Selected Writings

on

Experimental and Computational Mathematics

and

Related Topics

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Selected Writings on Experimental and Computational Mathematics and Somewhat Related Topics

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Abstract

This is a representative collection of most of my writings about the nature of mathematics over the past twenty-five years. Permissions will need to be collected for any items reprinted from this selection.

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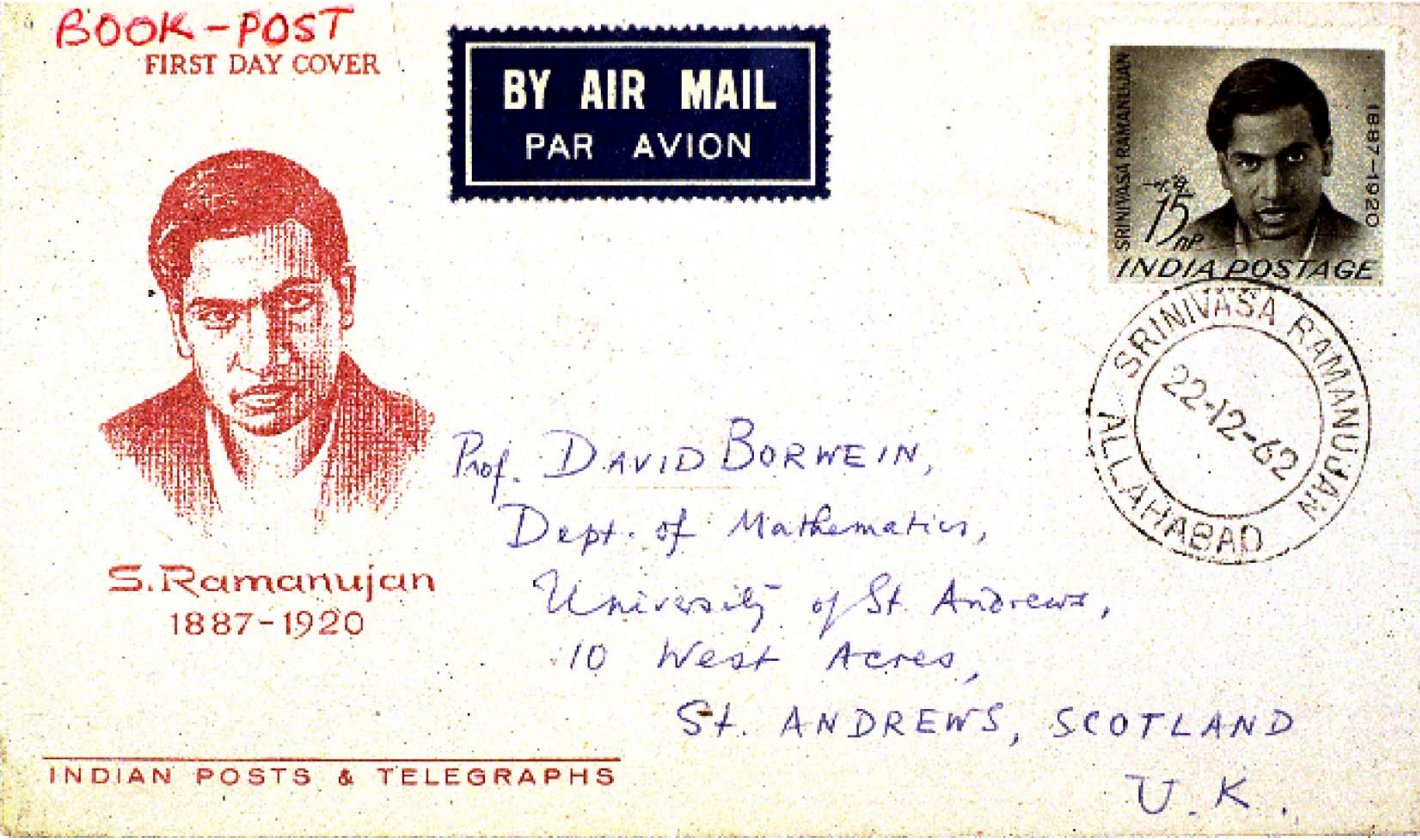
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THE ARITHMETIC-GEOMETRIC MEAN AND FAST COMPUTATION OF ELEMENTARY FUNCTIONS*

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Abstract. We produce a self contained account of the relationship between the Gaussian arithmetic-geometric mean iteration and the fast computation of elementary functions. A particularly pleasant algorithm for π is one of the by-products.

Introduction. It is possible to calculate 2^n decimal places of π using only n iterations of a (fairly) simple three-term recursion. This remarkable fact seems to have first been explicitly noted by Salamin in 1976 [16]. Recently the Japanese workers Y. Tamura and Y. Kanada have used Salamin's algorithm to calculate π to 2^{23} decimal places in 6.8 hours. Subsequently 2^{24} places were obtained ([18] and private communication). Even more remarkable is the fact that all the elementary functions can be calculated with similar dispatch. This was proved (and implemented) by Brent in 1976 [5]. These extraordinarily rapid algorithms rely on a body of material from the theory of elliptic functions, all of which was known to Gauss. It is an interesting synthesis of classical mathematics with contemporary computational concerns that has provided us with these methods. Brent's analysis requires a number of results on elliptic functions that are no longer particularly familiar to most mathematicians. Newman in 1981 stripped this analysis to its bare essentials and derived related, though somewhat less computationally satisfactory, methods for computing π and log. This concise and attractive treatment may be found in [15].

Our intention is to provide a mathematically intermediate perspective and some bits of the history. We shall derive implementable (essentially) quadratic methods for computing π and all the elementary functions. The treatment is entirely self-contained and uses only a minimum of elliptic function theory.

1. 3.14159265358979323846264338327950288 4197. The calculation of π to great accuracy has had a mathematical import that goes far beyond the dictates of utility. It requires a mere 39 digits of π in order to compute the circumference of a circle of radius 2×10^{25} meters (an upper bound on the distance travelled by a particle moving at the speed of light for 20 billion years, and as such an upper bound on the radius of the universe) with an error of less than 10^{-12} meters (a lower bound for the radius of a hydrogen atom).

Such a calculation was in principle possible for Archimedes, who was the first person to develop methods capable of generating arbitrarily many digits of π. He considered circumscribed and inscribed regular n-gons in a circle of radius 1. Using $n = 96$ he obtained

$$3.1405 \cdots = \frac{6336}{2017.25} < \pi < \frac{14688}{4673.5} = 3.1428.$$

If $1/A_n$ denotes the area of an inscribed regular 2^n-gon and $1/B_n$ denotes the area of a circumscribed regular 2^n-gon about a circle of radius 1 then

$$A_{n+1} = \sqrt{A_n B_n}, \qquad B_{n+1} = \frac{A_{n+1} + B_n}{2}. \tag{1.1}$$

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This two-term iteration, starting with $A_2 := \frac{1}{2}$ and $B_2 := \frac{1}{4}$, can obviously be used to calculate π. (See Edwards [9, p. 34].) A_{15}^{-1}, for example, is 3.14159266 which is correct through the first seven digits. In the early sixteen hundreds Ludolph von Ceulen actually computed π to 35 places by Archimedes' method [2].

Observe that $A_n := 2^{-n} \operatorname{cosec}(\theta/2^n)$ and $B_n := 2^{-n-1} \operatorname{cotan}(\theta/2^{n+1})$ satisfy the above recursion. So do $A_n := 2^{-n} \operatorname{cosech}(\theta/2^n)$ and $B_n := 2^{-n-1} \operatorname{cotanh}(\theta/2^{n+1})$. Since in both cases the common limit is $1/\theta$, the iteration can be used to calculate the standard inverse trigonometric and inverse hyperbolic functions. (This is often called Borchardt's algorithm [6], [19].)

If we observe that

$$A_{n+1} - B_{n+1} = \frac{1}{2(\sqrt{A_n}/\sqrt{B_n} + 1)}(A_n - B_n)$$

we see that the error is decreased by a factor of approximately four with each iteration. This is linear convergence. To compute n decimal digits of π, or for that matter arcsin, arcsinh or log, requires $O(n)$ iterations.

We can, of course, compute π from arctan or arcsin using the Taylor expansion of these functions. John Machin (1680–1752) observed that

$$\pi = 16 \arctan\left(\frac{1}{5}\right) - 4 \arctan\left(\frac{1}{239}\right)$$

and used this to compute π to 100 places. William Shanks in 1873 used the same formula for his celebrated 707 digit calculation. A similar formula was employed by Leonhard Euler (1707–1783):

$$\pi = 20 \arctan\left(\frac{1}{7}\right) + 8 \arctan\left(\frac{3}{79}\right).$$

This, with the expansion

$$\arctan(x) = \frac{y}{x}\left(1 + \frac{2}{3}y + \frac{2.4}{3.5}y^2 + \cdots\right)$$

where $y = x^2/(1 + x^2)$, was used by Euler to compute π to 20 decimal places in an hour. (See Beckman [2] or Wrench [21] for a comprehensive discussion of these matters.) In 1844 Johann Dase (1824–1861) computed π correctly to 200 places using the formula

$$\frac{\pi}{4} = \arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{5}\right) + \arctan\left(\frac{1}{8}\right).$$

Dase, an "idiot savant" and a calculating prodigy, performed this "stupendous task" in "just under two months." (The quotes are from Beckman, pp. 105 and 107.)

A similar identity:

$$\pi = 24 \arctan\left(\frac{1}{8}\right) + 8 \arctan\left(\frac{1}{57}\right) + 4 \arctan\left(\frac{1}{239}\right)$$

was employed, in 1962, to compute 100,000 decimals of π. A more reliable "idiot savant", the IBM 7090, performed this calculation in a mere 8 hrs. 43 mins. [17].

There are, of course, many series, products and continued fractions for π. However, all the usual ones, even cleverly evaluated, require $O(\sqrt{n})$ operations ($+$, $\times$, $\div$, $\sqrt{\ }$) to arrive at n digits of π. Most of them, in fact, employ $O(n)$ operations for n digits, which is essentially linear convergence. Here we consider only full precision operations. For a time complexity analysis and a discussion of time efficient algorithms based on binary splitting see [4].

The algorithm employed in [17] requires about 1,000,000 operations to compute 1,000,000 digits of π. We shall present an algorithm that reduces this to about 200 operations. The algorithm, like Salamin's and Newman's requires some very elementary elliptic function theory. The circle of ideas surrounding the algorithm for π also provides algorithms for all the elementary functions.

2. Extraordinarily rapid algorithms for algebraic functions. We need the following two measures of speed of convergence of a sequence (a_n) with limit L. If there is a constant C_1 so that

$$|a_{n+1} - L| \leqq C_1 |a_n - L|^2$$

for all n, then we say that (a_n) converges to L *quadratically*, or with *second order*. If there is a constant $C_2 > 1$ so that, for all n,

$$|a_n - L| \leqq C_2^{-2^n}$$

then we say that (a_n) converges to L *exponentially*. These two notions are closely related; quadratic convergence implies exponential convergence and both types of convergence guarantee that a_n and L will "agree" through the first $O(2^n)$ digits (provided we adopt the convention that .9999. . .9 and 1.000. . .0 agree through the required number of digits).

Newton's method is perhaps the best known second order iterative method. Newton's method computes a zero of $f(x) - y$ by

$$x_{n+1} := x_n - \frac{f(x_n) - y}{f'(x_n)} \tag{2.1}$$

and hence, can be used to compute f^{-1} quadratically from f, at least locally. For our purposes, finding suitable starting values poses little difficulty. Division can be performed by inverting $(1/x) - y$. The following iteration computes $1/y$:

$$x_{n+1} := 2x_n - x_n^2 y. \tag{2.2}$$

Square root extraction ($\sqrt{y}$) is performed by

$$x_{n+1} := \frac{1}{2}\left(x_n + \frac{y}{x_n}\right). \tag{2.3}$$

This ancient iteration can be traced back at least as far as the Babylonians. From (2.2) and (2.3) we can deduce that division and square root extraction are of the same order of complexity as multiplication (see [5]). Let $M(n)$ be the "amount of work" required to multiply two n digit numbers together and let $D(n)$ and $S(n)$ be, respectively, the "amount of work" required to invert an n digit number and compute its square root, to n digit accuracy. Then

$$D(n) = O(M(n))$$

and

$$S(n) = O(M(n)).$$

We are not bothering to specify precisely what we mean by work. We could for example count the number of single digit multiplications. The basic point is as follows. It requires

$O(\log n)$ iterations of Newton's method (2.2) to compute $1/y$. However, at the ith iteration, one need only work with accuracy $O(2^i)$. In this sense, Newton's method is self-correcting. Thus,

$$D(n) = O\left(\sum_{i=1}^{\log n} M(2^i)\right) = O(M(n))$$

provided $M(2^i) \geqq 2M(2^{i-1})$. The constants concealed beneath the order symbol are not even particularly large. Finally, using a fast multiplication, see [12], it is possible to multiply two n digits numbers in $O(n \log(n) \log\log(n))$ single digit operations.

What we have indicated is that, for the purposes of asymptotics, it is reasonable to consider multiplication, division and root extraction as equally complicated and to consider each of these as only marginally more complicated than addition. Thus, when we refer to operations we shall be allowing addition, multiplication, division and root extraction.

Algebraic functions, that is roots of polynomials whose coefficients are rational functions, can be approximated (calculated) exponentially using Newton's method. By this we mean that the iterations converge exponentially and that each iterate is itself suitably calculable. (See [13].)

The difficult trick is to find a method to exponentially approximate just one elementary transcendental function. It will then transpire that the other elementary functions can also be exponentially calculated from it by composition, inversion and so on.

For this Newton's method cannot suffice since, if f is algebraic in (2.1) then the limit is also algebraic.

The only familiar iterative procedure that converges quadratically to a transcendental function is the arithmetic-geometric mean iteration of Gauss and Legendre for computing complete elliptic integrals. This is where we now turn. We must emphasize that it is difficult to exaggerate Gauss' mastery of this material and most of the next section is to be found in one form or another in [10].

3. The real AGM iteration. Let two positive numbers a and b with $a > b$ be given. Let $a_0 := a$, $b_0 := b$ and define

$$a_{n+1} := \frac{1}{2}(a_n + b_n), \qquad b_{n+1} := \sqrt{a_n b_n} \tag{3.1}$$

for n in $\mathbb{N}$.

One observes, as a consequence of the arithmetic-geometric mean inequality, that $a_n \geqq a_{n+1} \geqq b_{n+1} \geqq b_n$ for all n. It follows easily that (a_n) and (b_n) converge to a common limit L which we sometimes denote by $AG(a, b)$. Let us now set

$$c_n := \sqrt{a_n^2 - b_n^2} \quad \text{for } n \in \mathbb{N}. \tag{3.2}$$

It is apparent that

$$c_{n+1} = \frac{1}{2}(a_n - b_n) = \frac{c_n^2}{4a_{n+1}} \leqq \frac{c_n^2}{4L}, \tag{3.3}$$

which shows that (c_n) converges quadratically to zero. We also observe that

$$a_n = a_{n+1} + c_{n+1} \quad \text{and} \quad b_n = a_{n+1} - c_{n+1} \tag{3.4}$$

which allows us to define a_n, b_n and c_n for negative n. These negative terms can also be generated by the *conjugate scale* in which one starts with $a_0' := a_0$ and $b_0' := c_0$ and defines

(a_n') and (b_n') by (3.1). A simple induction shows that for any integer n

$$a_n' = 2^{-n}a_{-n}, \quad b_n' = 2^{-n}c_{-n}, \quad c_n' = 2^{-n}b_{-n}. \tag{3.5}$$

Thus, backward iteration can be avoided simply by altering the starting values. For future use we define the quadratic *conjugate* $k' := \sqrt{1 - k^2}$ for any k between 0 and 1.

The limit of (a_n) can be expressed in terms of a *complete elliptic integral of the first kind,*

$$I(a, b) := \int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}. \tag{3.6}$$

In fact

$$I(a, b) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dt}{\sqrt{(a^2 + t^2)(b^2 + t^2)}} \tag{3.7}$$

as the substitution $t := a\tan\theta$ shows. Now the substitution of $u := \frac{1}{2}(t - (ab/t))$ and some careful but straightforward work [15] show that

$$I(a, b) = I\left(\left(\frac{a + b}{2}\right), \sqrt{ab}\right). \tag{3.8}$$

It follows that $I(a_n, b_n)$ is independent of n and that, on interchanging limit and integral,

$$I(a_0, b_0) = \lim_{n\to\infty} I(a_n, b_n) = I(L, L).$$

Since the last integral is a simple arctan (or directly from (3.6)) we see that

$$I(a_0, b_0) = \frac{\pi}{2}AG(a_0, b_0). \tag{3.9}$$

Gauss, of course, had to derive rather than merely verify this remarkable formula. We note in passing that $AG(\cdot, \cdot)$ is positively homogeneous.

We are now ready to establish the underlying limit formula.

PROPOSITION 1.

$$\lim_{k\to 0^+}\left[\log\left(\frac{4}{k}\right) - I(1, k)\right] = 0. \tag{3.10}$$

Proof. Let

$$A(k) := \int_0^{\pi/2} \frac{k'\sin\theta\, d\theta}{\sqrt{k^2 + (k')^2\cos^2\theta}}$$

and

$$B(k) := \int_0^{\pi/2} \sqrt{\frac{1 - k'\sin\theta}{1 + k'\sin\theta}}\, d\theta.$$

Since $1 - (k'\sin\theta)^2 = \cos^2\theta + (k\sin\theta)^2 = (k'\cos\theta)^2 + k^2$, we can check that

$$I(1, k) = A(k) + B(k).$$

Moreover, the substitution $u := k'\cos\theta$ allows one to evaluate

$$A(k) := \int_0^{k'} \frac{du}{\sqrt{u^2 + k^2}} = \log\left(\frac{1 + k'}{k}\right). \tag{3.11}$$

Finally, a uniformity argument justifies

$$\lim_{k\to 0^+} B(k) = B(0) = \int_0^{\pi/2} \frac{\cos\theta\, d\theta}{1+\sin\theta} = \log 2, \tag{3.12}$$

and (3.11) and (3.12) combine to show (3.10). □

It is possible to give various asymptotics in (3.10), by estimating the convergence rate in (3.12).

PROPOSITION 2. *For* $k \in (0, 1]$

$$\left| \log\left(\frac{4}{k}\right) - I(1,k) \right| \leqq 4k^2 I(1,k) \leqq 4k^2(8 + |\log k|). \tag{3.13}$$

Proof. Let

$$\Delta(k) := \log\left(\frac{4}{k}\right) - I(1,k).$$

As in Proposition 1, for $k \in (0, 1]$,

(3.14)
$$|\Delta(k)| \leqq \left| \log\left(\frac{2}{k}\right) - \log\left(\frac{1+k'}{k}\right) \right| + \left| \int_0^{\pi/2} \left[\sqrt{\frac{1-k'\sin\theta}{1+k'\sin\theta}} - \sqrt{\frac{1-\sin\theta}{1+\sin\theta}} \right] d\theta \right|.$$

We observe that, since $1 - k' = 1 - \sqrt{1-k^2} < k^2$,

$$\left| \log\left(\frac{2}{k}\right) - \log\left(\frac{1+k'}{k}\right) \right| = \left| \log\left(\frac{1+k'}{2}\right) \right| \leqq 1 - k' \leqq k^2. \tag{3.15}$$

Also, by the mean value theorem, for each θ there is a $\gamma \in [0, k]$, so that

$$\begin{aligned} 0 &\leqq \left[\sqrt{\frac{1-k'\sin\theta}{1+k'\sin\theta}} - \sqrt{\frac{1-\sin\theta}{1+\sin\theta}} \right] \\ &\leqq \left[\sqrt{\frac{1-(1-k^2)\sin\theta}{1+(1-k^2)\sin\theta}} - \sqrt{\frac{1-\sin\theta}{1+\sin\theta}} \right] \\ &= \left[\frac{\sqrt{1+(1-\gamma^2)\sin\theta}}{\sqrt{1-(1-\gamma^2)\sin\theta}} \cdot \frac{2\gamma\sin\theta}{(1+(1-\gamma^2)\sin\theta)^2} \right] k \\ &\leqq \frac{2\gamma k}{\sqrt{1-(1-\gamma^2)\sin\theta}} \leqq \frac{2k^2}{\sqrt{1-(1-k^2)\sin\theta}}. \end{aligned}$$

This yields

$$\left| \int_0^{\pi/2} \left[\sqrt{\frac{1-k'\sin\theta}{1+k'\sin\theta}} - \sqrt{\frac{1-\sin\theta}{1+\sin\theta}} \right] d\theta \right| \leqq 2k^2 \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k'\sin\theta}} \leqq 2\sqrt{2}\, k^2 I(1,k)$$

which combines with (3.14) and (3.15) to show that

$$|\Delta(k)| \leqq (1 + 2\sqrt{2}) k^2 I(1,k) \leqq 4k^2 I(1,k).$$

We finish by observing that

$$kI(1,k) \leqq \frac{\pi}{2}$$

allows us to deduce that

$$I(1, k) \leqq 2\pi k + \log\left(\frac{4}{k}\right). \qquad \square$$

Similar considerations allow one to deduce that

$$|\Delta(k) - \Delta(h)| \leqq 2\pi|k - h| \tag{3.16}$$

for $0 < k, h < 1/\sqrt{2}$.

The next proposition gives all the information necessary for computing the elementary functions from the AGM.

PROPOSITION 3. *The* AGM *satisfies the following identity (for all initial values)*:

$$\lim_{n\to\infty} 2^{-n}\frac{a_n'}{a_n}\log\left(\frac{4a_n}{c_n}\right) = \frac{\pi}{2}. \tag{3.17}$$

Proof. One verifies that

$$\begin{aligned}\frac{\pi}{2} &= \lim_{n\to\infty} a_n' I(a_0', b_0') &&\text{(by (3.9))}\\ &= \lim_{n\to\infty} a_n' I(a_{-n}', b_{-n}') &&\text{(by (3.8))}\\ &= \lim_{n\to\infty} a_n' I(2^n a_n, 2^n c_n) &&\text{(by 3.5)).}\end{aligned}$$

Now the homogeneity properties of $I(\cdot, \cdot)$ show that

$$I(2^n a_n, 2^n c_n) = \frac{2^{-n}}{a_n} I\left(1, \frac{c_n}{a_n}\right).$$

Thus

$$\frac{\pi}{2} = \lim_{n\to\infty} 2^{-n}\frac{a_n'}{a_n} I\left(1, \frac{c_n}{a_n}\right),$$

and the result follows from Proposition 1. $\square$

From now on we fix $a_0 := a_0' := 1$ and consider the iteration as a function of $b_0 := k$ and $c_0 := k'$. Let P_n and Q_n be defined by

$$P_n(k) := \left(\frac{4a_n}{c_n}\right)^{2^{1-n}}, \qquad Q_n(k) := \frac{a_n}{a_n'}, \tag{3.18}$$

and let $P(k) := \lim_{n\to\infty} P_n(k)$, $Q(k) := \lim_{n\to\infty} Q_n(k)$. Similarly let $a := a(k) := \lim_{n\to\infty} a_n$ and $a' := a'(k) := \lim_{n\to\infty} a_n'$.

THEOREM 1. *For* $0 < k < 1$ *one has*·

$$\begin{aligned}&\text{(a)}\quad P(k) = \exp(\pi Q(k)),\\ &\text{(b)}\quad 0 \leqq P_n(k) - P(k) \leqq \frac{16}{1-k^2}\left(\frac{a_n - a}{a}\right),\\ &\text{(c)}\quad |Q_n(k) - Q(k)| \leqq \frac{a'|a - a_n| + a|a' - a_n'|}{(a')^2}.\end{aligned} \tag{3.19}$$

Proof. (a) is an immediate rephrasing of Proposition 3, while (c) is straightforward.

To see (b) we observe that

$$P_{n+1} = P_n \cdot \left(\frac{a_{n+1}}{a_n}\right)^{2^{1-n}} \tag{3.20}$$

because $4a_{n+1}c_{n+1} = c_n^2$, as in (3.3). Since $a_{n+1} \leqq a_n$ we see that

$$\begin{aligned} O \leqq P_n - P_{n+1} &\leqq \left[1 - \left(\frac{a_{n+1}}{a_n}\right)^{2^{1-n}}\right] P_n \leqq \left(1 - \frac{a_{n+1}}{a_n}\right) P_0, \\ P_n - P_{n+1} &\leqq \left(\frac{a_n - a_{n+1}}{a}\right) P_0 \end{aligned} \tag{3.21}$$

since a_n decreases to a. The result now follows from (3.21) on summation. □

Thus, the theorem shows that both P and Q can be computed exponentially since (a_n) can be so calculated. In the following sections we will use this theorem to give implementable exponential algorithms for π and then for all the elementary functions.

