For Pi to Omega: Constants in Mathematics

Proposed Structure for a book to be matched with a current web project on **Top Numbers**

WE propose to write 10 inter-related but largely stand alone essays (combine log 2 and log_10) on the numbers on our web site. They would all have the same section structure

1. *History/uses* (4-5 pages) (Possibly by Tom Archibald)
2. *Mathematics* (8-15 pages)
3. *Curiosities* (1-2 pages)
4. *Key Formulas* (2-5 pages)
5. *Key Reference*s (1 page)

Total 16-28 pages = approximately 225 pages.

Web links and other references would all be on the partner web site
http://ddrive.cs.dal.ca:9999/page

All chapters would be written by JMB based, I hope, in part on notes from Keith Taylor and Tom Archibald. Potential other authors include Peter Borwein and Karl Dilcher.

Attached are the PDF files of our proposals to the Discovery Channel (they seem very interested) and of the current rough text sitting under the website.

Cheers, Jon

Keith Taylor, Dean of Science
and
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THE WORLD'S GREATEST NUMBERS. Our list of top numbers will begin with 12 contestants. Some are very old and some are new. Some are very easy to describe, others a little less so. They arise in algebra, analysis, number theory, graph theory, geometry, chemistry, physics and computer science. They occur constantly, often without being noticed, throughout science, industry and society.

- We shall provide a website with a zoom-able number line and a page about each number
  - its definition and meaning
  - its history and uses
  - some carefully chosen references and links highlighting the best of 'mathematics on the web'.
  - a few high-quality graphics, animations and applets (we may get *MathResources*, *AARMS*, *Mathematica* to help).
- On the first segment, we introduce all *twelve competitors* with a Grammy-like 20 seconds each (see the brief ideas after each number listed below). We should like to wait a week for the next four segments with the upcoming segments advertised each show of the intervening week. This is to allow the audience time to use the website to build their knowledge of, and interest in, the numbers.
- On the second segment, we judge and announce the *six finalists*.
- Each segment will finish with a puzzler about one of the remaining numbers. These will be placed on the web site — answers could be provided at the end of the series or a contest could be run.
- The third, fourth and fifth segments weed the field to three, then two and finally one *grand winner* — while each time discussing all the current contestants further.
- We propose a third judge, *Jay*, to decide when Jon and I *disagree* on which numbers should be cut.
- We will provide dramatizations as appropriate for each number such as rolling a bicycle wheel for π, zooming in on a pine cone for the golden ratio or standing in front of the beautiful Chase Building with proportions approximating the golden ratio.

The 12 contestants in increasing numerical order

1. Madelung's constant: $M_3 = -1.7475645946331821\cdots$

   - The numerical glue that holds common table salt together. Use model of NaCl crystal to define Madelung's constant as the electrostatic potential at one ion due to the location of all the other ions in the crystal.

2. Planck's constant: $h = 6.626068\cdots \times 10^{-34}$

   - A turning point for the era of modern physics. Laid the ground for uncertainty to be an unavoidable physical reality.

3. Euler's constant: $\gamma = 0.5772156649015329\cdots$

   - Start with the trick of stacking blocks to introduce the divergence of the harmonic series. Have a bassist illustrate the harmonic series on a double bass. γ makes precise the growth rate of the harmonic series.

4. The golden mean: $G = (\sqrt{5} - 1)/2 = 0.61803398874989\,50\cdots$

   - Has fascinated philosophers, artists, biologists and orthodontists since ancient times; star player in the *da Vinci Code*. Use pentagon to introduce, pine cone, elegant buildings, quasicrystal diffraction patterns.

5. The natural logarithm: $\log(2) = 0.6931471805599453\cdots$

   - Where does the banker's rule of 70 come from in calculating the doubling time for an investment at a fixed interest rate? Role of this number in connecting natural logarithm to logarithm with base 2. Computational complexity. Half life in radioactive decay.

6. Apery's constant: $\zeta(3) = 1.202056903159594\cdots$

   - Introduce the Riemann zeta function and the Riemann hypothesis. Talk about the $1,000,000 prize. Tell story of Roger Apery's fight as a reactionary against the *new math* reforms and his vindication, then triumph, with the proof that $\zeta(3)$ is irrational in 1977.

7. The first surd: $\sqrt{2} = 1.414213562373095\cdots$

   - Use a square, any square, to define. Give physical proof of its irrationality - the first known irrational number. The crushing impact on the Pythagoreans.

8. The common logarithm: $1/\log_{10}(e) = \log(10) = 2.302585092994046\cdots$

- Show old, well-worn, table of common logs. The most important calculation aid for almost 400 years. Enabled the emergence of the natural sciences by allowing precise calculations. Talk about Simon Newcomb, perhaps the greatest Canadian born scientist who is little known.

9. The exponential: $e = \exp(1) = 2.718281828459045\cdots$

   - Exponential growth: Show graph of actual world population for last 10,000 years. Why is e the natural base? Flash through the bell shaped curve, minimum uncertainty, continuous compounding of interest, etc.

10. Archimedes' constant: $\pi = 3.141592653589793\cdots$

    - Roll a wheel. Discuss π as a cultural icon, the Simpson's, super computers, ubiquitous nature of periodic processes. The phenomenal equation $e^{i\pi} + 1 = 0$.

11. Feigenbaum's constant: $F = 4.669210609102990\cdots$

    - Illustrate chaos in variety of situations. What can be common in all this seeming randomness? Extract Feigenbaum's constant as a common element.

12. Chaitin's binary constant: $\Omega = 0.007875474041752057\cdots$

    $(0.0000001000000100001000001000011101110011001001111000100100111000\cdots)$

    - A number that can be defined and described but never fully computed. What does this mean? Get to talk about Turing machines and the philosophical conundrums of the last century.

References

[1] J.M Borwein and P. Borwein, *Pi and the AGM*, John Wiley, 1987, 1998.

JMB and KT, November 2, 2005

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JMB and KT, October 18th, 2005

1 Research Assistant Search

We are looking for a student to help design, build and maintain a web site on the *Top Mathematical Constants* over the next year and perhaps longer as part of as part of an *AARMS* (`www.arms.math.ca`) outreach project and a proposal to the *Discovery channel*:

2 Programming proposal for *Daily Planet*: June 13, 2005

The proposal would be to run segments on 6 successive airings of Daily Planet under a title something like "Jon and Keith's Lists of the Top 5 Irrational Numbers."

Segment 1: Talk about the Real Numbers, rational and irrational; introduce a visual tool for exploring the number line and play with decimal representations of real numbers. We would also each give our criteria for ranking numbers. We would end with revealing our lists along with the promise that next week we would each justify our choices.

Segment 2: We each present the merits of the top number on our list and criticize the other's choice, when we are in disagreement.

Segments 3-6: We continue with the successive numbers on our lists. This is in the style that viewers will be familiar with in formats such as Ebert and Roeper might use for their lists of the top 5 movies in a year. Jon can criticize me for including a particular number or ranking something above another number or we might agree on some slots.

We have access to expert programmers who would produce clear and dynamic illustrations of the points we were making: as a preliminary idea of what such an illustration might look like, go to
`http://math.usask.ca/ code/javacat/rollcirc.html` If your computer is configured to accept Java applets of that vintage (now a little dated), click on the "roll circle" button and observe.

Keith's list of numbers might be something like this:

1. The ratio of the length of the diagonal of a square to the length of a side
2. The ratio of the circumference of a circle to its diameter
3. The natural base e
4. The Golden ratio
5. Madelung's constant for NaCl

Jon's list might be:

1. The ratio of the circumference of a circle to its diameter
2. The natural base e
3. The natural logarithm of ten
4. The Golden ratio
5. Feigenbaum's constant in chaos theory

As you can see there is some commonality, but we can disagree on order and some inclusions. We would expect some vigorous feedback from viewers because of our choices.

