

A NEW PROOF OF THE LYAPUNOV CONVEXITY THEOREM*

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Abstract. A new proof of the Lyapunov Theorem is given, based on the Shapley–Folkman Theorem, that does not require any tools of functional analysis.

Key words. vector-valued measures, Lyapunov Convexity Theorem, convex analysis, integrals of multi-functions

AMS(MOS) subject classifications. 28B05, 46G10

1. Introduction. The Lyapunov Theorem appeared for the first time in [14]. It states that a nonatomic measure that takes values in \mathbf{R}^n has closed and convex range. Most of its applications are to be found in the theory of optimal control and calculus of variations (see, e.g., [2], [8], [9], [18]). However it has also been fruitfully used in other fields, such as economics [16] and differential equations [19].

Because of its importance, many proofs of the Lyapunov Theorem have been given (see [3]–[5], [10]–[13], [20]). Nevertheless, the proofs appearing in the literature are rather involved or employ sophisticated theorems from functional analysis, such as the Krein–Milman Theorem, or compactness theorems in infinite-dimensional spaces.

Our purpose is to give a short proof of the Lyapunov Theorem that does not digress from measure theory and convex analysis. In fact the burden of the proof is carried by the Shapley–Folkman Theorem, a result of convex analysis (discovered by two economists) whose applications have unfortunately been largely restricted to mathematical economics [1], [17]. See, however, Appendix 1 of [7] for an interesting application to duality in mathematical programming.

SHAPLEY–FOLKMAN THEOREM. Consider a finite family $[C_i]_{i \in I}$ of subsets of \mathbf{R}^n . If

$$x \in \text{co} \sum_{i \in I} C_i,$$

then there exists a subset J of I , of cardinality at most n , such that

$$x \in \sum_{i \notin J} C_i + \text{co} \sum_{i \in J} C_i.$$

Here $\text{co } S$ denotes the convex hull of a set $S \subset \mathbf{R}^n$.

As Blackwell has noted [3], the convexity of the range of a nonatomic vector measure is only a special case of a more general fact, namely, the convexity of the integral of a vector-valued multifunction. Since no extra work is required, we prove the latter statement in Theorem 1. The remaining part of the Lyapunov Theorem, namely, closedness, is established in Theorem 2.

We finally remark that the nonatomicity assumption is not only sufficient but, in a sense, also necessary for a vector measure to have a convex range.

2. The Lyapunov Theorem. Let (T, Σ, μ) be a positive measure space. The measure μ is said to be nonatomic if for every $A \in \Sigma$ with $\mu(A) > 0$, there is $B \in \Sigma$, $B \subset A$, such

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that $0 < \mu(B) < \mu(A)$. Since Σ is a σ -algebra, the nonatomicity of μ is easily shown to be equivalent to the following ‘‘Darboux property’’ [6, p. 25]:

$$\forall \alpha \in [0, 1], \forall A \in \Sigma, \exists B \in \Sigma, B \subset A, \text{ such that } \mu(B) = \alpha\mu(A).$$

A function that associates with every t in T a subset $F(t)$ of \mathbf{R}^n is called a multifunction from T to \mathbf{R}^n . An integrable function from T to \mathbf{R}^n such that, for almost every t , $f(t) \in F(t)$ is called an integrable selection of F . The integral of F over T , denoted $\int_T F$, is defined as the set of all points in \mathbf{R}^n of the form $\int_T f(t) d\mu$, where f is an integrable selection of F .