We conclude this section by rephrasing (3.19a). By using (3.20) repeatedly we derive that

$$P = \frac{16}{1-k^2} \prod_{n=0}^{\infty} \left(\frac{a_{n+1}}{a_n}\right)^{2^{1-n}}. \tag{3.22}$$

Let us note that

$$\frac{a_{n+1}}{a_n} = \frac{a_n + b_n}{2a_n} = \frac{1}{2}\left(1 + \frac{b_n}{a_n}\right),$$

and $x_n := b_n/a_n$ satisfies the one-term recursion used by Legendre [14]

$$x_{n+1} := \frac{2\sqrt{x_n}}{x_n + 1} \qquad x_0 := k. \tag{3.23}$$

Thus, also

$$P_{n+1}(k) = \frac{16}{1-k^2} \prod_{j=0}^{n+1} \left(\frac{1 + x_j}{2}\right)^{2^{1-j}} = \left(\frac{1 + x_n}{1 - x_n}\right)^{2^{-n}}. \tag{3.24}$$

When $k := 2^{-1/2}$, $k = k'$ and one can explicitly deduce that $P(2^{-1/2}) = e^{\pi}$. When $k = 2^{-1/2}$ (3.22) is also given in [16].

4. Some interrelationships. A centerpiece of this exposition is the formula (3.17) of Proposition 3.

$$\lim_{n\to\infty} \frac{1}{2^n} \log\left(\frac{4a_n}{c_n}\right) = \frac{\pi}{2} \lim_{n\to\infty} \frac{a_n}{a_n'}, \tag{4.1}$$

coupled with the observation that both sides converge exponentially. To approximate $\log x$ exponentially, for example, we first find a starting value for which

$$\left(\frac{4a_n}{c_n}\right)^{1/2^n} \to x.$$

This we can do to any required accuracy quadratically by Newton's method. Then we compute the right limit, also quadratically, by the AGM iteration. We can compute exp analogously and since, as we will show, (4.1) holds for complex initial values we can also get the trigonometric functions.

There are details, of course, some of which we will discuss later. An obvious detail is that we require π to desired accuracy. The next section will provide an exponentially converging algorithm for π also based only on (4.1). The principle for it is very simple. If we differentiate both sides of (4.1) we lose the logarithm but keep the π!

Formula (3.10), of Proposition 1, is of some interest. It appears in King [11, pp. 13, 38] often without the "4" in the log term. For our purposes the "4" is crucial since without it (4.1) will only converge linearly (like $(\log 4)/2^n$). King's 1924 monograph contains a wealth of material on the various iterative methods related to computing elliptic integrals. He comments [11, p. 14]:

"The limit [(4.1) without the "4"] does not appear to be generally known, although an equivalent formula is given by Legendre (*Fonctions éliptiques,* t. I, pp. 94–101)."

King adds that while Gauss did not explicitly state (4.1) he derived a closely related series expansion and that none of this "appears to have been noticed by Jacobi or by subsequent writers on elliptic functions." This series [10, p. 377] gives (4.1) almost directly.

Proposition 1 may be found in Bowman [3]. Of course, almost all the basic work is to be found in the works of Abel, Gauss and Legendre [1], [10] and [14]. (See also [7].) As was noted by both Brent and Salamin, Proposition 2 can be used to estimate log given π. We know from (3.13) that, for $0 < k \leqq 10^{-3}$,

$$\left| \log\left(\frac{4}{k}\right) - I(1, k) \right| < 10k^2 |\log k|.$$

By subtraction, for $0 < x < 1$, and $n \geqq 3$,

$$|\log(x) - [I(1, 10^{-n}) - I(1, 10^{-n}x)]| < n\,10^{-2(n-1)} \tag{4.2}$$

and we can compute log exponentially from the AGM approximations of the elliptic integrals in the above formula. This is in the spirit of Newman's presentation [15]. Formula (4.2) works rather well numerically but has the minor computational drawback that it requires computing the AGM for small initial values. This leads to some linear steps (roughly $\log(n)$) before quadratic convergence takes over.

We can use (3.16) or (4.2) to show directly that π is exponentially computable. With $k := 10^{-n}$ and $h := 10^{-2n} + 10^{-n}$ (3.16) yields with (3.9) that, for $n \geqq 1$,

$$\left| \log(10^{-n} + 1) - \frac{\pi}{2}\left[\frac{1}{AG(1, 10^{-n})} - \frac{1}{AG(1, 10^{-n} + 10^{-2n})}\right] \right| \leqq 10^{1-2n}.$$

Since $|\log(x+1)/x - 1| \leqq x/2$ for $0 < x < 1$, we derive that

$$\left| \frac{2}{\pi} - \left[\frac{10^n}{AG(1, 10^{-n})} - \frac{10^n}{AG(1, 10^{-n} + 10^{-2n})}\right] \right| \leqq 10^{1-n}. \tag{4.3}$$

Newman [15] gives (4.3) with a rougher order estimate and without proof. This analytically beautiful formula has the serious computational drawback that obtaining n digit accuracy for π demands that certain of the operations be done to twice that precision.

Both Brent's and Salamin's approaches require *Legendre's relation*: for $0 < k < 1$

$$I(1, k)J(1, k') + I(1, k')J(1, k) - I(1, k)I(1, k') = \frac{\pi}{2} \tag{4.4}$$

where $J(a, b)$ is the *complete elliptic integral of the second kind* defined by

$$J(a, b) := \int_0^{\pi/2} \sqrt{a^2 \cos^2\theta + b^2 \sin^2\theta}\, d\theta.$$

The elliptic integrals of the first and second kind are related by

$$J(a_0, b_0) = \left(a_0^2 - \frac{1}{2}\sum_{n=0}^{\infty} 2^n c_n^2\right) I(a_0, b_0) \tag{4.5}$$

where, as before, $c_n^2 = a_n^2 - b_n^2$ and a_n and b_n are computed from the AGM iteration.

Legendre's proof of (4.4) can be found in [3] and [8]. His elegant elementary argument is to differentiate (4.4) and show the derivative to be constant. He then evaluates the constant, essentially by Proposition 1. Strangely enough, Legendre had some difficulty in evaluating the constant since he had problems in showing that $k^2 \log(k)$ tends to zero with k [8, p. 150].

Relation (4.5) uses properties of the ascending Landen transformation and is derived by King in [11].

From (4.4) and (4.5), noting that if k equals $2^{-1/2}$ then so does k', it is immediate that

$$\pi = \frac{[2AG(1, 2^{-1/2})]^2}{1 - \sum_{n=1}^{\infty} 2^{n+1} c_n^2}. \tag{4.6}$$

This concise and surprising exponentially converging formula for π is used by both Salamin and Brent. As Salamin points out, by 1819 Gauss was in possession of the AGM iteration for computing elliptic integrals of the first kind and also formula (4.5) for computing elliptic integrals of second kind. Legendre had derived his relation (4.4) by 1811, and as Watson puts it [20, p. 14] "in the hands of Legendre, the transformation [(3.23)] became a most powerful method for computing elliptic integrals." (See also [10], [14] and the footnotes of [11].) King [11, p. 39] derives (4.6) which he attributes, in an equivalent form, to Gauss. It is perhaps surprising that (4.6) was not suggested as a practical means of calculating π to great accuracy until recently.

It is worth emphasizing the extraordinary similarity between (1.1) which leads to linearly convergent algorithms for all the elementary functions, and (3.1) which leads to exponentially convergent algorithms.

Brent's algorithms for the elementary functions require a discussion of incomplete elliptic integrals and the Landen transform, matters we will not pursue except to mention that some of the contributions of Landen and Fagnano are entertainingly laid out in an article by G.N. Watson entitled "The Marquis [Fagnano] and the Land Agent [Landen]" [20]. We note that Proposition 1 is also central to Brent's development though he derives it somewhat tangentially. He also derives Theorem 1(a) in different variables via the Landen transform.

5. An algorithm for π. We now present the details of our exponentially converging algorithm for calculating the digits of π. Twenty iterations will provide over two million digits. Each iteration requires about ten operations. The algorithm is very stable with all the operations being performed on numbers between ½ and 7. The eighth iteration, for example, gives π correctly to 694 digits.

THEOREM 2. *Consider the three-term iteration with initial values*

$$\alpha_0 := \sqrt{2}, \quad \beta_0 := 0, \quad \pi_0 := 2 + \sqrt{2}$$

given by

(i) $\alpha_{n+1} := \frac{1}{2}(\alpha_n^{1/2} + \alpha_n^{-1/2})$,

(ii) $\beta_{n+1} := \alpha_n^{1/2}\left(\dfrac{\beta_n + 1}{\beta_n + \alpha_n}\right)$,

(iii) $\pi_{n+1} := \pi_n\beta_{n+1}\left(\dfrac{1 + \alpha_{n+1}}{1 + \beta_{n+1}}\right)$.

Then π_n converges exponentially to π and

$$|\pi_n - \pi| \leq \frac{1}{10^{2^n}}.$$

Proof. Consider the formula

$$\frac{1}{2^n}\log\left(4\frac{a_n}{c_n}\right) - \frac{\pi}{2}\frac{a_n}{a'_n} \tag{5.1}$$

which, as we will see later, converges exponentially at a uniform rate to zero in some (complex) neighbourhood of $1/\sqrt{2}$. (We are considering each of a_n, b_n, c_n, a'_n, b'_n, c'_n as being functions of a complex initial value k, i.e. $b_0 = k$, $b'_0 = \sqrt{1 - k^2}$, $a_0 = a'_0 = 1$.)

Differentiating (5.1) with respect to k yields

$$\frac{1}{2^n}\left(\frac{\dot{a}_n}{a_n} - \frac{\dot{c}_n}{c_n}\right) - \frac{\pi}{2}\frac{a_n}{a'_n}\left(\frac{\dot{a}_n}{a_n} - \frac{\dot{a}'_n}{a'_n}\right) \tag{5.2}$$

which also converges uniformly exponentially to zero in some neighbourhood of $1/\sqrt{2}$. (This general principle for exponential convergence of differentiated sequences of analytic functions is a trivial consequence of the Cauchy integral formula.) We can compute $\dot{a}_n$, $\dot{b}_n$ and $\dot{c}_n$ from the recursions

$$\begin{aligned} \dot{a}_{n+1} &:= \frac{\dot{a}_n + \dot{b}_n}{2}, \\ \dot{b}_{n+1} &:= \frac{1}{2}\left(\dot{a}_n\sqrt{\frac{b_n}{a_n}} + \dot{b}_n\sqrt{\frac{a_n}{b_n}}\right), \\ \dot{c}_{n+1} &:= \frac{1}{2}(\dot{a}_n - \dot{b}_n), \end{aligned} \tag{5.3}$$

where $\dot{a}_0 := 0$, $\dot{b}_0 := 1$, $a_0 := 1$ and $b_0 := k$.

We note that a_n and b_n map $\{z \mid Re(z) > 0\}$ into itself and that $\dot{a}_n$ and $\dot{b}_n$ (for sufficiently large n) do likewise.

It is convenient to set

$$\alpha_n := \frac{a_n}{b_n} \quad \text{and} \quad \beta_n := \frac{\dot{a}_n}{\dot{b}_n} \tag{5.4}$$

with

$$\alpha_0 := \frac{1}{k} \quad \text{and} \quad \beta_0 := 0.$$

We can derive the following formulae in a completely elementary fashion from the basic relationships for a_n, b_n and c_n and (5.3):

$$\dot{a}_{n+1} - \dot{b}_{n+1} = \frac{1}{2}(\sqrt{a_n} - \sqrt{b_n})\left(\frac{\dot{a}_n}{\sqrt{a_n}} - \frac{\dot{b}_n}{\sqrt{b_n}}\right), \tag{5.5}$$

$$(5.6)\qquad 1-\frac{a_{n+1}}{\dot{a}_{n+1}}\frac{\dot{c}_{n+1}}{c_{n+1}}=\frac{2(\alpha_n-\beta_n)}{(\alpha_n-1)(\beta_n+1)},$$

$$(5.7)\qquad \alpha_{n+1}=\frac{1}{2}(\alpha_n^{1/2}+\alpha_n^{-1/2}),$$

$$(5.8)\qquad \beta_{n+1}=\alpha_n^{1/2}\left(\frac{\beta_n+1}{\beta_n+\alpha_n}\right),$$

$$(5.9)\qquad \alpha_{n+1}-1=\frac{1}{2\alpha_n^{1/2}}(\alpha_n^{1/2}-1)^2,$$

$$(5.10)\qquad \alpha_{n+1}-\beta_{n+1}=\frac{\alpha_n^{1/2}}{2}\frac{(1-\alpha_n)(\beta_n-\alpha_n)}{\alpha_n(\beta_n+\alpha_n)},$$

$$(5.11)\qquad \frac{\alpha_{n+1}-\beta_{n+1}}{\alpha_{n+1}-1}=\frac{(1+\alpha_n^{1/2})^2}{(\beta_n+\alpha_n)}\cdot\frac{(\alpha_n-\beta_n)}{(\alpha_n-1)}.$$

From (5.7) and (5.9) we deduce that $\alpha_n \to 1$ uniformly with second order in compact subsets of the open right half-plane. Likewise, we see from (5.8) and (5.10) that $\beta_n \to 1$ uniformly and exponentially. Finally, we set

$$(5.12)\qquad \gamma_n := \frac{1}{2^n}\left(\frac{\alpha_n-\beta_n}{\alpha_n-1}\right).$$

We see from (5.11) that

$$(5.13)\qquad \gamma_{n+1}=\frac{(1+\alpha_n^{1/2})}{2(\beta_n+\alpha_n)}\gamma_n$$

and also from (5.6) that

$$(5.14)\qquad \frac{\gamma_n}{1+\beta_n}=\frac{1}{2^{n+1}}\left(1-\frac{a_{n+1}}{\dot{a}_{n+1}}\frac{\dot{c}_{n+1}}{c_{n+1}}\right).$$

Without any knowledge of the convergence of (5.1) one can, from the preceding relationships, easily and directly deduce the exponential convergence of (5.2), in $\{z \,||z-\tfrac{1}{2}| \leqq c < \tfrac{1}{2}\}$. We need the information from (5.1) only to see that (5.2) converges to zero.

The algorithm for π comes from multiplying (5.2) by $a_n/\dot{a}_n$ and starting the iteration at $k := 2^{-1/2}$. For this value of k $a'_n = a_n$, $(\dot{a}'_n) = -\dot{a}_n$ and

$$\frac{1}{2^{n+1}}\left(1-\frac{a_{n+1}}{\dot{a}_{n+1}}\frac{\dot{c}_{n+1}}{c_{n+1}}\right)\to\pi$$

which by (5.14) shows that

$$\pi_n := \frac{\gamma_n}{1+\beta_n}\to\pi.$$

Some manipulation of (5.7), (5.8) and (5.13) now produces (iii). The starting values for α_n, β_n and γ_n are computed from (5.4). Other values of k will also lead to similar, but slightly more complicated, iterations for π.

To analyse the error one considers

$$\frac{\gamma_{n+1}}{1+\beta_{n+1}}-\frac{\gamma_n}{1+\beta_n}=\left[\frac{(1+\alpha_n^{1/2})^2}{2(\beta_n+\alpha_n)(1+\beta_{n+1})}-\frac{1}{(1+\beta_n)}\right]\gamma_n$$

and notes that, from (5.9) and (5.10), for $n \geqq 4$,

$$|\alpha_n - 1| \leqq \frac{1}{10^{2^n+2}} \quad \text{and} \quad |\beta_n - 1| \leqq \frac{1}{10^{2^n+2}}.$$

(One computes that the above holds for $n = 4$.) Hence,

$$\left|\frac{\gamma_{n+1}}{1+\beta_{n+1}} - \frac{\gamma_n}{1+\beta_n}\right| \leqq \left|\frac{1}{10^{2^n+1}}\right| |\gamma_n|$$

and

$$\left|\frac{\gamma_n}{1+\beta_n} - \pi\right| \leqq \frac{1}{10^{2^n}}. \qquad \square$$

In fact one can show that the error is of order $2^n e^{-\pi 2^{n+1}}$.

If we choose integers in $[\delta, \delta^{-1}]$, $0 < \delta < 1/2$ and perform n operations $(+, -, \times, \div, \sqrt{\ })$ then the result is always less than or equal to δ^{2^n}. Thus, if $\gamma > \delta$, it is not possible, using the above operations and integral starting values in $[\delta, \delta^{-1}]$, for every n to compute π with an accuracy of $O(\gamma^{-2^n})$ in n steps. In particular, convergence very much faster than that provided by Theorem 2 is not possible.

The analysis in this section allows one to derive the Gauss-Salamin formula (4.6) without using Legendre's formula or second integrals. This can be done by combining our results with problems 15 and 18 in [11]. Indeed, the results of this section make quantitative sense of problems 16 and 17 in [11]. King also observes that Legendre's formula is actually equivalent to the Gauss–Salamin formula and that each may be derived from the other using only properties of the AGM which we have developed and equation (4.5).

This algorithm, like the algorithms of §4, is not self correcting in the way that Newton's method is. Thus, while a certain amount of time may be saved by observing that some of the calculations need not be performed to full precision it seems intrinsic (though not proven) that $O(\log n)$ full precision operations must be executed to calculate π to n digits. In fact, showing that π is intrinsically more complicated from a time complexity point of view than multiplication would prove that π is transcendental [5].

6. The complex AGM iteration. The AGM iteration

$$a_{n+1} := \frac{1}{2}(a_n + b_n), \qquad b_{n+1} := \sqrt{a_n b_n}$$

is well defined as a complex iteration starting with $a_0 := 1$, $b_0 := z$. Provided that z does not lie on the negative real axis, the iteration will converge (to what then must be an analytic limit). One can see this geometrically. For initial z in the right half-plane the limit is given by (3.9). It is also easy to see geometrically that a_n and b_n are always nonzero.

The iteration for $x_n := b_n/a_n$ given in the form (3.23) as $x_{n+1} := 2\sqrt{x_n}/x_{n+1}$ satisfies

$$(6.1) \qquad (x_{n+1} - 1) = \frac{(1-\sqrt{x_n})^2}{1+x_n}.$$

This also converges in the cut plane $\mathbb{C} - (-\infty, 0]$. In fact, the convergence is uniformly exponential on compact subsets (see Fig. 1). With each iteration the angle θ_n between x_n and 1 is at least halved and the real parts converge uniformly to 1.

It is now apparent from (6.1) and (3.24) that

$$(6.2) \qquad P_n(k) := \left(\frac{4a_n}{c_n}\right)^{2^{1-n}} = \left(\frac{1+x_n}{1-x_n}\right)^{2^{-n}}$$

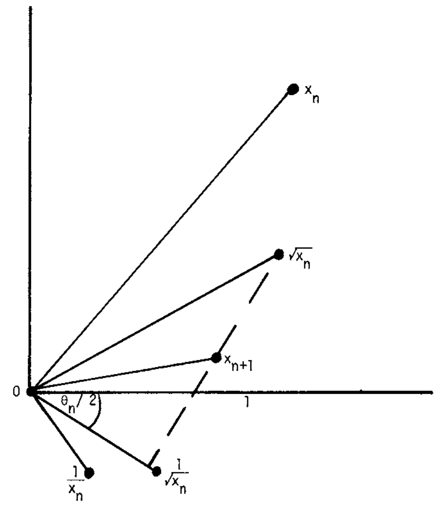

FIG. 1.

and also,

$$Q_n(k) := \frac{a_n}{a_n'}$$

converge exponentially to analytic limits on compact subsets of the complex plane that avoid

$$D := \{z \in \mathbb{C} \mid z \notin (-\infty, 0] \cup [1, \infty)\}.$$

Again we denote the limits by P and Q. By standard analytic reasoning it must be that (3.19a) still holds for k in D.

Thus one can compute the complex exponential—and so also cos and sin—exponentially using (3.19). More precisely, one uses Newton's method to approximately solve $Q(k) = z$ for k and then computes $P_n(k)$. The outcome is e^z. One can still perform the root extractions using Newton's method. Some care must be taken to extract the correct root and to determine an appropriate starting value for the Newton inversion. For example $k := 0.02876158$ yields $Q(k) = 1$ and $P_4(k) = e$ to 8 significant places. If one now uses k as an initial estimate for the Newton inversions one can compute $e^{1+i\theta}$ for $|\theta| \leq \pi/8$. Since, as we have observed, e is also exponentially computable we have produced a sufficient range of values to painlessly compute $\cos\theta + i\sin\theta$ with no recourse to any auxiliary computations (other than π and e, which can be computed once and stored). By contrast Brent's trigonometric algorithm needs to compute a different logarithm each time.

The most stable way to compute P_n is to use the fact that one may update c_n by

$$c_{n+1} = \frac{c_n^2}{4a_{n+1}}. \tag{6.3}$$

One then computes a_n, b_n and c_n to desired accuracy and returns

$$\left(\frac{4a_n}{c_n}\right)^{1/2^n} \quad \text{or} \quad \left(\frac{2(a_n + b_n)}{c_n}\right)^{1/2^n}.$$

This provides a feasible computation of P_n, and so of exp or log.

In an entirely analogous fashion, formula (4.2) for log is valid in the cut complex plane. The given error estimate fails but the convergence is still exponential. Thus (4.2) may also be used to compute all the elementary functions.

7. Concluding remarks and numerical data. We have presented a development of the AGM and its uses for rapidly computing elementary functions which is, we hope, almost entirely self-contained and which produces workable algorithms. The algorithm for π is particularly robust and attractive. We hope that we have given something of the flavour of this beautiful collection of ideas, with its surprising mixture of the classical and the modern. An open question remains. Can one entirely divorce the central discussion from elliptic integral concerns? That is, can one derive exponential iterations for the elementary functions without recourse to some nonelementary transcendental functions? It would be particularly nice to produce a direct iteration for e of the sort we have for π which does not rely either on Newton inversions or on binary splitting.

The algorithm for π has been run in an arbitrary precision integer arithmetic. (The algorithm can be easily scaled to be integral.) The errors were as follows:

Iterate	Digits correct	Iterate	Digits correct
1	3	6	170
2	8	7	345
3	19	8	694
4	41	9	1392
5	83	10	2788

Formula (4.2) was then used to compute 2 log (2) and log (4), using π estimated as above and the same integer package. Up to 500 digits were computed this way. It is worth noting that the error estimate in (4.2) is of the right order.

The iteration implicit in (3.22) was used to compute e^{π} in a double precision Fortran. Beginning with $k := 2^{-1/2}$ produced the following data:

Iterate	$P_n - e^{\pi}$	$a_n/b_n - 1$
1	1.6×10^{-1}	1.5×10^{-2}
2	2.8×10^{-9}	2.8×10^{-5}
3	1.7×10^{-20}	9.7×10^{-11}
4	$< 10^{-40}$	1.2×10^{-21}

Identical results were obtained from (6.3). In this case $y_n := 4a_n/c_n$ was computed by the two term recursion which uses x_n, given by (3.23), and

$$y_0^2 := \frac{16}{1-k^2}, \qquad y_{n+1} = \left(\frac{1+x_n}{2}\right)^2 y_n^2. \tag{7.1}$$

One observes from (7.1) that the calculation of y_n is very stable.

We conclude by observing that the high precision root extraction required in the AGM [18], was actually calculated by inverting $y = 1/x^2$. This leads to the iteration

$$x_{n+1} = \frac{3x_n - x_n^3 y}{2} \tag{7.2}$$

for computing $y^{-1/2}$. One now multiplies by y to recapture $\sqrt{y}$. This was preferred because it avoided division.

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003

ON THE COMPLEXITY OF FAMILIAR FUNCTIONS AND NUMBERS*

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Abstract. This paper examines low-complexity approximations to familiar functions and numbers. The intent is to suggest that it is possible to base a taxonomy of such functions and numbers on their computational complexity. A central theme is that traditional methods of approximation are often very far from optimal, while good or optimal methods are often very far from obvious. For most functions, provably optimal methods are not known; however the gap between what is known and what is possible is often small. A considerable number of open problems are posed and a number of related examples are presented.

Key words. elementary functions, pi, low-complexity approximation, reduced-complexity approximation, rational approximation, algebraic approximation, computation of digits, open problems

AMS(MOS) subject classifications. 68C25, 41A30, 10A30

1. Introduction. We examine various methods for evaluating familiar functions and numbers to high precision. Primarily, we are interested in the asymptotic behavior of these methods. The kinds of questions we pose are:

(1) How much work (by various types of computational or approximation measures) is required to evaluate n digits of a given function or number?

(2) How do analytic properties of a function relate to the efficacy with which it can be approximated?

(3) To what extent are analytically simple numbers or functions also easy to compute?

(4) To what extent is it easy to compute analytically simple functions?

Even partial answers to these questions are likely to be very difficult. Some, perhaps easier, specializations of the above are:

(5) Why is the function $\sqrt{x}$ easier to compute than exp? Why is it only marginally easier?