For Keith's first number, he would start with an animation like
`http://math.usask.ca/∼code/javacat/forsquar.html` to show that the ratio he is talking about for my first number is a real number whose square is 2. he would talk about how the discovery that this number was irrational had a profound impact on the Greek ideas about

numbers. (There is a story that, in their outrage, the Pythagoreans put the discoverer to death; but Keith doesn't buy it.)

Jon would then talk about his top number, illustrating its definition and arguing how it is so much more important and ubiquitous than the square root of 2. Our competing arguments would use up about a 7 or 8 minute segment. The next day we would go on to our second numbers.

If the top 5 numbers worked, we could take a break then do sets in a similar format on topics such as:

The top 5 mathematical revolutions

The top 5 current applications of mathematics

The 5 most beautiful theorems, etc.

After viewers got used to us, and assuming the reaction was positive, we could go on to debate the nature of mathematical knowledge: *Proof versus Experiment, Created versus Discovered*. See `www.economist.com/science/displayStory.cfm?story_id=3809661` for a very readable article on the nature of proof in mathematics in the presence of computational power. Jon has given a number of lectures on Experimental Mathematics and its implications. For example this spring at Tulane he gave the famous Clifford lectures. See `http://users.cs.dal.ca/~jborwein/clifford.html`

3 Revised Proposal: October 18th, 2005

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References

[1] J.M Borwein and P. Borwein, *Pi and the AGM*, John Wiley, 1987, 1998.

[2] E. Borowski and J. Borwein, *Mathematics: Web-Linked Dictionary* (Collins-Smithsonian Web-Linked Dictionary), 1987–2006.

APPENDIX 4: Forty Useful Constants

MATHEMATICAL CONSTANTS

1. $\sqrt{2} = 1.4142135623730950488$
2. $\sqrt{3} = 1.7320508075688772935$
3. $\sqrt{5} = 2.2360679774997896964$
4. Golden mean $\frac{\sqrt{5}-1}{2} = 0.61803398874989484820$
5. $\pi = 3.1415926535897932385$ and $1/\pi = .31830988618379067153$
6. $\pi/2 = 1.5707963267948966192$
7. $e = 2.7182818284590452354$ and $1/e = .36787944117144232160$
8. $e^\pi = 23.140692632779269007$
9. $\log 2 = .69314718055994530942$
10. $\log 10 = 2.3025850929940456840$
11. $\log_2 10 = 3.3219280948873623478$ and $\log_{10} 2 = .30102999566398119522$
12. $\log_2 3 = 1.5849625007211561815$
13. $\zeta(2) = 1.6449340668482264365$
14. Apéry's constant $\zeta(3) = 1.2020569031595942854$
15. $\zeta(5) = 1.0369277551433699263$
16. Catalan's constant $G = 0.91596559417721901505$
17. Euler's constant $\gamma = 0.57721566490153286061$
18. $\Gamma(1/2) = \sqrt{\pi} = 1.7724538509055160273$
19. $\Gamma(1/3) = 2.6789385347077476337$
20. $\Gamma(1/4) = 3.6256099082219083121$
21. Elliptic integral of the first kind $K(1/\sqrt{2}) = 1.8540746773013719184$
22. Elliptic Integral of the second kind $E(1/\sqrt{2}) = 1.3506438810476755025$

23. Chaitin's universal halting constant (in binary)
$\Omega = 0.0000001000000100001000001000011101110011001001111000100100111 00$

24. Feigenbaum's first bifurcation constant $\alpha = 4.669201609102990\ldots$

25. Feigenbaum's second bifurcation constant $\delta = 2.502907875095892\ldots$

26. Khintchine's continued fraction constant $K = 2.6854520010653064453$

27. Madelung's electrochemical constant $M_3 = \sum{}' (-1)^{i+j+k}/\sqrt{i^2 + j^2 + k^2} = 1.7475645946331821903$

...

PHYSICAL CONSTANTS

28. Avogadro's constant $N_A = 6.022, 141, 99(47) \times 1023\,\mathrm{mol}^{-1}$

29. Boltzmann's constant $k = 1.380, 6503(24) \times 1023\, J\, K^{-1}$

30. Fine structure constant $\alpha = 7.297, 352, 533(27) \times 10^{-3}$

31. Newton's constant $g = 6.67310 \times 10^{-11} m^3\, kg^{-1}\, s^{-2}$

32. Planck's constant $h = 6.62606876(52) \times x10^{-34}\, J\, s$

33. Speed of light in vacuum $c = 299, 792, 458 m\, s^{-1}$

34. Mass of an electron $m_e = 9.109, 381, 88(72) \times 10^{-31}\, kg$

35. Electron volt $eV = 1.602176462(63) \times 10^{-19}\, J$

36. Electron radios $r_e = 2.817, 940, 285(31) \times 10^{-15}\, m$

37. Proton/electron mass ratio $m_p/m_e = 1, 836.152, 6675(39)$

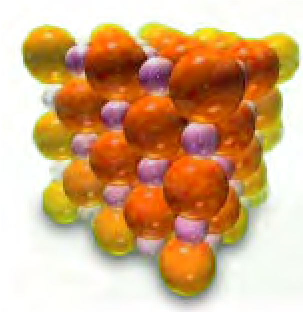
38. Diameter of earth $\approx 12.7 \times 10^6\, m$

39. Astronomical unit (average distance to Sun) $AU \approx 150 \times 10^9\, m$

40. Light-year $= 9.460528405106 \times 10^{15}\, m$

Wherever possible mathematical constants are given to 20 significant places.
Bracketed numbers for physical constants indicate uncertainty.
Definitions not in the text may be found at
`http://pauillac.inria.fr/algo/bsolve/constant/constant.html` and
`http://physics.nist.gov/cuu/Constants/index.html`.

M3 : Madelung's Constant



Charged and Ready

An ion is just another way to say that an atom is charged. It either has extra electrons, making it negatively charged, or it is missing electrons, making it positively charged. When you put a bunch of these ions together, sometimes they will form a very ordered structure called an ionic solid.

The Structure of Salt

A great example of an ionic solid is a crystal of salt. Have you ever wondered what makes all those little atoms in salt stick together? Salt is made up of Sodium ions that have a +1 charge and Chloride ions that have a -1 charge. So, salt is held together by the attraction between the Sodium and Chloride ions because they have different charges. It is also kept from collapsing on itself by the repulsion of the same charged ions. When it is all put together, the ions line up to make a nicely structured three dimensional lattice.

Packed with Energy

Since these ionic solids are stuck together, it takes energy to break them apart. This energy is called lattice energy.

Now, suppose we wish to know what this lattice energy is for a given crystal. A good place to start would be to find the energy of a single ion in that crystal. To find this, we would need to know the sum of all the charges that act on that ion.

The Model for Success

Erwin Madelung(1881-1972), a German Physicist, tackled this problem in the early 20th century. He managed to calculate what is known as Madelung's constant using the NaCl crystal known as rocksalt.

By defining a grid system with the origin at (0,0,0), he applied alternating charges at all the points (j,k,l) where j, k, and l are all whole numbers. This is a good model of what is going on in the crystal. The total charge at the origin, or the energy on that single ion, would have to be:

$$\sum_{j^2+k^2+l^2\neq 0} \frac{(-1)^{j+k+l}}{\sqrt{j^2+k^2+l^2}}$$

How Strong is Salt?

Using this mathematical model, Madelung calculated the energy of a single ion to be -1.74756495, known as Madelung's constant. Then, using Coulomb's Law, he determined the lattice energy of the entire crystal.

This theoretical lattice energy ended up being much larger than what was experimentally observed, so there was some concern about how well the model actually worked. As it turns out, the different answers were just caused by imperfections in the crystals.

