THIS IS A REVIEW, PREPARED FOR *IEEE CONTROL SYSTEMS*, OF *IMPLICIT FUNCTIONS AND SOLUTION MAPPINGS: A VIEW FROM VARIATIONAL ANALYSIS*, BY ASEN L. DONTCHEV, AND R. TYRRELL ROCKAFELLAR. IT APPEARED IN *SPRINGER MONOGRAPHS IN MATHEMATICS*, 2009.

1. Implicit Functions Then and Now

When I was first introduced to inverse and implicit functions as an undergraduate in the late sixties, students were drilled in the need to have functions as a prerequisite to doing analysis. Multivalued relations were to be avoided as much as possible. Indeed, Claude Berge's influential book [2] doing elementary point-set topology with multifunctions first appeared in English — finely translated by my father's St Andrew's topologist colleague — in 1963 and was largely ignored. It is still in print despite Frank Bonsall's review [4]:

> It is difficult to detect a consistent purpose behind the writing of this book, or a substantial class of readers for whom it is intended. The first half of the book is in some respects an excellent introduction to general topology, and I particularly like its thoroughness over elementary matters and its unusually explicit use of quantifiers. On the other hand, its utility for the beginner is surely greatly reduced by the author's insistence on allowing functions to be many-valued.

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While this attitude has not entirely disappeared, a modern undergraduate text *Real Analysis with Real Applications* [8] includes *Hausdorff metric*, *epigraph* and *subdifferential* in its index.

In advanced calculus the need to justify the existence of a suitable implicit function was usually dismissed with an argument about "having n equations in n unknowns"[1] but in the differential equation or introductory functional analysis courses the matter was treated more seriously and made the role of implicit function theorems crucial.

A representative inverse function theorem of the period is given on page 10 of the book under review along with two traditional proofs:

Theorem 1 (Inverse function theorem)**.** *Suppose that a function f maps a Euclidean space to itself and is continuously differentiable in a neighbourhood of a point x. If $\nabla f(x)$ is non-singular then f^{-1} admits a single valued localization s in a neighbourhood of $y = f(x)$. Moreover s is continuously differentiable in some neighbourhood of y and*

$$\nabla s(y) = \nabla f(s(y))^{-1}.$$

The first proof, based on a *modified Newton's method*, originates with Goursat in 1903 and the second relies on Banach's 1922 *contraction mapping principle*. Many of the requisite ideas had originated in order to prove theorems like the following, again due to Goursat in his influential 1903 text:

[1]This was famously the case in mathematical economics before the celebrated Arrow-Debreu theorem [1].

Theorem 2 (Initial value problem)**.** *Suppose F is continuous along with its first partial derivatives on some domain D. Consider an ordinary differential equation*

$$y^{(n)} = F(x, y, y^{'}, \ldots, y^{(n)}) \tag{1.1}$$

along with the initial conditions

$$f(x_0) = y_0, f^{'}(x_0) = y_0^{'}, \ldots, f^{(n)}(x_0) = y_0^{(n)}. \tag{1.2}$$

Then (1.1) admits locally unique solutions satisfying (1.2) for each vector $\left(y_0, y_0^{'}, \ldots, y_0^{(n)}\right)$ *in D.*

Theorem 3 (Banach closed graph and open mapping theorem)**.** *We have both:*

(1) *A linear mapping T between Banach spaces with a (norm)* closed graph *is continuous.*

(2) *A surjective linear mapping T with a closed graph is an* open mapping *(i.e., $0 \in \operatorname{int} T(B)$ where B is the unit ball).*

Typically one would be deduced and the other extracted as a consequence of a quotient argument so that both T and T^{-1} were *bona fide* functions. This both added spurious technical conditions, so as to assure that the quotient was a Banach space, and hid the complete symmetry of a modern treatment as done as it is for multifunctions in Chapter 5 of the book under review.

Seemingly unrelated were estimates for the growth of solutions to (linear) inequality systems — of the sort that the discovery of the simplex method for linear programming made important.

Theorem 4 (Hoffmann error bounds, 1952)**.** *Consider the finite dimensional inequality system*

$$S := S(b) = \{x\colon Ax \le a, Bx = b\}. \tag{1.3}$$

If $S \neq \emptyset$ *then for some constant* $K > 0$ *one has*

$$d(x,S) \le K\left(\|(Ax-a)_+\| + \|Bx-b\|\right). \tag{1.4}$$

Here $d_S(x) = d(x,S) := \inf_{x\in S}\|x-s\|$ is the metric *distance function* of x from S, "$\le$" represents the coordinate ordering, "$_+$" the coordinate-wise positive part, while if desired all norms in X are Euclidean. A fine modern treatment of such estimates is in given [12]. By the late 1970's, led by Stephen Robinson [13] researchers had started to connect these various topics coherently.

