A Uniform Treatment of Darboux's Method

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I. Introduction and Summary

DARBOUX showed in his famous memoir, [1], that if a function $F(t)$ was analytic at $t=0$, and if on its corresponding circle of convergence, $|t|=R$, $0<R<\infty$, it possessed only a finite number of singular points, all of which were algebraic in nature, then the rate of growth of its Maclaurin coefficients, $F^{(n)}(0)/n!$, could be deduced as $n\to\infty$. If these singular points, $\mathrm{R}e^{i\theta_1}$, $\mathrm{R}e^{i\theta_2}$, ..., were free to vary, it was assumed that they were restricted in such a manner that their essential configuration did not change as the θ_j varied. In practice, their configuration was preserved by restricting the θ_j to closed sub-intervals of $[0, 2\pi]$ whose boundaries were independent of n.

In this paper we consider the special case

$$\begin{aligned} F(t,\theta) &= \sum_{n=0}^{\infty} t^n A_n(\theta), \qquad |t|<1, \\ &= (1-t)^{-\lambda}[(e^{i\theta}-t)(e^{-i\theta}-t)]^{-\Delta} f(t,\theta), \end{aligned} \tag{1.1}$$

where λ and Δ are bounded quantities, those branches of $(1-t)^{-\lambda}$ and $[(e^{i\theta}-t)(e^{-i\theta}-t)]^{-\Delta}$ are taken which reduce to unity at $t=0$, and $f(t,\theta)$ is analytic in $|t|\leqq e^{\eta}(\eta>0)$, uniformly for θ in $[0,\pi]$. We show how DARBOUX's method can be extended in such a way that the rate of growth of $A_n(\theta)$, as $n\to\infty$, can be deduced in certain θ sub-intervals of $[0,\pi]$ which depend on n. Although this extension is a very special case, the more general case is quite analogous.

In Section II, two theorems are proved which characterize the asymptotic behavior of certain integrals which appear in the later development. The term "asymptotic" is used here in the sense of ERDÉLYI & WYMAN [4], which is more general than the usual POINCARÉ sense. This distinction is made clear in the theorems. Theorem 3 in Section III contains the complete analysis of (1.1), whereas Section IV contains several examples, including the ultraspherical polynomials, to which the results of Theorem 3 are applicable. An appendix is included which contains the evaluation of certain pertinent integrals.

II. Two Preliminary Theorems

The following notation will be used throughout this paper. We take N to be a large, positive parameter approaching $+\infty$. Unless specifically stated otherwise, the various real parameters, $s, r, \sigma, \ldots$, which occur will be taken as independent of N. When the individual value of such constants is of no interest, the letter κ will be used generically as a real parameter independent of N. If $0 \leqq \sigma \leqq \eta$, we define $C_v(\sigma, \eta)$ to be the contour in the v-plane which proceeds along the straight line from $\eta e^{-i\pi}$ to $\sigma e^{-i\pi}$, then counterclockwise around the circle $|v|=\sigma$ to $\sigma e^{i\pi}$, and from there along the straight line to $\eta e^{i\pi}$. Certain obvious modifications of this notation will be used, e.g., $\tau + C_v(\sigma, \eta)$ will denote the contour $C_v(\sigma, \eta)$ translated through a distance τ. We now set

$$\omega_a(v, \rho) = (-1)^a (v - \rho), \qquad a = 0 \quad \text{or} \quad 1. \tag{2.1}$$

Then for a fixed value of Δ and v restricted to $C_v(\sigma, \eta)$, a well defined branch of the multiple valued function, $[\omega_a(v, \rho)]^{-\Delta}$, may be determined by specifying $[\omega_a(\sigma, 2a\sigma)]^{-\Delta} = \exp(-\Delta \log \sigma)$ explicitly, and requiring that

$$\begin{aligned} &|\rho| \leqq \sigma - \varepsilon, \qquad \text{if } a = 0, \\ &|\rho| \geqq \sigma + \varepsilon, \quad |\arg \rho| \leqq \pi - \delta, \qquad \text{if } a = 1, \end{aligned} \tag{2.2}$$

where ε and δ are small positive numbers. The restrictions in (2.2) will be tacitly assumed in the following whenever such binomial factors occur. Finally, throughout this paper we let $A(v)$ be a generic notation for a positive valued function which is continuous, is bounded away from zero in the finite v-plane, and has only an algebraic rate of growth near infinity, i.e. $A(v)$ is $O(v^\gamma)$ as $|v| \to \infty$, for some real parameter γ. In particular, for $\sigma \geqq 0$, $e^v A(v)$ is absolutely integrable over $C_v(\sigma, \infty)$, and for $\sigma, s > 0$,

$$\begin{aligned} \int_{N^s}^{\infty} e^{-v} A(v)\, dv &= O\Big((N^s)^{1+\gamma} e^{-N^s} \int_0^{\infty} e^{-N^s u} (1+u)^\gamma\, du\Big), \\ &= O(N^\kappa e^{-N^s}), \qquad N \to \infty. \end{aligned} \tag{2.3}$$

We now prove a theorem analogous to Theorem (3.4) in [4].

Theorem 1. *Let Δ_1, Δ_2 and λ be arbitrary bounded parameters. Assume that a function $H(v, N) = H(v, N, \rho_1, \rho_2, \Delta_1, \Delta_2)$ possesses an expansion of the form*

$$H(v, N) = \sum_{k=0}^{m-1} N^{-k} Q_k(v) + H_m(v, N), \tag{2.4}$$

where the $Q_k(v)$ are polynomials in v, and

$$H_m(v, N) = O\big(N^{-m} [1 + |v|]^{m/s}\big), \qquad N \to \infty, \qquad s > 0, \tag{2.5}$$

uniformly in v for $0 \leqq |v| \leqq N^s$, and in the parameters $\rho_1, \rho_2, \Delta_1, \Delta_2$ and λ. Then for $\sigma > 0$, $a = 0$ or 1, $b = 0$ or 1, the function

$$F_{a,b}(N) = \frac{1}{2\pi i} \int_{C_v(\sigma, N^s)} e^v v^{-\lambda} [\omega_a(v, \rho_1)]^{-\Delta_1} [\omega_b(v, \rho_2)]^{-\Delta_2} H(v, N)\, dv, \tag{2.6}$$

has the asymptotic expansion

$$\begin{aligned} F_{a,b}(N) = \sum_{k=0}^{m-1} \frac{N^{-k}}{2\pi i} \int_{C_v(\sigma, \infty)} & e^v v^{-\lambda} [\omega_a(v, \rho_1)]^{-\Delta_1} [\omega_b(v, \rho_2)]^{-\Delta_2} Q_k(v)\, dv \\ & + O\big(N^{-m} |\rho_1|^{-a \operatorname{Re} \Delta_1} |\rho_2|^{-b \operatorname{Re} \Delta_2}\big) + O\big(N^\kappa \exp(-N^s)\big), \qquad N \to \infty. \end{aligned} \tag{2.7}$$

The particular branch of $[\omega_a(v, \rho_1)]^{-\Delta_1}([\omega_b(v, \rho_2)]^{-\Delta_2})$ *used in* (2.6) *is also used in* (2.7).

Proof. Substitution of (2.4) into (2.6) yields

$$F_{a,b}(N) = \sum_{k=0}^{m-1} \frac{N^{-k}}{2\pi i} \int_{C_v(\sigma, N^s)} e^v v^{-\lambda} [\omega_a(v, \rho_1)]^{-\Delta_1} [\omega_b(v, \rho_2)]^{-\Delta_2} Q_k(v)\, dv + R_m(N), \quad (2.8)$$

where

$$R_m(N) = \frac{1}{2\pi i} \int_{C_v(\sigma, N^s)} e^v v^{-\lambda} [\omega_a(v, \rho_1)]^{-\Delta_1} [\omega_b(v, \rho_2)]^{-\Delta_2} H_m(v, N)\, dv. \quad (2.9)$$

Elementary computations show that

$$|[\omega_a(v, \rho_1)]^{-\Delta_1} [\omega_b(v, \rho_2)]^{-\Delta_2}| \leqq |\rho_1|^{-a \operatorname{Re} \Delta_1} |\rho_2|^{-b \operatorname{Re} \Delta_2} A(v), \quad (2.10)$$

uniformly for v on $C_v(\sigma, \infty)$. It then follows directly from (2.3) and (2.10) that the error incurred by extending the contour of integration of the m explicit integrals in (2.8) to $C_v(\sigma, \infty)$ is $O(N^\kappa e^{-N^s})$, $N \to \infty$. If the integrand of (2.9) is analytic on and interior to the contour $C_v(\sigma, N^s)$, $R_m(N)$ is identically zero, and the first O symbol in (2.7) can be deleted. In general, however, $R_m(N)$ is not zero, and one uses (2.5) and (2.10) to obtain the estimate

$$\begin{aligned} R_m(N) &= O\Big(N^{-m} |\rho_1|^{-a \operatorname{Re} \Delta_1} |\rho_2|^{-b \operatorname{Re} \Delta_2} \int_{C_v(\sigma, N^s)} e^{\operatorname{Re} v} |v|^{-\operatorname{Re} \lambda} [1 + |v|]^{m/s} A(v)\, |dv|\Big), \\ &= O\Big(N^{-m} |\rho_1|^{-a \operatorname{Re} \Delta_1} |\rho_2|^{-b \operatorname{Re} \Delta_2} \int_{C_v(\sigma, \infty)} e^{\operatorname{Re} v} A(v)\, |dv|\Big), \quad (2.11) \\ &= O\big(N^{-m} |\rho_1|^{-a \operatorname{Re} \Delta_1} |\rho_2|^{-b \operatorname{Re} \Delta_2}\big), \qquad N \to \infty, \end{aligned}$$

which completes the theorem.