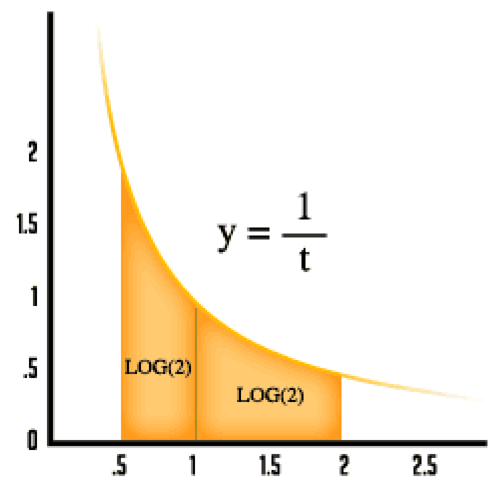
Results!

Madelung's constant can be used to determine the lattice energy for any crystal as long as you know the distance of an ion in the crystal to its nearest neighbor and the charges of all the ions in the crystal. It is because of Madelung that we can calculate how much energy is needed to break apart ionic crystals.

$\log_e 2$: The Natural Logarithm

Double Trouble

With the $\log_e 2$ at our side, we can find how long it takes for something to double in size. This could be anything from cell growth or population growth and decay, to the growth of money and the effects of inflation. We will explore what this number is and how it works.



WHAT IS IT?

The natural logarithm (sometimes ln for short) is just a logarithm with base e, the exponential constant. $\log_e 2$ in particular is equal to 0.69314718… It's an irrational and transcendental number with some interesting identities:

- $\log_e 2$ =
- $\log_e 2$ =
- The area under the curve 1/x from ½ to 1 is $\log_e 2$, and so is the area from 1 to 2.
- Choose a random integer n and look at all its prime factors. The probability that the largest one is greater than sqrt(n) is $\log_e 2$.

Why Do We Love It?

It is often useful to find a quick rule to estimate things, like how long it would take for your money or population to double at a certain rate. $\log_e 2$ can help us out with this! But first, we have to look at how money and population grow in general.

Modelling Growth

Growing population or money can be simply and accurately modeled by a constant growth rate that is compounded every period. OK, so maybe that sounds a bit complicated and scary, but it's really not! Let's work through an example.

A Growing Example

Say you have a annual growth rate of 5% per year. This means that at the end of the first year you get back the original amount plus an extra 5 percent. At the end of the next year you get another 5% on not only the amount you started with, but also the interest you received in the previous year. We call this compound growth, because you earn interest on interest on interest! Since compound growth is sometimes more difficult to work with, we can use exponential growth instead as an easy approximation.

Double or Nothin'

OK so now that we've covered the basic model, we can figure out how long it takes for money or population to double. First, consider the equation for exponential growth:

Pr=Po e^rt

Po is initial pop, Pr is final pop, e is 2.7182..., r is rate, and t is time.

By setting the initial population to 1 and the final population to 2 (effectively doubling our population or money):

2 = e^rt

and rearranging:

$\log_e 2$ = rt

t = $\log_e 2$ / r

we get that t equals $\log_e 2/r$. So, the time it takes for your initial amount to double is approximately $\log_e 2$ divided by the rate.

The Rule of 70

This is where the rule of 70 (sometimes the rule of 72) comes from. Bankers and financial advisors use this rule to find how long it takes your money to double.

Since $\log_e 2$ is approximately .693, they just round it up to .7, which is 70%. So, if you take 70 and divide it by the rate percentage, you get the doubling time of your money.

For instance, if your rate is 5%, 70/5 = 14. So at 5% annual interest rate, it will take about 14 years for your money to double. The exact answer is actually 14.2067, but our rule gets us a very close answer very quickly!

Sometimes 72 is used instead of 70, because it gives a better approximation for higher rates and has more whole numbers that divide evenly into it.

THE BABY BOOM

Cell growth is a process that is highly dependent on doubling. Each one of us starts out as a single cell, barely visible by the naked eye. This little cell collection is called an embryo, and it starts out only about 1/10 of a millimeter in size.

Before long, this cell splits into two, and each of the new cells split into two more cells, and after about a week, the embryo has doubled in size!

By week 3 it has doubled again. At week 4 the embryo is already about 4 mm in size. Cell growth continues as all the cells split and double in number. By week 10 the embryo is 30mm(1.2 inches), which increases to 3.2 inches by week 14, and 6 inches by week 18.

In only 37 weeks, we get a fully formed baby around 20 inches in length. This is the miracle of cell growth and doubling! We might start out as only 1 cell 1/10 of a millimeter in size, but with the power of doubling we end up with the 6 trillion cells that make up a 20 inch newborn baby.

THE INVENTION OF CHESS

Legend has it that the game of chess was invented by Sissa ben Dahir of the court of King Shiram. The king was so impressed with this invention that he promised to give Sissa any reward he wished for.

Being a clever chap, Sissa gave the king a choice. The king could either give him 10,000 rupees, or a payment of wheat based on the 64 square chessboard. To pay Sissa in wheat, the king would need to give him one grain of wheat for the first square, 2 grains of wheat for the second square, 4 for the third, 8 for the forth, and so on. Each square should have double the amount of wheat as the previous square.

Now, the kingdom was known for its wheat production, so King Shiram was more than willing to part with some wheat. He thought Sissa a fool to ask for such an insignificant award.

The gathering of Sissa's reward began. The king had a couple bags of wheat brought into the throne room and the servants started counting. 1, 2, 4, 8, 16, 32, 64, 128, 256, 512…and soon, the first bag was empty. 2 more bags were required, then 4, then 8. The amount needed was growing out of control! Soon, more grain was required then the kingdom could provide. In fact, the total sum of Sissa's reward amounts to (2^64)-1. That equals a whopping 18,446,744,073,709,551,615 grains of wheat!!

The king realized that he was the fool, not Sissa. The king had underestimated the power of doubling. The amount might not be so noticeable at first, but it can quickly explode out of control.

Zeta 3 : Apéry's Constant

Wanted: Dead or Alive

Apéry's constant is simply the value of the Riemann Zeta Function at 3 .

[GRAPHIC OF SUM]= 1.2020569….

This constant is named for Rodger Apéry who proved it was irrational in 1977.

The Riemann Zeta Function

The Riemann Zeta Function in general is:

[GRAPHIC]

Developed by Bernhard Riemann, this function helps predict the distribution of prime numbers through something called the Riemann Hypothesis. So, maybe the prime numbers are not as unpredictable as we previously thought! Although the hypothesis has not yet been proven, it works so well that mathematicians are quite certain that it's true.

Intriguing Tidbit

Apéry's constant shows up unexpectedly in a variety of mathematical places, such as in the solution to various complicated finite integrals, sums, and combinations of the two. I'm not going to list them here, as they aren't very exciting for the average person.

On the otherhand, Apery's Constant has an interesting application to Number Theory. $1/\ \zeta(3)$ is the probability that three randomly chosen whole numbers are relatively prime. This means that about 83% of the time, three random whole numbers have no factors in common greater than 1. It's really amazing that we are able to give this probability, given the infinitude of the numberline.

The Riemann Hypothesis

Some Starting Definitions

We need a few simple definitions before we talk about this hypothesis.

- A complex number: is just a special type of number that has not only a real part, but also an imaginary part (a multiple $\sqrt{-1}$).

- The zeros of the Zeta Function: are complex numbers that, when plugged into the Riemann Zeta Function, evaluate it to zero.

So, now that we know these terms, we can explain what the Riemann Hypothesis is.

The Riemann Hypothesis

Thus far, all the discovered zeros of the Zeta Function have a real part equal to ½. Is this just a coincidence, or is there a pattern here? The Riemann hypothesis guesses that it is a pattern, and that every single one of these zeros has the same real part of ½.

Progress!

