Jonathan Borwein, Matthew Skerritt

An Introduction to Modern Mathematical Computing

July 25, 2010

Springer

Preface

Thirty years ago mathematical, as opposed to applied numerical, computation was difficult to perform and so relatively little used. Three threads changed that:

- The emergence of the personal computer; identified with the iconic Macintosh but made ubiquitous by the IBM PC.
- The discovery of fibre-optics and the consequent development of the modern internet culminating with the foundation of the World Wide Web in 1989 made possible by the invention of hypertext earlier in the decade.
- The building of the *Three Ms*: *Maple, Mathematica* and MATLAB. Each of these is a complete mathematical computation workspace with a large and constantly expanding built-in "knowledge base". The first two are known as "computer algebra" or "symbolic computation" systems, sometimes written *CAS*. They aim to provide exact mathematical answers to mathematical questions like what is

  $$\int_{-\infty}^{\infty} e^{-x^2}\, dx,$$

  what is the real root of $x^3 + x = 1$, or what is the next prime number after $1,000,000,000$? The answers, respectively, are

  $$\sqrt{\pi},\ \frac{\sqrt[3]{108+12\sqrt{93}}}{6} - \frac{2}{\sqrt[3]{108+12\sqrt{93}}},\ \text{and}\ 1,000,000,007.$$

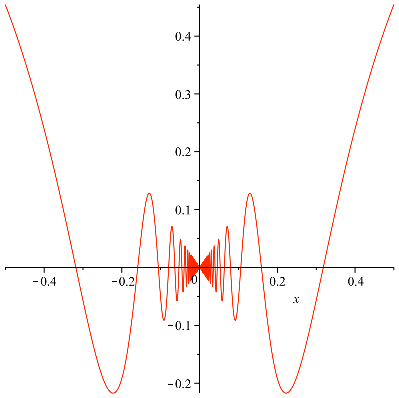
  The third 'M' is primarily numerically based. The distinction, however is not a simple one. Moreover, more-and-more modern mathematical computation requires a mixture of so-called *hybrid* numeric-symbolic computation and also relies on significant use of geometric, graphic and visualization tools. These we shall also introduce, both inside *Maple* and later in the course using the "interactive geometry" package, *Cinderella*. It is also possible to mix these technologies, for example to make use of MATLAB through a *Maple* interface, see also [5]. MATLAB is the preferred tool of many engineers and other scientist who need easy access to efficient numerical computation.

Of course each of these threads relies on earlier related events and projects, and there are many other open-source and commercial software packages. For example, *Sage* is an opensource CAS, *GeoGebra* an opensource interactive geometry package and *Octave* is an opensource counterpart of MATLAB. But, here is not the place to discuss the merits and demerits of opensource alternatives. For many purposes *Mathematica* and *Maple* are interchangeable as adjuncts to mathematical learning. We propose to teach you the latter. After this course, you should find it easy to pick up the requisite skills to use *Mathematica* [13] or MATLAB.

Many introductions to computer packages aim to teach you the *syntax* (rules and structure) and *semantics* (meaning) of the system as efficiently as possible [6, 7, 8, 12]. They assume you know why you want to learn such things. By contrast, we intend to persuade you that *Maple* and other like tools are worth knowing assuming only that you wish to be a mathematician, a mathematics educator, a computer scientist, an engineer or scientist, or anyone else who wishes/needs to use mathematics better. We also hope to explain to you how to become an 'experimental mathematician' while learning to be better at proving things. To accomplish this our material is divided into three main chapters followed by a postscript on:

- **Elementary number theory**. Using only mathematics that should be familiar from high school, we shall introduce most of the basic computational ideas behind *Maple*. By the end of this chapter the hope is that the student can learn new features of *Maple* while also learning more mathematics.
- **Calculus of one and several variables.** In this chapter we shall revisit ideas met in first year calculus and introduce the basic ways to plot and explore functions graphically in *Maple.* Many have been taught not to trust pictures in Mathematics. This is bad advice. Rather, one has to learn how to draw trustworthy pictures.

$> \mathit{plot}\left(x\sin\left(\frac{1}{x}\right), x = -\frac{1}{2}..\frac{1}{2}\right)$



- **Introductory linear algebra.** Similarly, in this chapter we will see how much of linear algebra can be animated (that is, brought to live) within a computer algebra system. While we suppose the underlying concepts will be familiar, this is not necessary. One of the powerful attractions of computer-assisted mathematics is that it allows for a lot of "learning while doing" especially by using the help files in the system and by learning how and when to consult internet mathematics resources such as *MathWorld, PlanetMath* or *Wikipedia.*
- **Visualisation and interactive geometric computation.** Finally, we shall explore more carefully how visual computing [9, 10] can help build mathematical intuition and knowledge. This is a theme we will emphasise throughout the course.