Obviously, for Theorem 1 to be of any practical use, one needs information on the integrals in (2.7). A complete listing of the possibilities for these integrals is given in the Appendix. Theorem 1 can be improved if $a = b = 0$. In particular, the following generalization of Theorem 1 will be used in Theorem 3.

Theorem 2. *Let* Δ *and* λ *be arbitrary bounded parameters, and* τ *a real, non-negative parameter which is* $o(N^s)$, *as* $N \to \infty$. *Assume that a function* $G(v, N) = G(v, N, \tau, \Delta, \lambda)$ *is analytic in* v, *and possesses an expansion of the form*

$$G(v, N) = \sum_{k=0}^{m-1} N^{-k} Q_k(v) + G_m(v, N), \quad (2.12)$$

where the $Q_k(v)$ *are polynomials in* v, *and*

$$G_m(v, N) = O\big(N^{-m} [1 + |v|]^{m/s}\big), \qquad N \to \infty, \quad s > 0, \quad (2.13)$$

uniformly in v *for* $0 \leqq |v| \leqq \beta N^s$, $\beta > 1$, *and in the parameters* τ, Δ *and* λ.

Then for $\sigma > 0$, *the function*

$$F(N) = \frac{1}{2\pi i} \int_{C_v(\tau + \sigma, N^s)} e^v v^{-\lambda} [v^2 + \tau^2]^{-\Delta} G(v, N)\, dv, \quad (2.14)$$

has the asymptotic expansion

$$F(N)=\sum_{k=0}^{m-1}\frac{N^{-k}}{2\pi i}\int_{C_v(\tau+\sigma,\,\infty)} e^v v^{-\lambda}[v^2+\tau^2]^{-\Delta}Q_k(v)\,dv+O\big(N^{\kappa}\exp(-N^s)\big)$$
$$+O\left(N^{-m}\frac{(1+\tau)^{-\mathrm{Re}\,(2\Delta)}}{\Gamma(\lambda)}\right)+O\left(N^{-m}\frac{(1+\tau)^{(m/s)-\mathrm{Re}\,(\lambda+\Delta)}}{\Gamma(\Delta)}\right),\quad N\to\infty. \tag{2.15}$$

The particular branch of $[v\pm i\tau]^{-\Delta}$ *used in* (2.14) *is also used in* (2.15). *If* λ *or* Δ *is an integer* $\leqq 0$, *then the second or third O symbol, respectively, in* (2.15) *can be deleted.*

Proof. If $\tau\leqq 2\sigma$, then $C_v(\tau+\sigma, N^s)$ and $C_v(\tau+\sigma,\infty)$ can be deformed into $C_v(3\sigma, N^s)$ and $C_v(3\sigma,\infty)$, respectively, and Theorem 2 reduces to a special case of Theorem 1 (see the remark preceding (2.11)). Therefore we assume $\tau\geqq 2\sigma$. Substituting (2.12) into (2.14), we have

$$F(N)=\sum_{k=0}^{m-1}\frac{N^{-k}}{2\pi i}\int_{C_v(\tau+\sigma,\,N^s)} e^v v^{-\lambda}[v^2+\tau^2]^{-\Delta}Q_k(v)\,dv+R_m(N), \tag{2.16}$$

where

$$R_m(N)=\frac{1}{2\pi i}\int_{C_v(\tau+\sigma,\,N^s)} e^v v^{-\lambda}[v^2+\tau^2]^{-\Delta}G_m(v,N)\,dv. \tag{2.17}$$

Since

$$|[v^2+\tau^2]^{-\Delta}|\leqq(1+\tau)^{-\mathrm{Re}\,(2\Delta)}A(v), \tag{2.18}$$

uniformly for v on $C_v(\tau+\sigma,\infty)$, it follows, exactly as in Theorem 1, that the error incurred by extending the contour of integration of the m explicit integrals in

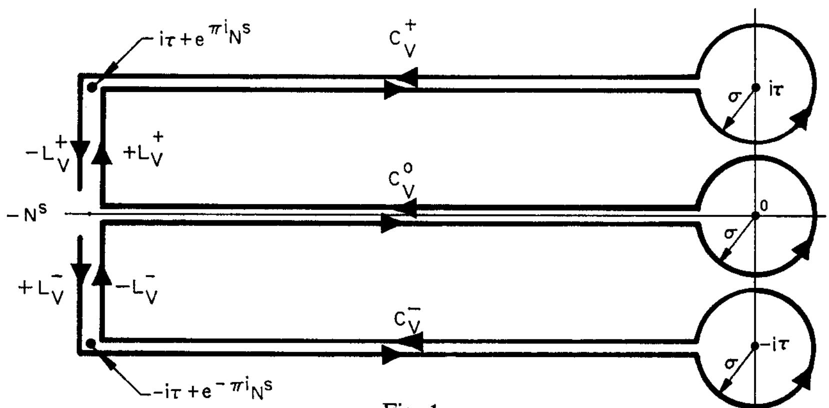

Fig. 1

(2.16) to $C_v(\tau+\sigma,\infty)$ is $O(N^{\kappa}e^{-N^s})$, as $N\to\infty$. To estimate $R_m(N)$, it is necessary to deform $C_v(\tau+\sigma, N^s)$ into a new contour, $C_v^*(\tau+\sigma, N^s)$, in such a way that the limiting contour, $C_v^*(\tau+\sigma,\infty)$, could be used to deduce the asymptotic behavior with respect to τ of $R_m(\infty)$ when $G_m(v,\infty)\equiv 1$ and $\tau\to+\infty$. One may take for $C_v^*(\tau+\sigma, N^s)$, the contour shown in Fig. 1.

The contour C_v^γ, $\gamma=-,0$ or $+$, is the contour $C_v(\sigma, N^s)$ translated through the distance $\gamma i\tau$, while the contours $\pm L_v^\gamma$, $\gamma=-$ or $+$, are straight line segments parallel to the imaginary axis. After performing this deformation on $C_v(\tau+\sigma, N^s)$, let $R_m(N; C_v^\gamma)$ and $R_m(N; \pm L_v^\gamma)$ denote the contributions to $R_m(N)$ from C_v^γ and $\pm L_v^\gamma$, respectively. Clearly $R_m(N)$ is the sum of such contributions. By symmetry, $R_m(N; \pm L_v^-)$ and $R_m(N; C_v^-)$ are of the same order as $R_m(N; \pm L_v^+)$ and $R_m(N; C_v^+)$, respectively. It is thus sufficient to consider in detail $R_m(N; \pm L_v^+)$, $R_m(N; C_v^0)$ and $R_m(N; C_v^+)$. First, for all v on $C_v^*(\tau+\sigma, N^s)$,

$$|v| \leqq |-N^s + i\tau| = N^s \sqrt{1+(N^{-s}\tau)^2} \leqq \beta N^s, \tag{2.19}$$

for N sufficiently large. Thus, $G_m(v, N)$ can be estimated on all of $C_v^*(\tau+\sigma, N^s)$ by (2.13). Simple computations then show that the $R_m(N; \pm L_v^+)$ terms are $O(N^\kappa e^{-N^s})$ as $N \to \infty$. Next, consider

$$R_m(N; C_v^0) = \frac{1}{2\pi i} \int_{C_v(\sigma, N^s)} e^v v^{-\lambda} [v^2 + \tau^2]^{-\Delta} G_m(v, N)\, dv. \tag{2.20}$$