So far, we are in good shape; the first 1,500,000,000 zeros all fit the pattern. Unfortunately, this can't be considered a proof. If the next zero was an exception to the rule, the Riemann Hypothesis would be false.

So, mathematicians are still looking for a way to prove this hypothesis, not only for the million dollar prize, but because it is considered one of the most important unsolved problems in the field of mathematics.

Primed and Ready

I can tell that at this point you are wondering: so what? Why is this problem so important to warrent so much attention? It all has to do with the prime numbers.

Primes are the building blocks of the number system. Every other whole number can be made by multiplying primes together. Mathematicians today are always on the hunt to find more of them.

Unpredictibility

Unfortunately, we have no simple way to find primes or predict when they will pop up. And even when we find primes, the only way to check their primality is to try and divide by all the numbers smaller than it. That's not very efficient or easy, especially when your prime numbers get very large. Think of how many divisors you would have to check!

A New Function

Now consider for a moment, a function that takes a number x and returns how many primes are beneath it. For example, if x was 8, the function would return 4, because 7, 5, 3, and 2 are all the primes less than 8. Since some values of x return the same number (9 also has 4 primes below it), a graph of this function looks like the steps of a staircase. This is why our function is referred to as a step function.

The Prime Staircase

It turns out that this step function is very close to being x/ln(x). Since x/ln(x) is smooth and our step function is not, this approximation is not exact, but it's close. If we could improve this function somehow, we would have a better probabilistic model of when the primes would appear.

The Riemann Connection

This is where the Riemann Hypothesis comes into the picture. Suppose we look at the smooth x/ln(x) graph, and we start adding frequencies to it that are the zeros of the Riemann Zeta Function. The function we are constructing gets closer and closer to the step function of the primes we were considering.

See an animation!

So, if we could find all the zeros of the Riemann Zeta Function, we could make a function that tells us the underlying distribution of the primes.

A New Found Order

All of a sudden, the number system is not so unpredictable. The primes still appear at random, but with an underlying probability, much like the probability of a flipped coin. There is no way to know whether you will get heads on the next flip, but over a lot of flips, we know that heads would occur about half of the time. The primes would be very much the same. This would be an amazing advance for mathematics. All we need is for the Riemann Hypothesis to be true.

A Tie to Quantum Physics

Tantalizingly Close

It seems we might be on the brink of solving the Riemann Hypothesis. In recent years, there has been strong evidence that the Riemann zeros correspond to the energy levels of a quantum mechanical system. This physical model dealing with subatomic particles like electrons and quarks could be the key to unlocking the Riemann Hypothesis!

Physics and Math Intertwined

If this link was firmly established, we would not only have a nice map of the prime numbers, we could also use the tie to predict the energies of all the atomic and subatomic particles.

More Applications!

All of this newly acquired knowledge would also give us a way to model chaotically bouncing light and sound waves. Potential applications include improving microwaves, fibre optics, and the acoustics of concert halls.

Math Could Make You Rich!

Back in the Day

On August 8, 1900, David Hilbert presented a list of 23 millennium problems at a conference in Paris which collected together what he believed to be the most interesting, relevant unsolved problems in mathematics. Most of these problems have been solved over the last 100 years, with a couple of exceptions. One of these exceptions is the Riemann Hypothesis.

The Millenium Prize Problems

In May 2000 in Paris, in honor of Hilbert and the new millennium, the Clay Mathematics Institute of Cambridge announced 7 Millennium Prize Problems that have resisted solution over the years and are key to the development of mathematics. There is a million dollar prize being offered for each of these millennium problems, the Riemann Hypothesis among them.

A Tantilizing Incentive

The first person that proves the Riemann Hypothesis gets a million dollars, not to mention international fame! Interestingly, there is no monetary prize for a counterexample that proves the hypothesis to be false.

Root 2 : The First Surd

Be There or be Square!

The square root of 2 is the positive real number that satisfies $x^2 -2=0$. It is most simply the number whose square is 2.

It is also the diagonal length of a right triangle who's side lengths are both 1. We can prove this without even using Pythagoras' theorem. Check out the animation below. The area in the middle is C^2.

[Flash omitted. Experimentation with animation on paper yielded poor results.]

Notice that when the triangles move, we can see that the area must also be the sum of two squares both of area 1. Thus, $C^2=2$. This means that C is root 2!

Interesting equalities:

- sqrt(2)= 2sin(π/4)
- sqrt(2)= $e^{.5\ln(2)}$

Root 2 is Discovered!

The idea of the square root of 2 first appeared sometime before 600 BCE in an ancient Indian math text called the Sulbasutras, written by the Vedic Hindu scholars. Root 2 was the first surd (root) ever discovered. They called root 2 “dvi-karani”, the Sanskrit word which literally means “that which produces 2”. The scholars came up with a pretty good estimation of root 2, correct to 5 decimal places:

$1 + 1/3 + 1/(3*4) - 1/(3*4*34) = 577/408 \approx 1.414215686$

Later, around 430 BCE, Pythagoras found that a square, whose sides are length 1, must have a diagonal of length root 2. As a result, root 2 is sometimes referred to as Pythagoras’ constant.

Mad Influence!

Root 2 has expanded our number system past simply adding, subtracting, multiplying, and dividing whole numbers. Before its discovery, the Greeks believed that every number could be expressed as a combination of whole numbers with those four operations. Root 2 broke the mold and showed us that there were more numbers than we ever imagined existing.

Root 2, You're Being Irrational!

The Discovery

The Pythagorean number system was based on only whole numbers and fractions of whole numbers, called rational numbers. So, when Hippasus geometrically proved (see geometric proof animation) that root 2 couldn’t be written as a fraction of whole numbers (that it was irrational), there was a great uproar.

Outrage

Pythagoras couldn’t accept the existence of irrational numbers, because he believed in the absoluteness of numbers. There is speculation that Pythagoras was so upset that he sentenced Hippasus to death by drowning. Another possibility is that the Pythagoreans while sailing were so outraged that they tossed him overboard.

A Revolution

Although they make for good stories, there is no reliable evidence to support that either of these events actually happened. What we do know is that Root 2 is the first known irrational number. It redefined and expanded what math could do and what it could be.

Phi : The Golden Ratio

Nature's Golden Boy

The golden ratio, sometimes known as the golden mean, golden number, or the divine proportion, is one of those magical numbers that appears in too many places to be overlooked.

[Nick's impressive spiral animation goes here.]

WHERE DID IT COME FROM?

Suppose we want to know how to make it such that the ratio of the whole to the large part is the same as the large part to the small part. So, if a is the large part and b is the small part, then we want:

[GRAPHICS!!......]

The unique value for a/b is known as the golden ratio. It is equal to (1+sqrt(5))/2.

Symbology!

The symbol for the golden ratio φ was first used at the beginning of the 20th century by Mark Barr to honour Phidias (ca. 490-430 BC) a Greek sculptor whose works made extensive use of the golden ratio.

Fibonacci: Mining for Gold

The golden ratio also has an interesting relation to Fibonacci numbers. The Fibonacci numbers are a series of numbers in which the first two numbers are both 1, and each of the following numbers is the sum of the previous two numbers. They start: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ….

Now what is interesting, is that if you take any number in this sequence and divide it by the previous one, you get a fraction close to the golden ratio. The larger the numbers you use, the closer the approximation gets. So, the Fibonacci numbers and the golden ratio are intimately related, because the limiting ratio of the Fibonacci numbers is the golden ratio.

Some interesting identities:

$\varphi^n = \varphi^{n-1} + \varphi^{n-2}$ consequently:

- $\varphi^2 = \varphi + 1$
- $1/\ \varphi = \varphi - 1$

[EQUATION GRAPHICS!!.....]

Fighting Over Toilet Paper

Grout and Tile

An n-fold rotational symmetry is when you can rotate an image around a point n times before you get back to the original, and the image looks the same every time as the original every time you rotate it. These n-fold symmetries are useful when tiling a surface.