THEOREM 1. *If μ is finite and nonatomic, we have*

$$\int_T F = \text{co} \int_T F.$$

Proof. We will show that if x_1 and x_2 belong to $\int_T F$, then the whole segment joining x_1 and x_2 is contained in $\int_T F$. Let $x_1 = \int_T f_1(t) d\mu$ and $x_2 = \int_T f_2(t) d\mu$, where f_1 and f_2 are integrable selections of F . We consider the multifunction $G(t) = \{f_1(t), f_2(t)\} \subset F(t)$. Because of the nonatomicity of μ , we can find a family $\{A_i\}_{1 \leq i \leq 2n}$ of elements of Σ such that $A_i \cap A_j = \emptyset$ for $i \neq j$, $\cup_{i=1,2n} A_i = T$, and $\mu(A_i) = (2n)^{-1}\mu(T)$ for $1 \leq i \leq 2n$. Given a point

$$x \in \text{co} \int_T G = \text{co} \sum_{i=1}^{2n} \int_{A_i} G$$

the Shapley–Folkman Theorem allows us to find a set I of n indices between 1 and $2n$, such that

$$x \in \sum_{i \notin I} \int_{A_i} G + \text{co} \sum_{i \in I} \int_{A_i} G.$$

The above relation is equivalent to

$$x \in \int_{S_1} G + \text{co} \int_{T \setminus S_1} G$$

where we have set $S_1 = \cup_{i \notin I} A_i$. We can therefore assert the existence of two points $x_1 \in \int_{S_1} G$ and $z_1 \in \text{co} \int_{T \setminus S_1} G$ such that

$$x = x_1 + z_1 \quad \text{and} \quad \mu(S_1) = \mu(T \setminus S_1) = 2^{-1}\mu(T).$$

The same argument can be iteratively repeated replacing T by $T \setminus S_1$, so that, by induction, at the m th step we obtain m measurable sets $\{S_i\}_{1 \leq i \leq m}$ and $m+1$ points $\{x_i\}_{1 \leq i \leq m}$ and z_m , with $x_i \in \int_{S_i} G$ and $z_m \in \text{co} \int_{T \setminus \cup_{i=1}^m S_i} G$ such that

$$x = \sum_{i=1}^m x_i + z_m \quad \text{and} \quad \mu(S_i) = 2^{-i}\mu(T), \quad 1 \leq i \leq m.$$

Observe now that $z_m \rightarrow 0$, since $\mu(T \setminus \cup_{i=1}^m S_i) = 2^{-m}\mu(T) \rightarrow 0$. Therefore we have

$$x = \sum_{i=1}^{\infty} x_i \in \int_{\cup_{i=1}^{\infty} S_i} G = \int_T G,$$

with the equality between the integrals holding because $\mu(\cup_{i=1}^{\infty} S_i) = \mu(T)$. We have thus shown that $\int_T G$ is convex. To complete the proof it is now sufficient to note that $x_1, x_2 \in \int_T G \subset \int_T F$. \square

Let $m = (\mu_1, \mu_2, \dots, \mu_n)$, where the μ_i are finite signed measures on the measurable space (T, Σ) . We say that m is nonatomic if the total variation $|\mu_i|$ of every μ_i is nonatomic.

Remark 1. An equivalent way of defining the nonatomicity of a measure is that of requiring that for every $A \in \Sigma$ there is a $B \subset A$, $B \in \Sigma$, such that $\mu(B) \neq 0$ and $\mu(B) \neq \mu(A)$. This definition can also be applied to the more general case of a measure taking its values in a topological vector space.

We will denote by

$$R(\Sigma) = \{m(A) : A \in \Sigma\}$$

the range of the vector measure m , with respect to Σ , and by

$$\Sigma_A = \{A \cap B : B \in \Sigma\}$$

the trace of the σ -algebra Σ on a set $A \in \Sigma$.

The dimension of a subset C of \mathbf{R}^n , denoted $\dim C$, is defined as the dimension of the smallest affine subspace of \mathbf{R}^n containing C .