If the integrand of (2.20) is analytic on and interior to $C_v(\sigma, N^s)$, e.g., as when λ is an integer $\leqq 0$, $R_m(N; C_v^0)$ is identically zero. More generally, however, we proceed by noting that (2.18) also holds uniformly for v on $C_v(\sigma, \infty)$, and hence, by computations similar to (2.11), that $R_m(N; C_v^0)$ can be estimated by the second O symbol in (2.15). Finally, consider

$$R_m(N; C_v^+) = \frac{1}{2\pi i} \int_{C_w(\sigma, N^s)} e^{i\tau+w} (i\tau+w)^{-\lambda} w^{-\Delta} (w+2i\tau)^{-\Delta} G_m(i\tau+w, N)\, dw. \tag{2.21}$$

As before, if the integrand of (2.21) is analytic on and interior to $C_w(\sigma, N^s)$, e.g., as when Δ is an integer $\leqq 0$, then $R_m(N; C_v^+)$ is equal to zero. If $R_m(N; C_v^+)$ is not zero, we use the simple estimate

$$|(i\tau+w)^{-\lambda} w^{-\Delta} (w+2i\tau)^{-\Delta} [1+|i\tau+w|]^{m/s}| \leqq (1+\tau)^{(m/s)-\mathrm{Re}(\Delta+\lambda)} A(w), \tag{2.22}$$

which is valid uniformly for w on $C_w(\sigma, N^s)$, to deduce, by computations similar to (2.11), that $R_m(N; C_v^+)$ can be estimated by the third O symbol in (2.15). The theorem is completed by combining the estimates for the various contributions to $R_m(N)$.

III. Main Theorem

We now analyze for large n and $0 \leqq \theta \leqq \pi$, the behavior of the coefficients $A_n(\theta)$ defined by (1.1), or using CAUCHY's theorem, the functions $A_n(\theta)$ defined by

$$A_n(\theta) = \frac{1}{2\pi i} \int_{C_t(0)} \frac{F(t,\theta)}{t^N}\, dt, \qquad N = n+1,$$
$$F(t,\theta) = (1-t)^{-\lambda} [(e^{i\theta}-t)(e^{-i\theta}-t)]^{-\Delta} f(t,\theta), \tag{3.1}$$

where $F(t, \theta)$ has the same interpretation as in (1.1), and $C_t(0)$ is a simple, closed, counterclockwise-oriented contour in $|t| \leqq e^\eta$, enclosing $t=0$ but not the singularities $t=e^{\pm i\theta}$ or $t=1$. Our choice for $C_t(0)$ will depend upon θ and N. In particular, we take $C_t(0)$ to be one of the contours shown in Figs. 2, 3 and 4.

These contours correspond, respectively, to the geometric configurations where all three singular points are "close", all three singular points are "well separated", and the configuration where the singular points $t=e^{\pm i\theta}$ are "close", but "well separated" from $t=1$. What is meant by the phrases "close" and "well separated" will be specified in each situation. Those portions of $C_t(0)$ near the singularities $t=e^{\pm i\theta}$ and $t=1$ are essentially $C_t(\sigma N^{-1}, N^{-\frac{1}{2}}$ or $N^{-\frac{1}{3}})$, properly translated and rotated, and only they contribute significantly to the value of $A_n(\theta)$. The radius of the large circle $|t|=R_0$ (of which the subcontours C_t form a part) will also depend on θ and N.

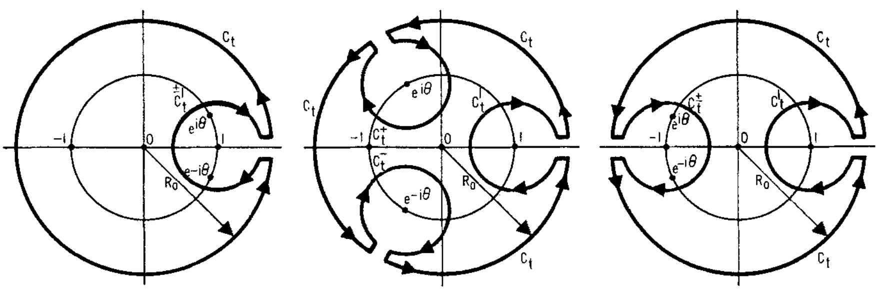

FIGURE 2 t-PLANE FIGURE 3 t-PLANE FIGURE 4 t-PLANE

Figs. 2—4

It is convenient to introduce the following notation. The θ-intervals A_α, B_β and C_α are defined as follows: $N=n+1$, $\frac{1}{8}\pi<\sigma<\frac{1}{6}\pi$,

$$\begin{aligned} &A_\alpha\colon\ 0\leqq\theta\leqq 5\sigma N^{-\frac{1}{3}-\frac{2}{3}\alpha}, && 0<\alpha\leqq 1;\\ &B_\beta\colon\ 4\sigma N^{-1+\beta}\leqq\theta\leqq\pi-2\sigma N^{-1+\beta}, && 0\leqq\beta\leqq 1;\\ &C_\alpha\colon\ 0\leqq\pi-\theta\leqq 3\sigma N^{-\frac{1}{3}-\frac{2}{3}\alpha}, && 0<\alpha\leqq 1. \end{aligned} \tag{3.2}$$

Clearly, if $2\alpha+3\beta\leqq 2$, the union of these intervals covers the θ-interval $[0,\pi]$. Figs. 2, 3 and 4 correspond to the situations when $\theta\in A_\alpha$, B_β and C_α respectively. The phrase "uniformly for θ in the interval I_γ, γ fixed" will be abbreviated by "u. $\theta\in I_\gamma$".

Next, we define the parameters, $N=n+1$,

$$\begin{aligned} \mu&=N\,2\sin\frac{\theta}{2},\\ \tau&=N\,2\cos\frac{\theta}{2},\\ \rho&=N\sin\theta. \end{aligned} \tag{3.3}$$

These parameters are obviously closely related, e.g. it follows from the elementary inequalities, $2\varphi\leqq\pi\sin\varphi\leqq\pi\varphi$, $0\leqq 2\varphi\leqq\pi$, that $N\leqq\mu$, u. $\theta\in C_\alpha$; $\rho^{\frac{1}{2}}<\mu^{\frac{3}{4}}$, u. $\theta\in B_\beta$; and $\rho^{-1}=O(N^{-\beta})$ as $N\to\infty$, u. $\theta\in B_\beta$. Thus $\mu,\rho\to\infty$, as $N\to\infty$, u. $\theta\in B_\beta$ $(\beta>0)$. Finally, we will need the function

$$\psi(w)=w\sqrt{1+\tfrac{1}{4}w^2}-\tfrac{1}{2}w^2,\qquad |w|<2. \tag{3.4}$$

Simple computations show that $w^2[1-\psi(w)]=[\psi(w)]^2$ and that $\psi(w)$ is monotone increasing for w positive.

Theorem 3. *If $A_n(\theta)$ is defined by* (3.1), *where λ and Δ are bounded quantities, if those branches of $(1-t)^{-\lambda}$ and $[(e^{i\theta}-t)(e^{-i\theta}-t)]^{-\Delta}$ are taken which reduce to unity at $t=0$, and if $f(t,\theta)$ is analytic in $|t|\leqq e^{\eta}(\eta>0)$,* u. $\theta\in A_\alpha$, B_β or C_α, *then as* $N=n+1\to\infty$,

$$A_n(\theta)=I_{\pm 1}(N)+O\left(N^{\kappa}\exp\left(-\tfrac{1}{2}N^{\frac{2}{3}}\right)\right),\qquad \text{u. } \theta\in A_\alpha, \tag{3.5}$$

$$=I_1(N)+I_+(N)+I_-(N)+O\left(N^{\kappa}\exp\left(-\rho^{\frac{1}{2}}\right)\right),\qquad \text{u. } \theta\in B_\beta(\beta>0), \tag{3.6}$$

$$=I_1(N)+I_\pm(N)+O\left(N^{\kappa}\exp\left(-\tfrac{1}{2}N^{\frac{2}{3}}\right)\right),\qquad \text{u. } \theta\in C_\alpha, \tag{3.7}$$

where

$$I_{\pm 1}(N)=\frac{1}{2\pi i}\int_{C_w((1+\mu)N^{-1},N^{-\frac{1}{3}})}\frac{F(1-\psi(w),\theta)}{[1-\psi(w)]^N}\frac{d\psi(w)}{dw}\,dw,\qquad \theta \text{ in } A_\alpha, \tag{3.8}$$

