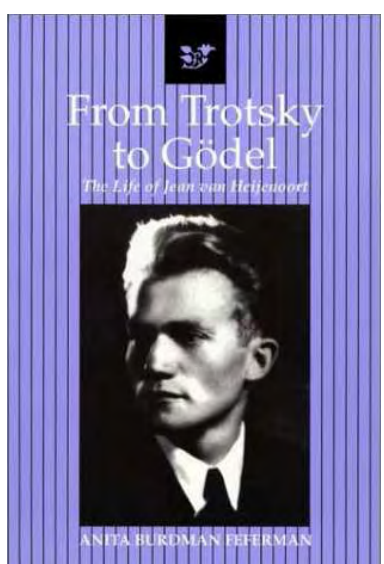
- *Euclidean geometry* $\Rightarrow$ *Dynamic geometry*

FOUR Humanist VIGNETTES

I. Revolutions

> *By 1948, the Marxist-Leninist ideas about the proletariat and its political capacity seemed more and more to me to disagree with reality ... I pondered my doubts, and for several years the study of mathematics was all that allowed me to preserve my inner equilibrium. Bolshevik ideology was, for me, in ruins. I had to build another life.*

Jean Van Heijenoort (1913-1986) *With Trotsky in Exile*, in Anita Feferman's *From Trotsky to Gödel*



- Dewey ran Trotsky's 'treason trial' in Mexico

II. It's Obvious . . .

Aspray: Since you both [Kleene and Rosser] had close associations with Church, I was wondering if you could tell me something about him. What was his wider mathematical training and interests? What were his research habits? I understood he kept rather unusual working hours. How was he as a lecturer? As a thesis director?

Rosser: In his lectures he was painstakingly careful. There was a story that went the rounds. *If Church said it's obvious, then everybody saw it a half hour ago. If Weyl says it's obvious, von Neumann can prove it. If Lefschetz says it's obvious, it's false.**

*One of several versions of this anecdote in *The Princeton Mathematics Community in the 1930s*. This one in Transcript Number 23 (**PMC23**)

III. The Evil of Bourbaki

"There is a story told of the mathematician Claude Chevalley (1909–84), who, as a true Bourbaki, was extremely opposed to the use of images in geometric reasoning.

He is said to have been giving a very abstract and algebraic lecture when he got stuck. After a moment of pondering, he turned to the blackboard, and, trying to hide what he was doing, drew a little diagram, looked at it for a moment, then quickly erased it, and turned back to the audience and proceeded with the lecture....

...The computer offers those less expert, and less stubborn than Chevalley, access to the kinds of images that could only be imagined in the heads of the most gifted mathematicians, ..."[a] (Nathalie Sinclair)

[a]Chapter in *Making the Connection: Research and Practice in Undergraduate Mathematics*, MAA Notes, 2004 in Press.

IV. The Historical Record

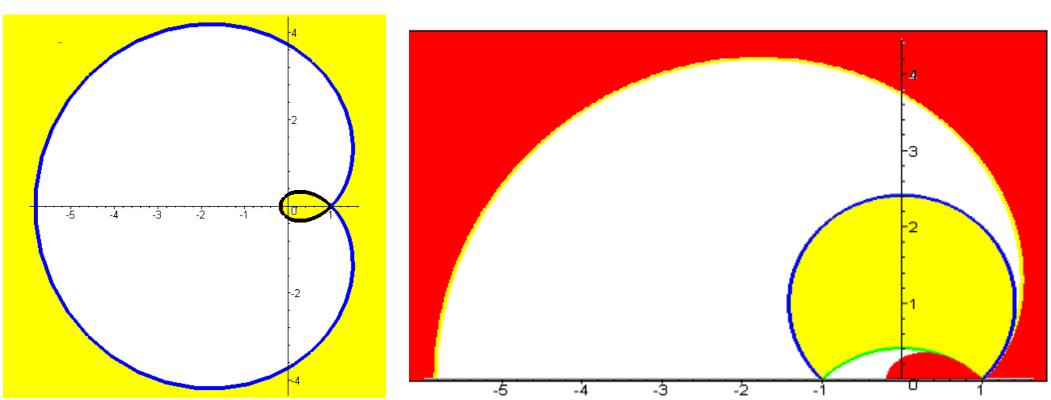
And it is one of the ironies of this entire field that were you to write a history of ideas in the whole of DNA, simply from the documented information as it exists in the literature - that is, a kind of Hegelian history of ideas - you would certainly say that Watson and Crick depended on Von Neumann, because von Neumann essentially tells you how it's done.