Likewise, the emergence of convex analysis, and soon thereafter of modern nonsmooth analysis, made inescapable the need to admit multivalued "derivatives" or *subdifferentials* [14, 15, 6, 5] such as the *convex subdifferential*

$$\partial f(\overline{x}) = \{y \in X^*\colon f(\overline{x}) + \langle y, x-\overline{x}\rangle \le f(x), \forall x \in X\}. \tag{1.5}$$

Forty years on, the utility of multifunctions is much better recognized and there are many fine expositions of their power and ubiquity. This has lead to a much better unified treatment of stability theorems, of inversion theorems, of measurable and topological *selection theorems*, and much more. I refer to [15] for a magisterial accounting in finite dimensions by Terry Rockafellar and Roger Wets [15], and to two recent books I have coauthored for more details in infinite dimensions [6, 5].

A simple informative illustration is that the square-root mapping

$$re^{i\theta} \mapsto \{-\sqrt{r}e^{i\theta/2}, \sqrt{r}e^{i\theta/2}\}$$

in the plane is nonexpansive in the Hausdorff metric but has no single-valued continuous selection, as observed by Nadler who first extended the Banach contraction principle to multifunctions in 1969 [11].

The book under review, *Implicit Functions and Solution Mappings*, which I will refer to as ("the book") is the first, and a very welcome, comprehensive modern treatment of implicit functions. Most satisfactorily as the approach has broadened, the results have both become easier and more generally applicable.

2. Inside this Book

The authors are distinguished members of the mathematical research community. Terry Rockafellar, alone and with others, has been a driving force in modern optimization and cognate areas since the appearance of his definitive *Convex Analysis* [14] forty years ago. Asen Dontchev has been an important contributor to the study of *perturbed* *optimization and control* problems and is also a skillful numerical analyst. Their overlapping but distinct skills are well displayed in the book.

The book commences with a helpful context-setting preface followed by six *Chapters* which I shall briefly discuss individually. Each *Chapter* starts with a useful preamble and concludes with a careful and instructive *Commentary*; while a good set of *References*, a *Notation* guide and a somewhat brief *Index* complete this valuable study.

2.1. **Ch. 1. Functions Defined Implicitly by Equations.** The authors begin by tracing the history of implicit function theorems from Dini in the late 19th century on. I wish all authors would so situate their work. I often encounter technically-expert post doctoral fellows who have no idea why they are studying the generalizations they care deeply about.

Dontchev and Rockafellar then introduce the key ideas of modern Lipschitz analysis such as *calmness* and so are ready for the main event:

2.2. **Ch. 2. Implicit Function Theorems for Variational Problems.** With appropriate recognition of Robinson's pioneering work [13] we are now introduced to parametric *generalized equations*:

$$0 \in f(p, x) + F(x), \tag{2.6}$$

and given the analytic tools necessary for their study. Here the variables x, p lie in different Euclidean spaces, and f is a single valued function, while F has sets for images and so can model tangency properties or capture inequality systems. A good example of such a multifunction is to consider $F = S$ of equation (1.3).

Robinson's implicit function theorem for the case of a *normal cone* $F(x) := N_C(x)$ (as defined below (2.10)) is given, after which many nice extensions and applications to optimization are explored. These include variational inequality theory and more. Through out, smoothness of the inverse is relaxed to Lipschitz continuity and explicit bounds are given.

2.3. **Ch. 3. Regularity Properties of Set-Valued Solution Mappings.** This chapter then turns to the various stability properties of the *solution set* as a function of the parameter p. To do this requires the introduction of various fundamental *set convergence* and *Lipschitz continuity* concepts.

A key notion is that of *metric regularity* at $\overline{x}$ for $\overline{y} \in F(\overline{x})$ which requires

$$d(x, F^{-1}(y)) \leq \kappa\, d(y, F(x)) \tag{2.7}$$

for some $\kappa > 0$ and for x, y close to $(\overline{x}, \overline{y})$.

This is a term I apparently coined in the mid-eighties (e.g., [9, p. 195], [10]) and holds precisely when the inverse mapping is appropriately continuous. Thus, in the perspective of the current book, metric regularity is an *openness* condition. Note that (2.7) is qualitatively similar to (1.4). Once (2.7) is known for a generalized equation of interest, much powerful analysis and numerical analysis can be performed as we then know that the distance to the feasible set (the LHS) is of no greater order that the error in the data (the RHS). This in essence yields an inverse function.

Of course to do this we must have effective — ideally quantitative — conditions that ensure metric regularity or one of its weaker or stronger variants. This is the goal of the next two chapters.

2.4. **Ch. 4. Regularity Properties Through Generalized Derivatives.** In Theorem 1 the sufficient condition for invertibility is the surjectivity of the derivative. So it will come as no surprise that one must first introduce appropriate *generalized derivatives* of multifunctions. This is a much studied topic and there is no magic bullet.

That said, there is an elegant theory of *graphical differentiation*. Once developed, the authors are able to use it to characterize regularity in terms of the local boundedness of the multifunctional inverse of the derivate. This leads to very complete and attractive results in Euclidean space. Appropriate notice is made of other developments due to Clarke [7] (*generalized Jacobians*) and Mordukhovich [10] (*coderivatives*) that do not fit entirely into the present development.