$$\begin{aligned}&=N^{\lambda+2\Delta-1}\left\{\sum_{k=0}^{m-1}\frac{N^{-k}}{2\pi i}\int_{C_v(1+\mu,\infty)}e^v v^{-\lambda+k}[v^2+\mu^2]^{-\Delta}P_k(v,\theta)\,dv\right.\\&\left.\quad+O\left(N^{-m}\frac{(1+\mu)^{-\mathrm{Re}(2\Delta)}}{\Gamma(\lambda-m)}\right)+O\left(N^{-m}\frac{(1+\mu)^{\frac{3}{2}m-\mathrm{Re}(\lambda+\Delta)}}{\Gamma(\Delta)}\right)\right\}\end{aligned} \tag{3.9}$$

$$+O\left(N^{\kappa}\exp\left(-N^{\frac{2}{3}}\right)\right),\quad N\to\infty,\qquad \text{u. } \theta\in A_\alpha,\quad P_0(v,\theta)=f(1,\theta),$$

$$I_1(N)=\frac{1}{2\pi i}\int_{C_w(\sigma N^{-1},N^{-1}\mu^{\frac{2}{3}})}\frac{F(1-\psi(w),\theta)}{[1-\psi(w)]^N}\frac{d\psi(w)}{dw}\,dw, \tag{3.10}$$

θ *in* B_β, *or in* C_α,

$$\begin{aligned}&=\mu^{-2\Delta}N^{\lambda+2\Delta-1}\left\{\sum_{k=0}^{m-1}\frac{\mu^{-k}}{2\pi i}\int_{C_v(\sigma,\infty)}e^v v^{-\lambda+k}Q_k(v,\theta)\,dv+O\left(\frac{\mu^{-m}}{\Gamma(\lambda-m)}\right)\right\}\\&\quad+O\left(N^{\kappa}\exp\left(-\mu^{\frac{2}{3}}\right)\right),\quad N\to\infty,\end{aligned} \tag{3.11}$$

u. $\theta\in B_\beta(\beta>0)$ *and* $\theta\in C_\alpha$, $Q_0(v,\theta)=f(1,\theta)$,

$$I_+(N)=\frac{e^{-in\theta}}{2\pi i}\int_{C_w(\sigma N^{-1},N^{-1}\rho^{\frac{1}{2}})}\frac{F(e^{i\theta}(1+e^{-\pi i}w),\theta)}{(1-w)^N}\,dw, \tag{3.12}$$

θ in B_β,

$$\begin{aligned}&=\mu^{-\lambda-\Delta}\tau^{-\Delta}N^{\lambda+3\Delta-1}\exp\left(-i\theta\left(n+\Delta+\frac{\lambda}{2}\right)+i\frac{\pi}{2}(\lambda+\Delta)\right)\\&\quad\cdot\left\{\sum_{k=0}^{m-1}\frac{\rho^{-k}}{2\pi i}\int_{C_v(\sigma,\infty)}e^v v^{-\Delta+k}S_k^+(v,\theta)\,dv+O\left(\frac{\rho^{-m}}{\Gamma(\Delta-m)}\right)\right\}\end{aligned} \tag{3.13}$$

$$+O\left(N^{\kappa}\exp\left(-\rho^{\frac{1}{2}}\right)\right),\quad N\to\infty,\qquad \text{u. } \theta\in B_\beta\ (\beta>0),\quad S_0^+(v,\theta)=f(e^{i\theta},\theta),$$

$$I_{\pm}(N)=\frac{(-1)^n}{2\pi i}\int_{C_w((1+\tau)N^{-1},N^{-\frac{1}{3}})}\frac{F(-1+\psi(w),\theta)}{[1-\psi(w)]^N}\frac{d\psi(w)}{dw}dw,\quad \theta \text{ in } C_\alpha, \tag{3.14}$$

$$=(-1)^n 2^{-\lambda} N^{2\Delta-1}\left\{\sum_{k=0}^{m-1}\frac{N^{-k}}{2\pi i}\int_{C_v(1+\tau,\infty)} e^v v^k[v^2+\tau^2]^{-\Delta}\,T_k(v,\theta)\,dv\right.$$
$$\left.+O\left(N^{-m}\frac{(1+\tau)^{\frac{3}{2}m-\mathrm{Re}\,\Delta}}{\Gamma(\Delta)}\right)\right\}+O(N^\kappa\exp(-N^{\frac{1}{3}})),\quad N\to\infty, \tag{3.15}$$
$$\text{u. } \theta\in C_\alpha,\quad T_0(v,\theta)=f(-1,\theta),$$

and $I_-(N)$ is $(-1)\,I_+(N)$ with $+i$ replaced by $-i$.

In the above, $\psi(w)$ is the function defined by (3.4), *and $P_k(v,\theta)$, $Q_k(v,\theta)$, $S_k^+(v,\theta)$ and $T_k(v,\theta)$ are polynomials in v, with coefficients dependent on θ, defined by* (3.36), (3.38), (3.39) *and* (3.37), *respectively. The degree in v of $P_k(v,\theta)$, $Q_k(v,\theta)$ and $T_k(v,\theta)$ is $[\frac{1}{2}k]$, while that of $S_k^+(v,\theta)$ is k. Under the stronger hypothesis that $f(t,\theta)$ is analytic in $|t|\leq e^\eta(\eta>0)$, uniformly for θ in* $[0,2\pi]$,

$$A_n(\theta)=O(N^\Omega),\quad N\to\infty,\quad \text{uniformly for } \theta \text{ in } [0,\pi],$$
$$\Omega=-1+\mathrm{Max}\{\mathrm{Re}(\lambda+2\Delta),\ \mathrm{Re}(\lambda),\ \mathrm{Re}(\Delta),\ \mathrm{Re}(2\Delta)\}. \tag{3.16}$$

Proof. Clearly, $F(t,\theta)$ is $O(N^\kappa)$, as $N\to\infty$, uniformly in t along any t-contour in $|t|\leq e^\eta$, which avoids the singular points, $e^{\pm i\theta}$ and $t=1$, by a distance only algebraically dependent on N. Thus, if we take

$$R_0=1-\psi(-N^{-\frac{1}{3}}), \tag{3.17}$$

for the analyses of the intervals A_α and C_α, it follows from the inequality $x-\frac{1}{2}x^2<\log(1+x)$, $0<x<1$, that as $N\to\infty$,

$$A_n(\theta)=\frac{1}{2\pi i}\int_{C_t^{\pm 1}}\frac{F(t,\theta)}{t^N}dt+O(N^\kappa\exp(-\tfrac{1}{2}N^{\frac{1}{3}})),\quad \text{u. } \theta\in A_\alpha, \tag{3.18}$$

$$=\frac{1}{2\pi i}\int_{C_t^1+C_t^{\pm}}\frac{F(t,\theta)}{t^N}dt+O(N^\kappa\exp(-\tfrac{1}{2}N^{\frac{1}{3}})),\quad \text{u. } \theta\in C_\alpha. \tag{3.19}$$

Then, the change of variable $t=1-\psi(w)$ in those integrals over $C_t^{\pm 1}$ and C_t^1, and the change of variable $t=-1+\psi(w)$ in the integral over $C_t^{\pm}$, reduces (3.18) to (3.5), and (3.19) to an equation which varies from (3.7) by terms of order, as $N\to\infty$,

$$\int_{1-\psi(-N^{-\frac{1}{3}})}^{1-\psi(-N^{-1}\mu^{\frac{2}{3}})}\frac{F(t,\theta)}{t^N}dt=O\left(N^\kappa\int_{1-\psi(-N^{-\frac{1}{3}})}^{\infty}\frac{dt}{t^N}\right)=O(N^\kappa\exp(-\tfrac{1}{2}N^{\frac{1}{3}}))\ \text{ u. } \theta\in C_\alpha. \tag{3.20}$$

The analysis for the interval B_β $(\beta>0)$ is similar, except that here we choose

$$R_0=1+N^{-1}\rho^{\frac{1}{2}}. \tag{3.21}$$

Thus, it follows from (3.1) that, as $N\to\infty$,

$$A_n(\theta)=\frac{1}{2\pi i}\int_{C_t^1+C_t^++C_t^-}\frac{F(t,\theta)}{t^N}dt+O(N^\kappa\exp(-\rho^{\frac{1}{2}})),\quad \text{u. } \theta\in B_\beta\ (\beta>0). \tag{3.22}$$

Then, setting $t=1-\psi(w)$ in the integral over C_t^1 in (3.22), $t=e^{i\theta}[1+e^{-i\pi}w]$ in the integral over C_t^+, and making a corresponding change of variable in the integral over C_t^-, one obtains an equation which varies from (3.6) by terms of order, as $N\to\infty$,

$$\int_{1+N^{-1}\rho^{\frac{1}{2}}}^{1-\psi(-N^{-1}\mu^{\frac{2}{3}})} \frac{F(t,\theta)}{t^N}\,dt = O\left(N^{\kappa}\int_{1+N^{-1}\rho^{\frac{1}{2}}}^{\infty}\frac{dt}{t^N}\right) = O\left(N^{\kappa}\exp(-\rho^{\frac{1}{2}})\right), \tag{3.23}$$

$$\text{u. } \theta\in B_\beta\ (\beta>0).$$

The first part of the theorem will follow if we can show that Theorems 1 and 2 of Section II can be applied to the various integrals. To see this, we prove the following lemmas.