THEOREM 2 (Lyapunov Theorem). *If m is nonatomic, $R(\Sigma)$ is closed and convex.*

Proof. The measures $\mu_1, \mu_2, \dots, \mu_n$ are all absolutely continuous with respect to the nonatomic measure $\mu = |\mu_1| + |\mu_2| + \dots + |\mu_n|$. Then, by the Radon-Nikodym Theorem, there is an integrable function f from T to \mathbf{R}^n such that $m(A) = \int_A f(t) d\mu$ for every $A \in \Sigma$. Let us consider the multifunction $F(t) = \{0, f(t)\}$. It is easy to see that the integrable selections of F are of the form $f \cdot \chi_A$, where χ_A is the characteristic function of $A \in \Sigma$. We then have

$$R(\Sigma) = \int_T F$$

and hence $R(\Sigma)$ is convex by Theorem 1.

Observe now that $R(\Sigma)$ is trivially closed when $\dim R(\Sigma) = 0$. We assume that it is closed when $\dim R(\Sigma) \leq n - 1$ and we will prove that the same thing is true when $\dim R(\Sigma) = n$. Assume there is a point $y \in \text{cl } R(\Sigma) \setminus R(\Sigma)$. Then by a standard separation argument, we can find $p \in \mathbf{R}^n$ such that

$$p \cdot y = \sup \{p \cdot x : x \in R(\Sigma)\}.$$

Let us consider the sets

$$R_k = \{t \in T : p \cdot f(t) < -1/k\}, \quad S_k = \{t \in T : p \cdot f(t) \geq -1/k\}, \quad k = 1, 2, \dots$$

Obviously,

$$R_k \cap S_k = \emptyset \quad \text{and} \quad R_k \cup S_k = T, \quad k = 1, 2, \dots$$

Let $\{A_n\}$ be a sequence of sets in Σ such that $m(A_n)$ converges to y . We then have

$$\lim_{n \rightarrow \infty} \mu(A_n \cap R_k) = 0, \quad k = 1, 2, \dots$$

Hence, for every k , we have

$$y = \lim_{n \rightarrow \infty} m(A_n) = \lim_{n \rightarrow \infty} \left(\int_{A_n \cap S_k} f(t) dt + \int_{A_n \cap R_k} f(t) dt \right) = \lim_{n \rightarrow \infty} m(A_n \cap S_k).$$

We can then find an increasing function $\sigma : \mathbf{N} \rightarrow \mathbf{N}$ such that, setting $B_n = A_{\sigma(n)} \cap S_n$, we have

$$\lim_{n \rightarrow \infty} m(B_n) = y.$$

Observe now that the sets B_n can be partitioned as follows:

$$B_n = B_n^+ \cup B_n^0 \cup B_n^-$$

where $B_n^+ \subset P = \{t \in T: p \cdot f(t) > 0\}$, $B_n^0 \subset Z = \{t \in T: p \cdot f(t) = 0\}$ and $B_n^- \subset N = \{t \in T: p \cdot f(t) < 0\}$. Since $B_n^- \subset \{t \in T: -1/n \leq p \cdot f(t) < 0\}$, we have $\lim_{n \rightarrow \infty} m(B_n^-) = 0$. Furthermore, $\lim_{n \rightarrow \infty} m(B_n^+) = m(P)$. Hence

$$y = m(P) + \lim_{n \rightarrow \infty} m(B_n^0)$$

so that $y \in m(P) + \text{cl } R(\Sigma_Z)$. The proof is easily completed observing that $\dim R(\Sigma_Z) < \dim R(\Sigma)$, and thus $R(\Sigma_Z)$ is closed by the induction hypothesis. \square

Remark 2. It has been shown by Halmos [12] that $R(\Sigma)$ is actually closed even without the assumption of nonatomicity of m .

It can be easily observed that the nonatomicity assumption on m implies not only the convexity of $R(\Sigma)$ but also the convexity of $R(\Sigma_A)$ for every $A \in \Sigma$. Conversely, taking into account Remark 1, it is clear that if $R(\Sigma_A)$ is convex for every $A \in \Sigma$, the measure m is nonatomic. These remarks are summarized in the following proposition.

PROPOSITION 3. *The measure m is nonatomic if and only if $R(\Sigma_A)$ is convex for every $A \in \Sigma$.*

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