2.5. **Ch. 5. Regularity in Infinite Dimensions.** The chapter starts with a recapture of the open mapping theorem and moves on to *sublinear multifunctions* — those whose graphs are closed and convex cones. For these and various strictly differentiable generalizations a wonderfully complete and useful extension of the classical results is possible. These include very general versions of the *Kuhn-Tucker* theorem, see (2.11).

In stark contrast to the finite dimensional setting, it is impossible to get such precise and exact regularity measures without some restriction. The unit ball is no longer norm-compact and so much of the familiar mode of limiting argument is unavailable; and Baire category and weak compactness arguments can only partially remedy the situation. That said, the authors show that regularity ideas can be profitably extended even to complete metric spaces.

Let us illustrate one of the simplest but still puissant instances of metric regularity:

Example 1 (Metric regularity for two convex sets)**.** Suppose $C \subset H_1, D \subset H_2$ are closed and convex subsets with $x_0 \in A(C) \cap D \neq \emptyset$, where $A\colon H_1 \to H_2$ is a bounded linear mapping between Hilbert spaces H_1, H_2 with adjoint A^*. We consider the multifunction

$$\Omega(x) =: \begin{cases} Ax - D & x \in C \\ \emptyset & x \notin C. \end{cases} \tag{2.8}$$

Then $\Omega(H_1) = A(C) - D$ and we discover that the openness/regularity condition: $0 \in \operatorname{int} \Omega(H_1)$ guarantees metric regularity and so (2.7) allows us to write

$$d_{C\cap A^{-1}D}(x) \leq \kappa \left\{(d_C(x)) + d_D(Ax)\right\} \tag{2.9}$$

for x near $\overline{x}$ and some $\kappa > 0$. In particular this holds if the classical *Slater condition* [14] $A(C) \cap \operatorname{int} D \neq \emptyset$ holds.

If we view Ω as a solution mapping then the perturbed solution set is $\Omega^{-1}(y) = C \cap A^{-1}(y + D)$ and $\Omega^{-1}(0) = C \cap A^{-1}D$. We have thus written a powerful error bound as an open mapping theorem for a natural multifunction. Given (2.9) with $A = I$ we can now establish the geometric convergence of the familiar *Von Neumann alternating projection* algorithm to find a point in $C \cap D$ and much more [3]. □

Example 2 (Normal cones for convex systems)**.** Moreover, taking subgradients in (2.9) leads to

$$\partial d_{C\cap A^{-1}D}(\overline{x}) \subset \kappa\,\partial d_C(\overline{x}) + \kappa\, A^*\partial d_D(A\overline{x}), \tag{2.10}$$

and so to

$$N_{C\cap A^{-1}D}(\overline{x}) \subset N_C(\overline{x}) + A^* N_D(A\overline{x}), \tag{2.11}$$

since the *normal cone* to C at $\overline{x}$, $N_C(\overline{x})$, can be defined to be the cone generated by $\partial d_C(\overline{x})$. Applied to linearizations of a smooth optimization problem — so that A becomes $\nabla g(\overline{x})$ for some g — this is precisely the sort of result that in classical Kuhn-Tucker theory is based on the inverse function theorem.

It is worth considering $A = I$, and C and D to be two tangent discs in the plane at the point of tangency, where metric regularity fails and (2.10), (2.11) are not valid. □

So we have exposed the underlying geometry in a most effective and natural way.

2.6. **Ch. 6. Applications in Numerical Variational Analysis.** In the final chapter, the authors give additional tastes of the power of the tools they have presented. They start by describing beautiful modern understanding of *condition numbers* for inequality and more general systems. They then turn to iterative — Newton type — processes for the solution of generalized equations, and finish with an application to numerical optimal control.

3. Final Comments

While reviewing this book, I asked a recent PhD well versed with the material for a comment. He wrote

> That's a hard question for me: I've seen this book growing from the very beginning, so it's hard to be objective. But I'll do my best.
>
> I think it's important the way they extend the ideas from the implicit function theorem by relaxing some assumptions, like differentiability. They carry the ideas from classical analysis into modern variational analysis, providing thus a motivation for introducing the regularity properties in Chapter 3 (and later Chapter 5). Overall, I think that's the main difference with respect to other books.
>
> In my (biased) opinion, it's the best reference I've seen to learn about metric regularity for someone who doesn't know anything about it, like I was. They do not forget to explain the graphical differentiation viewpoint of metric regularity in Chapter 4, and they provide some applications of the results in the book to numerical analysis in the last chapter. I also like the way it's organized, with clear and detailed proofs, providing some examples that help to understand the notions and results.

Like my younger colleague, I unreservedly recommend this book to all practitioners and graduate students interested in modern optimization theory or control theory or to those just engaged by beautiful analysis cleanly described.

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