Lemma 1. *If* $\psi(w)$ *is defined by* (3.4), *then*

$$e^{-Nw}[1-\psi(w)]^{-N} = \exp(-Nw^3 g(w)), \tag{3.24}$$

where $g(0)=\frac{1}{24}$ *and the function* $g(w)$ *is an even function of* w, *analytic in the region* $|w|<w_0=\frac{1}{2}(\sqrt{5}-1)$.

Proof. Since

$$|\psi(w)| \leqq \frac{|w|^2}{2} + |w|\sqrt{1+\frac{|w|^2}{4}} \leqq |w|^2+|w|<1, \qquad |w|<w_0, \tag{3.25}$$

$w+\log[1-\psi(w)]$ can be expanded in a power series in w for $|w|<w_0$. That this series is an odd function of w follows from the fact that $[1-\psi(w)]\,[1-\psi(-w)]=1$. The rest follows by explicit computation.

Lemma 2. *Suppose* r *is an integer* $\geqq 1$, *and* $h(w)$, $g(w)$ *are functions analytic in* $|w|\leqq\eta\,(>0)$. *Let*

$$K(v,\zeta) = h(v\zeta)\exp\left(v(v\zeta)^r g(v\zeta)\right), \qquad |v\zeta|\leqq\eta, \tag{3.26}$$

$$A=\left\{(v,\zeta)\,\middle|\, |\zeta v^q|\leqq\left(\frac{4}{3}\right)^q,\ |\zeta|\leqq\zeta_0=\left(\frac{4}{9}\right)^q\left(\frac{\eta}{3}\right)^{r+1}\right\}, \qquad q=1+\frac{1}{r}. \tag{3.27}$$

Then $K(v,\zeta)$ *can be written in the form,* m *arbitrary,*

$$K(v,\zeta)=\sum_{k=0}^{m-1}(\zeta v)^k Q_k(v) + (\zeta v)^m K_m(v,\zeta), \qquad Q_0(v)=h(0), \tag{3.28}$$

where $Q_k(v)$ *is a polynomial in* v *of degree*

$$\leqq\left[\frac{k}{r}\right] \qquad \left(=\left[\frac{k}{r}\right] \text{ if } h(0)\,g(0)\neq 0\right),$$

and where $K_m(v,\zeta)$, *for* $(v,\zeta)\in A$, *is an analytic function of* v *which satisfies an inequality of the form*

$$|K_m(v,\zeta)|\leqq M[1+|v|]^{m/r}, \qquad (v,\zeta)\in A, \tag{3.29}$$

M being a positive number independent of v and ζ.

Proof. The existence of the expansion in (3.28) and the degree of the $Q_k(v)$ are obvious. It follows from CAUCHY's estimate for the remainder of a series, that for $|\zeta|<\rho\leqq\eta\,|v|^{-1}$,

$$|v^m K_m(v,\zeta)|\leqq\frac{1}{\rho^{m-1}(\rho-|\zeta|)}\operatorname*{Max}_{|\xi|=\rho}|h(v\xi)\exp\big(v(v\xi)^r g(v\xi)\big)|. \tag{3.30}$$

For $(v,\zeta)\in\Lambda$ we may take

$$\rho=\left(\frac{v_0+|v|}{3}\right)^{-q},\qquad v_0=3\left(\frac{\eta}{3}\right)^{-r}, \tag{3.31}$$

as

$$|v\rho|=\frac{3|v|}{v_0+|v|}\left(\frac{v_0+|v|}{3}\right)^{-(1/r)}\leqq 3\left(\frac{v_0}{3}\right)^{-(1/r)}=\eta, \tag{3.32}$$

and

$$\begin{aligned}\frac{|\zeta|}{\rho}&=|\zeta|\left(\frac{v_0+|v|}{3}\right)^q,\\ &\leqq|\zeta v^q|\left(\frac{2}{3}\right)^q\leqq\left(\frac{8}{9}\right)^q<1,\qquad |v|\geqq v_0,\\ &\leqq|\zeta|(v_0)^q\left(\frac{2}{3}\right)^q\leqq\left(\frac{8}{9}\right)^q<1,\qquad |v|\leqq v_0,\ |\zeta|\leqq\zeta_0.\end{aligned} \tag{3.33}$$

Since $|v(v\rho)^r|\leqq 3^{r+1}$ and $v_0+|v|\leqq(1+v_0)(1+|v|)$, it follows from (3.30), that

$$|v^m K_m(v,\zeta)|\leqq M_1[1+|v|]^{mq},\qquad (v,\zeta)\in\Lambda, \tag{3.34}$$

M_1 being a positive number independent of v and ζ. Under the additional assumption, $|v|\geqq v_0$, (3.29) can be deduced from (3.34). Since $(v,\zeta)\in\Lambda$ implies

$$|v\zeta|\leqq\left(\frac{8}{9}\right)^q|v\rho|\leqq\left(\frac{8}{9}\right)^q\eta, \tag{3.35}$$

$K_m(v,\zeta)$ is an analytic function of both v and ζ. Thus by continuity, $K_m(v,\zeta)$ is uniformly bounded on the compact (v,ζ) subset $\{(v,\zeta)|\ |v|\leqq v_0,\ |\zeta|\leqq\zeta_0\}$, which is the content of (3.29) for $|v|\leqq v_0$. Combining the inequalities for $|v|\geqq v_0$ and $|v|\leqq v_0$, we arrive at (3.29), and the lemma.

We now complete the proof of the theorem. To evaluate asymptotically the integrals defining $I_{\pm 1}(N)$, $I_{\pm}(N)$ and $I_1(N)$, we proceed as follows. After making the change of variable $w=vN^{-1}$ in (3.8) and (3.14), we see from the lemmas that the functions

$$\begin{aligned}&e^{-v}[1-\psi(vN^{-1})]^{-N-\Delta}f(1-\psi(vN^{-1}),\theta)\left[\frac{\psi(vN^{-1})}{vN^{-1}}\right]^{-\lambda}\frac{d\psi(vN^{-1})}{d(vN^{-1})}\\ &\qquad=\sum_{k=0}^{\infty}N^{-k}v^k P_k(v,\theta),\qquad P_0(v,\theta)=f(1,\theta),\end{aligned} \tag{3.36}$$

$$\begin{aligned}&e^{-v}[1-\psi(vN^{-1})]^{-N-\Delta}\left[1-\frac{\psi(vN^{-1})}{2}\right]^{-\lambda}f(-1+\psi(vN^{-1}),\theta)\frac{d\psi(vN^{-1})}{d(vN^{-1})}\\ &\qquad=\sum_{k=0}^{\infty}N^{-k}v^k T_k(v,\theta),\qquad T_0(v,\theta)=f(-1,\theta),\end{aligned} \tag{3.37}$$

are functions of the type $G(v, N)$ in Theorem 2 with $s=\frac{2}{3}$. Moreover, since μ and τ are $O(N^{\frac{2}{3}-\frac{2}{3}\alpha})$ as $N\to\infty$, u. $\theta\in A_\alpha$ and C_α, respectively, Theorem 2 is applicable and yields (3.9) and (3.15). For $I_1(N)$, we make the change of variable $w=vN^{-1}=v2(\sin\theta/2)\,\mu^{-1}$ in (3.10) and then use the lemmas to deduce that

$$\begin{aligned} e^{-v}[1-\psi(vN^{-1})]^{-N-\Delta}f(1-\psi(vN^{-1}),\theta)\left[\frac{\psi(vN^{-1})}{vN^{-1}}\right]^{-\lambda}\frac{d\psi(vN^{-1})}{d(vN^{-1})} \\ \cdot\left[1+\frac{v^2}{\mu^2}\right]^{-\Delta}=\sum_{k=0}^{\infty}\mu^{-k}v^k Q_k(v,\theta),\qquad Q_0(v,\theta)=f(1,\theta), \end{aligned} \tag{3.38}$$