*But of course no one knew anything about the other. It's a great paradox to me that this connection was not seen. Of course, all this leads to a real distrust about what historians of science say, especially those of the history of ideas.** (Sidney Brenner)

*The 2002 Nobelist talking about von Neumann's essay on The General and Logical Theory of Automata on pages 35–36 of My life in Science as told to Lewis Wolpert.

POLYA and HEURISTICS

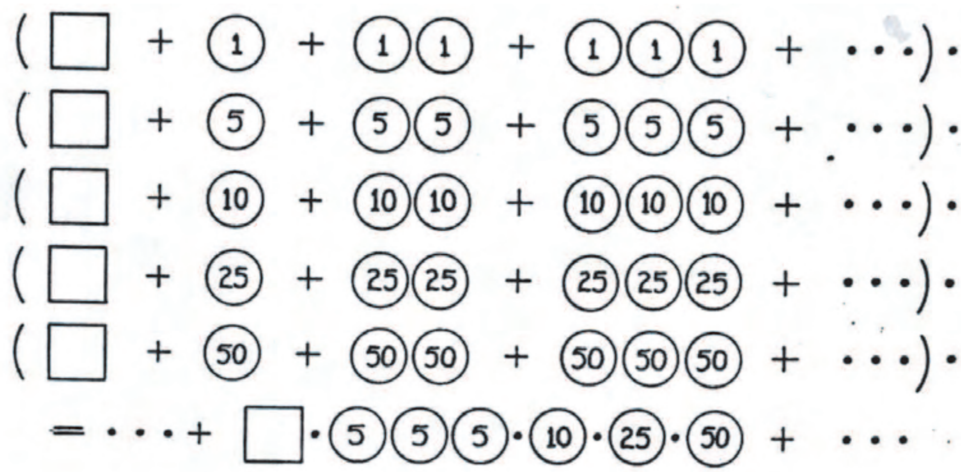
*"[I]ntuition comes to us much earlier and with much less outside influence than formal arguments which we cannot really understand unless we have reached a relatively high level of logical experience and sophistication."** (George Polya)



Scatter-plot discovery of a cardioid

*In *Mathematical Discovery: On Understanding, Learning and Teaching Problem Solving*, *1968.*

Polya on Picture-writing



Polya's illustration of the change solution*

Polya, in a 1956 *American Mathematical Monthly* article provided three provoking examples of converting pictorial representations of problems into generating function solutions. We discuss the first one.

1. *In how many ways can you make change for a dollar?*

*Illustration courtesy Mathematical Association of America

This leads to the (US currency) *generating function*

$$\sum_{k\geq 0} P_k x^k = \frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})(1-x^{50})}$$

which one can easily expand using a *Mathematica* command,

```
Series[1/((1-x)*(1-x^5)*(1-x^10)*(1-x^25)*(1-x^50)),
       {x,0,100}]
```

to obtain $P_{100} = 292$ (**243** for Canadian currency, which lacks a 50 cent piece but has a dollar coin in common circulation).

- Polya's diagram is shown in the Figure

- To see why, we use geometric series and consider the so called *ordinary generating function*

$$\frac{1}{1-x^{10}} = 1 + x^{10} + x^{20} + x^{30} + \cdots$$

for dimes and

$$\frac{1}{1-x^{25}} = 1 + x^{25} + x^{50} + x^{75} + \cdots$$

for quarters etc.