It is very easy to tile a flat surface nicely with 4 equilateral triangles that have 3-fold symmetry. Likewise, you can have 4-fold symmetry with 4 squares, and 6-fold symmetry with 7 regular hexagons. Until the 1970s, it was thought that you couldn't accomplish a tiling with 5-fold symmetry.

Penrose Tilings

The mathematician Sir Rodger Penrose, knighted in 1994, is a recipient of the 1988 Wolf Prize for Physics for his work with Stephen W. Hawking. He is a professor at Oxford and the author of some well recieved popular science books about the physics behind human consciousness. He also developed Penrose tilings.

Penrose discovered in 1974 that you could tile a flat surface with only two shapes and get 5-fold symmetry. Before, this feat was thought to be impossible.

By drawing a five pointed star inside of a regular pentagram with sides length phi, you can pull out the resulting triangles and use them to create the basic pieces for making the two different Penrose Tilings: the tiling with darts and kites, and the one with two rhombuses (flattened squares).

[Graphics!!]

The intriguing part about these tilings is that they are aperiodic, which means that they never repeat. It is impossible to take a section of the tiling and repeat it to create the rest of the tiling.

Interestingly enough, the same sort of 5-fold symmetries occur in the 3rd dimension with quasicrystals. It was originally thought that crystals were either entirely symmetric being the same repeated pattern throughout, or entirely asymmetric. But in 1984, the first quasicrystals were observed in an aluminiun-manganese alloy (Al6Mn). When cross sections are taken of these crystals it reveals a two-dimensional penrose tiling.

Fighting Over Toilet Paper

When Rodger Penrose's wife brought home some Kleenex toilet paper made by the British division of the Kimberly-Clark Corporation that appeared to be quilted with the same pattern Penrose had developed and patented back in the 1970s, he was a little less than pleased. The company explained that the quilted aperiodic pattern made it so that the toilet paper sat evenly on the roll.

Legal Action!

Penrose and Pentaplex Ltd., who owns the licensing rights to Penrose's work, filed a lawsuit against the corporation for a breach of copyright.

"When it comes to the population of Great Britain being invited by a multinational to wipe their bottoms on what appears to be the work of a Knight of the Realm without his permission, then a last stand must be taken," maintained a director of Pentaplex.

Penrose dropped the lawsuit with Kimberly-Clark upon coming to a mutually beneficial arrangement.

Appearances in Art

The golden ratio appears in all kinds of paintings, sculptures, and architecture. Whether this frequency is by coincidence or not is up for debate. The Greeks believed that the golden ratio makes for some very pleasing shapes; it's a nice balance between symmetry and asymmetry. The ratio, when applied to the dimensions of a rectangle, creates one that is very regular and pleasing to the eye. Even so, studies have shown that people have no significant preference for the golden rectangles over thinner or fatter ones.

[PIC OF GOLDEN RECTANGLE]

The golden ratio is also close to ratios in the human face, and it is thought that the perfect face would contain the golden ratio exactly. In 1509, Leonardo Da Vinci did the illustrations for Luca Pacioli's Divina Proportione which explored the math behind the golden ratio and its uses in architecture and art. There is suspicion that Da Vinci went on to apply the golden ratio to his painting of the Mona Lisa.

Appearances in Nature

A Natural Ratio

Although appearances of the golden mean in art, architecture, and human proportions are largely unsubstantiated, the golden ratio genuinely appears in many elements of nature.

[PIC OF NAUTILUS SHELL]

In Nautilus shells for example, there is a logarithmic spiral that is governed by the golden ratio. Draw a line from the center of the shell out in any direction and take two successive points where the shell spiral crosses the line. The distance of the farther point from the center will be about the golden ratio bigger than the distance of the closer point from the center. In other words, the shell has grown about 1.618 times bigger in one turn.

A similar logarithmic spiral governs the Pergrine falcon's prey snatching flight path. Because the spiral is based on the golden ratio, it is a very good approximation of the golden spiral, a spiral based on a series of boxes the size of Fibonacci numbers. The golden spiral can be found in hurricanes and the shape of galaxies.

[PICS OF GALAXY AND HURRICANE]

Spiraling Outta Control!

The number of spirals that can be outlined in pineapples, cauliflower, pinecones, cactus spines, and the seeds of sunflowers are all Fibonacci numbers. Ok, but how do we know that all these occurrences in nature are not purely coincidental? These distributions of the respective units and spirals they form can actually be recreated.

[PROLIFERATE WITH PICTURES AND ANIMATIONS OF PINECONES PINEAPPLES GROWTH PATTERNS ETC]

Growing Perfection

Spin a source point (or growth point) at a constant speed and release a seed every $1/\varphi \approx 0.618$ turn of the circle. You end up not only with spiral patterns (Fibonacci in number), but also you get a distribution of seeds that is the most evenly spaced distribution possible. It looks exactly like the seeds in the sunflower. There is no overcrowding and no strange spaces or holes. The golden ratio results in a perfectly spaced pattern containing a Fibonacci number of spirals.

Coincidence??

There is a brilliant and logical explanation for all this. Nature is governed by efficiency. It wants to pack the most amount of parts in the least amount of space, but not put everything so close that it can't function. The golden ratio accomplishes this goal by being the key angle of rotation to make the most efficient, equally spaced packing.

The leaf arrangements of plants also have a spiraling pattern based on the golden ratio. If the plant only produces a leaf about every 0.618 of a turn about the stem (1.618 leafs per turn), then we get the best and most even distribution of leaves. The maximum amount of space is allowed between every leaf that is directly above another one on the stem, so shadowing is minimized. The leaves make the best use of the space they have to capture sunlight in.

Counting on Plants

Because a plant's growth acts this way, with high dependence on the golden ratio, it makes sense that the Fibonacci numbers would pop up. If you look at the number of petals in flowers, more often than not, they are Fibonacci numbers. Lilies, irises, and trillium all have 3 petals. Buttercups, geraniums, pansies, primroses, rhododendrons, and tomato blossoms all have 5 petals, the most common number of petals for a flower to have. Delphiniums have 8, marigold and ragwort have 13, and black-eyed susans ,chicory, and asters all have 21. Finally daisies often have 34, 55, or even 89 petals.

1/log e : The Common Logarithm

The Computing Wonder

What is a Logarithm?

A logarithm is just an operation like addition or multiplication. It undoes exponents, in much the same way that division undoes multiplication.

So, a logarithm is written as $\log_a b$, where a and b are just numbers, and a is called the base. What this means, is that if $\log_a b=x$, we are asking what x makes $a^x=b$. So, for example, $\log_2 8$ asks what number x makes $2^x=8$? Since $2^3=8$, $\log_2 8=3$.

Common Logarithms

The common logarithm is a special log with base 10. Since it is used so often, when you write just logb without a base, we assume the base to be 10, in much the same was that we assume the radical sign √ with no number to mean the square root, and not any other root. So, for example, log1000= 3 because 10^3=1000.

Connecting to the Natural Log

A natural log is just a logarithm that has the exponential constant e as a base. Suppose we have a common log of a number and we want to convert that to the natural log instead. Changing bases is not actually all that complicated! All we need the connection factor 1/loge.

ln b * log e = log b

Going the other way is just as easy. To convert from the natural log to a common log, we just multiply by that same factor loge instead of dividing.

log b* 1/log e = ln b

Benford's Law

Simon Newcomb is a mathematician and accomplished astronomer born in Wallace, Nova Scotia. Although he had no formal education aside from his father and a brief apprenticeship with an herbalist, he went on to be a well respected professor of mathematical astronomy, writing several textbooks, becoming a founding member and the first president of the American Astronomical Society, and serving as president of the American Mathematical society.