(6) Why is the Taylor series often the wrong way to compute familiar functions?

(7) Why is the number $\sqrt{2}$ easier to compute than e or π? Why is it only marginally easier?

(8) Why is the number $.1234567891011\cdots$ computationally easier than π or e?

(9) Why is computing just the nth digit of $\exp(x)$ really no easier than computing all the first n digits?

(10) Why is computing just the nth digit of π really no easier than computing all the first n digits?

Answers to (7) and (10) are almost certainly far beyond the scope of current number-theoretic techniques. Partial answers to some of the remaining questions are available.

The traditional way to compute elementary functions, such as exp or log, is to use a partial sum of the Taylor series or a related polynomial or rational approximation. These are analytically tractable approximations, and over the class of such

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approximations are often optimal or near optimal. For example, the nth partial sums to exp are asymptotically the best polynomial approximations in the uniform norm on the unit disc in the complex plane, in the sense that if s_n is the nth partial sum of the Taylor expansion and p_n is any polynomial of degree n, then for large n,

$$\|\exp(z) - s_n(z)\|_D < \left[1 + \frac{4}{n}\right] \|\exp(z) - p_n(z)\|_D.$$

Here $\| \ \|_D$ denotes the supremum norm over the unit disc in the complex plane (see [5]). If the measure of the amount of work is the degree of the approximation, as it has been from a conventional point of view, then the story for exp might end here.

Questions (1)–(3) above have a very elegant answer for polynomial approximation in the form of the Bernstein–Jackson theorems [11]. These, for example, tell us that a function is entire if and only if the error in best uniform polynomial approximation of degree n on an interval tends to zero faster than geometrically, with a similar exact differentiability classification of a function in terms of the rate of polynomial approximation.

If we wish to compute n digits of $\log(x)$ using a Taylor polynomial then we employ a polynomial of degree n and perform $O(n)$ rational operations, while for $\exp(x)$ we require $O(n/\log n)$ rational operations to compute n digits. The slight improvement for exp reflects the faster convergence rate of the Taylor series. Padé approximants, best rational approximants and best polynomial approximants all behave in roughly the same fashion, except that the constants implicit in the order symbol change [5], [8].

A startling observation is that there exist rational functions that give n digits of log, exp, or any elementary function but require only $O((\log n)^k)$ rational operations to evaluate. These approximants are of degree $O(n)$ but can be evaluated in $O((\log n)^k)$ infinite-precision arithmetic operations. The simplest example of such a function is x^n which can be evaluated in $O(\log n)$ arithmetic operations by repeated squaring. While we cannot very explicitly construct these low-rational-complexity approximations to exp or log, it is clear that much of their simplicity results from squarings of intermediate terms. The moral is that it is appropriate and useful to view x^n as having the complexity of a general polynomial of degree $\log n$, not of degree n.

The existence of such approximants is a consequence of the construction of low-bit-complexity algorithms for log and π resting on the Arithmetic-Geometric Mean (AGM) iteration of Gauss, Lagrange, and Legendre (see §2 for definitions). These algorithms were discovered and examined by Beeler, Gosper, and Schroeppel [3], Brent [9], and Salamin [21] in the 1970s. A complete exposition is available in [5]. These remarkable algorithms are both theoretically and practically faster than any of the traditional methods for extended precision evaluation of elementary functions. The exact point at which they start to outperform the usual series expansions depends critically on implementation; the switchover comes somewhere in the 100- to 1000-digit range.

The main purpose of this paper is to catalogue the known results on complexity of familiar functions. We now appear to know enough structure to at least speculate on the existence of a reasonable taxonomy of functions based on their computational complexity. Here we have in mind something that relates computational properties of functions to their analytic or algebraic properties, something vaguely resembling the Bernstein–Jackson theorems in the polynomial case.

Likewise we would like to suggest the possibility of a taxonomy of numbers based on their computational nature. Here, we are looking for something that resembles

Mahler's classification of transcendendentals in terms of their rate of algebraic approximation [15].

It is not our intention to provide a taxonomy; this must await further progress in the field. We do hope, however, to present enough examples and pose enough interesting questions to persuade the reader that it is fruitful to pursue such an end.

2. Definitions. We consider four notions of complexity.

(1) *Rational complexity.* We say that a function f has *rational complexity* $O_{\text{rat}}(s(n))$ on a set A if there exists a sequence of rational functions R_n so that

(a) $|R_n(x) - f(x)| < 10^{-n}$ for all $x \in A$;

(b) asymptotically, R_n can be evaluated using no more than $O(s(n))$ rational operations (i.e., infinite-precision additions, subtractions, multiplications, and divisions).

That exp has rational complexity $O_{\text{rat}}(\log^3 n)$ means that there is a sequence of rational functions, the nth being evaluable in roughly $\log^3 n$ arithmetic operations, giving an n-digit approximation to exp. The subscript on the order symbol is for emphasis.

We will sometimes use Ω and Ω_{rat} as the lower bound order symbols. Whenever we talk about "n-digit precision" or "computing n digits" we mean computing to an accuracy of 10^{-n}.

(2) *Algebraic complexity.* We say that a function f has *algebraic complexity* $O_{\text{alg}}(s(n))$ on a set A if there exists a sequence of algebraic functions A_n so that

(a) $|A_n(x) - f(x)| < 10^{-n}$ for all $x \in A$;

(b) asymptotically, all the A_n can be evaluated using no more than $O(s(n))$ algebraic operations (i.e., infinite-precision solutions of a fixed number of prespecified algebraic equations).

This algebraic complexity measure allows us, for example, to use square root extractions in the calculation of the approximants and to count them on an equal footing with the rational operations. This is often appropriate because, from a bit-complexity point of view, root extraction is equivalent to multiplication (see §4). Note that we allow only a finite number of additional algebraic operations—so while we might allow for computing square roots, cube roots, and seventeenth roots, we would not allow an infinite number of different orders of roots.

Neither of the above measures takes account of the fact that low-precision operations are easier than high-precision operations.

(3) *Bit complexity.* We say that a function f has *bit complexity* $O_{\text{bit}}(s(n))$ on a set A if there exists a sequence of approximations B_n so that

(a) $|B_n(x) - f(x)| < 10^{-n}$ for all $x \in A$;

(b) B_n is the output of an algorithm (given input n and x) that evaluates the B_n to n-digit accuracy using $O(s(n))$ *single-digit* operations ($+, -, \times$).

This is the appropriate measure of time complexity on a serial machine. (See [1] for more formal definitions.)

We wish to capture in the next definition the notion of how complex it is to compute only the nth digit of a function.

(4) *Digit complexity.* We say that a function f has *digit complexity* $O_{\text{dig}}(s(n))$ on a set A if there exists a sequence of approximations D_n so that

(a) $D_n(x)$ gives the nth digit of $f(x)$. By this we mean that $D_n(x)$ differs from the n through $(n+k)$th digits of $f(x)$ by at most 10^{-k} for any preassigned fixed k;

(b) D_n is the output of an algorithm (given input n and x) that evaluates the D_n to k digits using $O(s(n))$ *single-digit* operations.

This definition of agreement of nth digits takes account of the fact that sequences of repeated nines can occur. We really want to say that .19999⋯ and .2000⋯ agree in the first digit. As it stands, the definition above exactly computes only the nth digit to a probability dependent on k.

It is also assumed that accessing the kth through nth digit of input of x is an $O_{\text{bit}}(\max(n-k, \log k))$ operation, so that accessing the first n digits is $O_{\text{bit}}(n)$ while accessing just the nth bit is $O_{\text{bit}}(\log n)$.

Addition is $O_{\text{rat}}(1)$, $O_{\text{alg}}(1)$, $O_{\text{bit}}(n)$, and $O_{\text{dig}}(\log n)$. Here we take the set A, where we seek a uniform algorithm, to be the unit square in $\mathbb{R}^2$. The usual addition algorithm gives the upper bounds shown above. Addition is one of the very few cases where we know the exact result. Trivial uniqueness considerations show that addition is $\Omega_{\text{bit}}(n)$, and hence all the above orders are exact.

It comes as a major surprise of this side of theoretical computer science that the usual way of multiplying is far from optimal from a bit-complexity point of view. The usual multiplication algorithm has bit complexity $\Omega_{\text{bit}}(n^2)$. However, it is possible to construct a multiplication which is $O_{\text{bit}}(n \log n \log\log n)$. This is based on the Fast Fourier Transform and is due to Schönhage and Strassen (see [1], [16]). The extent to which the log terms are necessary is not known. Given a standard model of computation the best known lower bound is the trivial one, $\Omega_{\text{bit}}(n)$. We will denote the *bit complexity of multiplication* by $M(n)$.

3. A table of results. The state of our current knowledge is contained in Table 1. The orders of the various measures of complexities for computing n digits (or in the final case the nth digit) compose the columns. In each case, except addition, the only upper bound we know for the digit complexity is the same as the bit-complexity bound. When we deal with functions, we assume that we are on a compact region of the domain of the given function that is bounded away from any singularities and that contains an interval. Numbers may be considered as functions whose domain is a singleton.

For our purposes *hypergeometric functions* are functions of the form

$$f(x) := \sum a_n x^n \quad \text{where } a_n/a_{n-1} = R(n)$$

and R is a fixed rational function (with coefficients in $\mathbb{Q}$).

TABLE 1

Type of function	O_{rat}	O_{alg}	O_{bit}	Ω_{dig}
(1) Addition	1	1	n	$\log n$
(2) Multiplication	1	1	$n \log n \log\log n$	n
(3) Algebraic (nonlinear)	$\log n$	1	$M(n)$	n
(4) log (complete elliptic integrals)	$\log^2 n$	$\log n$	$(\log n)M(n)$	n
(5) exp	$\log^3 n$	$\log^2 n$	$(\log n)M(n)$	n
(6) Elementary (nonlinear)	$\log^k n$	$\log^k n$	$(\log n)M(n)$	n
(7) Hypergeometric (over $\mathbb{Q}$)	$n^{1/2+}$	$n^{1/2+}$	$(\log^2 n)M(n)$	n
(8) Gamma and zeta	$n^{1/2+}$	$n^{1/2+}$	$n^{1/2+}M(n)$	n
(9) Gamma and zeta on $\mathbb{Q}$	$n^{1/2+}$	$n^{1/2+}$	$(\log^2 n)M(n)$	$\log n$
(10) pi, log (2), $\Gamma(\frac{1}{3})$	$\log^2 n$	$\log n$	$(\log n)M(n)$	$\log n$
(11) Euler's constant (Catalan's constant)	$n^{1/2+}$	$n^{1/2+}$	$(\log^2 n)M(n)$	$\log n$

Elementary functions are functions built from rational functions (with rational coefficients) exp and log by any number of additions, multiplications, compositions, and solutions of algebraic equations.

A number of techniques are employed in deriving Table 1. Our intention is to indicate the most useful of these without going into too much detail. The next four sections outline the derivations of most of the bounds.

4. Newton's method. The calculation of algebraic functions, given that we have algorithms for addition and multiplication, is entirely an exercise in applying Newton's method to solving equations of the form $f(x)-y=0$. Newton's method for $1/x-y=0$ gives the iteration

(a) $x_{n+1}:=2x_n-yx_n^2$,

while for $x^2-y=0$ the iteration is

(b) $x_{n+1}:=(x_n+y/x_n)/2$.

These two iterations converge quadratically. Thus $O(\log n)$ iterations give n digits of $1/y$ and $\sqrt{y}$, respectively, and we have given an $O_{\text{rat}}(\log n)$ algorithm for square root extraction.

The quadratic rate of convergence is only half the story. Because Newton's method is *self-correcting*, in the sense that a small perturbation in x_n does not change the limit, it is possible to start with a single-digit estimate and double the precision with each iteration. Thus the bit complexity of root extraction is

$$O(M(1)+M(2)+M(4)+\cdots+M(n))=O(M(n)).$$

This leads to $O_{\text{bit}}(M(n))$ algorithms for root extraction and division, and a similar analysis works for any algebraic function. This explains most of (1)–(3) in Table 1. We also have the interesting result that the computation of digits of any algebraic number is asymptotically no more complicated than multiplication. (These results on the complexity of algebraic functions may be found in [5] and [9].)

The approximation in (a), x_n, is in fact the (2^n-1)st Taylor polynomial to $1/y$ at 1. In (b), x_n is in fact the $(2^n, 2^n-1)$st Padé approximant to $\sqrt{y}$ at 1. (See [5] or [11] for further material on Padé approximants.) This is one of the very few cases where Newton's method generates familiar approximants.

Newton's method is also useful for inverting functions. The inverse of f is computed from the iteration

$$x_{n+1}:=x_n-[f(x_n)-y]/f'(x_n).$$

For any reasonable f this gives the same bit complexity estimate for f^{-1} as for f. Inverting by Newton's method multiplies the rational and algebraic complexities by $\log n$.

5. The AGM. The two-term iteration with starting values $a_0:=x\in(0,1]$ and $b_0:=1$ given by

$$a_{n+1}:=(a_n+b_n)/2, \qquad b_{n+1}:=\sqrt{(a_n\cdot b_n)}$$

converges quadratically to $m(1,x)$, where

$$\frac{1}{m(1,x)}=\frac{2}{\pi}\int_0^{\pi/2}\frac{dt}{\sqrt{1-(1-x^2)\sin^2 t}}.$$

This is the arithmetic-geometric mean iteration of Gauss, Lagrange, and Legendre. This latter complete elliptic integral is $2K'(x)/\pi$ and is a nonelementary transcendental function with complexity

$$O_{\text{alg}}(\log(n)), \quad O_{\text{rat}}(\log^2(n)), \quad O_{\text{bit}}(\log(n)M(n)).$$

It is also essentially the only identifiable nonelementary limit of a quadratically converging fixed iteration and as such is of central importance [5].

One way to get a low complexity algorithm for log is to use the logarithmic asymptote of K' at 0. This gives the estimate

$$|(2/\pi)\log x - 1/m(1,10^{-n}) + 1/m(1,x10^{-n})| < n10^{-2(n-1)}, \quad n>3, \quad x\in[.5,1].$$

Up to computing π, this allows for the derivation of algorithms with the complexity of entry (4) in Table 1. Algorithms for π can be derived from the same kinds of considerations (see [4], [5], [9], [18], [21]). Probably the fastest known algorithm for π is the quartic example given below [5], [2].

ALGORITHM. Let $\alpha_0 := 6 - 4\sqrt{2}$ and $y_0 := \sqrt{2} - 1$. Let

$$y_{n+1} := [1-(1-y_n^4)^{1/4}]/[1+(1-y_n^4)^{1/4}]$$

and

$$\alpha_{n+1} := (1+y_{n+1})^4\alpha_n - 2^{2n+3}y_{n+1}(1+y_{n+1}+y_{n+1}^2).$$

Then $1/\alpha_n$ tends to π quartically and

$$0 < \alpha_n - \frac{1}{\pi} < 16 \cdot 4^n \exp(-2 \cdot 4^n \pi).$$

The exponential function may be derived from log by inverting using Newton's method. This continues to work for appropriate complex values. The elementary functions are now built from log and exp and the solution of algebraic equations in these quantities. The constant k in the rational- and bit-complexity estimates depends on the number of these equations that require solution. This explains entries (5), (6), and (10) in Table 1, except for $\Gamma(\frac{1}{3})$. (This and a few other values of Γ arise as algebraic combinations of complete elliptic integrals and pi.) (Substantial additional material on this section is to be found in [5].)

6. FFT methods. The Fast Fourier Transform (FFT) is a way of solving the following two problems:

(a) Given the coefficients of a polynomial of degree $n-1$, evaluate the polynomial at all n of the nth roots of unity.

(b) Given the values of a polynomial of degree $n-1$ at the nth roots of unity, compute the coefficients of the polynomial.

These two problems are actually equivalent (see [1], [5], [16]). The important observation made by Cooley and Tukey in the 1960s is that both of these problems are solvable with rational complexity $O_{\text{rat}}(n\log n)$, rather than the complexity of $\Omega_{\text{rat}}(n^2)$ that the usual methods require (i.e., Horner's method). This is an enormously useful algorithm.

We can multiply two polynomials of degree n with complexity $O_{\text{rat}}(n\log n)$ by using the FFT three times. First we compute the values of the two polynomials at $2n+1$ roots of unity. Then we work out the coefficients of the polynomial of degree $2n$ that agrees with the product at these roots.

Variations on this technique allow for the evaluation of a rational function of degree n at n points in $O_{\text{rat}}(n\log^2 n)$ and $O_{\text{bit}}(n\log^2 n\, M(k))$, where k is the precision to which we are working [5].

Fast multiplications are constructed by observing that multiplication of numbers is much like multiplication of polynomials whose coefficients are the digits, the additional complication being the "carries."

How does this give reduced-complexity algorithms? We illustrate with $\log(1-x)$. Let

$$s_{n^2}(x) := \sum_{k=1}^{n^2} \frac{x^k}{k}$$

and write

$$s_{n^2}(x) = \sum_{k=0}^{n-1} x^k p(kn) \quad \text{where}\, p(y) := \sum_{j=1}^{n} \frac{x^j}{j+y}.$$

Now evaluate $p(0)$, $p(n)$, $\cdots$, $pn(n-1)$ using FFT methods, and then evaluate s_{n^2}. This gives an $O_{\text{rat}}(n^{1/2}(\log n)^2)$ and $O_{\text{bit}}(n^{1/2}(\log n)^2 M(n))$ algorithm for log. At any fixed rational value r, we get an $O_{\text{bit}}((\log n)^2 M(n))$ for $\log r$. For this final estimate we must take advantage of the reduced precision possible for intermediate calculations.

This is not as good an estimate as the AGM estimates for log. It is, however, a much more generally applicable method. We can orchestrate the calculation, much as above, for any hypergeometric function. This is how the estimates in line (7) in Table 1 are deduced. Schroeppel [3], [22] shows how a similar circle of ideas can be used to give $O_{\text{bit}}(\log^k n\, M(n))$ algorithms for the solutions of linear differential equations whose coefficients are rational functions with coefficients in $\mathbb{Q}$.[1]

The gamma function, Γ, can be computed from the estimate

$$\left| \Gamma(x) - N^x \sum_{k=0}^{6N} \frac{(-1)^k N^k}{k!(x+k)} \right| < 2Ne^{-N}, \qquad x \in [1,2]$$

(see [5] for details). The zeta function, ζ, is then computable from Riemann's integral [24]:

$$\zeta(x)\Gamma\left(\frac{x}{2}\right)\pi^{-x/2} - \frac{1}{x(x-1)} = \int_1^\infty \frac{t^{(1-x)/2} + t^{x/2}}{t} \sum_{n=1}^{\infty} e^{-n^2\pi t}\, dt.$$

We truncate both the integral and the sum. These two formulae explain lines (8) and (9) of Table 1.

Catalan's constant

$$G := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$$

can be computed from Ramanujan's sum

$$\frac{8}{3}G = \frac{\pi}{3}\log(2+\sqrt{3}) + \sum_{m=0}^{\infty} \frac{m!\,m!}{(2m+1)^2(2m)!},$$

while Euler's constant, γ, can be computed from the asymptotic expansion

$$\gamma = -\log x - \sum_{k=1}^{\infty} \frac{(-x)^k}{k \cdot k!} + O(\exp(-x)), \qquad x > 1.$$

[1] Chudnovsky and Chudnovsky [26] provide a low-bit complexity approach to solutions of linear differential equations in [26].

This gives line (11) of Table 1. Some of the details may be found in [5] and [6].

A variation of the above method for computing γ has been used by Brent and McMillan [10] to compute over 29,000 partial quotients of the continued fraction of γ. From this computation it follows that if γ is rational its denominator exceeds $10^{15,000}$.

7. Digit complexity. The aim of this section is to explain the last column in Table 1. The main observation is that the digit complexity of computing the mth digit ($m \leqq n$) of the product of two n-digit numbers is $\Omega_{\text{dig}}(m)$. This is essentially just a uniqueness argument the details of which may be pursued in [7].

Now suppose that f is analytic around zero (C^3 suffices). Then

$$f(x) = a + bx + cx^2 + O(x^3)$$

or equivalently

$$cx^2 = f(x) - a - bx + O(x^3).$$

If f is of low-digit complexity then, as above, truncating after one term gives a low-complexity algorithm for $a + bx$. Recall that addition is $O_{\text{dig}}(\log n)$. This in turn gives a low-digit complexity evaluation of cx^2 in a neighborhood of zero, but evaluation of cx^2 is essentially equivalent to multiplication. Once again, the details are available in [7]. Thus, if f is any nonlinear C^3 function it is $\Omega_{\text{dig}}(n)$, or we would have too good an algorithm for calculating the mth digit of multiplication.

We now have the following type of theorem.

THEOREM. *If f is a nonlinear elementary function (on an interval) then f is*

$$O_{\text{bit}}(n(\log n)^k) \quad \textit{and} \quad \Omega_{\text{dig}}(n).$$

This is now close to an exact result. Actually we can say considerably more. For example, we have the following theorem.

THEOREM. *If f is a nonlinear C^3 function (on an interval) then the set of x for which the digit complexity of $f(x)$ is $o(n)$ by <u>any</u> algorithm is of the first Baire category.*

A set of first Baire category is small in a topological sense (see [25]).

We define the class of *sublinear numbers* by calling a number x sublinear if the digit complexity of x is $O_{\text{dig}}(n^{1-})$. Call α a *sublinear multiplier* if the function αx is sublinear for all $x \in [0, 1]$ (given both α and x as input).

THEOREM. *The set of sublinear multipliers is a nonempty set of the first Baire category.*

Two more definitions are useful in relation to numbers of very low digit complexity. We say that x is *sparse* if x has digit complexity $O_{\text{dig}}(n^\delta)$ for all $\delta > 0$, and we say that α is a *sparse multiplier* if αx is sparse for all $x \in [0, 1]$. Sparse multipliers have sparse digits. Indeed, let $S := \{x \mid \#(\text{nonzero digits of } x \text{ among the first } n \text{ digits}) = O(n^\delta) \text{ for all } \delta > 0\}$.

THEOREM. *The set of sparse multipliers is exactly the set S.*

Thus there are uncountably many sparse multipliers and hence also uncountably many sublinear multipliers.

These are base-dependent notions. The previous theorem shows that $\frac{1}{2}$ is a sparse multiplier base 2 but not base 3. We can prove directly that irrational sparse multipliers must be transcendental. Various questions concerning these matters will be raised in the next sections.

8. Questions on the complexity of functions. The hardest problems associated with Table 1 of §3 concern the almost complete lack of nontrivial lower bound estimates. This reflects the current state of affairs in theoretical computer science. Not only is the question of whether $P = NP$ still open, it is still not resolved that any NP problems are nonlinear. Friedman [13], for example, shows that we can take maxima over the class of polynomially computable functions if and only if $P = NP$ and that we can integrate over this class if and only if $P = \#P$. While these notions are somewhat tangential to our concerns they do indicate that some of our problems are likely to be hard.

One of the reasons for looking at the rational complexity is that it is likely to be a little more amenable to analysis. We can show that exp and log *cannot* have rational complexity $o(\log n)$. This is a consequence of the known estimates in approximating exp and log by rational functions of degree n [5], [8]. Note that n rational operations can generate a rational function of at most degree 2^n. Thus there is only a small gap between the known and best possible rational complexity estimates for log.

Question 1. Does log have rational complexity $O_{\text{rat}}(\log n)$?

The extra power of log in the rational complexity of exp over that of log is almost certainly an artifact of the method. So at least one power of log ought to be removable.

Question 2. Show that exp has rational complexity $O_{\text{rat}}(\log^2 n)$. Does exp have rational complexity $O_{\text{rat}}(\log n)$?

The low-complexity approximants to exp and log are constructed indirectly. It would be valuable to have a direct construction.

Question 3. Construct, as explicitly as possible, approximants to exp and log with complexity $O_{\text{rat}}(\log^k n)$.

There is a big difference in the rational complexity of exp and of Γ. It is tempting to speculate that this is artificial.

Question 4. Does Γ have rational complexity $O_{\text{rat}}(\log^k n)$?

Ideally we would like to identify those functions with this complexity.

Question 5. Classify (analytic) functions with rational complexity $O_{\text{rat}}(\log^k n)$.

This last question is almost certainly very hard.

We would expect there to be little difference between rational complexity and algebraic complexity.

Question 6. Does any of exp, log, or K have rational complexity essentially slower than its algebraic complexity?

In the case of bit complexity, there are no nontrivial lower bounds. At best we can say that the bit complexity is always at least that of multiplication. Thus a crucial first step is the content of the next question.

Question 7. Show that exp, log, or any of the functions we have considered is *not* $O_{\text{bit}}(M(n))$.

It is easy to construct entire functions with very low bit complexity; we simply use very rapidly converging power series. Thus there exist nonalgebraic analytic functions with bit complexity $O_{\text{bit}}(a_n M(n))$, where a_n is any sequence tending to infinity. However, the following question appears to be open.