- We multiply these two together and compare coefficients

$$\begin{aligned} \frac{1}{1-x^{10}} \frac{1}{1-x^{25}} &= 1 + x^{10} + x^{20} + x^{25} \\ &+ x^{30} + x^{35} + x^{40} + x^{45} \\ &+ 2\,x^{50} + x^{55} + 2\,x^{60} + \cdots \end{aligned}$$

We argue that the *coefficient* of x^{60} on the right is precisely the number of ways of making 60 cents out of identical dimes and quarters.

- This is easy to check with a handful of change or a calculator, The general question with more denominations is handled similarly.

- I leave it open whether it is easier to decode the generating function from the picture or vice versa
  - in any event, symbolic and graphic experiment provide abundant and mutual reinforcement and assistance in concept formation.

"*In the first place, the beginner must be convinced that proofs deserve to be studied, that they have a purpose, that they are interesting.*" (George Polya)

While by 'beginner' George Polya intended young school students, I suggest this is equally true of anyone engaging for the first time with an unfamiliar topic in mathematics.

Our MOTIVATION and GOALS

INSIGHT − demands speed ≡ **micro-parallelism**

- For rapid verification.

- For validation; proofs *and* refutations; "monster barring".

★ *What is "easy" changes:* HPC & HPN blur, merging disciplines and collaborators — democratizing math but challenging authenticity.

- **Parallelism** ≡ more space, speed & stuff.

- **Exact** ≡ hybrid ≡ symbolic '+' numeric (*Maple meets NAG*, Matlab calls *Maple*).

- In analysis, algebra, geometry & topology.

. . . Moreover

- Towards an Experimental **Mathodology**— philosophy and practice.

▶ **Intuition is acquired** — mesh computation and mathematics.

- **Visualization** — **3** is a lot of dimensions.

▶ "Monster-barring" (Lakatos) and "Caging" (JMB):

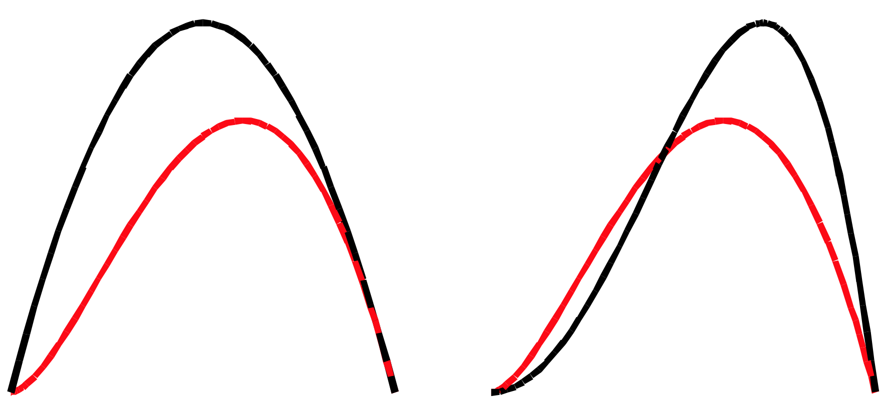
  – randomized checks: equations, linear algebra, primality.

  – graphic checks: equalities, inequalities, areas.

. . . Graphic Checks

- Comparing $y - y^2$ and $y^2 - y^4$ to $-y^2 \ln(y)$ for $0 < y < 1$ pictorially is a much more rapid way to divine which is larger than traditional analytic methods.

- It is clear that in the later case they cross, it is futile to try to prove one majorizes the other. In the first case, evidence is provided to motivate a proof.