is a function of the type $H(v, N)$ envisaged in Theorem 1 with $s=\frac{2}{3}$, but with μ now playing the role of N. This leads to (3.11). Finally, to evaluate asymptotically the integral defining $I_+(N)$, we make the change of variable $w=vN^{-1}=v(\sin\theta)\rho^{-1}$ in (3.12) and use the above lemmas to deduce that

$$\begin{aligned} & e^{-v}[1-vN^{-1}]^{-N}f(e^{i\theta}(1+e^{-i\pi}vN^{-1}),\theta) \\ & \cdot\left(1+\frac{ive^{i\theta}}{2\rho}\right)^{-\Delta}\left(1+\frac{ive^{i\frac{\theta}{2}}}{\rho}\cos\frac{\theta}{2}\right)^{-\lambda} \\ & =\sum_{k=0}^{\infty}\rho^{-k}v^k S_k^+(v,\theta),\qquad S_0(v,\theta)=f(e^{i\theta},\theta), \end{aligned} \tag{3.39}$$

is a function of the type $H(v, N)$ in Theorem 1 with $s=\frac{1}{2}$, but with ρ now playing the role of N. This yields (3.13). The last statement in the theorem, (3.16), clearly follows from the preceeding statements, and completes the proof of the theorem.

Remark 1. The integrals in (3.11) and (3.13) are just sums of gamma functions, see (A.3), whereas the integrals in (3.9) and (3.15) are, in hypergeometric notation, sums of ${}_1F_2(-\mu^2/4)$'s, see (A.12). The ${}_1F_2(-\mu^2/4)$'s which occur in (3.15) are of the type where the numerator parameter differs from one or the other of the denominator parameters by a non-negative integer. It then follows from the formula (FIELDS [10]),

$${}_1F_2\left(\begin{matrix}a+m\\a,b\end{matrix}\middle|x\right)=\sum_{k=0}^{m}\frac{(-m)_k(-x)^k}{(a)_k(b)_k k!}\,{}_0F_1(k+b\,|\,x), \tag{3.40}$$

that the ${}_1F_2$'s in (3.15) can be written as a finite sum of ${}_0F_1$'s of the same argument. These ${}_0F_1$'s are essentially Bessel functions of the first kind, since

$${}_0F_1\left(1+\nu\middle|-\frac{\tau^2}{4}\right)=\Gamma(1+\nu)\left(\frac{\tau}{2}\right)^{-\nu}J_\nu(\tau). \tag{3.41}$$

If λ is an integer, $\leqq 0$, these same remarks are applicable to the integrals of (3.9).

Remark 2. The relative simplicity of the above expansions for $A_n(\theta)$ in the various θ-intervals stems from the fortuitous circumstance that the coefficient of w^2 in $\log(1-\psi(w))$ is zero. This circumstance allows us to approximate $A_n(\theta)$ in the "large" A_α and C_α intervals, $\alpha<1$. If this coefficient were not zero, the above analysis would be sufficient to approximate $A_n(\theta)$ only in the "small" A_1 and C_1

intervals, and it would be necessary to design a more complicated analysis to approximate $A_n(\theta)$ over the "huge" B_0 interval. Such an analysis can actually be made, but only at the expense of utilizing much more complicated functions than gamma or Bessel functions. For completeness, we sketch such a B_0 analysis. The choice $R_0=1+N^{-\frac{1}{2}}$ leads to an expansion of $I_+(N)$ in powers of N^{-1}, whose coefficients are sums of integrals of the form

$$\begin{gathered}\frac{1}{2\pi i}\int_{C_v(\sigma,\infty)} e^v v^{k-\Delta}[\rho^+(\theta)-v]^{-\Delta}[\rho^+(\theta/2)-v]^{-\lambda}\,dv;\\ \rho^+(\theta),\quad \rho^+(\theta/2) \text{ exterior to } C_v(\sigma,\infty),\\ \rho^+(\theta)=N\,2(\sin\theta)\,e^{i(\frac{\pi}{2}-\theta)},\end{gathered} \tag{3.42}$$

which are confluent hypergeometric functions of the two variables $x=\rho^+(\theta/2)$ and $y=\rho^+(\theta)$, see (A.8). When $4\sigma N^{-1}\leqq\theta\leqq\theta_1<\frac{2}{3}\pi$, then $|x|<|y|$ and $|y-x| <|y|$, and the double series expansions given in (A.8) are applicable. For $0<\theta_0\leqq \theta\leqq\pi-2\sigma N^{-1}$, $|\rho^+(\theta/2)|\geqq N2\sin(\theta_0/2)$, and the second binomial factor in (3.42) can be expanded in powers of N^{-1}. After some rearrangement, one obtains an expansion of $I_+(N)$ in powers of N^{-1}, whose coefficients are sums of integrals of the form

$$\frac{1}{2\pi i}\int_{C_v(\sigma,\infty)} e^v v^{k-\Delta}[\rho^+(\theta)-v]^{-\Delta}\,dv;\qquad \rho^+(\theta) \text{ exterior to } C_v(\sigma,\infty), \tag{3.43}$$

which are confluent hypergeometric functions of the one variable $\rho^+(\theta)$, see (A.5). The $I_-(N)$ analysis is clearly conjugate to that for $I_+(N)$. The above choice of R_0 also leads to an expansion of $I_1(N)$ in powers of N^{-1}, whose coefficients are sums of integrals of the form

$$\frac{1}{2\pi i}\int_{C_v(\sigma,\infty)} e^v v^{k-\lambda}[\mu^2+v^2]^{-\Delta}\,dv;\qquad \pm i\mu \text{ exterior to } C_v(\sigma,\infty), \tag{3.44}$$

which are Meijer G-functions which can be written as a sum of three ${}_1F_2(-\mu^2/4)$'s see (A.13).

IV. Applications and Comments

It is clear that Theorem 3 is applicable to the ultraspherical polynomials $P_n^{(\Delta)}(x)$, when $-1\leqq x\leqq 1$, as they can be defined by the generating function (SZEGÖ [13], (4.7.23)),

$$[(e^{i\theta}-t)(e^{-i\theta}-t)]^{-\Delta}=\sum_{n=0}^{\infty} t^n P_n^{(\Delta)}(\cos\theta),\qquad |t|<1, \tag{4.1}$$

$$P_n^{(\Delta)}(x)=\frac{\Gamma(n+2\Delta)}{\Gamma(2\Delta)\Gamma(n+1)}\,{}_2F_1\left(\begin{matrix}-n,\ n+2\Delta\\ \Delta+\frac{1}{2}\end{matrix}\,\middle|\,\frac{1-x}{2}\right). \tag{4.2}$$

For special values of Δ, limits need to be taken in (4.1) and (4.2). The above results are also applicable to the polynomials

$$S_n^{(\Delta)}(x)=\sum_{k=0}^{n} P_k^{(\Delta)}(x), \tag{4.3}$$

when $-1 \leqq x \leqq 1$, since

$$(1-t)^{-1}[(e^{i\theta}-t)(e^{-i\theta}-t)]^{-\Delta}=\sum_{n=0}^{\infty} t^n S_n^{(\Delta)}(\cos\theta), \qquad |t|<1. \tag{4.4}$$

Next, we point out that in Theorem 3, the explicit algebraic form of $F(t, \theta)$ as given in (3.1) was used only in a neighborhood of radius $O(N^{-\frac{1}{2}}$ or $N^{-\frac{1}{3}})$ of the singular points $t=e^{\pm i\theta}$ and $t=1$. Thus, if

$$G(t,\theta)=\sum_{n=0}^{\infty} t^n B_n(\theta), \qquad |t|<1, \tag{4.5}$$

and $G(t, \theta)$ can be extended analytically into $|t| \leqq e^{\eta}(\eta>0)$, uniformly for θ in $[0, \pi]$ except for isolated singular points at $t=e^{\pm i\theta}$ and $t=1$, then it is only necessary that the singular terms of $G(t, \theta)$ be of the form $F(t, \theta)$ in proper neighborhoods of the singular points $t=e^{\pm i\theta}$ and $t=1$ for completely analogous results to Theorem 3 to hold for $B_n(\theta)$, as $n \to \infty$. For example, define the Jacobi polynomials, $P_n^{(\alpha,\beta)}(x)$, by the generating function (SZEGÖ [13], FIELDS [9]),

$$\begin{aligned} G(t,\theta)&=(1-t)^{-\lambda}\,{}_2F_1\left(\begin{matrix}\frac{\lambda}{2},\ \frac{\lambda+1}{2}\\ 1+\beta\end{matrix}\middle|\chi(t,\theta)\right)\\ &=\sum_{n=0}^{\infty}(-t)^n\frac{(\lambda)_n}{(1+\beta)_n}P_n^{(\alpha,\beta)}(\cos\theta), \end{aligned} \tag{4.6}$$

$$\lambda=\alpha+\beta+1, \qquad \chi(t,\theta)=-4t(1-t)^{-2}\left(\sin\frac{\theta}{2}\right)^2,$$

$$P_n^{(\alpha,\beta)}(x)=(-1)^n\frac{(1+\beta)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n,\ n+\alpha+\beta+1\\ 1+\beta\end{matrix}\middle|\frac{1+x}{2}\right). \tag{4.7}$$