Each chapter has three main sections each of which forms the basis for a week's lectures. The fourth section of each chapter has exercises and additional examples. It is assumed the student will try many of them; indeed they will from the basis for the lab-test assessments. The final section of each chapter is entitled further explorations. It has two main intentions. One is to provide extra material for more mathematically advanced students. The other is as a set of possible topics for the final project which each student taking the course must produce.

A more detailed discussion relating to many these brief remarks may be followed up in [4], in [2, 3] or [1], and in the references given therein.

Additional reading and references.

We also supply a list of, largely recent, books at various levels that you may find useful or stimulating. Some are technical and some are more general. Some may be useful during a final project.

1. George Boros and Victor Moll, *Irresistible Integrals*, Cambridge University Press, New York, 2004.
2. Jonathan M. Borwein and Peter B. Borwein, *Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity*, John Wiley & Sons, New York, 1987 (Paperback, 1998).
3. Christian S. Calude, *Randomness And Complexity, from Leibniz To Chaitin*, World Scientific Press, Singapore, 2007.
4. Gregory Chaitin and Paul Davies, *Thinking About Gödel and Turing: Essays on Complexity, 1970-2007,* World Scientific, 2007.
5. Richard Crandall and Carl Pomerance, P*rime Numbers: a Computational Perspective,* Springer, New York, 2001
6. Philip J. Davis, *Mathematics and Common Sense: A Case of Creative Tension,* A.K. Peters, Natick, MA, 2006.
7. Stephen R. Finch, *Mathematical Constants*, Cambridge University Press, Cambridge, 2003.
8. Marius Giaguinto, *Visual Thinking in Mathematics,* Oxford University, Oxford, 2007.
9. Ronald L. Graham, Donald E. Knuth, and Oren Patashnik, *Concrete Mathematics*, Addison-Wesley, Boston, 1994.
10. Bonnie Gold and Roger Simons (Eds.), *Proof and Other Dilemmas: Mathematics and Philosophy*, Mathematical Association of America, Washington, DC, in press, 2008.
11. Richard K. Guy, *Unsolved Problems in Number Theory*, Springer-Verlag, Heidelberg, 1994.
12. Reuben Hersh, *What is Mathematics Really?* Oxford University Press, Oxford, 1999.
13. J. Havil, *Gamma: Exploring Euler's Constant*, Princeton University Press, Princeton, NJ, 2003.
14. Steven G. Krantz, *The Proof Is in the Pudding: A Look at the Changing Nature of Mathematical Proof*, Springer, 2010.
15. Marko Petkovsek, and Herbert Wilf, and Doron Zeilberger, *A=B,* A.K. Peters, Natick, MA, 1996.
16. Nathalie Sinclair, David Pimm, and William Higginson (Eds.), *Mathematics and the Aesthetic. New Approaches to an Ancient Affinity*, CMS Books in Math, Springer-Verlag, New York, 2007.
17. J. M. Steele, *The Cauchy-Schwarz Master Class*, Mathematical Association of America, Washington, DC, 2004.
18. Karl R. Stromberg, *An Introduction to Classical Real Analysis*, Wadsworth, Belmont, CA, 1981.

19. Richard P. Stanley, *Enumerative Combinatorics*, Volumes 1 and 2, Cambridge University Press, New York, 1999.
20. Terence Tao, *Solving Mathematical Problems*, Oxford University Press, New York, 2006.
21. Nico M. Temme, *Special Functions, an Introduction to the Classical Functions of Mathematical Physics*, John Wiley, New York, 1996.
22. Fernando R. Villegas, *Experimental Number Theory*, Oxford University Press, New York, 2007.

Finally many useful links are maintained by the authors of *Mathematics by Experiment* [2] at `www.experimentalmath.info`.

Jonathan Borwein and Matthew Skerritt, July 25, 2010.

Contents

Conventions and Notation

Maple

Maple provides a wealth of different options for its interface, and mechanisms for entering commands. It is not within the scope of this document to deal with all of them. For the entirety of this document, we present *Maple* input as if it is entered in *Worksheet* mode with *2D Math* as both the input and output display settings. All three of these settings may be changed to be defaults (or not, of course) through *Maple*'s preferences.

This input scheme has the advantage that it looks a lot more like "regular" mathematics, making *Maple* worksheets easier to read than the historic text-only input scheme used in earlier versions. Using a worksheet mode instead of document mode also makes input, output and explanatory text generally much clearer to discern and is, in this author's opinion, superior when working on mathematical problems—although the document mode is probably preferable when trying to write results in a more human readable format.