Graphical comparison of
$y - y^2$ and $y^2 - y^4$ to $-y^2 \ln(y)$ (red)

MINIMAL POLYNOMIALS of MATRICES

Consider matrices A, B, C, M:

$$A := \left[(-1)^{k+1}\binom{2n-j}{2n-k}\right], \quad B := \left[(-1)^{k+1}\binom{2n-j}{k-1}\right]$$

$$C := \left[(-1)^{k+1}\binom{j-1}{k-1}\right]$$

$(k, j = 1, \ldots, n)$ and set

$$\boxed{M := A + B - C}$$

- In work on *Euler Sums* we needed to prove M invertible: actually

$$\boxed{M^{-1} = \frac{M+I}{2}}$$

- The key is discovering

$$\begin{aligned} A^2 &= C^2 = I \\ B^2 &= CA, \qquad AC = B. \end{aligned} \tag{2}$$

$\therefore$ *The group generated by* A,B,C *is* S_3

◇ Once discovered, the combinatorial proof of this is routine – for a human or a computer ('$A = B$', Wilf-Zeilberger).

One now easily shows using (2)

$$\boxed{M^2 + M = 2I}$$

as formal algebra since $M = A + B - C$.

- In truth I started in Maple with cases of

$$`minpoly(M, x)`$$

and then emboldened I typed

$$`minpoly(B, x)` \ldots$$

  – Random matrices have full degree *minimal polynomials*.

  – *Jordan Forms* uncover Spectral Abscissas.

OUR EXPERIMENTAL MATHODOLOGY

1. Gaining *insight* and intuition

2. Discovering new *patterns* and relationships

3. *Graphing* to expose math principles

4. Testing and especially *falsifying* conjectures

5. Exploring a possible result to see if it *merits* formal proof

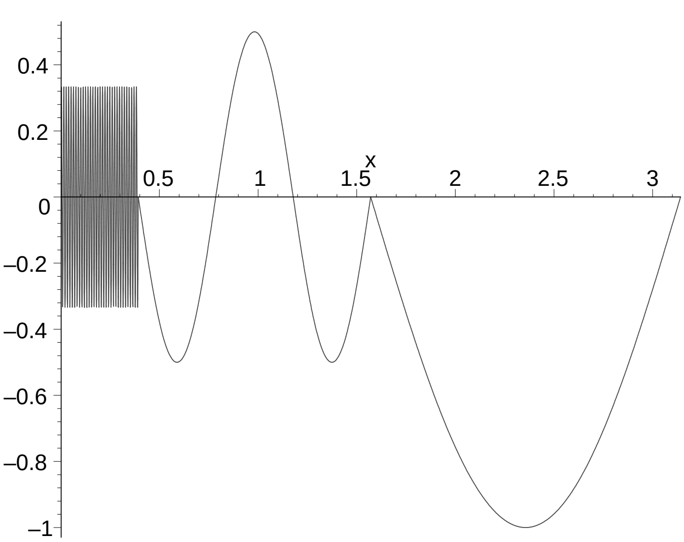
6. Suggesting approaches for *formal proof*

7. *Computing* replacing lengthy hand derivations

8. *Confirming* analytically derived results

A BRIEF HISTORY OF RIGOUR

- **Greeks**: trisection, circle squaring, cube doubling and $\sqrt{2}$
- **Newton and Leibniz**: fluxions/infinitesimals
- **Cauchy and Fourier**: limits and continuity
- **Frege and Russell, Gödel and Turing**: paradoxes and types, proof and truth
- **ENIAC and COQ**: verification and validation



For continuous functions **Fourier series need not converge:** in 1810, 1860 or 1910?

THE PHILOSOPHIES OF RIGOUR

- Everyman: Platonism—stuff exists (1936)
- Hilbert: Formalism—math is invented; formal symbolic games without meaning
- Brouwer: Intuitionism-—many variants; ('embodied cognition')
- Bishop: Constructivism—tell me how big; (not 'social constructivism')

℧ Last two deny *excluded middle*: $A \vee \tilde{A}$ and resonate with computer science—as does some of formalism.