Since

$$(1-t)^2+2t\left(\sin\frac{\theta}{2}\right)^2=(e^{i\theta}-t)(e^{-i\theta}-t), \tag{4.8}$$

$G(t, \theta)$ has singularities at $t=e^{\pm i\theta}$ and $t=1$. Instead of considering the full interval, $-1 \leqq x \leqq 1$, it follows from the relation ([13], (4.1.3)),

$$P_n^{(\alpha,\beta)}(x)=(-1)^n P_n^{(\beta,\alpha)}(-x), \tag{4.9}$$

that it is sufficient to consider only $-1 \leqq x \leqq 0$, or $\frac{1}{2}\pi \leqq \theta \leqq \pi$. The singularity of $G(t, \theta)$ at $t=1$ is only an apparent one, as can be seen from the analytic continuation formula for the ${}_2F_1(\chi)$ in (4.6), when χ is near infinity. Similarly, the behavior of $G(t, \theta)$ for t near $e^{\pm i\theta}$ can be deduced from the analytic continuation formula for the ${}_2F_1(\chi)$ when χ is near one. Full details of this development will be included in a subsequent paper which will give uniform expansions of the generalized Jacobi polynomials (FIELDS and LUKE [7, 8], FIELDS [9])

$${}_{p+2}F_{p+1}\left(\begin{matrix}-n,\ n+\lambda,\ \alpha_1,\ldots,\alpha_p\\ \beta_1,\ldots,\beta_{p+1}\end{matrix}\middle| z\right)$$

in the interval $0 \leqq z \leqq 1$, as $n \to \infty$. SZEGÖ [11, 12] gave similar analyses for the ultraspherical and Jacobi polynomials.

Finally, we point out that when the problem of finding uniform asymptotic expansions of the Jacobi polynomials, $P_n^{(\alpha,\beta)}(x)$, for $-1 \leqq x \leqq 1$, is attacked from the standpoint of the differential equation satisfied by the $P_n^{(\alpha,\beta)}(x)$, i.e. ([13], (4.2.1)),

$$\begin{gathered}\{(1-x^2)D^2+[(\beta-\alpha)-(\alpha+\beta+2)x]D+n(n+\alpha+\beta+1)\}P_n^{(\alpha,\beta)}(x)=0,\\ D=\frac{d}{dx},\end{gathered} \tag{4.10}$$

one is led to turning point problems of the type which are introduced by poles as opposed to zeros. Thus, the above extension of DARBOUX's method gives a simple, alternate approach to certain types of turning point problems. This approach becomes especially important in the case of the generalized Jacobi polynomials, because they satisfy a differential equation similar to (4.10), but of order $(p+2)$, and explicit computation is quite difficult.

Appendix

Here we list the various possibilities for the integrals occuring in (2.7), i.e.,

$$I_{a,b}(x,y,\Delta_1,\Delta_2,\lambda)=\frac{1}{2\pi i}\int_{C_v(\sigma,\infty)} e^{v-\lambda}[\omega_a(v,x)]^{-\Delta_1}[\omega_b(v,y)]^{-\Delta_2}\,dv, \tag{A.1}$$

$$\begin{aligned}\omega_0(v,\rho)&=v-\rho, \quad |\rho|\leqq\sigma-\varepsilon,\\ \omega_1(v,\rho)&=\rho-v, \quad |\rho|\geqq\sigma+\varepsilon, \quad |\arg\rho|\leqq\pi-\delta\end{aligned} \tag{A.2}$$

where ε and δ are small positive numbers. The principal branch of each binomial factor is used in the following. If neither variable occurs explicitly in (A.1), the integral is essentially the Hankel representation for the gamma function, i.e.,

$$I_{a,b}(x,y,0,0,\lambda)=\frac{1}{\Gamma(\lambda)}. \tag{A.3}$$

If only one of the variables does not occur explicitly, the resulting integrals are confluent hypergeometric functions of one variable, i.e.,

$$I_{0,b}(x,y,\Delta_1,0,\lambda)=\frac{1}{\Gamma(\lambda+\Delta_1)}\,{}_1F_1\left(\begin{matrix}\Delta_1\\ \lambda+\Delta_1\end{matrix}\middle|x\right); \tag{A.4}$$

$$\begin{aligned}I_{a,1}(x,y,0,\Delta_2,\lambda)&=\frac{\Gamma(1-\lambda-\Delta_2)}{\Gamma(\lambda)\Gamma(1-\lambda)}\,{}_1F_1\left(\begin{matrix}\Delta_2\\ \lambda+\Delta_2\end{matrix}\middle|y\right)\\ &\quad+\frac{\Gamma(-1+\lambda+\Delta_2)}{\Gamma(\lambda)\Gamma(\Delta_2)}\,y^{1-\lambda-\Delta_2}\,{}_1F_1\left(\begin{matrix}1-\lambda\\ 2-\lambda-\Delta_2\end{matrix}\middle|y\right); \qquad \text{(A.5)}\\ &\sim\frac{1}{\Gamma(\lambda)}\,y^{-\Delta_2}\,{}_2F_0\left(\Delta_2,1-\lambda\middle|-\frac{1}{y}\right), \quad y\to\infty, \quad |\arg y|<\pi.\end{aligned}$$

The function, $I_{a,1}(x, y, 0, \Delta_2, \lambda)$, can also be written in terms of TRICOMI's Ψ function, see [2], as $[\Gamma(\lambda)]^{-1}\Psi(\Delta_2, \lambda+\Delta_2|y)$.

In general, both variables actually occur, and we have the following formulae

$$I_{0,0}(x, y, \Delta_1, \Delta_2, \lambda) = \frac{1}{\Gamma(\lambda+\Delta_1+\Delta_2)}\Phi_2(\Delta_1, \Delta_2; \lambda+\Delta_1+\Delta_2; x, y); \tag{A.6}$$

$$\begin{aligned} I_{0,1}(x, y, \Delta_1, \Delta_2, \lambda) &= \frac{\Gamma(\lambda)\Gamma(1-\lambda)}{\Gamma(1-\lambda-\Delta_1)\Gamma(\lambda+\Delta_1)} I_{1,1}(x, y, \Delta_1, \Delta_2, \lambda) \\ &+ \frac{\Gamma(1-\lambda)}{\Gamma(\Delta_1)\Gamma(2-\lambda-\Delta_1)} x^{1-\lambda-\Delta_1} y^{-\Delta_2} \Phi_1\left(1-\lambda, \Delta_2, 2-\lambda-\Delta_1; \frac{x}{y}, y\right), \\ & |x|<|y|; \end{aligned} \tag{A.7}$$

$$\begin{aligned} I_{1,1}(x, y, \Delta_1, \Delta_2, \lambda) &= \frac{\Gamma(1-\lambda-\Delta_1-\Delta_2)}{\Gamma(\lambda)\Gamma(1-\lambda)}\Phi_2(\Delta_1, \Delta_2; \lambda+\Delta_1+\Delta_2; x, y) \\ &- \frac{\Gamma(\lambda+\Delta_1+\Delta_2)\Gamma(1-\lambda-\Delta_1-\Delta_2)\Gamma(1-\Delta_1)}{\Gamma(\lambda)\Gamma(\Delta_1+\Delta_2)\Gamma(1-\Delta_1-\Delta_2)\Gamma(2-\lambda-\Delta_1)} x^{1-\lambda-\Delta_1} y^{-\Delta_2} \\ &\cdot \Phi_1\left(1-\lambda, \Delta_2, 2-\lambda-\Delta_1; \frac{x}{y}, x\right) \\ &- \frac{\Gamma(\lambda+\Delta_1+\Delta_2)\Gamma(1-\lambda-\Delta_1-\Delta_2)\Gamma(1-\Delta_1)}{\Gamma(\lambda)\Gamma(\Delta_2)\Gamma(1-\lambda)\Gamma(2-\Delta_1-\Delta_2)} y^{-\lambda}(y-x)^{1-\Delta_1-\Delta_2} e^y \\ &\cdot \Phi_1\left(1-\Delta_2, \lambda, 2-\Delta_1-\Delta_2; 1-\frac{x}{y}, x-y\right), \quad |x|<|y|, \quad |x-y|<|y|; \\ &\sim \frac{1}{\Gamma(\lambda)} x^{-\Delta_1} y^{-\Delta_2} \Phi_4\left(1-\lambda, \Delta_1, \Delta_2; -\frac{1}{x}, -\frac{1}{y}\right), \\ & x, y \to \infty, \quad |\arg x|, \quad |\arg y|<\pi, \end{aligned} \tag{A.8}$$