Question 8. Does there exist a nonalgebraic analytic function with bit complexity $O_{\text{bit}}(M(n))$?

A negative answer to this question would also resolve the question preceding it.

A very natural class to examine is the class of functions that satisfy algebraic differential equations (not necessarily linear). Almost all familiar functions arise in this context. Even an unlikely example like the theta function

$$\theta_3(q) := \sum_{n\in\mathbb{Z}} q^{n^2},$$

satisfies a nonlinear algebraic differential equation, as Jacobi showed (see [20]).

Question 9. How do solutions of algebraic differential equations fit into the complexity table?

We end with a question on digit complexity.

Question 10. Does there exist an analytic function whose digit complexity is essentially faster than its bit complexity? Does there exist an analytic function with digit complexity $O_{\text{dig}}(n)$?

There exist functions of the form

$$\sum a_n |x - b_n|$$

with low-digit complexity, where a_n and b_n are low-digit complexity numbers. Possibly we can construct nowhere differentiable functions that are sublinear, in the sense of digit complexity.

9. Questions on the complexity of numbers. Questions concerning the transcendence of functions tend to be easier than questions on the transcendence of individual numbers. In much the same way, questions on the complexity of functions tend to be easier than those on the complexity of specific numbers. The intent of this section is to pose various problems that suggest the link between complexity and transcendence. Such questions, while raised before, tend to have been concerned just with the notion of computability rather than also considering the rate of the computation (see [14]).

The class of sublinear numbers, defined in §7, contains all rational numbers; it also contains known transcendents such as

$$\alpha := .12345678910111213 \cdots .$$

However, while the rationals are in this class in a base-independent fashion, it is not at all clear that the above number α is sublinear in bases relatively prime to 10. The 10^{10}th digit, base 10, is 1. What is it in base 2?

Question 11. Are there any irrational numbers that are sublinear in every base?

It is easy to generate numbers that are sublinear in particular bases. Numbers such as

$$a := .d_1 d_2 \cdots \qquad \begin{array}{ll} d_i := 1 & \text{if } i \text{ is a square,} \\ d_i := 0 & \text{otherwise,} \end{array}$$

or

$$b := .d_1 d_2 \cdots \qquad \begin{array}{ll} d_i := 1 & \text{if } i \text{ is a power of 2,} \\ d_i := 0 & \text{otherwise,} \end{array}$$

are sublinear in whatever base is specified. It is tempting to conjecture that the next question has a positive answer.

Question 12. Must an irrational number that is sublinear (in all bases) be transcendental?

Loxton and van der Poorten [17] show that a particular very special class of sublinear numbers, namely those generated by finite automata, are either rational or transcendental. These are numbers for which computation of the nth digit essentially requires no memory of the preceding digits. The base dependence of these numbers is discussed in [12].

Question 13. Is either of π or e sublinear (in any base)?

Almost certainly the answer to this question is no. There is an interesting observation relating to this. Consider the series

$$\frac{1}{\pi}=\sum_{n=0}^{\infty}\binom{2n}{n}^3\frac{42n+5}{2^{12n+4}}.$$

This series due to Ramanujan [5], [19] has numerators that grow roughly, e.g., 2^{6n}, while the denominators are powers of 2. Thus, as has been observed, we can compute the second length n block of binary digits of $1/\pi$ without computing the first block. Likewise, in base 10, we can compute the second block of length n of decimal digits of $\sqrt{(\frac{5}{3})}$ from the series

$$\frac{1}{\sqrt{1-4x}}=\sum_{n=0}^{\infty}\binom{2n}{n}x^n.$$

In neither case, however, is there any reduction in the order of complexity.

It seems likely that computing the nth digit of π is an $\Omega_{\text{dig}}(n)$ calculation. Thus, we might make the strong conjecture that no one will ever compute the 10^{1000}th digit of π. This number arises from an (over)estimate of the number of electrons in the known universe and as such almost certainly overestimates the amount of storage that will ever be available for such a calculation.

The set of sparse multipliers is a subset of the sublinear numbers that can be shown directly to contain no irrational algebraics. We do not know this about sparse numbers, though we strongly suspect it to be true.

Recall that a sparse multiplier has mostly zero digits and observe that a nonintegral rational cannot possess a terminating expansion in two relatively prime bases. This suggests the following question.

Question 14. Do there exist irrationals that are sparse multipliers in two relatively prime bases? Do there exist irrationals whose digits are asymptotically mostly zeros in two relatively prime bases?

Many of these questions are at least partly related to questions on normality [23]. Virtually nothing is known about the normality of familiar numbers. The following is a somewhat related question by Mahler.

Question 15 (Mahler [15]). Does there exist a nonrational function

$$f(x):=\sum_{n=0}^{\infty}a_nx^n$$

where the a_n are a bounded sequence of positive integers, that maps algebraic numbers in the unit disc to algebraic numbers?

Suppose that such an example exists, and suppose the a_n are bounded by 9. Then

$$f(1/1000)=a_0.00a_1 00a_2\cdots$$

is a thoroughly nonnormal irrational algebraic. Thus, in some sense, Mahler's question is a very weak conjecture concerning normality. Note also that, if in such an example the a_n were sublinearly computable, we would have produced sublinearly computable algebraic irrationalities.

Perhaps we will be able to distinguish rational numbers by their digit complexity. What can we hope to say about algebraic numbers? A natural class to look at is the class of numbers that are *linear* (*in multiplication*), that is, numbers with bit complexity $O_{\text{bit}}(M(n))$. This class contains all algebraic numbers in a base-independent fashion. It also contains numbers such as

$$a := \sum_{n=0}^{\infty} \frac{1}{3^{3^n}} \quad \text{and} \quad b := \prod_{n=0}^{\infty} (1 + 3^{-3^n}),$$

also in a base-independent fashion.

Question 16. Can we identify the class of numbers that are linear in multiplication?

This is almost certainly hard. As is the following question.

Question 17. Are either e or π linear in multiplication?

A negative answer to the above would include a proof of the transcendence of π. The place to start might be with the following.

Question 18. Can we construct any natural nonlinear number?

Our current state of knowledge is that γ and G have bit complexity $O_{\text{bit}}(\log^2 nM(n))$.

Question 19. Are γ and G both $O_{\text{bit}}(\log nM(n))$?

We might expect that elementary functions cannot take sublinear numbers to sublinear numbers.

Question 20. Does there exist a number $a \neq 0$ so that both a and $\exp(a)$ are sublinear (in some base)? Can a and $\exp(a)$ both be linear in multiplication?

It seems likely that the answer is no. Question 20 should also be asked about other elementary transcendental functions.

For simple nonelementary functions Question 20 has a positive answer. Consider the function $F := (2/\pi)K$, where K is the complete elliptic integral of the first kind. Then F satisfies a linear differential equation of order 2 and is a nonelementary transcendental function. However, if

$$k(q) := q^{1/2}\left(\sum_{n\in\mathbb{Z}} q^{n^2+n}\right)^2 \Big/ \left(\sum_{n\in\mathbb{Z}} q^{n^2}\right)^2,$$

then

$$F(k(q)) = \left(\sum_{n\in\mathbb{Z}} q^{n^2}\right)^2,$$

and when $q := 1/10^{2k}$ both $F(k(q))$ and $k(q)$ are linear in multiplication, at least in base 10. (This is because the series above have particularly low complexity for $q := 1/10^{2k}$.)

Note also that the function

$$\theta_3(q) := \sum_{n\in\mathbb{Z}} q^{n^2},$$

which satisfies a nonlinear algebraic differential equation, takes sublinear numbers of the form $q := 1/10^n$ to sublinear numbers (base 10).

10. Conclusion. Many issues have not been touched upon at all. One such issue is the overhead costs of these low-complexity algorithms. This amounts to a discussion of the constants buried in the asymptotic estimates. Sometimes the theoretically low-complexity algorithms are also of low complexity practically. This is the case for AGM-related algorithms for complete elliptic integrals. These are probably the algorithms of choice in any precision. The AGM-related algorithms for log and exp will certainly not outperform more traditional methods in the usual ranges in which we compute (less than 100 digits). Some of the FFT-related algorithms are probably of only theoretical interest, even for computing millions of digits, because the overhead constants are so large. In other cases, such as multiplication or the computation of π, an FFT-related method is vital for very high precision computations.

We have not succeeded in completely answering any of the questions in the Introduction. In large part, this is because we have virtually no methods for handling lower bounds for such problems. The questions raised in this paper seem to be fundamental. The partial answers have provided a number of substantial surprises. For these reasons we believe these questions are deserving of study.

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Ramanujan and Pi

Some 75 years ago an Indian mathematical genius developed ways of calculating pi with extraordinary efficiency. His approach is now incorporated in computer algorithms yielding millions of digits of pi

by Jonathan M. Borwein and Peter B. Borwein

Pi, the ratio of any circle's circumference to its diameter, was computed in 1987 to an unprecedented level of accuracy: more than 100 million decimal places. Last year also marked the centenary of the birth of Srinivasa Ramanujan, an enigmatic Indian mathematical genius who spent much of his short life in isolation and poor health. The two events are in fact closely linked, because the basic approach underlying the most recent computations of pi was anticipated by Ramanujan, although its implementation had to await the formulation of efficient algorithms (by various workers including us), modern supercomputers and new ways to multiply numbers.

Aside from providing an arena in which to set records of a kind, the quest to calculate the number to millions of decimal places may seem rather pointless. Thirty-nine places of pi suffice for computing the circumference of a circle girdling the known universe with an error no greater than the radius of a hydrogen atom. It is hard to imagine physical situations requiring more digits. Why are mathematicians and computer scientists not satisfied with, say, the first 50 digits of pi?

Several answers can be given. One is that the calculation of pi has become something of a benchmark computation: it serves as a measure of the sophistication and reliability of the computers that carry it out. In addition, the pursuit of ever more accurate values of pi leads mathematicians to intriguing and unexpected niches of number theory. Another and more ingenuous motivation is simply "because it's there." In fact, pi has been a fixture of mathematical culture for more than two and a half millenniums.

Furthermore, there is always the chance that such computations will shed light on some of the riddles surrounding pi, a universal constant that is not particularly well understood, in spite of its relatively elementary nature. For example, although it has been proved that pi cannot ever be exactly evaluated by subjecting positive integers to any combination of adding, subtracting, multiplying, dividing or extracting roots, no one has succeeded in proving that the digits of pi follow a random distribution (such that each number from 0 to 9 appears with equal frequency). It is possible, albeit highly unlikely, that after a while all the remaining digits of pi are 0's and 1's or exhibit some other regularity. Moreover, pi turns up in all kinds of unexpected places that have nothing to do with circles. If a number is picked at random from the set of integers, for instance, the probability that it will have no repeated prime divisors is six divided by the square of pi. No different from other eminent mathematicians, Ramanujan was prey to the fascinations of the number.

The ingredients of the recent approaches to calculating pi are among the mathematical treasures unearthed by renewed interest in Ramanujan's work. Much of what he did, however, is still inaccessible to investigators. The body of his work is contained in his "Notebooks," which are personal records written in his own nomenclature. To make matters more frustrating for mathematicians who have studied the "Notebooks," Ramanujan generally did not include formal proofs for his theorems. The task of deciphering and editing the "Notebooks" is only now nearing completion, by Bruce C. Berndt of the University of Illinois at Urbana-Champaign.

To our knowledge no mathematical redaction of this scope or difficulty has ever been attempted. The effort is certainly worthwhile. Ramanujan's legacy in the "Notebooks" promises not only to enrich pure mathematics but also to find application in various fields of mathematical physics. Rodney J. Baxter of the Australian National University, for example, acknowledges that Ramanujan's findings helped him to solve such problems in statistical mechanics as the so-called hard-hexagon model, which considers the behavior of a system of interacting particles laid out on a honeycomblike grid. Similarly, Carlos J. Moreno of the City University of New York and Freeman J. Dyson of the Institute for Advanced Study have pointed out that Ramanujan's work is beginning to be applied by physicists in superstring theory.

Ramanujan's stature as a mathematician is all the more astonishing when one considers his limited formal education. He was born on December 22, 1887, into a somewhat impoverished family of the Brahmin caste in the town of Erode in southern India and grew up in Kumbakonam, where his father was an accountant to a clothier. His mathematical precocity was recognized early, and at the age of seven he was given a scholarship to the Kumbakonam Town High School. He is said to have recited mathematical formulas to his schoolmates—including the value of pi to many places.

When he was 12, Ramanujan mastered the contents of S. L. Loney's rather comprehensive *Plane Trigonometry*, including its discussion of the sum and products of infinite sequences, which later were to figure prominently in his work. (An infinite sequence is an unending string of terms, often generated by a simple formula. In this context the interesting sequences are those whose terms can be added or multiplied to yield an identifiable, finite value. If the terms are added, the resulting expression is called a series; if they are multiplied, it is called a product.) Three years later he borrowed the *Synopsis of Elementary Results in Pure Mathematics*, a listing of some 6,000 theorems (most of them given without proof) compiled by G. S. Carr, a tutor at the University of Cambridge. Those two books were the basis of Ramanujan's mathematical training.

In 1903 Ramanujan was admitted to a local government college. Yet total absorption in his own mathematical diversions at the expense of everything else caused him to fail his examinations, a pattern repeated four years later at another college in Madras. Ramanujan did set his avocation aside—if only temporarily—to look for a job after his marriage in 1909. Fortunately in 1910 R. Ramachandra Rao, a well-to-do patron of mathematics, gave him a monthly stipend largely on the strength of favorable recommendations from various sympathetic Indian mathematicians and the findings he already had jotted down in the "Notebooks."

In 1912, wanting more conventional work, he took a clerical position in the Madras Port Trust, where the chairman was a British engineer, Sir Francis Spring, and the manager was V. Ramaswami Aiyar, the founder of the Indian Mathematical Society. They encouraged Ramanujan to communicate his results to three prominent British mathematicians. Two apparently did not respond; the one who did was G. H. Hardy of Cambridge, now regarded as the foremost British mathematician of the period.

Hardy, accustomed to receiving crank mail, was inclined to disregard Ramanujan's letter at first glance the day it arrived, January 16, 1913. But after dinner that night Hardy and a close colleague, John E. Littlewood, sat down to puzzle through a list of 120 formulas and theorems Ramanujan had appended to his letter. Some hours later they had reached a verdict: they were seeing the work of a genius and not a crackpot. (According to his own "pure-talent scale" of mathematicians, Hardy was later to rate Ramanujan a 100, Littlewood a 30 and himself a 25. The German mathematician David Hilbert, the most influential figure of the time, merited only an 80.) Hardy described the revelation and its consequences as the one romantic incident in his life. He wrote that some of Ramanujan's formulas defeated him completely, and yet "they must be true, because if they were not true, no one would have had the imagination to invent them."

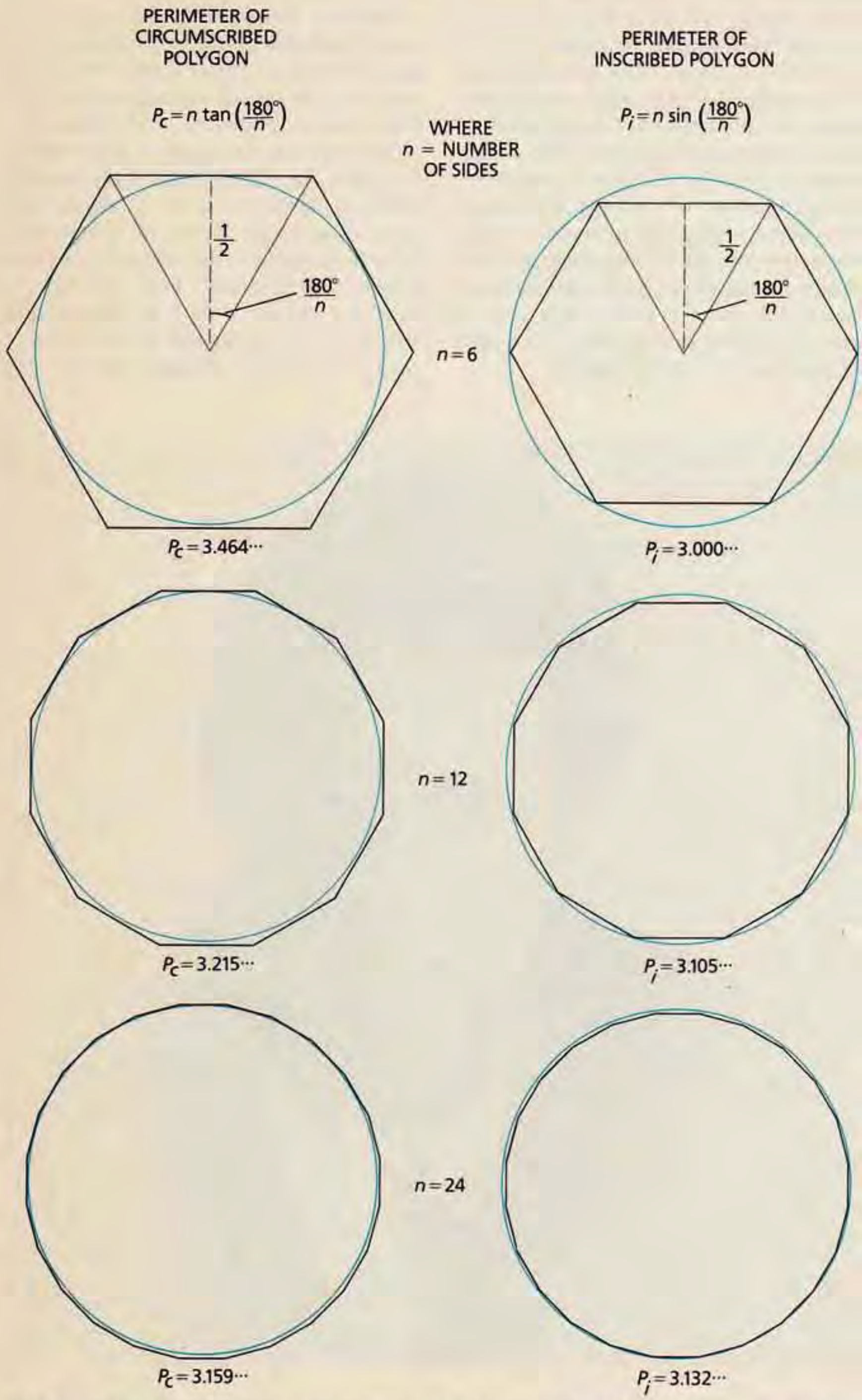
Hardy immediately invited Ramanujan to come to Cambridge. In spite of his mother's strong objections as well as his own reservations, Ramanujan set out for England in March of 1914. During the next five years Hardy and Ramanujan worked together at Trinity College. The blend of Hardy's technical expertise and Ramanujan's raw brilliance produced an unequaled collaboration. They published a series of seminal papers on the properties of various arithmetic functions, laying the groundwork for the answer to such questions as: How many prime divisors is a given number likely to have? How many ways can one express a number as a sum of smaller positive integers?

In 1917 Ramanujan was made a Fellow of the Royal Society of London and a Fellow of Trinity College—the first Indian to be awarded either honor. Yet as his prominence grew his health deteriorated sharply, a decline perhaps accelerated by the difficulty of maintaining a strict vegetarian diet in war-rationed England. Although Ramanujan was in and out of sanatoriums, he continued to pour forth new results. In 1919, when peace made travel abroad safe again, Ramanujan returned to India. Already an icon for young Indian intellectuals, the 32-year-old Ramanujan died on April 26, 1920, of what was then diagnosed as tuberculosis but now is thought to have been a severe vitamin deficiency. True to mathematics until the end, Ramanujan did not slow down during his last, pain-racked months, producing the re-

SRINIVASA RAMANUJAN, born in 1887 in India, managed in spite of limited formal education to reconstruct almost single-handedly much of the edifice of number theory and to go on to derive original theorems and formulas. Like many illustrious mathematicians before him, Ramanujan was fascinated by pi: the ratio of any circle's circumference to its diameter. Based on his investigation of modular equations (*see box on page 114*), he formulated exact expressions for pi and derived from them approximate values. As a result of the work of various investigators (including the authors), Ramanujan's methods are now better understood and have been implemented as algorithms.

markable work recorded in his so-called "Lost Notebook."

Ramanujan's work on pi grew in large part out of his investigation of modular equations, perhaps the most thoroughly treated subject in the "Notebooks." Roughly speaking, a modular equation is an algebraic relation between a function expressed in terms of a variable x—in mathematical notation, $f(x)$—and the same function expressed in terms of x raised to an integral power, for example $f(x^2)$, $f(x^3)$ or $f(x^4)$. The "order" of the modular equation is given by the integral power. The simplest modular equation is the second-order one: $f(x) = 2\sqrt{f(x^2)}\,/\,[1 + f(x^2)]$. Of course, not every function will satisfy a modular equation, but there is a class of functions, called modular functions, that do. These functions have various surprising symmetries that give them a special place in mathematics.



ARCHIMEDES' METHOD for estimating pi relied on inscribed and circumscribed regular polygons (polygons with sides of equal length) on a circle having a diameter of one unit (or a radius of half a unit). The perimeters of the inscribed and circumscribed polygons served respectively as lower and upper bounds for the value of pi. The sine and tangent functions can be used to calculate the polygons' perimeters, as is shown here, but Archimedes had to develop equivalent relations based on geometric constructions. Using 96-sided polygons, he determined that pi is greater than $3^{10}/_{71}$ and less than $3^1/_7$.

Ramanujan was unparalleled in his ability to come up with solutions to modular equations that also satisfy other conditions. Such solutions are called singular values. It turns out that solving for singular values in certain cases yields numbers whose natural logarithms coincide with pi (times a constant) to a surprising number of places [*see box on page 114*]. Applying this general approach with extraordinary virtuosity, Ramanujan produced many remarkable infinite series as well as single-term approximations for pi. Some of them are given in Ramanujan's one formal paper on the subject, *Modular Equations and Approximations to π*, published in 1914.

Ramanujan's attempts to approximate pi are part of a venerable tradition. The earliest Indo-European civilizations were aware that the area of a circle is proportional to the square of its radius and that the circumference of a circle is directly proportional to its diameter. Less clear, however, is when it was first realized that the ratio of any circle's circumference to its diameter and the ratio of any circle's area to the square of its radius are in fact the same constant, which today is designated by the symbol π. (The symbol, which gives the constant its name, is a latecomer in the history of mathematics, having been introduced in 1706 by the English mathematical writer William Jones and popularized by the Swiss mathematician Leonhard Euler in the 18th century.)

Archimedes of Syracuse, the greatest mathematician of antiquity, rigorously established the equivalence of the two ratios in his treatise *Measurement of a Circle*. He also calculated a value for pi based on mathematical principles rather than on direct measurement of a circle's circumference, area and diameter. What Archimedes did was to inscribe and circumscribe regular polygons (polygons whose sides are all the same length) on a circle assumed to have a diameter of one unit and to consider the polygons' respective perimeters as lower and upper bounds for possible values of the circumference of the circle, which is numerically equal to pi [*see illustration on opposite page*].

This method of approaching a value for pi was not novel: inscribing polygons of ever more sides in a circle had been proposed earlier by Antiphon, and Antiphon's contemporary, Bryson of Heraclea, had added circumscribed polygons to the procedure. What was novel was Archimedes' correct determination of the effect of doubling the number of sides on both the circumscribed and the inscribed polygons. He thereby developed a procedure that, when repeated enough times, enables one in principle to calculate pi to any number of digits. (It should be pointed out that the perimeter of a regular polygon can be readily calculated by means of simple trigonometric functions: the sine, cosine and tangent functions. But in Archimedes' time, the third century B.C., such functions were only partly understood. Archimedes therefore had to rely mainly on geometric constructions, which made the calculations considerably more demanding than they might appear today.)

Archimedes began with inscribed and circumscribed hexagons, which yield the inequality $3 < \pi < 2\sqrt{3}$. By doubling the number of sides four times, to 96, he narrowed the range of pi to between $3\frac{10}{71}$ and $3\frac{1}{7}$, obtaining the estimate $\pi \approx 3.14$. There is some evidence that the extant text of *Measurement of a Circle* is only a fragment of a larger work in which Archimedes described how, starting with decagons and doubling them six times, he got a five-digit estimate: $\pi \approx 3.1416$.

Archimedes' method is conceptually simple, but in the absence of a ready way to calculate trigonometric functions it requires the extraction of roots, which is rather time-consuming when done by hand. Moreover, the estimates converge slowly to pi: their error decreases by about a factor of four per iteration. Nevertheless, all European attempts to calculate pi before the mid-17th century relied in one way or another on the method. The 16th-century Dutch mathematician Ludolph van Ceulen dedicated much of his career to a computation of pi. Near the end of his life he obtained a 32-digit estimate by calculating the perimeter of inscribed and circumscribed polygons having 2^{62} (some 10^{18}) sides. His value for pi, called the Ludolphian number in parts of Europe, is said to have served as his epitaph.