Ξ *Absolutism versus Fallibilism.*

SOME SELF PROMOTION

- Today *Experimental Mathematics* is being discussed quite widely

MATH

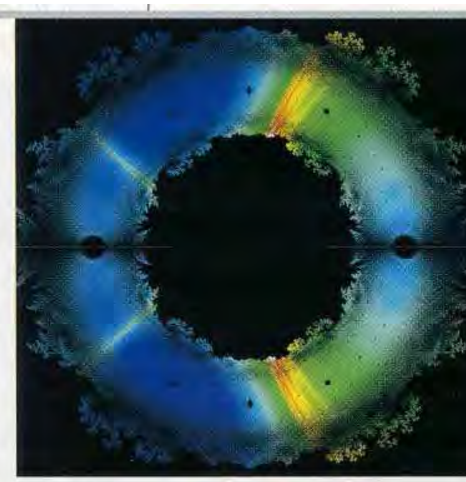
A Digital Slice of Pi

THE NEW WAY TO DO PURE MATH: EXPERIMENTALLY BY W. WAYT GIBBS

"**One of the greatest ironies** of the information technology revolution is that while the computer was conceived and born in the field of pure mathematics, through the genius of giants such as John von Neumann and Alan Turing, until recently this marvelous technology had only a minor impact within the field that gave it birth." So begins *Experimentation in Mathematics*, a book by Jonathan M. Borwein and David H. Bailey due out in September that documents how all that has begun to change. Computers, once looked on by mathematical researchers with disdain as mere calculators, have gained enough power to enable an entirely new way to make fundamental discoveries: by running experiments and observing what happens.

The first clear evidence of this shift emerged in 1996. Bailey, who is chief technologist at the National Energy Research Scientific Computing Center in Berkeley, Calif., and several colleagues developed a computer program that could uncover integer relations among long chains of real numbers. It was a problem that had long vexed mathematicians. Euclid discovered the first integer relation scheme—a way to work out the greatest common divisor of any two integers—around 300 B.C. But it wasn't until 1977 that Helaman Ferguson and Rodney W. Forcade at last found a method to detect relations among an arbitrarily large set of numbers. Building on that work, in 1995 Bailey's group turned its computers loose on some of the fundamental constants of math, such as log 2 and pi.

To the researchers' great surprise, after months of calculations the machines came up with novel formulas for these and other nat-



COMPUTER RENDERINGS of mathematical constructs can reveal hidden structure. The bands of color that appear in this plot of all solutions to a certain class of polynomials (specifically, those of the form $\pm 1 \pm x \pm x^2 \pm x^3 \pm \ldots \pm x^n = 0$, up to $n = 18$) have yet to be explained by conventional analysis.

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From Scientific American, May 2003

MATH LAB

Computer experiments are transforming mathematics

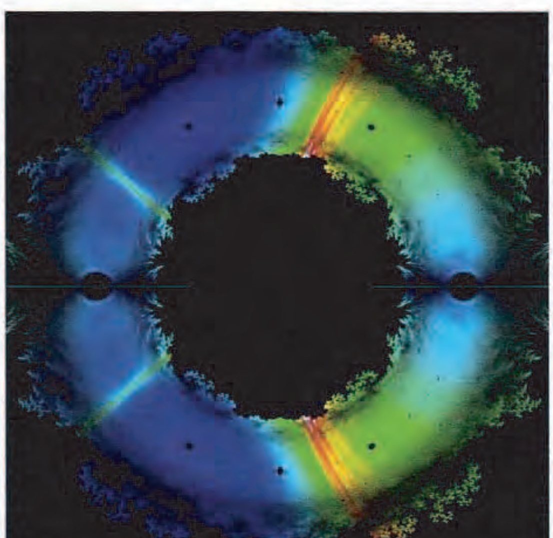
BY ERICA KLARREICH

Many people regard mathematics as the crown jewel of the sciences. Yet math has historically lacked one of the defining trappings of science: laboratory equipment. Physicists have their particle accelerators; biologists, their electron microscopes; and astronomers, their telescopes. Mathematics, by contrast, concerns not the physical landscape but an idealized, abstract world. For exploring that world, mathematicians have traditionally had only their intuition.