In 1881, Newcomb was looking at a book of log tables, and he noticed that certain pages were more worn than others. The pages with numbers of leading digit 1 seemed to be used more often. Newcomb theorized that given a set of measurements, numbers with 1 as a leading digit occur more often than numbers of any other leading digit. In fact, the probability of a leading digit n is log(n+1/n). So the larger the leading digit, the less likely it was to occur. This discovery was met with a lack of interest and was largely overlooked.

This first digit law was later revived in 1938 by a physicist named Frank Benford. He verified the phenomenon on varied and numerous sets of data. He investigated this property so thoroughly that they named this law Benford's law. So if you collect all the measurements you can find on river lengths, river areas, baseball batting averages, country areas, the figures in Bill Clinton's tax returns, properties of chemicals, or even the house numbers of everyone named john, numbers starting with 1 will occur about 30% of the time. So everyday sources of naturally generated numbers follow this pattern outlined by Benford's law. In fact, it is scale invariant- it doesn't matter if you measure something in meters or feet or inches or furlongs, the law continues to hold, and we would expect this property from a law like this. But this is mind boggling! Why should this work? I would have expected that the numbers starting with one should occur as often as any other number, with a probability of 1/9. Think of it this way. If you have a measurement of 1000, you would need a 100% increase that measurement from 1000 to 2000. But, to make it from 2000 to 3000, you would only need a 50% increase, and from 3000 to 4000 only about 33%. This amount of increase from one leading digit to the next gradually decreases, with finally 9000 needing only 11% to get back to 10000. And again, you would need to increase 10,000 by 100% to get to the next leading digit. So this is why leading digits are distributed in this way. So, the larger the leading digit, the closer it is to its neighbor, probabilistically.

Now, I can see what you are thinking, maybe this is the key to getting rich: only choose lottery numbers that start with 1. Well, not exactly. Lottery numbers are truly random, with each digit having equal probability of getting chosen. These numbers are not naturally occurring data, their properties are controlled. Ok, so where else does Benford's law not apply? Look at the heights of all the males in Canada: it averages about 70 inches (5'10"). I can comfortably say with few exceptions that most males are between 60 inches and 80 inches. So probably 99% of these heights have a leading digit of 6 or 7, and you would be extremely hard pressed to find someone who is 10 inches or 100 inches tall. This is because male heights are normally distributed about a mean (average value), and the heights do not span much more than 2 leading digits with any frequency. So, Benford's law doesn't apply to purely random numbers like lottery numbers, or random numbers that are normally distributed about a mean. You can only look at naturally occurring measurements. But what practical applications could possibly result from Benford's law? Benford's law is used by forensic accountants as a method to spot fraud. If we look at a company's account information, all the figures should follow the leading digit distribution outlined by this law. A deviation from this distribution could be evidence that a company is cooking its books, and that further investigation is warranted.

Light and Sound

Why's the Common Log So Special?

Why should we use a logarithm with base 10 as opposed to any other base? First, 10 is just a nice number to use. Our whole number system is based on the number 10. This is because there are 10 digits, namely 0 through 9. Base 10 is a very natural and easy choice.

A great property of the common log is that if you take the common log of some data, the large values get squished together, but the small ones stay spread apart. This nice behaviour makes the common log very useful for graphing a bunch of small values that are close together and a couple of scattered outliers.

When you transform and graph the data, you can still see all the individual points, but now the graph will fit all on one page. Common logs in particular make it very easy to convert these values back and forth, because our entire number system is based on values of 10.

Look and Listen

The eyes and ears respond logarithmically to brightness and loudness. There is also some evidence that pigeons logarithmically perceive time.

What exactly does it mean to respond logarithmically? It means that it is really easy to notice small differences in the small values, but it's really hard to notice small differences in large values.

For example, if you are in a room with only one candle for light, and a person lights another one, you can easily tell that the room is much brighter. If instead you have a hundred candles in a room and you light one more, it is not so easy to tell that the room is brighter.

Revolutionizing Calculation

The History of Logarithms

John Napier invented logarithms in the early 1600s. They were a major development in the abilities of computation, but this revolution did not just come out of nowhere.

Like all great inventions, there existed not only a need, but inspiration. Mathematicians of this time were mostly concerned with easing the complicated calculations of trigonometry, especially in astronomy and navigation. The method of prosthaphaeresis revolved around this formula:

$\sin(a) * \sin(b) = [\cos(a - b)-\cos(a + b)]/2$

This formula is just an identity that reduces a very difficult trig multiplication to simple additions, subtractions, and one division by 2. It is possible that it is this idea that inspired Napier to look for a similar method, one with applications beyond trigonometry. The logarithm was born!

Spreading the News

Napier first used the term artificial number, but he later decided on the word logarithm which comes from a Greek phrase meaning ratio number. He published the first table of answers to logarithms and introduced them to the world in 1614 in a small volume entitled *Mirifici Logarithmorum Canonis Descriptio*.

A Ripple of Excitement

The Descriptio excited many mathematicians of the time including Johann Kepler and Henry Briggs. Briggs assisted Napier in refining these tables, using the much simpler base of 10.

A Revolution

Using these tables, people could now do square roots and cube roots with ease. Hard multiplications and divisions could be changed into easy additions and subtractions.

By 1644, logarithms had spread to become a worldwide phenomenon. Explorers could now make quick calculations which allowed them to accurately navigate and also map out their discoveries like never before. Astronomers could now calculate planetary orbits that would have otherwise taken them years to do. The invention of the logarithm is considered by many to be the most influential leap in computation until perhaps the invention of the modern computer.

Napier's Bones

Napier also invented a device to do large multiplications and divisions easily by using printed rods and simple additions and subtractions. "Napier's Bones," as they are called, were extremely popular all over Europe in the 1600s.

They work much like the multiplication tables that many have us encountered in elementary school. In fact, they were still being used in elementary schools in Britain in the mid-1960s to help children learn multiplication.

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e : The Exponential Constant

Growing Outta Control

e might seem like an odd number at first, but it has a lot of special properties that makes it worth knowing about. It can be used to construct perfect arches, calculate continuous interest, and plot a trip where the distance you have traveled is always the same as your speed, which is always the same as your acceleration. It creates the ultimate exponential, growing without bound.

[Exciting e visualization.]

e's Spiffy Properties

- the first mention of e occurs in 1618
- e is used to find interest that is accumulated continuously, the method commonly used by banks
- an equation involving e describes a catenary, the shape of hanging cords that are flexible but not stretchy
- the Gateway Arch in St Louis, Missouri has the shape of an upside down catenary
- e is involved in the equation for the statistical bell curve
- The sum 1+1/2!+1/3!+1/4!+……= e
- The area underneath 1/x from 1 to e is 1
- e is irrational, because it cannot be expressed as a fraction of whole numbers

- e is transcendental, because there is no polynomial with rational coefficients of which e is a root
- the value of e^x at any point is the same as its slope at that point
- the infinite digits of e have no pattern

Where'd e come from?

The Key Players:

John Napier (1550-1617)

John Napier a Scottish mathematician, astronomer, and physicist is known for popularizing the use of the decimal point and inventing logarithms. Within his work on logarithms published in 1618 appears the first notion of e.

Gottfried Leibniz (1646-1716)

Gottfried Leibniz a German mathematician and philosopher was the first known user of the constant in 1690.

Jakob Bernoulli (1654-1705)

Jakob Bernoulli a Swiss mathematician and scientist further developed the notion of e with his study of compound interest.

Leonhard Euler(1707-1783)

Leonhard Euler a Swiss mathematician and physicist started using the letter e to describe the exponential constant around 1727. While other letters were used, e eventually became the standard.

Why e?

Why is this special constant the letter e and not some other letter? For a while there was no standard, and researchers often used the letters b or c to represent the number. Euler was the one that started using the letter e consistently, which later became convention.