The development of calculus, largely by Isaac Newton and Gottfried Wilhelm Leibniz, made it possible to calculate pi much more expeditiously. Calculus provides efficient techniques for computing a function's derivative (the rate of change in the function's value as its variables change) and its integral (the sum of the function's values over a range of variables). Applying the techniques, one can demonstrate that inverse trigonometric functions are given by integrals of quadratic functions that describe the curve of a circle. (The inverse of a trigonometric function gives the angle that corresponds to a particular value of the function. For example, the inverse tangent of 1 is 45 degrees or, equivalently, $\pi/4$ radians.)

(The underlying connection between trigonometric functions and algebraic expressions can be appreciated by considering a circle that has a radius of one unit and its center at the origin of a Cartesian *x-y* plane. The equation for the circle—whose area is numerically equal to pi—is $x^2 + y^2 = 1$, which is a restatement of the Pythagorean theorem for a right triangle with a hypotenuse equal to 1. Moreover, the sine and cosine of the angle between the positive *x* axis and any point on the circle are equal respectively to the point's coordinates, *y* and *x*; the angle's tangent is simply *y*/*x*.)

Of more importance for the purposes of calculating pi, however, is the fact that an inverse trigonometric function can be "expanded" as a series, the terms of which are computable from the derivatives of the function. Newton himself calculated pi to 15 places by adding the first few terms of a series that can be derived

WALLIS' PRODUCT (1665)

$$\frac{\pi}{2} = \frac{2\times2}{1\times3}\times\frac{4\times4}{3\times5}\times\frac{6\times6}{5\times7}\times\frac{8\times8}{7\times9}\times\cdots = \prod_{n=1}^{\infty}\frac{4n^2}{4n^2-1}$$

GREGORY'S SERIES (1671)

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$$

MACHIN'S FORMULA (1706)

$$\frac{\pi}{4} = 4\arctan(1/5) - \arctan(1/239), \quad \text{where } \arctan X = X - \frac{X^3}{3} + \frac{X^5}{5} - \frac{X^7}{7} + \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{X^{(2n+1)}}{2n+1}$$

RAMANUJAN (1914)

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9{,}801}\sum_{n=0}^{\infty}\frac{(4n)!\,[1{,}103 + 26{,}390n]}{(n!)^4\,396^{4n}}, \quad \text{where } n! = n \times (n-1) \times (n-2) \times \cdots \times 1 \text{ and } 0! = 1$$

BORWEIN AND BORWEIN (1987)

$$\frac{1}{\pi} = 12\sum_{n=0}^{\infty}\frac{(-1)^n(6n)!\,[212{,}175{,}710{,}912\sqrt{61} + 1{,}657{,}145{,}277{,}365 + n(13{,}773{,}980{,}892{,}672\sqrt{61} + 107{,}578{,}229{,}802{,}750)]}{(n)!^3(3n)!\,[5{,}280(236{,}674 + 30{,}303\sqrt{61})]^{(3n+3/2)}}$$

TERMS OF MATHEMATICAL SEQUENCES can be summed or multiplied to yield values for pi (divided by a constant) or its reciprocal. The first two sequences, discovered respectively by the mathematicians John Wallis and James Gregory, are probably among the best-known, but they are practically useless for computational purposes. Not even 100 years of computing on a supercomputer programmed to add or multiply the terms of either sequence would yield 100 digits of pi. The formula discovered by John Machin made the calculation of pi feasible, since calculus allows the inverse tangent (arc tangent) of a number, *x*, to be expressed in terms of a sequence whose sum converges more rapidly to the value of the arc tangent the smaller *x* is. Virtually all calculations for pi from the beginning of the 18th century until the early 1970's have relied on variations of Machin's formula. The sum of Ramanujan's sequence converges to the true value of 1/π much faster: each successive term in the sequence adds roughly eight more correct digits. The last sequence, formulated by the authors, adds about 25 digits per term; the first term (for which *n* is 0) yields a number that agrees with pi to 24 digits.

as an expression for the inverse of the sine function. He later confessed to a colleague: "I am ashamed to tell you to how many figures I carried these calculations, having no other business at the time."

In 1674 Leibniz derived the formula $1-1/3+1/5-1/7\ldots=\pi/4$, which is the inverse tangent of 1. (The general inverse-tangent series was originally discovered in 1671 by the Scottish mathematician James Gregory. Indeed, similar expressions appear to have been developed independently several centuries earlier in India.) The error of the approximation, defined as the difference between the sum of n terms and the exact value of $\pi/4$, is roughly equal to the $n+1$th term in the series. Since the denominator of each successive term increases by only 2, one must add approximately 50 terms to get two-digit accuracy, 500 terms for three-digit accuracy and so on. Summing the terms of the series to calculate a value for pi more than a few digits long is clearly prohibitive.

An observation made by John Machin, however, made it practicable to calculate pi by means of a series expansion for the inverse-tangent function. He noted that pi divided by 4 is equal to 4 times the inverse tangent of 1/5 minus the inverse tangent of 1/239. Because the inverse-tangent series for a given value converges more quickly the smaller the value is, Machin's formula greatly simplified the calculation. Coupling his formula with the series expansion for the inverse tangent, Machin computed 100 digits of pi in 1706. Indeed, his technique proved to be so powerful that all extended calculations of pi from the beginning of the 18th century until recently relied on variants of the method.

Two 19th-century calculations deserve special mention. In 1844 Johann Dase computed 205 digits of pi in a matter of months by calculating the values of three inverse tangents in a Machin-like formula. Dase was a calculating prodigy who could multiply 100-digit numbers entirely in his head—a feat that took him roughly eight hours. (He was perhaps the closest precursor of the modern supercomputer, at least in terms of memory capacity.) In 1853 William Shanks outdid Dase by publishing his computation of pi to 607 places, although the digits that followed the 527th place were wrong. Shank's task took years and was a rather routine, albeit laborious, application of Machin's formula. (In what must itself be some kind of record, 92 years passed before Shank's error was detected, in a comparison between his value and a 530-place approximation produced by D. F. Ferguson with the aid of a mechanical calculator.)

The advent of the digital computer saw a renewal of efforts to calculate ever more digits of pi, since the machine was ideally suited for lengthy, repetitive "number crunching." ENIAC, one of the first digital computers, was applied to the task in June, 1949, by John von Neumann and his colleagues. ENIAC produced 2,037 digits in 70 hours. In 1957 G. E. Felton attempted to compute 10,000 digits of pi, but owing to a machine error only the first 7,480 digits were correct. The 10,000-digit goal was reached by F. Genuys the following year on an IBM 704 computer. In 1961 Daniel Shanks and John W. Wrench, Jr., calculated 100,000 digits of pi in less than nine hours on an IBM 7090. The million-digit mark was passed in 1973 by Jean Guilloud and M. Bouyer, a feat that took just under a day of computation on a CDC 7600. (The computations done by Shanks and Wrench and by Guilloud and Bouyer were in fact carried out twice using different inverse-tangent identities for pi. Given the history of both human and machine error in these calculations, it is only after such verification that modern "digit hunters" consider a record officially set.)

Although an increase in the speed of computers was a major reason ever more accurate calculations for pi could be performed, it soon became clear that there were inescapable limits. Doubling the number of digits lengthens computing time by at least a factor of four, if one applies the traditional methods of performing arithmetic in computers. Hence even allowing for a hundredfold increase in computational speed, Guilloud and Bouyer's program would have required at least a quarter century to produce a billion-digit value for pi. From the perspective of the early 1970's such a computation did not seem realistically practicable.

Yet the task is now feasible, thanks

MODULAR FUNCTIONS AND APPROXIMATIONS TO PI

A modular function is a function, $\lambda(q)$, that can be related through an algebraic expression called a modular equation to the same function expressed in terms of the same variable, q, raised to an integral power: $\lambda(q^p)$. The integral power, p, determines the "order" of the modular equation. An example of a modular function is

$$\lambda(q) = 16q \prod_{n=1}^{\infty} \left(\frac{1+q^{2n}}{1+q^{2n-1}}\right)^8 .$$

Its associated seventh-order modular equation, which relates $\lambda(q)$ to $\lambda(q^7)$, is given by

$$\sqrt[8]{\lambda(q)\lambda(q^7)} + \sqrt[8]{[1-\lambda(q)][1-\lambda(q^7)]} = 1 .$$

Singular values are solutions of modular equations that must also satisfy additional conditions. One class of singular values corresponds to computing a sequence of values, k_p, where

$$k_p = \sqrt{\lambda(e^{-\pi\sqrt{p}})}$$

and p takes integer values. These values have the curious property that the logarithmic expression

$$\frac{-2}{\sqrt{p}} \log\left(\frac{k_p}{4}\right)$$

coincides with many of the first digits of pi. The number of digits the expression has in common with pi increases with larger values of p.

Ramanujan was unparalleled in his ability to calculate these singular values. One of his most famous is the value when p equals 210, which was included in his original letter to G. H. Hardy. It is

$$k_{210} = (\sqrt{2}-1)^2(2-\sqrt{3})(\sqrt{7}-\sqrt{6})^2(8-3\sqrt{7})(\sqrt{10}-3)^2(\sqrt{15}-\sqrt{14})(4-\sqrt{15})^2(6-\sqrt{35}) .$$

This number, when plugged into the logarithmic expression, agrees with pi through the first 20 decimal places. In comparison, $k_{2^{40}}$ yields a number that agrees with pi through more than one million digits.

Applying this general approach, Ramanujan constructed a number of remarkable series for pi, including the one shown in the illustration on the preceding page. The general approach also underlies the two-step, iterative algorithms in the top illustration on the opposite page. In each iteration the first step (calculating y_n) corresponds to computing one of a sequence of singular values by solving a modular equation of the appropriate order; the second step (calculating α_n) is tantamount to taking the logarithm of the singular value.

not only to faster computers but also to new, efficient methods for multiplying large numbers in computers. A third development was also crucial: the advent of iterative algorithms that quickly converge to pi. (An iterative algorithm can be expressed as a computer program that repeatedly performs the same arithmetic operations, taking the output of one cycle as the input for the next.) These algorithms, some of which we constructed, were in many respects anticipated by Ramanujan, although he knew nothing of computer programming. Indeed, computers not only have made it possible to apply Ramanujan's work but also have helped to unravel it. Sophisticated algebraic-manipulation software has allowed further exploration of the road Ramanujan traveled alone and unaided 75 years ago.

One of the interesting lessons of theoretical computer science is that many familiar algorithms, such as the way children are taught to multiply in grade school, are far from optimal. Computer scientists gauge the efficiency of an algorithm by determining its bit complexity: the number of times individual digits are added or multiplied in carrying out an algorithm. By this measure, adding two n-digit numbers in the normal way has a bit complexity that increases in step with n; multiplying two n-digit numbers in the normal way has a bit complexity that increases as n^2. By traditional methods, multiplication is much "harder" than addition in that it is much more time-consuming.

Yet, as was shown in 1971 by A. Schönhage and V. Strassen, the multiplication of two numbers can in theory have a bit complexity only a little greater than addition. One way to achieve this potential reduction in bit complexity is to implement so-called fast Fourier transforms (FFT's). FFT-based multiplication of two large numbers allows the intermediary computations among individual digits to be carefully orchestrated so that redundancy is avoided. Because division and root extraction can be reduced to a sequence of multiplications, they too can have a bit complexity just slightly greater than that of addition. The result is a tremendous saving in bit complexity and hence in computation time. For this reason all recent efforts to calculate pi rely on some variation of the FFT technique for multiplication.

Yet for hundreds of millions of dig-

(a) Let $y_0 = \frac{1}{\sqrt{2}} \quad \alpha_0 = \frac{1}{2}$

and

$$y_{n+1} = \frac{1 - \sqrt{1 - y_n^2}}{1 + \sqrt{1 - y_n^2}} \qquad \alpha_{n+1} = [(1 + y_{n+1})^2 \alpha_n] - 2^{n+1} y_{n+1}$$

(b) Let $y_0 = \sqrt{2} - 1 \quad \alpha_0 = 6 - 4\sqrt{2}$

and

$$y_{n+1} = \frac{1 - \sqrt[4]{1 - y_n^4}}{1 + \sqrt[4]{1 - y_n^4}}$$

$$\alpha_{n+1} = [(1 + y_{n+1})^4 \alpha_n] - 2^{2n+3} y_{n+1}(1 + y_{n+1} + y_{n+1}^2)$$

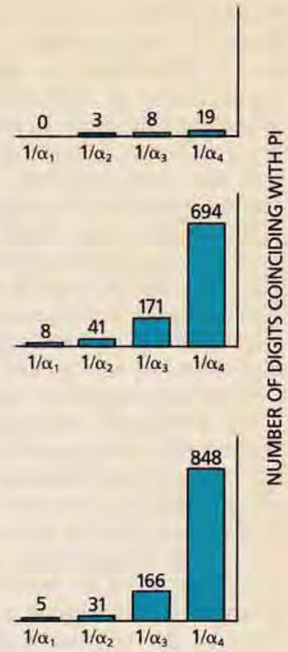
(c) Let $S_0 = 5(\sqrt{5} - 2) \quad \alpha_0 = \frac{1}{2}$

and

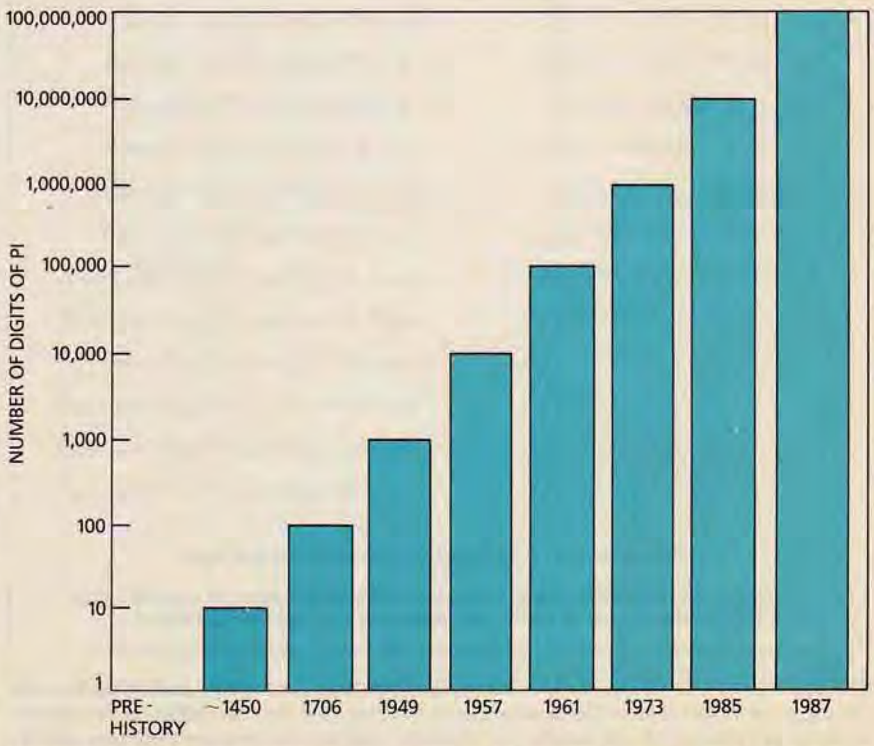
$$S_{n+1} = \frac{25}{S_n(Z + X/Z + 1)^2}, \text{ where } X = \frac{5}{S_n} - 1, \quad Y = (X - 1)^2 + 7$$

$$\text{and } Z = \sqrt[5]{\frac{X(Y + \sqrt{Y^2 - 4X^3})}{2}}$$

$$\alpha_{n+1} = S_n^2 \alpha_n - 5^n \left[\frac{S_n^2 - 5}{2} + \sqrt{S_n(S_n^2 - 2S_n + 5)}\right]$$



ITERATIVE ALGORITHMS that yield extremely accurate values of pi were developed by the authors. (An iterative algorithm is a sequence of operations repeated in such a way that the ouput of one cycle is taken as the input for the next.) Algorithm *a* converges to $1/\pi$ quadratically: the number of correct digits given by α_n more than doubles each time *n* is increased by 1. Algorithm *b* converges quartically and algorithm *c* converges quintically, so that the number of coinciding digits given by each iteration increases respectively by more than a factor of four and by more than a factor of five. Algorithm *b* is possibly the most efficient known algorithm for calculating pi; it was run on supercomputers in the last three record-setting calculations. As the authors worked on the algorithms it became clear to them that Ramanujan had pursued similar methods in coming up with his approximations for pi. In fact, the computation of s_n in algorithm *c* rests on a remarkable fifth-order modular equation discovered by Ramanujan.



NUMBER OF KNOWN DIGITS of pi has increased by two orders of magnitude (factors of 10) in the past decade as a result of the development of iterative algorithms that can be run on supercomputers equipped with new, efficient methods of multiplication.

its of pi to be calculated practically a beautiful formula known a century and a half earlier to Carl Friedrich Gauss had to be rediscovered. In the mid-1970's Richard P. Brent and Eugene Salamin independently noted that the formula produced an algorithm for pi that converged quadratically, that is, the number of digits doubled with each iteration. Between 1983 and the present Yasumasa Kanada and his colleagues at the University of Tokyo have employed this algorithm to set several world records for the number of digits of pi.

We wondered what underlies the remarkably fast convergence to pi of the Gauss-Brent-Salamin algorithm, and in studying it we developed general techniques for the construction of similar algorithms that rapidly converge to pi as well as to other quantities. Building on a theory outlined by the German mathematician Karl Gustav Jacob Jacobi in 1829, we realized we could in principle arrive at a value for pi by evaluating integrals of a class called elliptic integrals, which can serve to calculate the perimeter of an ellipse. (A circle, the geometric setting of previous efforts to approximate pi, is simply an ellipse with axes of equal length.)

Elliptic integrals cannot generally be evaluated as integrals, but they can be easily approximated through iterative procedures that rely on modular equations. We found that the Gauss-Brent-Salamin algorithm is actually a specific case of our more general technique relying on a second-order modular equation. Quicker convergence to the value of the integral, and thus a faster algorithm for pi, is possible if higher-order modular equations are used, and so we have also constructed various algorithms based on modular equations of third, fourth and higher orders.

In January, 1986, David H. Bailey of the National Aeronautics and Space Administration's Ames Research Center produced 29,360,000 decimal places of pi by iterating one of our algorithms 12 times on a Cray-2 supercomputer. Because the algorithm is based on a fourth-order modular equation, it converges on pi quartically, more than quadrupling the number of digits with each iteration. A year later Kanada and his colleagues carried out one more iteration to attain 134,217,000 places on an NEC SX-2 supercomputer and thereby verified a similar computation they had done earlier using the Gauss-Brent-Salamin algorithm. (Iterating our algorithm twice more—a feat entirely feasible if one could somehow monopolize a supercomputer for a few weeks—would yield more than two billion digits of pi.)

HOW TO GET TWO BILLION DIGITS OF PI WITH A CALCULATOR*

Let

$$y_0 = \sqrt{2} - 1 \qquad \alpha_0 = 6 - 4\sqrt{2}$$

$$y_1 = [1 - \sqrt[4]{1 - y_0^4}]/[1 + \sqrt[4]{1 - y_0^4}] \qquad \alpha_1 = (1 + y_1)^4 \alpha_0 - 2^3 y_1(1 + y_1 + y_1^2)$$

$$y_2 = [1 - \sqrt[4]{1 - y_1^4}]/[1 + \sqrt[4]{1 - y_1^4}] \qquad \alpha_2 = (1 + y_2)^4 \alpha_1 - 2^5 y_2(1 + y_2 + y_2^2)$$

$$y_3 = [1 - \sqrt[4]{1 - y_2^4}]/[1 + \sqrt[4]{1 - y_2^4}] \qquad \alpha_3 = (1 + y_3)^4 \alpha_2 - 2^7 y_3(1 + y_3 + y_3^2)$$

$$y_4 = [1 - \sqrt[4]{1 - y_3^4}]/[1 + \sqrt[4]{1 - y_3^4}] \qquad \alpha_4 = (1 + y_4)^4 \alpha_3 - 2^9 y_4(1 + y_4 + y_4^2)$$

$$y_5 = [1 - \sqrt[4]{1 - y_4^4}]/[1 + \sqrt[4]{1 - y_4^4}] \qquad \alpha_5 = (1 + y_5)^4 \alpha_4 - 2^{11} y_5(1 + y_5 + y_5^2)$$

$$y_6 = [1 - \sqrt[4]{1 - y_5^4}]/[1 + \sqrt[4]{1 - y_5^4}] \qquad \alpha_6 = (1 + y_6)^4 \alpha_5 - 2^{13} y_6(1 + y_6 + y_6^2)$$

$$y_7 = [1 - \sqrt[4]{1 - y_6^4}]/[1 + \sqrt[4]{1 - y_6^4}] \qquad \alpha_7 = (1 + y_7)^4 \alpha_6 - 2^{15} y_7(1 + y_7 + y_7^2)$$

$$y_8 = [1 - \sqrt[4]{1 - y_7^4}]/[1 + \sqrt[4]{1 - y_7^4}] \qquad \alpha_8 = (1 + y_8)^4 \alpha_7 - 2^{17} y_8(1 + y_8 + y_8^2)$$

$$y_9 = [1 - \sqrt[4]{1 - y_8^4}]/[1 + \sqrt[4]{1 - y_8^4}] \qquad \alpha_9 = (1 + y_9)^4 \alpha_8 - 2^{19} y_9(1 + y_9 + y_9^2)$$

$$y_{10} = [1 - \sqrt[4]{1 - y_9^4}]/[1 + \sqrt[4]{1 - y_9^4}] \qquad \alpha_{10} = (1 + y_{10})^4 \alpha_9 - 2^{21} y_{10}(1 + y_{10} + y_{10}^2)$$

$$y_{11} = [1 - \sqrt[4]{1 - y_{10}^4}]/[1 + \sqrt[4]{1 - y_{10}^4}] \qquad \alpha_{11} = (1 + y_{11})^4 \alpha_{10} - 2^{23} y_{11}(1 + y_{11} + y_{11}^2)$$

$$y_{12} = [1 - \sqrt[4]{1 - y_{11}^4}]/[1 + \sqrt[4]{1 - y_{11}^4}] \qquad \alpha_{12} = (1 + y_{12})^4 \alpha_{11} - 2^{25} y_{12}(1 + y_{12} + y_{12}^2)$$

$$y_{13} = [1 - \sqrt[4]{1 - y_{12}^4}]/[1 + \sqrt[4]{1 - y_{12}^4}] \qquad \alpha_{13} = (1 + y_{13})^4 \alpha_{12} - 2^{27} y_{13}(1 + y_{13} + y_{13}^2)$$

$$y_{14} = [1 - \sqrt[4]{1 - y_{13}^4}]/[1 + \sqrt[4]{1 - y_{13}^4}] \qquad \alpha_{14} = (1 + y_{14})^4 \alpha_{13} - 2^{29} y_{14}(1 + y_{14} + y_{14}^2)$$

$$y_{15} = [1 - \sqrt[4]{1 - y_{14}^4}]/[1 + \sqrt[4]{1 - y_{14}^4}] \qquad \alpha_{15} = (1 + y_{15})^4 \alpha_{14} - 2^{31} y_{15}(1 + y_{15} + y_{15}^2)$$

$1/\alpha_{15}$ agress with π for more than two billion decimal digits

*Of course, the calculator needs to have a two-billion-digit display; on a pocket calculator the computation would not be very interesting after the second iteration.

EXPLICIT INSTRUCTIONS for executing algorithm *b* in the top illustration on the preceding page makes it possible in principle to compute the first two billion digits of pi in a matter of minutes. All one needs is a calculator that has two memory registers and the usual capacity to add, subtract, multiply, divide and extract roots. Unfortunately most calculators come with only an eight-digit display, which makes the computation moot.