Now, computers are starting to give mathematicians the lab instrument that they have been missing. Sophisticated software is enabling researchers to travel further and deeper into the mathematical universe. They're calculating the number pi with mind-boggling precision, for instance, or discovering patterns in the contours of beautiful, infinite chains of spheres that arise out of the geometry of knots.

Experiments in the computer lab are leading mathematicians to discoveries and insights that they might never have reached by traditional means. "Pretty much every [mathematical] field has been transformed by it," says Richard Crandall, a mathematician at Reed College in Portland, Ore. "Instead of just being a number-crunching tool, the computer is becoming more like a garden shovel that turns over rocks, and you find things underneath."

At the same time, the new work is raising unsettling questions about how to regard experimental results



UNSOLVED MYSTERIES — A computer experiment produced this plot of all the solutions to a collection of simple equations in 2001. Mathematicians are still trying to account for its many features.

"I have some of the excitement that Leonardo of Pisa must have felt when he encountered Arabic arithmetic. It suddenly made certain calculations flabbergastingly easy," Borwein says. "That's what I think is happening with computer experimentation today."

EXPERIMENTERS OF OLD In one sense, math experiments are nothing new. Despite their field's reputation as a purely deductive science, the great mathematicians over the centuries have never limited themselves to formal reasoning and proof.

For instance, in 1666, sheer curiosity and love of numbers led Isaac Newton to calculate directly the first 16 digits of the number pi, later writing, "I am ashamed to tell you to how many figures I carried these computations, having no other business at the time."

Carl Friedrich Gauss, one of the towering figures of 19th-century mathematics, habitually discovered new mathematical results by experimenting with numbers and looking for patterns. When Gauss was a teenager, for instance, his experiments led him to one of the most important conjectures in the history of number theory: that the number of prime numbers less than a number x is roughly equal to x divided by the logarithm of x.

Gauss often discovered results experimentally long before he could prove them formally. Once, he complained, "I have the result, but I do not yet know how to get it."

In the case of the prime number theorem, Gauss later refined his conjecture but never did figure out how to prove it. It took more than a century for mathematicians to come up with a proof.

Like today's mathematicians, math experimenters in the late 19th century used computers—but in those days, the word referred to people with a special facility for calcu-

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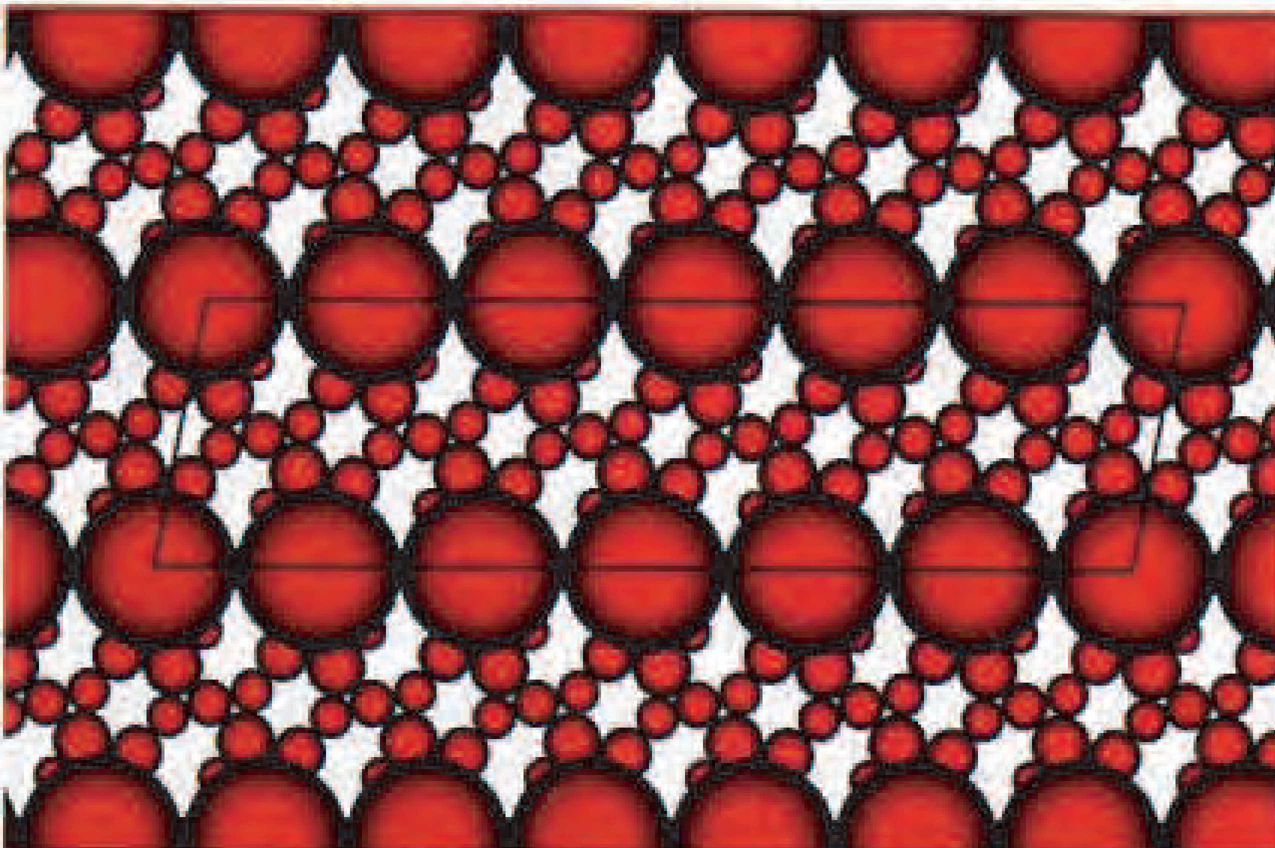