No one knows for sure, but there is spectulation on why Euler chose e. Possibly he used it because it was the next vowel after a, which he was already using for some other quantity. Others speculate that he chose e because it was the first letter of the word exponential. It is probably just a coincidence that the first letter of his name is also an e. He was a modest man who tried to give credit where credit was due.

Growing Money

The Price of Money

When you borrow money, you get charged a yearly percent of the amount you owe (the decimal interest rate r), and you get charged that interest maybe every day, week, or month. This is how often you compound interest (n times per year).

e's Interest in Money

To determine the amount you owe after t years just multiply how much you borrowed by (1+r/n)^(nt). Now this is where e comes in. If you let how often you calculate and add in the interest to the loan become so frequent that you are doing it all the time, you end up with something called continuously compounded interest.

This is what banks use when you take out a loan of money. Bernoulli found that if you let n go all the way to infinity, that formula above becomes simply e^rt.

So for example, if you earned a yearly interest rate r of 100% (ie double your money) and you wanted to continuously compound that interest all day and every day for that year, you would end up with e times what you started with.

Shaping the World

Being Flexible

Imagine a cable or a rope that is completely flexible and the exactly the same along its entire length. When we attach the ends of it to solid unmoving things directly across from one another and let the cable hang and dip in the middle it has a certain shape. This shape is called a catenary.

We can actually describe this shape with a formula involving e. With y as the vertical distance and x as the horizontal distance,

y= a/2 (e^(x/a) + e^(-x/a))

where a is a constant value depending on how close the ends are together and how long the rope is.

e in Architecture

So, there is a fundamental natural shape to hanging cables that we can describe with an equation involving e. The Gateway Arch in Saint Louis, Missouri has this same caternary shape only upside down. It is the ideal shape for an arch that only has to support its own weight because it has no shear forces.

A More Theoretical Curve

e is also an important part of the equation for the infamous statistical bell curve which characterizes a normal distribution and is often used to describe how exam grades are spread out.

e and His Imaginary Friend

Euler's Equation

If i is the square root of -1 which is an imaginary number, then…

$e^{ix} = \cos(x) + i \sin(x)$

This is Euler's equation which is a very useful identity to avoid working with imaginary numbers coupled with trigonometric functions.

Relating the Famous Five

The following identity is one which relates what some refer to as the "famous five" numbers, e, π, i, 1, and 0.

$e^{i*\pi} + 1 = 0$

This equation is easy to prove by substituting pi for x in Euler's equation mentioned above.

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pi : Archimedes' Constant

Running Circles 'Round the Competition.

π has everything to do with a perfect circle. It is the ratio of the circumference of the circle to its diameter, and it is needed for practically every calculation involving circles. It is an integral part of trigonometry, and it is half the period of the sinx and cosx wave functions. We now know pi to over a trillion decimal places, and are continuously developing new algorithms to make the calculations quicker and more elegant.

Square-free Integers

A square-free integer is one with only unique prime divisors. For example, 18 is not square free, because one of its factors is 9, and 9 is a perfect square. The prime factor 3 appears twice. On the other hand, 30=5*3*2 is square free, because all of its prime divisors are unique. It does not contain any squares.

π has a surprising circle free application to probability. The probability of randomly selecting a square-free integer out of the entire number line is 6/(pi^2).

Relatively Prime Natural Numbers

The probability of two natural numbers (0,1,2,3,etc) being relatively prime is also 6/(pi^2). Two numbers are considered relatively prime if they have no factors in common greater than 1.

For example, 15 and 8 are relatively prime because they have no common factors. 15 and 9 are not relatively prime, because they share the common factor of 3.

Finding the Digits

How Do You Find the Digits of π?

You might be tempted to thing that finding the digits of π has an obvious solution: just measure it! Well, let's try it. Get out a string or measuring tape of somekind, and using an actual circle, find the circumference and divide it by the diameter.

Experimental Results

You will definitely be able to tell that the ratio is a little more than 3. But how much more? If you make really good measurements, you might be able to find that it's about an extra tenth, but you would be hard pressed to get a better measurement than that. So, how is it that we have such an accurate representation of π?

Methods of Approximation

1)Archimedes method involving polygons:

Archimedes (287-212 BCE) came up with the approximation for π of 22/7 such a long time ago in a time before computers or calculators or calculus. These were simple times mathematically speaking, so how did he manage it?

He found this fractional approximation by inscribing a circle inside a 96 sided polygon and another 96 sided polygon within the circle. A polygon with this many sides looks very much like a circle, but it is easier to deal with. The perimeter and area of these figures are very easy to find, since they are basically made up of a bunch of identical triangles.

By finding the perimeter of each polygon and comparing it to their respective areas, Archimedes could put very accurate bounds on the ratio of π. Although he hadn't found it exactly, he pinned it down to a very small window.

2) Probabilistic model:

Randomly drop points in a unit square with a quarter of a circle of radius 1 inscribed in it. Then take 4*number of points in the circle section/total points dropped to get π. This method can either be done physically or simulated by a computer.

3)<u>Buffon's needle problem</u>:

Drop needles on a flat piece of paper with parallel lines on it and count how many of the needles cross any parallel line on the paper. Then take:

(2 * *#drops* * *needle length*) / (*#hits* * *distance between the parallel lines*)

to get π.

4) <u>Bounding Power Series</u>:

Use two power series approximations of pi (a type of function) that are very close together, but one is an upperbound and one is a lowerbound. So as long as they agree, we can be certain that the digit of pi is correct.

The History of π

π Comes on the Scene

π is the ratio of the circumference of a circle to its diameter. We have thought about π for a very long time. First appearing around 2000 BC, the Babylonian and Egyptians used reasonable approximations of the number. There is even a reference in the bible

Approximations!

Almost 1800 years later, Archimedes (287-212 BCE) came up with the approximation for π of 22/7 by inscribing a circle inside a 96 sided polygon and another 96 sided polygon within the circle. By finding the length of the perimeter of each polygon and comparing it to their respective areas, Archimedes could put bounds on the ratio π. This is called Archimedes' method.

By around 1430 there was a decimal estimation of π that was correct to 14 places. With the help of computers, we can now determine π to over a trillion digits.

Do We Really Need That Many Digits?

Knowing that many digits of π isn't exactly useful in any of the calculations that we will ever do. Calculating the digits of π has instead become a way to test the speed and efficiency of new super computers.

Why the Letter π?

π wasn't always known as π. Sometimes referred to as Archimedes' constant or Ludolph's number, the usage of the greek letter π to represent this constant first appears in 1706 by William Jones and was later popularized by Euler.

The usage of π is probably derived from the Greek words περιφέρεια (periphery/ circumference) or from περίμετρον (perimetron/ perimeter).

Properties and Formulae

Proving the Properties of Pi

Johann Lambert proved in 1761 that π is irrational and also in 1768 that it cannot be a repeating pattern of digits.

Ferdinand von Lindemann proved in 1882 that π is transcendental (is not a root of any polynomial with rational coefficients), and used this result to prove that it is impossible using only a compass and a straightedge to make a square with the same area as a circle.

Shape Formulae Involving π:

circle:

- Circumference= $2\pi r$
- Area= πr^2

sphere:

- Volume= $4/3 \pi r^3$
- Surface Area=$4\pi r^2$

cone:

- Volume= $1/3 \pi r^2 h$
- Surface Area= $\pi r (r + \sqrt{(r^2 + h^2)})$

Other Formulae

- $\cos(\pi)=-1$
- $\sin(\pi)=0$
- integral(-1 to 1) $dx/\sqrt{(1-x^2)} = \pi$
- integral(-inf to inf) $e^{(-x^2)}dx=\sqrt{(\pi)}$
- The formula for the normal statistical bell curve involves π.

F : Feigenbaum's Constant

Fig Trees and Butterflys

Systems like the weather, population growth, chemical reactions, and snowflake formation all have something in common: small changes in the starting conditions create huge changes in the final outcome.