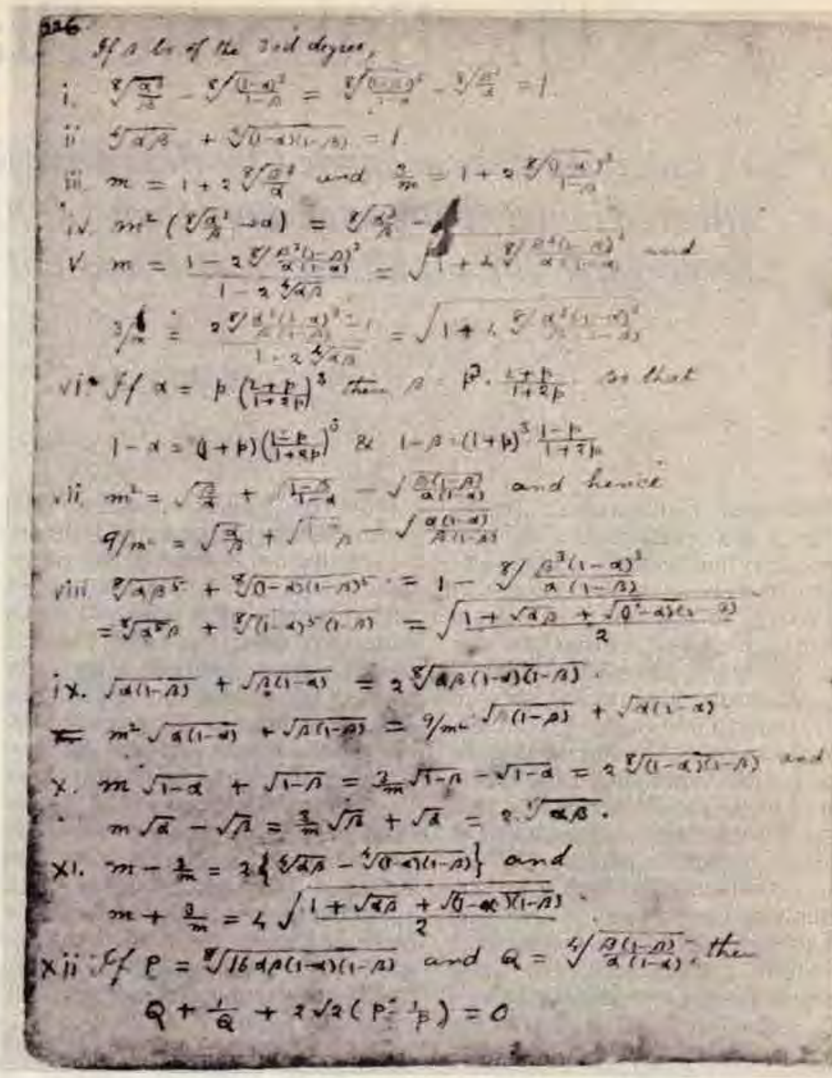
Iterative methods are best suited for calculating pi on a computer, and so it is not surprising that Ramanujan never bothered to pursue them. Yet the basic ingredients of the iterative algorithms for pi—modular equations in particular—are to be found in Ramanujan's work. Parts of his original derivation of infinite series and approximations for pi more than three-quarters of a century ago must have paralleled our own efforts to come up with algorithms for pi. Indeed, the formulas he lists in his paper on pi and in the "Notebooks" helped us greatly in the construction of some of our algorithms. For example, although we were able to prove that an 11th-order algorithm exists and knew its general formulation, it was not until we stumbled on Ramanujan's modular equations of the same order that we discovered its unexpectedly simple form.

Conversely, we were also able to derive all Ramanujan's series from the general formulas we had developed. The derivation of one, which converged to pi faster than any other series we knew at the time, came about with a little help from an unexpected source. We had justified all the quantities in the expression for the series except one: the coefficient 1,103, which appears in the numerator of the expression [*see illustration on page 113*]. We were convinced—as Ramanujan must have been—that 1,103 had to be correct. To prove it we had either to simplify a daunting equation containing variables raised to powers of several thousand or to delve considerably further into somewhat arcane number theory.

By coincidence R. William Gosper, Jr., of Symbolics, Inc., had decided in 1985 to exploit the same series of Ramanujan's for an extended-accuracy value for pi. When he carried out the calculation to more than 17 million digits (a record at the time), there was to his knowledge no proof that the sum of the series actually converged to pi. Of course, he knew that millions of digits of his value coincided with an earlier Gauss-Brent-Salamin calculation done by Kanada. Hence the possibility of error was vanishingly small.

As soon as Gosper had finished his calculation and verified it against Kanada's, however, we had what we needed to prove that 1,103 was the number needed to make the series true to within one part in $10^{10,000,000}$. In much the same way that a pair of integers differing by less than 1 must be equal, his result sufficed to specify the number: it is precisely 1,103. In effect, Gosper's computation became part of our proof. We knew that the series (and its associated algorithm) is so sensitive to slight inaccuracies that if Gosper had used any other value for the coefficient or, for that matter, if the computer had introduced a single-digit error during the calculation, he would have ended up with numerical nonsense instead of a value for pi.

Ramanujan-type algorithms for approximating pi can be shown to be very close to the best possible. If all the operations involved in the execution of the algorithms are totaled (assuming that the best techniques known for addition, multiplication and root extraction are applied), the bit complexity of computing n digits of pi is only marginally greater than that of multiplying two n-digit numbers. But multiplying two n-digit numbers by means of an FFT-based technique is only marginally more complicated than summing two n-digit numbers, which is the simplest of the arithmetic operations possible on a computer.



RAMANUJAN'S "NOTEBOOKS" were personal records in which he jotted down many of his formulas. The page shown contains various third-order modular equations—all in Ramanujan's nonstandard notation. Unfortunately Ramanujan did not bother to include formal proofs for the equations; others have had to compile, edit and prove them. The formulas in the "Notebooks" embody subtle relations among numbers and functions that can be applied in other fields of mathematics or even in theoretical physics.

Mathematics has probably not yet felt the full impact of Ramanujan's genius. There are many other wonderful formulas contained in the "Notebooks" that revolve around integrals, infinite series and continued fractions (a number plus a fraction, whose denominator can be expressed as a number plus a fraction, whose denominator can be expressed as a number plus a fraction, and so on). Unfortunately they are listed with little—if any—indication of the method by which Ramanujan proved them. Littlewood wrote: "If a significant piece of reasoning occurred somewhere, and the total mixture of evidence and intuition gave him certainty, he looked no further."

The herculean task of editing the "Notebooks," initiated 60 years ago by the British analysts G. N. Watson and B. N. Wilson and now being completed by Bruce Berndt, requires providing a proof, a source or an occasional correction for each of many thousands of asserted theorems and identities. A single line in the "Notebooks" can easily elicit many pages of commentary. The task is made all the more difficult by the nonstandard mathematical notation in which the formulas are written. Hence a great deal of Ramanujan's work will not become accessible to the mathematical community until Berndt's project is finished.

Ramanujan's unique capacity for working intuitively with complicated formulas enabled him to plant seeds in a mathematical garden (to borrow a metaphor from Freeman Dyson) that is only now coming into bloom. Along with many other mathematicians, we look forward to seeing which of the seeds will germinate in future years and further beautify the garden.

Ramanujan, Modular Equations, and Approximations to Pi or How to Compute One Billion Digits of Pi

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Preface. The year 1987 was the centenary of Ramanujan's birth. He died in 1920 Had he not died so young, his presence in modern mathematics might be more immediately felt. Had he lived to have access to powerful algebraic manipulation software, such as MACSYMA, who knows how much more spectacular his already astonishing career might have been.

This article will follow up one small thread of Ramanujan's work which has found a modern computational context, namely, one of his approaches to approximating pi. Our experienec has been that as we have come to understand these pieces of Ramanujan's work, as they have become mathematically demystified, and as we have come to realize the intrinsic complexity of these results, we have come to realize how truly singular his abilities were. This article attempts to present a considerable amount of material and, of necessity, little is presented in detail. We have, however, given much more detail than Ramanujan provided. Our intention is that the circle of ideas will become apparent and that the finer points may be pursued through the indicated references.

1. Introduction. There is a close and beautiful conncetion between the transformation theory for elliptic integrals and the very rapid approximation of pi. This connection was first made explicit by Ramanujan in his 1914 paper "Modular Equations and Approximations to π" **[26]**. We might emphasize that Algorithms 1 and 2 are not to be found in Ramanujan's work, indeed no recursive approximation of π is considered, but as we shall see they are intimately related to his analysis. Three central examples are:

Sum 1. (Ramanujan)

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4} \frac{[1103 + 26390n]}{396^{4n}}.$$

Algorithm 1. Let $\alpha_0 := 6 - 4\sqrt{2}$ and $y_0 := \sqrt{2} - 1$.
Let

$$y_{n+1} := \frac{1 - (1 - y_n^4)^{1/4}}{1 + (1 - y_n^4)^{1/4}}$$

and

$$\alpha_{n+1} := (1 + y_{n+1})^4 \alpha_n - 2^{2n+3} y_{n+1} (1 + y_{n+1} + y_{n+1}^2).$$

Then

$$0 < \alpha_n - 1/\pi < 16 \cdot 4^n e^{-2 \cdot 4^n \pi}$$

and α_n converges to $1/\pi$ *quartically* (that is, with order four).

Algorithm 2. Let $s_0 := 5(\sqrt{5} - 2)$ and $\alpha_0 := 1/2$.
Let

$$s_{n+1} := \frac{25}{(z + x/z + 1)^2 s_n},$$

where

$$x := 5/s_n - 1 \qquad y := (x-1)^2 + 7$$

and

$$z := \left[\frac{1}{2}x\left(y + \sqrt{y^2 - 4x^3}\right)\right]^{1/5}.$$

Let

$$\alpha_{n+1} := s_n^2 \alpha_n - 5^n \left\{ \frac{s_n^2 - 5}{2} + \sqrt{s_n\left(s_n^2 - 2s_n + 5\right)} \right\}.$$

Then

$$0 < \alpha_n - \frac{1}{\pi} < 16 \cdot 5^n e^{-5^n \pi}$$

and α_n converges to $1/\pi$ *quintically* (that is, with order five).

Each additional term in Sum 1 adds roughly eight digits, each additional iteration of Algorithm 1 quadruples the number of correct digits, while each additional iteration of Algorithm 2 quintuples the number of correct digits. Thus a mere thirteen iterations of Algorithm 2 provide in excess of one billion decimal digits of pi. In general, for us, pth-order convergence of a sequence (α_n) to α means that α_n tends to α and that

$$|\alpha_{n+1} - \alpha| \leq C|\alpha_n - \alpha|^p$$

for some constant $C > 0$. Algorithm 1 is arguably the most efficient algorithm currently known for the extended precision calculation of pi. While the rates of convergence are impressive, it is the subtle and thoroughly nontransparent nature of these results and the beauty of the underlying mathematics that intrigue us most.

Watson **[37]**, commenting on certain formulae of Ramanujan, talks of

> *a thrill which is indistinguishable from the thrill which I feel when I enter the Sagrestia Nuovo of the Capella Medici and see before me the austere beauty of the four statues representing "Day," "Night," "Evening," and "Dawn" which Michelangelo has set over the tomb of Giuliano de'Medici and Lorenzo de'Medici.*

Sum 1 is directly due to Ramanujan and appears in **[26]**. It rests on a modular identity of order 58 and, like much of Ramanujan's work, appears without proof and with only scanty motivation. The first complete derivation we know of appears in [**11**]. Algorithms 1 and 2 are based on modular identities of orders 4 and 5, respectively. The underlying quintic modular identity in Algorithm 2 (the relation for s_n) is also due to Ramanujan, though the first proof is due to Berndt and will appear in [**7**].

One intention in writing this article is to explain the genesis of Sum 1 and of Algorithms 1 and 2. It is not possible to give a short self-contained account without assuming an unusual degree of familiarity with modular function theory. Also, parts of the derivation involve considerable algebraic calculation and may most easily be done with the aid of a symbol manipulation package (MACSYMA, MAPLE, REDUCE, etc.). We hope however to give a taste of methods involved. The full details are available in [**11**].

A second intention is very briefly to describe the role of these and related approximations in the recent extended precision calculations of pi. In part this entails a short discussion of the complexity and implementation of such calculations. This centers on a discussion of multiplication by fast Fourier transform methods. Of considerable related interest is the fact that these algorithms for π are provably close to the theoretical optimum.

2. The State of Our Current Ignorance. Pi is almost certainly the most natural of the transcendental numbers, arising as the circumference of a circle of unit diameter. Thus, it is not surprising that its properties have been studied for some twenty-five hundred years. What is surprising is how little we actually know.

We know that π is irrational, and have known this since Lambert's proof of 1771 (see [**5**]). We have known that π is transcendental since Lindemann's proof of 1882 [**23**]. We also know that π is not a Liouville number. Mahler proved this in 1953. An irrational number β is *Liouville* if, for any n, there exist integers p and q so that

$$0 < \left|\beta - \frac{p}{q}\right| < \frac{1}{q^n}.$$

Liouville showed these numbers are all transcendental. In fact we know that

$$\left|\pi - \frac{p}{q}\right| > \frac{1}{q^{14.65}} \tag{2.1}$$

for p, q integral with q sufficiently large. This *irrationality estimate*, due to Chudnovsky and Chudnovsky [**16**] is certainly not best possible. It is likely that 14.65 should be replaced by $2 + \varepsilon$ for any $\varepsilon > 0$. Almost all transcendental numbers satisfy such an inequality. We know a few related results for the rate of algebraic approximation. The results may be pursued in [**4**] and [**11**].

We know that e^{π} is transcendental. This follows by noting that $e^{\pi} = (-1)^{-i}$ and applying the Gelfond-Schneider theorem [**4**]. We know that $\pi + \log 2 + \sqrt{2}\log 3$ is transcendental. This result is a consequence of the work that won Baker a Fields Medal in 1970. And we know a few more than the first two hundred million digits of the decimal expansion for π (Kanada, see Section 3).

The state of our ignorance is more profound. We do not know whether such basic constants as $\pi + e$, π/e, or $\log \pi$ are irrational, let alone transcendental. The best we can say about these three particular constants is that they cannot satisfy any polynomial of degree eight or less with integer coefficients of average size less than 10^9 [**3**]. This is a consequence of some recent computations employing the

Ferguson-Forcade algorithm **[17]**. We don't know anything of consequence about the single continued fraction of pi, except (numerically) the first 17 million terms, which Gosper computed in 1985 using Sum 1. Likewise, apart from listing the first many millions of digits of π, we know virtually nothing about the decimal expansion of π. It is possible, albeit not a good bet, that all but finitely many of the decimal digits of pi are in fact 0's and 1's. Carl Sagan's recent novel *Contact* rests on a similar possibility. Questions concerning the normality of or the distribution of digits of particular transcendentals such as π appear completely beyond the scope of current mathematical techniques. The evidence from analysis of the first thirty million digits is that they are very uniformly distributed **[2]**. The next one hundred and seventy million digits apparently contain no surprises.

In part we perhaps settle for computing digits of π because there is little else we can currently do. We would be amiss, however, if we did not emphasize that the extended precision calculation of pi has substantial application as a test of the "global integrity" of a supercomputer. The extended precision calculations described in Section 3 uncovered hardware errors which had to be corrected before those calculations could be successfully run. Such calculations, implemented as in Section 4, are apparently now used routinely to check supercomputers before they leave the factory. A large-scale calculation of pi is entirely unforgiving; it soaks into all parts of the machine and a single bit awry leaves detectable consequences.

3. Matters Computational

I am ashamed to tell you to how many figures I carried these calculations, having no other business at the time.

Isaac Newton

Newton's embarrassment at having computed 15 digits, which he did using the arcsinlike formula

$$\pi = \frac{3\sqrt{3}}{4} + 24\left(\frac{1}{12} - \frac{1}{5\cdot 2^5} - \frac{1}{28\cdot 2^7} - \frac{1}{72\cdot 2^9} - \cdots\right)$$
$$= \frac{3\sqrt{3}}{4} + 24\int_0^{\frac{1}{4}}\sqrt{x - x^2}\,dx,$$

is indicative both of the spirit in which people calculate digits and the fact that a surprising number of people have succumbed to the temptation **[5]**.

The history of efforts to determine an accurate value for the constant we now know as π is almost as long as the history of civilization itself. By 2000 B.C. both the Babylonians and the Egyptians knew π to nearly two decimal places. The Babylonians used, among others, the value 3 1/8 and the Egyptians used 3 13/81. Not all ancient societies were as accurate, however—nearly 1500 years later the Hebrews were perhaps still content to use the value 3, as the following quote suggests.

Also, he made a molten sea of ten cubits from brim to brim, round in compass, and five cubits the height thereof; and a line of thirty cubits did compass it round about.

Old Testament, 1 Kings 7:23

Despite the long pedigree of the problem, all nonempirical calculations have employed, up to minor variations, only three techniques.

Then

$$0 < \alpha_n - 1/\pi < 16 \cdot 4^n e^{-2 \cdot 4^n \pi}$$

and α_n converges to $1/\pi$ *quartically* (that is, with order four).

Algorithm 2. Let $s_0 := 5(\sqrt{5} - 2)$ and $\alpha_0 := 1/2$.
Let

$$s_{n+1} := \frac{25}{(z + x/z + 1)^2 s_n},$$

where

$$x := 5/s_n - 1 \qquad y := (x-1)^2 + 7$$

and

$$z := \left[\frac{1}{2}x\left(y + \sqrt{y^2 - 4x^3}\right)\right]^{1/5}.$$

Let

$$\alpha_{n+1} := s_n^2 \alpha_n - 5^n \left\{ \frac{s_n^2 - 5}{2} + \sqrt{s_n\left(s_n^2 - 2s_n + 5\right)} \right\}.$$

Then

$$0 < \alpha_n - \frac{1}{\pi} < 16 \cdot 5^n e^{-5^n \pi}$$

and α_n converges to $1/\pi$ *quintically* (that is, with order five).

Each additional term in Sum 1 adds roughly eight digits, each additional iteration of Algorithm 1 quadruples the number of correct digits, while each additional iteration of Algorithm 2 quintuples the number of correct digits. Thus a mere thirteen iterations of Algorithm 2 provide in excess of one billion decimal digits of pi. In general, for us, pth-order convergence of a sequence $\{\alpha_n\}$ to α means that α_n tends to α and that

$$|\alpha_{n+1} - \alpha| \le C|\alpha_n - \alpha|^p$$

for some constant $C > 0$. Algorithm 1 is arguably the most efficient algorithm currently known for the extended precision calculation of pi. While the rates of convergence are impressive, it is the subtle and thoroughly nontransparent nature of these results and the beauty of the underlying mathematics that intrigue us most.

Watson [37], commenting on certain formulae of Ramanujan, talks of

> *a thrill which is indistinguishable from the thrill which I feel when I enter the Sagrestia Nuovo of the Capella Medici and see before me the austere beauty of the four statues representing "Day," "Night," "Evening," and "Dawn" which Michelangelo has set over the tomb of Giuliano de'Medici and Lorenzo de'Medici.*

Sum 1 is directly due to Ramanujan and appears in [26]. It rests on a modular identity of order 58 and, like much of Ramanujan's work, appears without proof and with only scanty motivation. The first complete derivation we know of appears

i) The first technique due to Archimedes of Syracuse (287–212 B.C.) is, recursively, to calculate the length of circumscribed and inscribed regular $6 \cdot 2^n$-gons about a circle of diameter 1. Call these quantities a_n and b_n, respectively. Then $a_0 := 2\sqrt{3}$, $b_0 := 3$ and, as Gauss's teacher Pfaff discovered in 1800,

$$a_{n+1} := \frac{2a_n b_n}{a_n + b_n} \quad \text{and} \quad b_{n+1} := \sqrt{a_{n+1} b_n}\,.$$

Archimedes, with $n = 4$, obtained

$$3\tfrac{10}{71} < \pi < 3\tfrac{1}{7}.$$

While hardly better than estimates one could get with a ruler, this is the first method that can be used to generate an arbitrary number of digits, and to a nonnumerical mathematician perhaps the problem ends here. Variations on this theme provided the basis for virtually all calculations of π for the next 1800 years, culminating with a 34 digit calculation due to Ludolph van Ceulen (1540–1610). This demands polygons with about 2^{60} sides and so is extraordinarily time consuming.

ii) Calculus provides the basis for the second technique. The underlying method relies on Gregory's series of 1671

$$\arctan x = \int_0^x \frac{dt}{1+t^2} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \qquad |x| \le 1$$

coupled with a formula which allows small x to be used, like

$$\frac{\pi}{4} = 4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right).$$

This particular formula is due to Machin and was employed by him to compute 100 digits of π in 1706. Variations on this second theme are the basis of all the calculations done until the 1970's including William Shanks' monumental hand-calculation of 527 digits. In the introduction to his book **[32]**, which presents this calculation, Shanks writes:

> *Towards the close of the year 1850 the Author first formed the design of rectifying the circle to upwards of 300 places of decimals. He was fully aware at that time, that the accomplishment of his purpose would add little or nothing to his fame as a Mathematician though it might as a Computer; nor would it be productive of anything in the shape of pecuniary recompense.*

Shanks actually attempted to hand-calculate 707 digits but a mistake crept in at the 527th digit. This went unnoticed until 1945, when D. Ferguson, in one of the last "nondigital" calculations, computed 530 digits. Even with machine calculations mistakes occur, so most record-setting calculations are done twice—by sufficiently different methods.

The advent of computers has greatly increased the scope and decreased the toil of such calculations. Metropolis, Reitwieser, and von Neumann computed and analyzed 2037 digits using Machin's formula on ENIAC in 1949. In 1961, Dan Shanks and Wrench calculated 100,000 digits on an IBM 7090 **[31]**. By 1973, still using Machin-like arctan expansions, the million digit mark was passed by Guillard and Bouyer on a CDC 7600.

iii) The third technique, based on the transformation theory of elliptic integrals, provides the algorithms for the most recent set of computations. The most recent records are due separately to Gosper, Bailey, and Kanada. Gosper in 1985 calculated over 17 million digits (in fact over 17 million terms of the continued fraction) using a carefully orchestrated evaluation of Sum 1.

Bailey in January 1986 computed over 29 million digits using Algorithm 1 on a Cray 2 [**2**]. Kanada, using a related quadratic algorithm (due in basis to Gauss and made explicit by Brent [**12**] and Salamin [**27**]) and using Algorithm 1 for a check, verified 33,554,000 digits. This employed a HITACHI S-810/20, took roughly eight hours, and was completed in September of 1986. In January 1987 Kanada extended his computation to 2^{27} decimal places of π and the hundred million digit mark had been passed. The calculation took roughly a day and a half on a NEC SX2 machine. Kanada's most recent feat (Jan. 1988) was to compute 201,326,000 digits, which required only six hours on a new Hitachi S-820 supercomputer. Within the next few years many hundreds of millions of digits will no doubt have been similarly computed. Further discussion of the history of the computation of pi may be found in [**5**] and [**9**].

4. Complexity Concerns. One of the interesting morals from theoretical computer science is that many familiar algorithms are far from optimal. In order to be more precise we introduce the notion of *bit complexity*. Bit complexity counts the number of single operations required to complete an algorithm. The single-digit operations we count are $+, -, \times$. (We could, if we wished, introduce storage and logical comparison into the count. This, however, doesn't affect the order of growth of the algorithms in which we are interested.) This is a good measure of time on a serial machine. Thus, addition of two n-digit integers by the usual method has bit complexity $O(n)$, and straightforward uniqueness considerations show this to be asymptotically best possible.

Multiplication is a different story. Usual multiplication of two n-digit integers has bit complexity $O(n^2)$ and no better. However, it is possible to multiply two n-digit integers with complexity $O(n(\log n)(\log\log n))$. This result is due to Schönhage and Strassen and dates from 1971 [**29**]. It provides the best bound known for multiplication. No multiplication can have speed better than $O(n)$. Unhappily, more exact results aren't available.

The original observation that a faster than $O(n^2)$ multiplication is possible was due to Karatsuba in 1962. Observe that

$$(a + b10^n)(c + d10^n) = ac + [(a-b)(c-d) - ac - bd]10^n + bd10^{2n},$$

and thus multiplication of two $2n$-digit integers can be reduced to three multiplications of n-digit integers and a few extra additions. (Of course multiplication by 10^n is just a shift of the decimal point.) If one now proceeds recursively one produces a multiplication with bit complexity

$$O(n^{\log_2 3}).$$

Note that $\log_2 3 = 1.58\ldots < 2$.

We denote by $M(n)$ the bit complexity of multiplying two n-digit integers together by any method that is at least as fast as usual multiplication.

The trick to implementing high precision arithmetic is to get the multiplication right. Division and root extraction piggyback off multiplication using Newton's method. One may use the iteration

$$x_{k+1} = 2x_k - x_k^2 y$$

to compute $1/y$ and the iteration

$$x_{k+1} = \frac{1}{2}\left(x_k + \frac{y}{x_k}\right)$$

to compute $\sqrt{y}$. One may also compute $1/\sqrt{y}$ from

$$x_{k+1} = \frac{x_k(3 - yx_k^2)}{2}$$

and so avoid divisions in the computation of $\sqrt{y}$. Not only do these iterations converge quadratically but, because Newton's method is self-correcting (a slight perturbation in x_k does not change the limit), it is possible at the kth stage to work only to precision 2^k. If division and root extraction are so implemented, they both have bit complexity $O(M(n))$, in the sense that n-digit input produces n-digit accuracy in a time bounded by a constant times the speed of multiplication. This extends in the obvious way to the solution of any algebraic equation, with the startling conclusion that every algebraic number can be computed (to n-digit accuracy) with bit complexity $O(M(n))$. Writing down n-digits of $\sqrt{2}$ or $3\sqrt{7}$ is (up to a constant) no more complicated than multiplication.