STRAIGHT CIRCLES — When mathematicians Colin Adams and Eric Schoenfeld created this image while playing with the computer program Snappea last year, they were stunned to see perfectly straight chains of spheres. The observation led them to an unexpected discovery about knots.

A Discovery in SnapPea

Bailey and Jonathan Borwein advance the controversial thesis that mathematics should move toward a more empirical approach. In it, formal proof would not be the only acceptable way to establish mathematical knowledge.

Mathematicians, Bailey and Borwein argue, should be free to work more like other scientists do, developing hypotheses through experimentation and then testing them in further experiments. Formal proof is still the ideal, they say, but it is not the only path to mathematical truth.

"When I started school, I thought mathematics was about proofs, but now I think it's about having secure mathematical knowledge," Borwein says. "We claim that's not the same thing."

Bailey and Borwein point out that mathematical proofs can run to hundreds of pages and require such specialized knowledge that only a few people are capable of reading and judging them.

"We feel that in many cases, computations constitute very strong evidence, evidence that is at least as compelling as some of the more complex formal proofs in the literature," Bailey and Borwein say in *Mathematics by Experiment.*

"One thing that's happening is you can discover many more things than you can explain."

— JONATHAN BORWEIN
DALHOUSIE UNIVERSITY

CONCLUSION

Serving a Silicon Master

Mathematics by Experiment: Plausible Reasoning in the 21st Century. Jonathan Borwein and David Bailey. x + 288 pp. A K Peters, 2004. $45.

Experimentation in Mathematics: Computational Paths to Discovery. Jonathan Borwein, David Bailey and Roland Girgensohn. x + 357 pp. A K Peters, 2004. $49.

Once upon a time, in ancient Greece, science was platonic and *a priori*. The Sun revolved around the Earth in a perfect circle, because the circle is such a perfect figure; there were four elements, because four is such a nice number, and so forth. Then along came Bacon, Boyle, Galileo, Kepler, Lavoisier, Newton and their buddies, and revolutionized science, making it experimental and empirical.

But math remains *a priori* and platonic to this day. Kant even went to excruciating lengths to "show" that geometry, although synthetic, is nevertheless *a priori*. Sure, all mathematicians, great and small, conducted experiments (until recently, using paper and pencil), but they kept their diaries and notebooks well hidden in the closet.