Maple examples in this document are formatted to look like they would in a *Maple* worksheet using the above assumptions, and look like the following:

```
> Input; Input
  Input
                                 Output
                                 Output
                                 Output
                        Warnings and Information
                                 Errors
```

Each input—and its associated output, warnings and errors—are enclosed by a giant bracket (a "[", sometimes erroneously referred to as a "square bracket"), with a red > symbol acting as an input prompt. Multiple commands may be input together at the same prompt, and the input may even be spread over multiple lines. Input is coloured black and is left aligned. Output is blue, and centred. Warnings are blue in a monospaced "typewriter" font. Finally errors are red in a monospaced "typewriter" font. This is almost identical to *Maple* (given the above assumptions), with the only difference that *Maple*'s errors are more of a purple-like colour. For the sakes of simplicity (and not too many colours) we have adopted red for this document.

Maple Input Basics

It is, unfortunately, not immediately obvious which keystrokes produce which input effects. As such we include Table 0.1 which shows the non-obvious keystrokes. Note that *Maple* will automatically format the text as mathematics for you at the moment it is typed. *Maple* will also attempt to make sensible decisions, based around order of operations, as to which parts of the input will be effected. If *Maple* makes an incorrect decision, then parentheses (`(` and `)`) are needed to make the expression unambiguous.

Operation	**Keys**	**Example**	**Keystrokes**
Multiplication	`*`	$a \cdot b$	`a*b`
	`␣`	$a\,b$	`a␣b`
Powers	`^`	a^b	`a^b`
Subscripts	`_`	a_b	`a_b`
Fractions	`/`	$\frac{a}{b}$	`a/b`
Not Equal	`<>`	$a \neq b$	`a<>b`
Less than or Equal	`<=`	$a \leq b$	`a<=b`
Greater than or Equal	`>=`	$a \geq b$	`a>=b`
Function	`->`	$a \to b$	`a->b`
Multiple Line Commands	`shift-enter`	$\begin{matrix} a \\ b \end{matrix}$	`a shift-enter b`

Table 0.1 *Maple* Input Keystrokes

Note that whilst multiplication may be produced using a space (␣) as well as the asterisk (`*`), the dot · which *Maple* places in space of the asterisk leaves no ambiguity as to the nature of the calculation, whereas a space might easily be missed. As such the reader should feel free to use whichever method they prefer, but should be aware that the *Maple* examples in this document will use the · notation to denote multiplication, unless there can be no ambiguity (for instance, with polynomial coefficients).

Useful Functions

We list here, for convenience, a small list of *Maple* functions which are arguably essential to know. These are functions to perform elementary mathematical operations (for example, square roots) or for simplification. The list is in Table 0.2. Consult the help files on these functions for more specific details on their use. Note that many more functions are introduced and explained within the main text of this document.

Some comments:

- The **log** command may also be directed to perform logs to a base other than e if desired, but will perform natural logs unless specifically told otherwise. For example, the command log[2](3) will calculate the log of 3 to the base 2.
- The *Maple* command **Pi** is used for the *value* of π, however the *Maple* commands **pi** and **PI** respectively represent the symbols π and Π, but *hold no value whatsoever*, and may be used as variable names. Confusion may result because *Maple* prints the symbol π in the case of both the value **Pi** and the symbol **pi**.
- The commands for algebraic rearrangement work primarily as one would expect from their names. However, it should be noted that when faced with a complicated answer from *Maple* it is sometimes the case that **factor**, **expand** or even **combine**

Desired Effect	**Command**
Square root of a number ($\sqrt{x}$)	**sqrt**
n^{th} root of a number ($\sqrt[n]{x}$)	**root** **surd**
Power of e (e^x)	**exp**
Natural logarithm ($\log x$)	**log** **ln**
Value of π	**Pi**
Value of ∞	**infinity**
Rearrange an algebraic expression	**simplify** **factor** **expand** **combine**
Conversion between forms	**convert**

Table 0.2 Essential *Maple* functions

will produce a more simplified answer than the **simplify** command does. Furthermore, these commands used in conjunction with each other can produce superior results.

- The **convert** function is a many-faceted function which allows many different types of conversions to be performed (see Exercise 4). Of particular interest and utility, in conjunction with the algebraic rearrangement above, is the variant that allows conversion of an expression to one involving standard functions: $convert(\,\cdot\,, StandardFunctions)$.

Mathematics

Included in Table 0.3 is a list of mathematical notation, and its meaning.

Notation	**Meaning**
$\mathbb{N}$	The set of natural numbers
$\mathbb{Z}$	The set of integers
$\mathbb{Q}$	The set of rational numbers
$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of complex numbers
$\log x$	The *natural* logarithm of x
$\log_b x$	The logarithm of x (base b)
$\sin x$	The sine function applied to x
$\sin^n x$	The n^{th} power of the sine function applied to x. That is $(\sin x)^n$.
$\sin^{-1} x$	The inverse-sine function applied to x
$\arcsin x$	The inverse-sine function applied to x
$\cos x$	The cosine function applied to x
$\cos^n x$	The n^{th} power of the cosine function applied to x. That is $(\cos x)^n$.
$\cos^{-1} x$	The inverse-cosine function applied to x
$\arccos x$	The inverse-cosine function applied to x

Table 0.3 Mathematical Notation