The Schönhage-Strassen multiplication is hard to implement. However, a multiplication with complexity $O((\log n)^{2+\varepsilon} n)$ based on an ordinary complex (floating point) fast Fourier transform is reasonably straightforward. This is Kanada's approach, and the recent records all rely critically on some variations of this technique.

To see how the fast Fourier transform may be used to accelerate multiplication, let $x := (x_0, x_1, x_2, \dots, x_{n-1})$ and $y := (y_0, y_1, y_2, \dots, y_{n-1})$ be the representations of two high-precision numbers in some radix b. The radix b is usually selected to be some power of 2 or 10 whose square is less than the largest integer exactly representable as an ordinary floating-point number on the computer being used. Then, except for releasing each "carry," the product $z := (z_0, z_1, z_2, \dots, z_{2n-1})$ of x and y may be written as

$$\begin{aligned}
z_0 &= x_0 y_0 \\
z_1 &= x_0 y_1 + x_1 y_0 \\
z_2 &= x_0 y_2 + x_1 y_1 + x_2 y_0 \\
&\vdots \\
z_{n-1} &= x_0 y_{n-1} + x_1 y_{n-2} + \cdots + x_{n-1} y_0 \\
&\vdots \\
z_{2n-3} &= x_{n-1} y_{n-2} + x_{n-2} y_{n-1} \\
z_{2n-2} &= x_{n-1} y_{n-1} \\
z_{2n-1} &= 0.
\end{aligned}$$

Now consider x and y to have n zeros appended, so that x, y, and z all have length $N = 2n$. Then a key observation may be made: the product sequence z is precisely the discrete convolution $C(x, y)$:

$$z_k = C_k(x, y) = \sum_{j=0}^{N-1} x_j y_{k-j},$$

where the subscript $k - j$ is to be interpreted as $k - j + N$ if $k - j$ is negative.

Now a well-known result of Fourier analysis may be applied. Let $F(x)$ denote the *discrete Fourier transform* of the sequence x, and let $F^{-1}(x)$ denote the inverse discrete Fourier transform of x:

$$F_k(x) := \sum_{j=0}^{N-1} x_j e^{-2\pi ijk/N}$$

$$F_k^{-1}(x) := \frac{1}{N} \sum_{j=0}^{N-1} x_j e^{2\pi ijk/N}.$$

Then the "convolution theorem," whose proof is a straightforward exercise, states that

$$F[C(x, y)] = F(x)F(y)$$

or, expressed another way,

$$C(x, y) = F^{-1}[F(x)F(y)].$$

Thus the entire multiplication pyramid z can be obtained by performing two forward discrete Fourier transforms, one vector complex multiplication and one inverse transform, each of length $N = 2n$. Once the real parts of the resulting complex numbers have been rounded to the nearest integer, the final multiprecision product may be obtained by merely releasing the carries modulo b. This may be done by starting at the end of the z vector and working backward, as in elementary school arithmetic, or by applying other schemes suitable for vector processing on more sophisticated computers.

A straightforward implementation of the above procedure would not result in any computational savings—in fact, it would be several times more costly than the usual "schoolboy" scheme. The reason this scheme is used is that the discrete Fourier transform may be computed much more rapidly using some variation of the well-known "fast Fourier transform" (FFT) algorithm [**13**]. In particular, if $N = 2^m$, then the discrete Fourier transform may be evaluated in only $5m2^m$ arithmetic operations using an FFT. Direct application of the definition of the discrete Fourier transform would require 2^{2m+3} floating-point arithmetic operations, even if it is assumed that all powers of $e^{-2\pi i/N}$ have been precalculated.

This is the basic scheme for high-speed multiprecision multiplication. Many details of efficient implementations have been omitted. For example, it is possible to take advantage of the fact that the input sequences x and y and the output sequence z are all purely real numbers, and thereby sharply reduce the operation count. Also, it is possible to dispense with complex numbers altogether in favor of performing computations in fields of integers modulo large prime numbers. Interested readers are referred to [**2**], [**8**], [**13**], and [**22**].

When the costs of all the constituent operations, using the best known techniques, are totalled both Algorithms 1 and 2 compute n digits of π with bit complexity $O(M(n)\log n)$, and use $O(\log n)$ full precision operations.

The bit complexity for Sum 1, or for π using any of the arctan expansions, is between $O((\log n)^2 M(n))$ and $O(nM(n))$ depending on implementation. In each case, one is required to sum $O(n)$ terms of the appropriate series. Done naively, one obtains the latter bound. If the calculation is carefully orchestrated so that the terms are grouped to grow evenly in size (as rational numbers) then one can achieve the former bound, but with no corresponding reduction in the number of operations.

The Archimedean iteration of section 2 converges like $1/4^n$ so in excess of n iterations are needed for n-digit accuracy, and the bit complexity is $O(nM(n))$.

Almost any familiar transcendental number such as e, γ, $\zeta(3)$, or Catalan's constant (presuming the last three to be nonalgebraic) can be computed with bit complexity $O((\log n)M(n))$ or $O((\log n)^2 M(n))$. None of these numbers is known to be computable essentially any faster than this. In light of the previous observation that algebraic numbers are all computable with bit complexity $O(M(n))$, a proof that π cannot be computed with this speed would imply the transcendence of π. It would, in fact, imply more, as there are transcendental numbers which have complexity $O(M(n))$. An example is $0.10100100001\ldots$.

It is also reasonable to speculate that computing the nth digit of π is not very much easier than computing all the first n digits. We think it very probable that computing the nth digit of π cannot be $O(n)$.

5. The Miracle of Theta Functions

When I was a student, abelian functions were, as an effect of the Jacobian tradition, considered the uncontested summit of mathematics, and each of us was ambitious to make progress in this field. And now? The younger generation hardly knows abelian functions.

Felix Klein **[21]**

Felix Klein's lament from a hundred years ago has an uncomfortable timelessness to it. Sadly, it is now possible never to see what Bochner referred to as "the miracle of the theta functions" in an entire university mathematics program. A small pieec of this miracle is required here **[6]**, **[11]**, **[28]**. First some standard notations. The *complete elliptic integrals of the first and second kind*, respectively,

$$K(k) := \int_0^{\pi k} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}} \tag{5.1}$$

and

$$E(k) := \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 t}\, dt. \tag{5.2}$$

The second integral arises in the rectification of the ellipse, hence the name elliptic integrals. The *complementary modulus* is

$$k' := \sqrt{1 - k^2}$$

and the *complementary integrals* K' and E' are defined by

$$K'(k) := K(k') \quad \text{and} \quad E'(k) := E(k').$$

The first remarkable identity is *Legendre's relation* namely

$$E(k)K'(k) + E'(k)K(k) - K(k)K'(k) = \frac{\pi}{2} \tag{5.3}$$

(for $0 < k < 1$), which is pivotal in relating these quantities to pi. We also need to define two *Jacobian theta functions*

$$\Theta_2(q) := \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2} \tag{5.4}$$

and

$$\Theta_3(q) := \sum_{n=-\infty}^{\infty} q^{n^2}. \tag{5.5}$$

These are in fact specializations with ($t = 0$) of the general theta functions. More generally

$$\Theta_3(t, q) := \sum_{n=-\infty}^{\infty} q^{n^2} e^{\operatorname{im} t} \qquad (\operatorname{im} t > 0)$$

with similar extensions of Θ_2. In Jacobi's approach these general theta functions provide the basic building blocks for elliptic functions, as functions of t (see [**11**], [**39**]).

The complete elliptic integrals and the special theta functions are related as follows. For $|q| < 1$

$$K(k) = \frac{\pi}{2}\Theta_3^2(q) \tag{5.6}$$

and

$$E(k) = (k')^2\left[K(k) + k\frac{dK(k)}{dk}\right], \tag{5.7}$$

where

$$k := k(q) = \frac{\Theta_2^2(q)}{\Theta_3^2(q)}, \qquad k' := k'(q) = \frac{\Theta_3^2(-q)}{\Theta_3^2(q)} \tag{5.8}$$

and

$$q = e^{-\pi K'(k)/K(k)}. \tag{5.9}$$

The *modular function* λ is defined by

$$\lambda(t) := \lambda(q) := k^2(q) := \left[\frac{\Theta_2(q)}{\Theta_3(q)}\right]^4, \tag{5.10}$$

where

$$q := e^{i\pi t}.$$

We wish to make a few comments about modular functions in general before restricting our attention to the particular modular function λ. *Modular functions* are functions which are meromorphic in H, the upper half of the complex plane, and which are invariant under a group of linear fractional transformations, G, in the sense that

$$f(g(z)) = f(z) \qquad \forall g \in G.$$

[Additional growth conditions on f at certain points of the associated fundamental region (see below) are also demanded.] We restrict G to be a subgroup of the *modular group* Γ where Γ is the set of all transformations w of the form

$$w(t) = \frac{at+b}{ct+d},$$

with a, b, c, d integers and $ad - bc = 1$. Observe that Γ is a group under composition. A *fundamental region* F_G is a set in H with the property that any element in H is uniquely the image of some element in F_G under the action of G. Thus the behaviour of a modular function is uniquely determined by its behaviour on a fundamental region.

Modular functions are, in a sense, an extension of elliptic (or doubly periodic) functions—functions such as *sn* which are invariant under linear transformations and which arise naturally in the inversion of elliptic integrals.

The definitions we have given above are not complete. We will be more precise in our discussion of λ. One might bear in mind that much of the theory for λ holds in considerably greater generality.

The *fundamental region* F we associate with λ is the set of complex numbers

$$F := \{\operatorname{im} t \geq 0\} \cap \big[\{|\operatorname{re} t| < 1 \quad \text{and} \quad |2t \pm 1| > 1\} \cup \{\operatorname{re} t = -1\} \cup \{|2t+1| = 1\}\big].$$

The λ-*group* (or theta-subgroup) is the set of linear fractional transformations w satisfying

$$w(t) := \frac{at+b}{ct+d},$$

where a, b, c, d are integers and $ad - bc = 1$, while in addition a and d are odd and b and c are even. Thus the corresponding matrices are unimodular. What makes λ a λ-modular function is the fact that λ is meromorphic in $\{\operatorname{im} t > 0\}$ and that

$$\lambda(w(t)) := \lambda(t)$$

for all w in the λ-group, plus the fact that λ tends to a definite limit (possibly infinite) as t tends to a vertex of the fundamental region (one of the three points $(0, -1)$, $(0, 0)$, (i, ∞)). Here we only allow convergence from within the fundamental region.

Now some of the miracle of modular functions can be described. Largely because every point in the upper half plane is the image of a point in F under an element of the λ-group, one can deduce that any λ-modular function that is bounded on F is constant. Slightly further into the theory, but relying on the above, is the result that any two modular functions are algebraically related, and resting on this, but further again into the field, is the following remarkable result. Recall that q is given by (5.9).

THEOREM 1. *Let z be a primitive pth root of unity for p an odd prime. Consider the pth order modular equation for λ as defined by*

$$W_p(x, \lambda) := (x - \lambda_0)(x - \lambda_1) \cdots (x - \lambda_p), \tag{5.11}$$

where

$$\lambda_i := \lambda(z^i q^{1/p}) \qquad i < p$$

and

$$\lambda_p := \lambda(q^p).$$

*Then the function W_p is a polynomial in x and λ (**independent** of q), which has integer coefficients and is of degree $p + 1$ in both x and λ.*

The modular equation for λ usually has a simpler form in the associated variables $u := x^{1/8}$ and $v := \lambda^{1/8}$. In this form the 5th-order modular equation is given by

$$\Omega_5(u, v) := u^6 - v^6 + 5u^2v^2(u^2 - v^2) + 4uv(1 - u^4v^4). \tag{5.12}$$

In particular

$$\frac{\Theta_2(q^p)}{\Theta_3(q^p)} = v^2 \quad \text{and} \quad \frac{\Theta_2(q)}{\Theta_3(q)} = u^2$$

are related by an algebraic equation of degree $p + 1$.

The miracle is not over. The *pth-order multiplier* (for λ) is defined by

$$M_p(k(q), k(q^p)) := \frac{K(k(q^p))}{K(k(q))} = \left[\frac{\Theta_3(q^p)}{\Theta_3(q)}\right]^2 \tag{5.13}$$

and turns out to be a rational function of $k(q^p)$ and $k(q)$.

One is now in possession of a pth-order algorithm for K/π, namely: Let $k_i := k(q^{p^i})$. Then

$$\frac{2K(k_0)}{\pi} = M_p^{-1}(k_0, k_1) M_p^{-1}(k_1, k_2) M_p^{-1}(k_2, k_3) \cdots.$$

This is an entirely algebraic algorithm. One needs to know the pth-order modular equation for λ to compute k_{i+1} from k_i and one needs to know the rational multiplier M_p. The speed of convergence ($O(c^{p^i})$, for some $c < 1$) is easily deduced from (5.13) and (5.9).

The function $\lambda(t)$ is 1-1 on F and has a well-defined inverse, λ^{-1}, with branch points only at 0, 1 and ∞. This can be used to provide a one line proof of the "big" Picard theorem that a nonconstant entire function misses at most one value (as does exp). Indeed, suppose g is an entire function and that it is never zero or one; then $\exp(\lambda^{-1}(g(z)))$ is a bounded entire function and is hence constant.

Littlewood suggested that, at the right point in history, the above would have been a strong candidate for a 'one line doctoral thesis'.

6. Ramanujan's Solvable Modular Equations. Hardy [**19**] commenting on Ramanujan's work on elliptic and modular functions says

> *It is here that both the profundity and limitations of Ramanujan's knowledge stand out most sharply.*

We present only one of Ramanujan's modular equations.

THEOREM 2.

$$\frac{5\Theta_3(q^{25})}{\Theta_3(q)} = 1 + r_1^{1/5} + r_2^{1/5}, \tag{6.1}$$

where for $i = 1$ *and* 2

$$r_1 := \tfrac{1}{2}x\left(y \pm \sqrt{y^2 - 4x^3}\right)$$

with

$$x := \frac{5\Theta_3(q^5)}{\Theta_3(q)} - 1 \quad \text{and} \quad y := (x-1)^2 + 7.$$

This is a slightly rewritten form of entry 12(iii) of Chapter 19 of Ramanujan's *Second Notebook* (see [7], where Berndt's proofs may be studied). One can think of Ramanujan's quintic modular equation as an equation in the multiplier M_p of (5.13). The initial surprise is that it is solvable. The quintic modular relation for λ, W_5, and the related equation for $\lambda^{1/8}$, Ω_5, are both nonsolvable. The Galois group of the sixth-degree equation Ω_5 (see (5.12)) over $\mathbb{Q}(v)$ is A_5 and is nonsolvable. Indeed both Hermite and Kronecker showed, in the middle of the last century, that the solution of a general quintic may be effected in terms of the solution of the 5th-order modular equation (5.12) and the roots may thus be given in terms of the theta functions.

In fact, in general, the Galois group for W_p of (5.11) has order $p(p+1)(p-1)$ and is never solvable for $p \geq 5$. The group is quite easy to compute, it is generated by two permutations. If

$$q := e^{i\pi t}, \quad \text{then} \quad \tau \to \tau + 2 \quad \text{and} \quad \tau \to \frac{\tau}{(2\tau+1)}$$

are both elements of the λ-group and induce permutations on the λ_i of Theorem 1. For any fixed p, one can use the q-expansion of (5.10) to compute the effect of these transformations on the λ_i, and can thus easily write down the Galois group.

While W_p is not solvable over $\mathbb{Q}(\lambda)$, it is solvable over $\mathbb{Q}(\lambda, \lambda_0)$. Note that λ_0 is a root of W_p. It is of degree $p + 1$ because W_p is irreducible. Thus the Galois group for W_p over $\mathbb{Q}(\lambda, \lambda_0)$ has order $p(p-1)$. For $p = 5$, 7, and 11 this gives groups of order 20, 42, and 110, respectively, which are obviously solvable and, in fact, for general primes, the construction always produces a solvable group.

From (5.8) and (5.10) one sees that Ramanujan's modular equation can be rewritten to give λ_5 solvable in terms of λ_0 and λ. Thus, we can hope to find an explicit solvable relation for λ_p in terms of λ and λ_0. For $p = 3$, W_p is of degree 4 and is, of course, solvable. For $p = 7$, Ramanujan again helps us out, by providing a solvable seventh-order modular identity for the closely related *eta function* defined by

$$\eta(q) := q^{\frac{1}{12}} \prod_{n=1}^{\infty} (1 - q^{2n}).$$

The first interesting prime for which an explicit solvable form is not known is the "endecadic" ($p = 11$) case. We consider only prime values because for nonprime values the modular equation factors.

This leads to the interesting problem of mechanically constructing these equations. In principle, and to some extent in practice, this is a purely computational problem. Modular equations can be computed fairly easily from (5.11) and even more easily in the associated variables u and v. Because one knows a priori bounds on the size of the (integer) coefficients of the equations one can perform these calculations exactly. The coefficients of the equation, in the variables u and v, grow at most like 2^n. (See [**11**].) Computing the solvable forms and the associated computational problems are a little more intricate—though still in principle entirely mechanical. A word of caution however: in the variables u and v the endceadic modular equation has largest coefficient 165, a three digit integer. The endecadic modular equation for the intimately related function J (Klein's *absolute invariant*) has coefficients as large as

$$27090964785531389931563200281035226311929052227303 \times 2^{92}3^{19}5^{20}11^{2}53.$$

It is, therefore, one thing to solve these equations, it is entirely another matter to present them with the economy of Ramanujan.

The paucity of Ramanujan's background in complex analysis and group theory leaves open to speculation Ramanujan's methods. The proofs given by Berndt are difficult. In the seventh-order case, Berndt was aided by MACSYMA—a sophisticated algebraic manipulation package. Berndt comments after giving the proof of various seventh-order modular identities:

> *Of course, the proof that we have given is quite unsatisfactory because it is a verification that could not have been achieved without knowledge of the result. Ramanujan obviously possessed a more natural, transparent, and ingenious proof.*

7. Modular Equations and Pi. We wish to connect the modular equations of Theorem 1 to pi. This we contrive via the function *alpha* defined by:

$$\alpha(r) := \frac{E'(k)}{K(k)} - \frac{\pi}{(2K(k))^2}, \tag{7.1}$$

where

$$k := k(q) \quad \text{and} \quad q := e^{-\pi\sqrt{r}}.$$

This allows one to rewrite Legendre's equation (5.3) in a one-sided form without the conjugate variable as

$$\frac{\pi}{4} = K\left[\sqrt{r}\,E - \left(\sqrt{r} - \alpha(r)\right)K\right]. \tag{7.2}$$

We have suppressed, and will continue to suppress, the k variable. With (5.6) and (5.7) at hand we can write a q-expansion for α, namely,

$$\alpha(r) = \frac{\dfrac{1}{\pi} - \sqrt{r}\,4\dfrac{\sum_{n=-\infty}^{\infty} n^2(-q)^{n^2}}{\sum_{n=-\infty}^{\infty} (-q)^{n^2}}}{\left[\sum_{n=-\infty}^{\infty} q^{n^2}\right]^4}, \tag{7.3}$$

and we can see that as r tends to infinity $q = e^{-\pi\sqrt{r}}$ tends to zero and $\alpha(r)$ tends to $1/\pi$. In fact

$$\alpha(r) - \frac{1}{\pi} \approx 8\left(\sqrt{r} - \frac{1}{\pi}\right)e^{-\pi\sqrt{r}}. \tag{7.4}$$

The key now is iteratively to calculate α. This is the content of the next theorem.

THEOREM 3. *Let $k_0 := k(q), k_1 := k(q^p)$ and $M_p := M_p(k_0, k_1)$ as in (5.13). Then*

$$\alpha(p^2r) = \frac{\alpha(r)}{M_p^2} - \sqrt{r}\left[\frac{k_0^2}{M_p^2} - pk_1^2 + \frac{pk_1'^2k_1\dot{M}_p}{M_p}\right],$$

where represents the full derivative of M_p with respect to k_0. In particular, α is algebraic for rational arguments.

We know that $K(k_1)$ is related via M_p to $K(k)$ and we know that $E(k)$ is related via differentiation to K. (See (5.7) and (5.13).) Note that $q \to q^p$ corresponds to $r \to p^2r$. Thus from (7.2) some relation like that of the above theorem must exist. The actual derivation requires some careful algebraic manipulation. (See [**11**], where it has also been made entirely explicit for $p := 2, 3, 4, 5$, and 7, and where numerous algebraic values are determined for $\alpha(r)$.) In the case $p := 5$ we can specialize with some considerable knowledge of quintic modular equations to get:

THEOREM 4. *Let $s := 1/M_5(k_0, k_1)$. Then*

$$\alpha(25r) = s^2\alpha(r) - \sqrt{r}\left[\frac{(s^2-5)}{2} + \sqrt{s(s^2-2s+5)}\right].$$

This couples with Ramanujan's quintic modular equation to provide a derivation of Algorithm 2.

Algorithm 1 results from specializing Theorem 3 with $p := 4$ and coupling it with a quartic modular equation. The quartic equation in question is just two steps of the corresponding quadratic equation which is Legendre's form of the *arithmetic geometric mean iteration*, namely:

$$k_1 = \frac{2\sqrt{k}}{1+k}.$$

An algebraic pth-order algorithm for π is derived from coupling Theorem 3 with a pth-order modular equation. The substantial details which are skirted here are available in [**11**].

8. Ramanujan's sum. This amazing sum,

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801}\sum_{n=0}^{\infty}\frac{(4n)!}{(n!)^4}\frac{[1103 + 26390n]}{396^{4n}}$$

is a specialization ($N = 58$) of the following result, which gives reciprocal series for π in terms of our function alpha and related modular quantities.

THEOREM 5.

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{3}{4}\right)_n d_n(N)}{(n!)^3} x_N^{2n+1}, \tag{8.1}$$

where,

$$x_N := \frac{4k_N(k_N')^2}{(1+k_N^2)^2} := \left(\frac{g_N^{12}+g_N^{-12}}{2}\right)^{-1},$$

with

$$d_n(N) = \left[\frac{\alpha(N)x_N^{-1}}{1+k_N^2} - \frac{\sqrt{N}}{4}g_N^{-12}\right] + n\sqrt{N}\left(\frac{g_N^{12}-g_N^{-12}}{2}\right)$$

and

$$k_N := k\left(e^{-\pi\sqrt{N}}\right), \qquad g_N^{12} = (k_N')^2/(2k_N).$$

Here $(c)_n$ is the rising factorial: $(c)_n := c(c+1)(c+2)\cdots(c+n-1)$.

Some of the ingredients for the proof of Theorem 5, which are detailed in [**11**], are the following. Our first step is to write (7.2) as a sum after replacing the E by K and dK/dk using (5.7). One then uses an identity of Clausen's which allows one to write the square of a hypergeometric function ${}_2F_1$ in terms of a generalized hypergeometric ${}_3F_2$, namely, for all k one has

$$(1+k^2)\left[\frac{2K(k)}{\pi}\right]^2 = {}_3F_2\left(\frac{1}{4}, \frac{3}{4}, \frac{1}{2}, 1, 1 : \left(\frac{2}{g^{12}+g^{-12}}\right)^2\right)$$

$$= \sum_{n=0}^{\infty} \frac{\left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n \left(\frac{1}{2}\right)_n}{(1)_n(1)_n} \frac{\left(\frac{2}{g^{12}+g^{-12}}\right)^{2n}}{n!}.$$

Here g is related to k by

$$\frac{4k(k')^2}{(1+k^2)^2} = \left(\frac{g^{12}+g^{-12}}{2}\right)^{-1}$$

as required in Theorem 5. We have actually done more than just use Clausen's identity, we have also transformed it once using a standard hypergeometric substitution due to Kummer. Incidentally, Clausen was a nineteenth-century mathematician who, among other things, computed 250 digits of π in 1847 using Machin's formula. The desired formula (8.1) is obtained on combining these pieces.

Even with Theorem 5, our work is not complete. We still have to compute

$$k_{58} := k\left(e^{-\pi\sqrt{58}}\right) \quad \text{and} \quad \alpha_{58} := \alpha(58).$$

In fact

$$g_{58}^2 = \left(\frac{\sqrt{29}+5}{2}\right)$$

is a well-known *invariant* related to the fundamental solution to Pell's equation for 29 and it turns out that

$$\alpha_{58} = \left(\frac{\sqrt{29}+5}{2}\right)^6 (99\sqrt{29} - 444)(99\sqrt{2} - 70 - 13\sqrt{29}).$$

One can, in principle, and for $N = 58$, probably in practice, solve for k_N by directly solving the Nth-order equation

$$W_N\left(k_N^2, 1 - k_N^2\right) = 0.$$

For $N = 58$, given that Ramanujan [**26**] and Weber [**38**] have calculated g_{58} for us, verification by this method is somewhat easier though it still requires a tractable form of W_{58}. Actually, more sophisticated number-theoretic techniques exist for computing k_N (these numbers are called *singular moduli*). A description of such techniques, including a reconstruction of how Ramanujan might have computed the various singular moduli he presents in [**26**]; is presented by Watson in a long series of papers commencing with [**36**]; and some more recent derivations are given in [**11**] and [**30**]. An inspection of Theorem 5 shows that all the constants in Series 1 are determined from g_{58}. Knowing α is equivalent to determining that the number 1103 is correct.