where Φ_1, Φ_2 and Φ_4 are the confluent hypergeometric functions of two variables defined by the convergent series

$$\Phi_1(\alpha, \beta, \gamma; x, y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_{m+n}} \frac{x^m}{m!}\frac{y^n}{n!}, \quad |x|<1, \tag{A.9}$$

$$\Phi_2(\alpha, \beta; \gamma; x, y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_m(\beta)_n}{(\gamma)_{m+n}} \frac{x^m}{m!}\frac{y^n}{n!}, \tag{A.10}$$

and the formal series

$$\Phi_4\left(\alpha, \beta, \gamma; -\frac{1}{x}, -\frac{1}{y}\right) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\gamma)_n}{m!\,n!}\left(\frac{-1}{x}\right)^m\left(\frac{-1}{y}\right)^n. \tag{A.11}$$

These results are taken from ERDÉLYI [3, 5, 6].

Some specializations of the foregoing are of particular interest, i.e.,

$$I_{0,0}(i\tau, -i\tau, \Delta, \Delta, \lambda) = \frac{1}{\Gamma(\lambda+2\Delta)}\, {}_1F_2\left(\begin{matrix} \Delta \\ \Delta+\frac{\lambda}{2},\ \Delta+\frac{1+\lambda}{2} \end{matrix}\middle| -\frac{\tau^2}{4}\right);$$

$$\sim \frac{\tau^{-2\Delta}}{\Gamma(\lambda)}\, {}_3F_0\left(\Delta, \frac{2-\lambda}{2}, \frac{1-\lambda}{2}\middle| -\frac{4}{\tau^2}\right)$$

$$+2^{1-\Delta}\tau^{-\lambda-\Delta}\left\{\frac{1}{\Gamma(\Delta)}\cos\left[\tau-\frac{\pi}{2}(\lambda+\Delta)+O(\tau^{-1})\right]+O(\tau^{-2})\right\}, \tag{A.12}$$

for $\tau\to\infty$, $|\arg\tau|<\frac{\pi}{2}$;

and

$$I_{1,1}(i\mu, -i\mu, \Delta, \Delta, \lambda) = \frac{1}{\Gamma(\lambda)\,\Gamma(\Delta)\,\Gamma\left(\frac{1-\lambda}{2}\right)\Gamma\left(\frac{2-\lambda}{2}\right)}$$

$$\cdot\, \mu^{-2\Delta}\, G_{13}^{31}\left(\frac{\mu^2}{4}\middle|\begin{matrix} 1 \\ \Delta, \frac{1-\lambda}{2}, \frac{2-\lambda}{2}\end{matrix}\right); \tag{A.13}$$

$$\sim \frac{1}{\Gamma(\lambda)}\mu^{-2\Delta}\, {}_3F_0\left(\Delta, \frac{1-\lambda}{2}, \frac{2-\lambda}{2}\middle| -\frac{4}{\mu^2}\right), \quad \mu\to\infty,\ |\arg\mu|<\frac{\pi}{2}.$$

The Meijer G-function in (A.13), see [2], can be expanded in the ordinary way as the sum of three ${}_1F_2(-\mu^2/4)$'s.

The derivations of these formulae fall into three broad classes. If both x and y are interior to the contours, the product of the binomial factors can be expanded in a double series in x/v and y/v, and term-by-term integration can be justified. If one or more of the x and y are exterior to the contour, the ascending expansions can be deduced as in the following example. Consider $I_{0,0}(x, y, \Delta_1, \Delta_2, \lambda)$ with $0<x<y$ and the various parameters properly restricted. Then $C_v(\sigma, \infty)$ can be deformed and broken up into the straight line segments $[-\infty, 0]$, $[0, x]$, $[x, y]$, $[y, x]$, $[x, 0]$, $[0, -\infty]$. The arguments of v, $x-v$ and $y-v$ are then constant on each line segment, though varying from segment to segment. Pairing off these contributions to $I_{0,0}(x, y, \Delta_1, \Delta_2, \lambda)$ in the obvious way, $I_{0,0}(x, y, \Delta_1, \Delta_2, \lambda)$ can be written as the sum of three ordinary straight line integrals, two of which can be evaluated in terms of

$$\Phi_1(\alpha, \beta, \gamma; x, y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\gamma-\alpha)}\int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-xu)^{-\beta}e^{uy}\,du,$$

$$\mathrm{Re}(\gamma)>\mathrm{Re}(\alpha)>0, \quad |\arg(1-x)|<\pi, \quad x\neq 1, \tag{A.14}$$

and the third in terms of $I_{1,1}(x, y, \Delta_1, \Delta_2, \lambda)$. Once $I_{1,1}(x, y, \Delta_1, \Delta_2, \lambda)$ is identified in terms of Φ_1 and Φ_2, the parameter and variable restrictions can then be relaxed by analytic continuation. The various asymptotic formulae follow directly from WATSON's lemma when the paths of integration are rotated in a proper manner. The G-function identification in (A.13) follows when $I_{1,1}(i\mu, -i\mu, \Delta, \Delta, \lambda)$

is represented as a straight line integral, the binomial factors are replaced by their Mellin-Barnes integral representations, and the order of integration is interchanged.

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Bibliography

[1] DARBOUX, G., «Mémoire sur l'approximation des fonctions de très-grands nombres et sur une classe étendue de développements en série», *J. de Mathematiques pures et appliquées*, Series 3, **4**, 5—56, 377—416 (1878).
[2] ERDÉLYI, A., W. MAGNUS, F. OBERHETTINGER, and G. TRICOMI, "*Higher Transcendental Functions*," Vol. 1. New York: McGraw-Hill 1953.
[3] ERDÉLYI, A., W. MAGNUS, F. OBERHETTINGER, and F. TRICOMI, *Tables of Integral Transforms*, Vol. 1. New York: McGraw-Hill 1954.
[4] ERDÉLYI, A., & M. WYMAN, "The asymptotic evaluation of certain integrals," *Archive for Rational Mechanics and Analysis* **14**, 217—260 (1963).
[5] ERDÉLYI, A., "Integration of a certain system of linear partial differential equations of hypergeometric type," *Proc. Roy. Soc.*, Edinburgh **59**, 224—241 (1939).
[6] ERDÉLYI, A., "Some confluent hypergeometric functions of two variables," *Proc. Roy. Soc.*, Edinburgh **60**, 344—361 (1940).
[7] FIELDS, J., & Y. LUKE, "Asymptotic expansions of a class of hypergeometric polynomials with respect to the order," *J. Math. Anal. Appl.* **6**, 394—403 (1963).
[8] FIELDS, J., & Y. LUKE, "Asymptotic expansions of a class of hypergeometric polynomials with respect to the order, II," *J. Math. Anal. Appl.* **7**, 440—450 (1963).
[9] FIELDS, J., "Asymptotic expansions of a class of hypergeometric polynomials with respect to the order, III," *J. Math. Anal. Appl.* **12**, 593—601 (1965).
[10] FIELDS, J., "Confluent expansions," *Math. Comp.* **21**, 189—197 (1967).
[11] SZEGÖ, G., „Über einige Asymptotische Entwicklungen der Legendreschen Funktionen," *Proc. London Math. Soc.* (2) **36**, 427—450 (1932).
[12] SZEGÖ, G., „Asymptotische Entwicklungen der Jacobischen Polynome," *Schriften der Königsberger Gelehrten Gesellschaft*, Naturwissenschaftliche Klasse **10**, 35—112 (1933).
[13] SZEGÖ, G., "*Orthogonal Polynomials*," *Am. Math. Soc. Colloq. Publ.* **23** (1959).

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