But stand by for a paradigm shift: Thanks to Its Omnipotence the Computer, math—that last stronghold of dear Plato—is becoming (overtly!) experimental, *a posteriori* and even contingent.

But what are poor pure mathematicians to do? Their professional weltanschauung—in other words, their philosophy—and more important, their working habits—in other words, their methodology—never prepared them for serving this new silicon master. Some of them, like the conceptual genius Alexander Grothendieck, even consider the computer (seriously!) the devil. But although many pure mathematicians strongly dislike and mistrust

bandwagons, is here to stay) had better read both *Mathematics by Experiment* and *Experimentation in Mathematics*. Traditionalists may get annoyed, since the authors (Jonathan Borwein, David Bailey and Roland Girgensohn) don't make any bones about "math by experiment" being truly a paradigm shift. They even dedicate a whole section to the Kuhnian notion of paradigm shift, quoting Max Planck ("the transfer of allegiance from paradigm to paradigm is a conversion experience that cannot be forced") to make the point that we can't hasten acceptance of the new perspective, we can only be patient and wait for the old guard to die.

These are such fun books to read! Actually, calling them *books* does not do

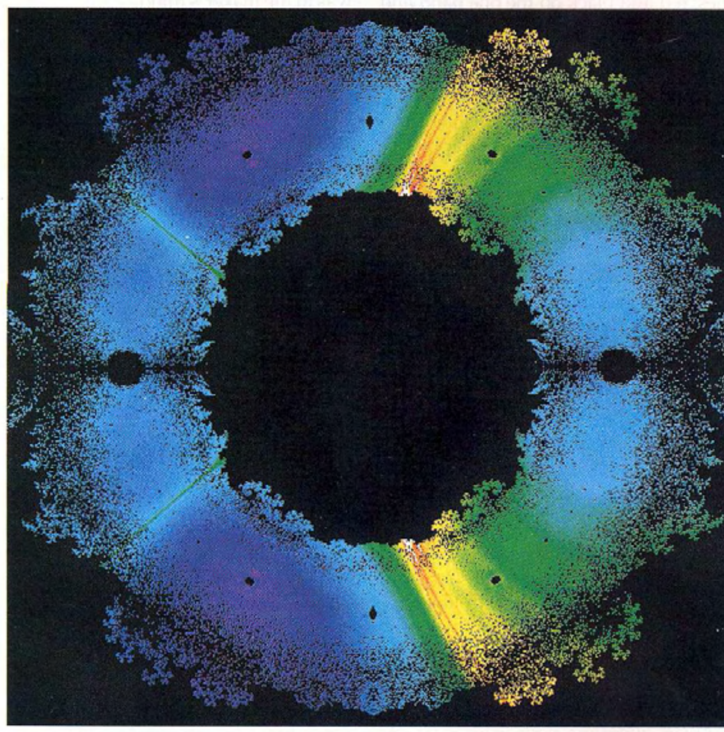
osmosis how to become an experimental mathematician.

One of the many highlights is a detailed behind-the-scenes account of the discovery of the amazing Borwein-Bailey-Plouffe (BBP) formula for π:

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i}\left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6}\right)$$

(By the way, the Bailey is David, but the Borwein is Jonathan's brother Peter. Simon Plouffe, a latter-day Ramanujan, is the webmaster of the celebrated Inverse Symbolic Calculator site.)

The BBP formula allows one to compute the billion-and-first digit of π (in base 2) without computing the first billion digits. It was discovered with the aid



This figure plots all roots of polynomials, β_N, with coefficients in {0, 1, –1} up to degree N=18. The zeroes are colored by their local density normalized to the range of densities, from red (low) to yellow (high). The fractal structures and holes around the roots come in different shapes and have precise locations. From *Experimentation in Mathematics*.

From American Scientist, March 2005

- In the next Lecture we will return to these themes more mathematically.