This is because each stage depends only on the previous stage, so small differences get amplified into big differences. We call these systems chaotic, because the final outcome is so unpredictable.

Chaos Discovered

When Mitchell Feigenbaum was looking at animal populations in the mid 1970's, he found that the population size was predictable only when a certain constant describing the birth and death rate was small, but it became chaotic when this constant was increased beyond a certain point. By studying this system, Feigenbaum discovered a fundamental value, now called Feigenbaum's constant, that can be used to predict the point at which any such system becomes chaotic.

Trivia

- Feigenbaum means "fig tree" in German.
- Feigenbaum solved for his constant using an HP-65, the first fully programmable, handheld calculator. This calculator was also used by astronauts in the final Apollo mission.
- Every snowflake is unique, because snowflake formation is chaotic.
- The Butterfly Effect describes the surprising results of weather being chaotic.
- Feigenbaum's constant also appears in the fractal created by the Mandelbrot set.

The Butterfly Effect

Modeling the Weather

Edward Lorenz was a mathematician and meteorologist who had the idea to use computers to help him make a predictive model of the weather in the 1960s. He came up with 12 equations describing basic weather quantities such as air pressure, temperature, and wind.

Making Predictions

Although the equations were pretty simple, they were nonlinear differential equations, so they couldn't be solved outright. The only way to know the eventual behavior of the system was to keep plugging in values. So, Lorenz set up his computer to take some initial conditions and use the equations to determine the changing quantities through time. He got printouts of all the data, and the model worked remarkably well. It seemed that Lorenz was on to something.

Taking Another Look

One day in 1961, Lorenz decided to reexamine some data that he had already considered. Instead of starting right from the beginning, to save some time, he picked up the simulation somewhere in the middle. Plugging in these conditions determined from the previous printouts, he reran just the later half of the simulation. While the new data was the same as the old at first, the new numbers started to wildly deviate from the original running. This was extremely puzzling, because these equations created a deterministic system- everything should be exactly the same.

Troubleshooting

After checking the computer to make sure it was functioning properly and pondering the possible differences between the first and second run, Lorenz realized that there was in fact one difference. He had used the numbers from the printouts which had 3 decimal places to save space, while the computer calculations carried 6 decimal places. But surely this seemingly insignificant difference would only cause small differences in the final answer.

Small Problems Become Large

Unfortunately this wasn't the case. The very small error had been amplified into huge differences in the long term outcome. It seems that there is actually no hope in making long term weather predictions. Small discrepancies in initial measurements lead to so much variety in the final outcome that we would have to have infinitely precise measurements to have any hope of making long term predictions.

Tornados in Texas

So, Lorenz determined that weather is a chaotic system, because small changes in temperature, air pressure, wind, and humidity can cause drastically different weather later on. This is often referred to as the butterfly effect, an idea Lorenz came up with and presented at conferences in the mid 1960s. The idea is that a butterfly flapping its wings in Brazil could cause small changes in the atmosphere, and these unnoticeable changes could set off a chain of events that could lead to a tornado in Texas. The idea of chaos theory was born.

Delving Deeper

Further investigation was warranted. To simplify the problem, Lorenz reduced the 12 equations to 3, making something that was graphable in 3-space. The graph of this new system is called the Lorenz Butterfly and is a great way to demonstrate a system extremely sensitive to its initial conditions.

[LORENZ BUTTERFLY GRAPHICS]

The path of the equations for any set of initial conditions never repeats itself and never crosses itself. Also, two sets of initial conditions that are really close to each other end up being extremely far away from each other later on. This is characteristic of a chaotic system.

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h : Plank's Constant

Enlightenment

Although very very small, Planck's constant has some fantastic results. At the beginning of the 20th century, Maxwell Planck was looking at blackbody radiation, which is what happens when you heat a black object and it emits light, like the burner on your stove or the sun.

The Discovery!

He discovered that the light energy given off cannot take on any possible value it wants, but only a defined set of values. He came up with Planck's constant as a way of relating the frequency of light emitted to the amount of energy in that light.

So, if we take the amount of radiated light energy and divide it by both Planck's constant and the frequency, we get a whole number.

Light Packets

This finding later led Einstein to propose that light energy comes in little packets or particles called photons. This whole number that we get by dividing the light energy by the frequency and Planck's constant is actually the number of photons that are emitted. This dramatically changed the way we think about light.

But light also acts like a wave, which was shown by the *link: double slit experiment*.

Dual Personality

So, light behaves both like a particle and a wave, appropriately called the wave-particle duality of light. Light has mass (an unbelievably small amount), but it also has an amplitude, frequency, and wavelength.

The Double Slit Experiment

Big Questions

Thomas Young was the first to perform the double slit experiment around 1805. His aim was to determine whether light behaves like a particle or a wave.

The Basic Setup

A beam of light is shone through two very narrow slits placed a certain distance apart in an otherwise solid barrier. On the other side of the barrier there is a screen where we can see the resulting light pattern.

What We Expect

The patterns on the other side will reveal if light behaves like a particle or a wave. If light behaves like a particle, then it must travel in a straight line. On the other side of the barrier, we would expect to see the individual beams of light as they travel through the slits. Plain and simple.

If, on the other hand, light behaves like a wave, we expect to see something called an interference pattern. Interference patterns happen when waves meet eachother and combine their energies.

All waves have peaks which are the high points, and troughs which are the low points. So, when two waves meet, some of the wave energies cancel eachother out because the peaks in one wave meet the troughs of the other. When peaks meet peaks and troughs meet troughs, the wave energy is added together to create an extra big peak or an extra big trough. This interference pattern in the case of light will represent itself as alternating bright and dark spots.

Results!

When the slits are are a good distance apart, we see what we would expect to see in either case. The light has made it through these two slits and we see two vertical sections of light on the other side. They are not close enough to interact.

Now, move these slits closer together. When they get close enough, the pattern on the other side changes a bit. The bands of light have now combined in a way that shows alternating bright spots and dark spots. The crests of the light waves have added together to create bright spots and the troughs have added together to create more bright spots. The dark spots result from the crests of one wave and the troughs of the other wave canceling each other out. This alternating light and dark pattern is the interference pattern. So light behaves like a wave!

Another Wave Demonstration

A nice way to see interference patterns in action is by making a similar set up with water.

Send a wave of water towards a divider with two spaces for the waves to get through. On the otherside, the ripples come through at the two different points and these two new ripples interfere with eachother! This is the same thing that is happened with our light experiment.

Heisenburg's Uncertainty Principle

Tricky Little Particles

Planck's constant h is later used in Heisenburg's uncertainty principle. The idea is that for very small particles, we cant really say exactly where it is and what momentum it has. If Δx is the uncertainty of position and $\Delta \varrho$ is the uncertainty of momentum, then

$\Delta x \, \Delta \varrho \geq h/4\pi$.

So to get more accuracy in position we have to give up some certainty in momentum and visa versa. We can only be so sure of where a very small particle is located or what momentum it has, but our uncertainty in bounded!

Extending Uncertainty

Since we can never be totally positive of where very small things are and what they're doing, we describe them as having a superposition of states. The particle is here, but it's also over there at the same time. The particle is moving this fast, but it's also moving that fast. It is in a variety of different states at the same time.

Schrödinger's cat

Schrödinger's cat is thought experiment that explains Heisenburg's uncertainty principle. Imagine that you have a radioactive material that has a 50% chance of decaying in 1 hour and emitting a radioactive particle. This particle will trigger the release of poison that is in a sealed box with a cat.

So if we seal the box, there is an equal chance that after an hour the cat is alive or the cat is dead. So we say that the cat is both alive and dead at the same time. This is a superposition of states. It is not until we open the box that the states collapse, and we know for certain that the cat is either alive or dead.