It is less clear how one explicitly calculates α_{58} in algebraic form, except by brute force, and a considerable amount of brute force is required; but a numerical calculation to any reasonable accuracy is easily obtained from (7.3) and 1103 appears! The reader is encouraged to try this to, say, 16 digits. This presumably is what Ramanujan observed. Ironically, when Gosper computed 17 million digits of π using Sum 1, he had no mathematical proof that Sum 1 actually converged to $1/\pi$. He compared ten million digits of the calculation to a previous calculation of Kanada et al. This verification that Sum 1 is correct to ten million places also provided the first complete proof that α_{58} is as advertised above. A nice touch—that the calculation of the sum should prove itself as it goes.

Roughly this works as follows. One knows enough about the exact algebraic nature of the components of $d_n(N)$ and x_N to know that if the purported sum (of positive terms) were incorrect, that before one reached 3 million digits, this sum must have ceased to agree with $1/\pi$. Notice that the components of Sum 1 are related to the solution of an equation of degree 58, but virtually no irrationality remains in the final packaging. Once again, there are very good number-theoretic reasons, presumably unknown to Ramanujan, why this must be so (58 is at least a good candidate number for such a reduction). Ramanujan's insight into this marvellous simplification remains obscure.

Ramanujan [**26**] gives 14 other series for $1/\pi$, some others almost as spectacular as Sum 1—and one can indeed derive some even more spectacular related series.* He gives almost no explanation as to their genesis, saying only that there are "corresponding theories" to the standard theory (as sketched in section 5) from which they follow. Hardy, quoting Mordell, observed that "it is unfortunate that Ramanujan has not developed the corresponding theories." By methods analogous

*(Added in proof) Many related series due to Borwein and Borwein and to Chudnovsky and Chudnovsky appear in papers in *Ramanujan Revisited*, Academic Press, 1988.

to those used above, all his series can be derived from the classical theory [**11**]. Again it is unclear what passage Ramanujan took to them, but it must in some part have diverged from ours.

We conclude by writing down another extraordinary series of Ramanujan's, which also derives from the same general body of theory,

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \binom{2n}{n}^3 \frac{42n+5}{2^{12n+4}}.$$

This series is composed of fractions whose numerators grow like 2^{6n} and whose denominators are exactly $16 \cdot 2^{12n}$. In particular this can be used to calculate the second block of n binary digits of π without calculating the first n binary digits. This beautiful observation, due to Holloway, results, disappointingly, in no intrinsic reduction in complexity.

9. Sources. References [**7**], [**11**], [**19**], [**26**], [**36**], and [**37**] relate directly to Ramanujan's work. References [**2**], [**8**], [**9**], [**10**], [**12**], [**22**], [**24**], [**27**], [**29**], and [**31**] discuss the computational concerns of the paper.

Material on modular functions and special functions may be pursued in [**1**], [**6**], [**9**], [**14**], [**15**], [**18**], [**20**], [**28**], [**34**], [**38**], and [**39**]. Some of the number-theoretic concerns are touched on in [**3**], [**6**], [**9**], [**11**], [**16**], [**23**], and [**35**].

Finally, details of all derivations are given in [**11**].

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A Singular Genius

The Man Who Knew Infinity. A Life of the Indian Genius Ramanujan. ROBERT KANIGEL. Scribner, New York, 1991. x, 438 pp. + plates. $27.95.

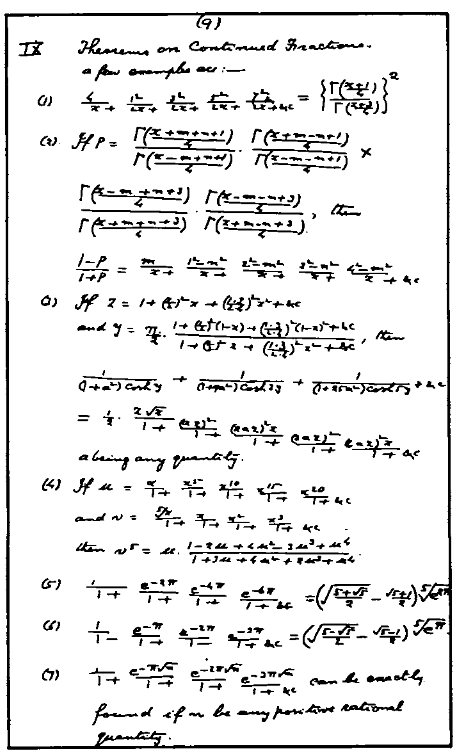
This is the romantic and ultimately tragic story of the singular mathematical genius Srinivasa Ramanujan. An easy tale to tell badly; over the years the story has been much contaminated by apocrypha and misinterpretation. The outline, however, is straightforward enough. Born in 1887 into a poor but high-caste (Brahmin) family from Kumbakonam in southern India, Ramanujan, against seemingly impossible odds, became a major mathematical figure. He was inadequately and incompletely educated, and though he exhibited a precocious gift for mathematics he was unable to complete an orthodox mathematical training. He lived in poverty and disease and bore the scars of smallpox. Too poor at times to afford paper, he did much of his mathematics with chalk and slate. Yet by the age of 25 he had, in relative isolation, discovered and rediscovered a tremendous body of mathematics. Ramanujan communicated these results to some of the leading English mathematicians of the day, probably including H. F. Baker and E. W. Hobson, who never responded, presumably dismissing Ramanujan as a crank. But the preeminent English mathematician of the period, G. H. Hardy (1877–1947), and his great collaborator, J. E. Littlewood, recognized in Ramanujan a touch of genius. Hardy would later write of the results Ramanujan had mailed in early 1913 that some of them

> defeated me completely; I had never seen anything the least like them before. A single look at them is enough to show that they could only be written down by a mathematician of the highest class. They must have been true because, if they were not true, no one would have had the imagination to invent them.

Hardy initiated serious efforts to bring Ramanujan to Trinity College, Cambridge. Another Cambridge analyst, E. H. Neville, traveled to India in 1913 to lecture and to secure Ramanujan's agreement—an agreement made difficult because of prevailing Brahmin taboos on travel. Funding was also lacking for a stay originally planned for two years, and it is significant that most of the money was provided by Indians and Anglo-Indians, not by Cambridge.

Thus in 1914 Ramanujan arrived in England: 26 years old, a devout Brahmin and vegetarian. He was unready for Cambridge ritual and English reserve. A brief but wonderfully fruitful collaboration with Hardy followed. It married Hardy's superb technical skills and knowledge to Ramanujan's intuition and uncanny capacity to divine identities. From 1914 to 1919 they produced a number of important and beautiful joint papers on number theory.

While the collaboration flourished Ramanujan's physical and mental health decayed. Most of 1917 and 1918 were spent in sanatoria. Ramanujan was diagnosed as having tuberculosis, no doubt exacerbated by wartime rationing and his strict vegetarian diet. In 1917 he was turned down for a Trinity fellowship and for membership in the Royal Society. An unsuccessful suicide attempt followed early in 1918 (he jumped in front of a London Underground train). His declining health may have precipitated a change of heart on the part of the Royal Society. Ramanujan learned in late February 1918 that he would become an F.R.S., and that autumn Littlewood succeeded in having him elected to a fellowship in Trinity in the face of opposition, some of an openly racist nature. When the war ended he returned to India, where he died prematurely in 1920. His extraordinary final work, produced while he lay dying, is now often and controversially identified as the "Lost Notebook." (Neither a notebook nor lost, it consisted of almost impenetrable notes on loose pages in Trinity's library; it was "discovered" and mathematically illuminated by George Andrews in 1976.)


(9)
IX Theorems on Continued Fractions.
a few examples are :—

A page of Ramanujan's first letter to G. H. Hardy. [From *The Man Who Knew Infinity*; Syndics of Cambridge University Library]

Ramanujan's legacy includes his famous "Notebooks": two large handwritten books densely packed with strange and exotic formulas, usually without much derivation and usually in his own nonstandard terminology. (A sample of the notebooks would have been a pleasant addition to this work.) The task of fleshing out the details in these notes has occupied some very talented mathematicians over the decades and is only now nearing completion. This work covers a profusion of results in the theory of series, integrals, asymptotic analysis, and elliptic and modular functions. It is appearing as three substantial volumes (two of which are already out) edited by Bruce Berndt, with complete proofs provided. Working mathematicians are often reminded of Ramanujan's impact on mathematics by the functions, series, and conjectures that bear his name.

This is the rough cloth of the Ramanujan fabric; the embroidery is more elaborate. All too often Ramanujan is reconstructed as some kind of divinely inspired mystic who rediscovered several millennia of mathematics while walking the dusty roads of southern India. Or, worse, he is painted as an idiot savant and a calculating prodigy. Getting the fabric right is hard, and here *The Man Who Knew Infinity* is most successful.

No, Ramanujan did not recreate all pre-20th-century mathematics by himself, but his education was far from mainstream. His primary source, Carr's 1886 *A Synopsis of Results in Pure and Applied Mathematics*, was a compilation of some 5000 formulas and theorems that covered large parts of 19th-century mathematics. As in Ramanujan's notebooks, little is proved. Still, most of the familiar objects of Ramanujan's mathematical hope chest are introduced and examined by Carr. Nor was Ramanujan entirely self-educated. He did attend college for a period in both Kumbakonam and Madras, failing because of inattention to the nonmathematical curriculum.

Yes, Ramanujan was enormously gifted, particularly in the formal manipulation of series, continued fractions, and the like. But even here he had historical peers, albeit very few, perhaps only Euler and Jacobi.

It is only by the delicate thread of Hardy that Ramanujan escaped falling to obscurity. Had Hardy not recognized Ramanujan, who would have? Hardy called Ramanujan "the one Romantic incident in my life," and perhaps rightly, but the sophisticated, exquisitely educated, and iconoclastic Hardy is almost as interesting a study as Ramanujan himself. Hardy didn't need Ramanujan. Indeed, Ramanujan wasn't even his most famous collaboration. The works of Hardy and Littlewood are so pervasive that it has been said that there were three great English mathematicians of the period: Hardy, Littlewood, and Hardy-Littlewood. But Ramanujan needed Hardy, and as the two stories cannot be separated, Kanigel also provides us with an intriguing portrait of the earlier parts of Hardy's somewhat eccentric life.

Where does Ramanujan belong in history? In raw ability, Hardy rated Ramanujan at 100 and Hilbert at 80, while Littlewood scored 30 and Hardy 25. But Hardy's and Littlewood's individual effects on the stream of mathematics were more profound, as of course were Hilbert's. Nonetheless, Ramanujan is a great figure who had a brief four or five years on the world stage to make his mark. As these years overlapped perfectly with the First World War, contact with Europe was impossible and activity in England was much reduced.

Hardy writing in 1940 concluded of Ramanujan's work:

> It has not the simplicity and inevitableness of the very greatest work; it would be greater if it were less strange. One gift it has which no one can deny, profound and invincible originality. He would probably have been a greater mathematician if he had been caught and tamed in his youth; he would have discovered more that was new, and no doubt, of greater importance. On the other hand he would have been less a Ramanujan, and more of a European professor and the loss might have been greater than the gain.

Today the results seem equally original but perhaps a little less strange.

As Kanigel puts it: "Cut cruelly short, Ramanujan's life bore something of the frustration that a checked swing does in baseball; it lacked follow-through, roundedness, completion." Hardy, an avid sports fan, might have liked this metaphor. Kanigel asks, "Would he have become the next Gauss or Newton?" and wonders whether his genius was built of "sheer intellectual

Indian stamp issued in 1962 to honor Ramanujan. [From *The Man Who Knew Infinity*]

power, different only in degree" from the normal or if it was "steeped in something of the mystical." Reasonably, he equivocates:

> In each case, the evidence left ample room to see it either way. In this sense, Ramanujan's life was like the Bible, or Shakespeare—a rich fund of data, that holds up a mirror to ourselves or our age.

Kanigel both provides the data and holds up the mirror in this superbly crafted biography. The hardest part of mathematical biography is including the mathematics, giving it content and life, without destroying the story. Kanigel does succeed in giving a taste of Ramanujan the mathematician, but his exceptional triumph is in the telling of this wonderful human story.

As children of a mathematician (from Hardy's school), we grew up knowing the rudiments of this story. As mathematicians we have had occasion to work in Ramanujan's garden—to use Freeman Dyson's lovely metaphor. For us this book was a pleasure to read. We hope it is for many others. It is a thoughtful and deeply moving account of a signal life.

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A Gendered Life

Jessie Bernard. The Making of a Feminist. ROBERT C. BANNISTER. Rutgers University Press, New Brunswick, NJ, 1991. xii, 276 pp. + plates. $27.95.

The sociologist Jessie Bernard, now in her late 80s, had already passed the conventional age of retirement when the feminist movement of the late '60s radically transformed her intellectual perspectives and inspired her to begin a new phase of her career. Between the ages of 68 and 84 she published six books (including *The Future of Marriage* and *The Female World*) and dozens of articles, works that are generally viewed as her most original and brilliant. It was in this late period that she achieved eminence in her profession, and it would not be an exaggeration to say that she has been canonized as a "founding mother" of sociology.

A study of Bernard's life and work is a worthy project for several reasons: as a window into the history of 20th-century sociology, as a case study of obstacles that women encounter in academe, as an account of one social scientist's deepening insights about gender. Unfortunately, her present biographer does not display a genuine appreciation or understanding of his subject. His treatment of her life is not only dismissive of her work and excessively focused on her early marriage but mean-spirited in its method and approach.

Bannister announces his opinion of Bernard's work in the introduction, when he explains his book is "not an intellectual history of the analytic or internal variety" because "Bernard has not been a deep thinker." In fact, Bannister typically deals with Bernard's work by providing brief summaries of her books followed by extensive quotations and arguments from her most negative reviewers. One might mistakenly conclude from Bannister's evidence that Bernard never found an appreciative audience. Throughout the book, Bannister characterizes Bernard as intellectually superficial and timid, an ambitious seeker of recognition who was always ready to jump on the latest bandwagon. He minimizes Bernard's later and widely admired work as not being especially revolutionary and observes that she was unable to keep up with the more demanding and current feminist theorists. The best he has to say about Bernard is when, trying to account for her appeal, he grants her "openness to new ideas, an ability to articulate issues before others have done so, and an engaging frankness concerning her own shortcomings."

Bannister misunderstands Bernard's importance for a number of reasons. First, he does not recognize that in her later work she was not following fashion but was well ahead of her time and willing to engage in controversial subjects others ducked. Her insights about the darker sides of marriage and the different worlds occupied by women and men even when they share households were highly original and have had a significant and lasting influence on younger scholars. Her thoughts about the impact of gender on the ways social scientists conceptualize and conduct their work opened up debates that are still of

Computers and Mathematics

Edited by Keith Devlin

This month's column

Experimental mathematics is the theme of this month's feature article, written by the Canadian mathematical brothers, Jonathan and Peter Borwein. This is followed by a number of review articles and a couple of announcements. Paul Abbott compares *Maple* and *Mathematica*. (See also the benchmark test results presented by Barry Simon in the previous column in the September *Notices*.) J. S. Milne provides an update on some reviews he wrote for this column back in October 1990 on scientific word processors. Louis Grey looks at the program *Numbers*, and Tevian Dray reports on the programs *4-dimensional Hypercube* and $f(z)$.

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Some Observations on Computer Aided Analysis

Jonathan Borwein* and Peter Borwein*

Preamble

Over the last quarter Century and especially during the last decade, a dramatic "re-experimentalization" of mathematics has begun to take place. In this process, fueled by advances in hardware, software, and theory, the computer plays a laboratory role for pure and applied mathematicians; a role which, in the eighteenth and nineteenth centuries, the physical sciences played much more fully than in our century.

Operations previously viewed as nonalgorithmic, such as indefinite integration, may now be performed within powerful symbolic manipulation packages like *Maple*, *Mathematica*, *Macsyma*, and *Scratchpad* to name a few. Similarly, calculations previously viewed as "practically" nonalgorithmic or certainly not worth the effort, such as large symbolic Taylor expansions, are computable with very little programming effort.

New subjects such as computational geometry, fractal geometry, turbulence, and chaotic dynamical systems have sprung up. Indeed, many second-order phenomena only become apparent after considerable computational experimentation. Classical subjects like number theory, group theory, and logic have received new infusions. The boundaries between mathematical physics, knot theory, topology, and other pure mathematical disciplines are more blurred than in many generations. Computer assisted proofs of "big" theorems are more and more common: witness the 1976 proof of the Four Colour theorem and the more recent 1989 proof of the non-existence of a projective plane of order ten (by C. Lam et al at Concordia).

There is also a cascading profusion of sophisticated computational and graphical tools. Many mathematicians use them but there are still many who do not. More importantly, expertise is highly focused: researchers in partial differential equations may be at home with numerical finite element packages, or with the NAG or IMSL Software Libraries, but may have little experience with symbolic or graphic languages. Similarly, optimizers may be at home with non-linear programming packages or with *Matlab*. The learning curve for many of these tools is very steep and researchers and students tend to stay with outdated but familiar resources long after these have been superceded by newer software. Also, there is very little methodology for the use of the computer as a general adjunct to research rather than as a means of solving highly particular problems.

We are currently structuring "The Simon Fraser Centre for Experimental and Constructive Mathematics" to provide a focal point for Mathematical research on such questions as

"How does one use the computer:

- to build intuition?
- to generate hypotheses?
- to validate conjectures or prove theorems?

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– to discover nontrivial examples and counterexamples?"

(Since we will be offering a number of graduate student, postdoctoral, and visiting fellowships, we are keen to hear from interested people.)

0. Introduction

Our intention is to display three sets of analytic results which we have obtained over the past few years entirely or principally through directed computer experimentation. While each set in some way involves π, our main interest is in the role of directed discovery in the analysis. The results we display either could not or would not have been obtained without access to high-level symbolic computation. In our case we primarily used *Maple*, but the precise vehicle is not the point. We intend to focus on the pitfalls and promises of what Lakatos called "quasi-inductive" mathematics.

1. Cubic Series for π

The Mathematical Component. Ramanujan [10] produced a number of remarkable series for $1/\pi$ including

$$(1.1)\qquad \frac{1}{\pi} = \frac{2\sqrt{2}}{9801}\sum_{n=0}^{\infty}\frac{(4n)!}{4^{4n}(n!)^4}\frac{[1103+26390n]}{99^{4n}}.$$

This series adds roughly eight digits per term and was used by Gosper in 1985 to compute 17 million terms of the continued fraction for π. Such series exist because various modular invariants are rational (which is more-or-less equivalent to identifying those imaginary quadratic fields with class number 1), see [3]. The larger the discriminant of such a field the greater the rate of convergence. Thus with $d = -163$ we have the largest of the class number 1 examples

$$(1.2)\qquad \frac{1}{\pi} = 12\sum_{n=0}^{\infty}(-1)^n\frac{(6n)!}{(n!)^3(3n)!}\frac{13591409+n545140134}{(640320^3)^{n+1/2}},$$

a series first displayed by the Chudnovskys [10]. The underlying approximation also produces

$$\pi \sim 3\log(640320)/\sqrt{163}$$

and is correct to 16 places.

Quadratic versions of these series correspond to class number two imaginary quadratic fields. The most spectacular and largest example has $d = -427$ and

$$(1.3)\qquad \frac{1}{\pi} = 12\sum_{n=0}^{\infty}\frac{(-1)^n(6n)!}{(n!)^3(3n)!}\frac{(A+nB)}{C^{n+1/2}}$$

where

$$A := 212175710912\sqrt{61} + 1657145277365$$
$$B := 13773980892672\sqrt{61} + 107578229802750$$
$$C := [5280(236674 + 30303\sqrt{61})]^3.$$

This series adds roughly twenty-five digits per term, $\sqrt{C}/(12A)$ already agrees with pi to twenty-five places [3]. The last two series are of the form

$$(*)\qquad \sum_{n=0}^{\infty}(a(t)+nb(t))\frac{(6n)!}{(3n)!(n!)^3}\frac{1}{(j(t))^n} = \frac{\sqrt{-j(t)}}{\pi}$$

where

$$b(t) = (t(1728 - j(t)))^{1/2},$$
$$a(t) = \frac{b(t)}{6}\left(1 - \frac{E_4(t)}{E_6(t)}\left(E_2(t) - \frac{6}{\pi\sqrt{t}}\right)\right),$$
$$j(t) = \frac{1728E_4^3(t)}{E_4^3(t) - E_6^2(t)}.$$

Here t is the appropriate discriminant, j is the "absolute invariant", and E_2, E_4, and E_6 are Eisenstein series.

For a further discussion of these, see [2], where many such quadratic examples are considered. Various of the recent record setting calculations of π have been based on these series. In particular, the Chudnovskys computed over two billion digits of π using the second series above.

There is an unlimited number of such series with increasingly more rapid convergence. The price one pays is that one must deal with more complicated algebraic irrationalities. Thus a class number p field will involve p^{th} degree algebraic integers as the constants $A = a(t)$, $B = b(t)$, and $C = c(t)$ in the series. The largest class number three example of $(*)$ corresponds to $d = -907$ and gives 37 or 38 digits per term. It is

$$(1.4)\qquad \frac{\sqrt{-C^3}}{\pi} = \sum_{n=0}^{\infty}\frac{(6n)!}{(3n)!(n!)^3}\frac{A+nB}{C^{3n}}$$

where

$$\begin{aligned}C = {}& 4320 * 2^{2/3} * 3^{1/3}(-471154444666161787306297086 3\\ & + 52735595419633 * 2721^{1/2})^{1/3} - 4320 * 2^{2/3}\\ & *3^{1/3}(4711544446661617873062970863 + 52735595419633\\ & *2721^{1/2})^{1/3} - 16580537033280\end{aligned}$$

$$\begin{aligned}A = {}& 27136(258100259167071465008428932350120206716329872 1\\ & +99780432501542041707016500 * 2721^{1/2})^{1/3}\\ & -27136(-2581002591670714650084289323501202067163298721\\ & +99780432501542041707016500 * 2721^{1/2})^{1/3}\\ & + 37222766169818947772\end{aligned}$$

$$\begin{aligned}B = {}& 193019904 * 907^{1/3}\\ & (66968860315135056482751353840919736 12\\ & +22970050316722125 * 2721^{1/2})^{1/3} - 193019904 * 907^{1/3}\\ & (-6696886031513505648275135384091973612\\ & +22970050316722125 * 2721^{1/2})^{1/3}\\ & + 352177949360400206551 2\end{aligned}$$

The series we computed of largest discriminant was the class number four example with $d = -1555$. Then

$$\begin{aligned}C = {}& -214772995063512240 - 96049403338648032 * 5^{1/2}\\ & -1296 * 5^{1/2}(10985234579463550323713318473\\ & +4912746253692362754607395912 * 5^{1/2})^{1/2}\end{aligned}$$

$$\begin{aligned}A = {}& 63365028312971999585426220\\ & +28337702140800842046825600 * 5^{1/2}\end{aligned}$$

$+384 * 5^{1/2}(10891728551171178200467436212395209160385656017$
$+4870929086578810225077338534541688721351255040 * 5^{1/2})^{1/2}$

$B = 7849910453496627210289749000$
$+3510586678260932028965606400 * 5^{1/2}$
$+2515968 * 3110^{1/2}(6260208323789001636993322654444020882161$
$+ 2799650273060444296577206890718825190235 * 5^{1/2})^{1/2}$

The series (1.4) with these constants gives 50 additional digits per term.

The Computational Component. The absolute invariant, and so the coefficients A, B, and C satisfy polynomial equations of known degree and height. Thus the problem of determining the coefficients of each series reduces to algebra and can be entirely automated. This is really the dream case for computer aided analysis. Indeed from the expressions for $j(t)$, $a(t)$, $b(t)$ we straightforwardly computed their values to several hundred digits. The lattice basis reduction algorithm, as implemented in *Maple*, now provides the minimal polynomials for each quantity. In addition, a higher precision calculation actually provides a proof of the claimed identity. This last step requires knowing *a priori* bounds on the degrees and heights of the invariants. While somewhat mathematically sophisticated, the computation required is fairly easy though a little slow.

2. Frauds and Identities

2a. The Mathematical Component. Gregory's series for π, truncated at 500, 000 terms, gives to forty places

(2.1)
$$4\sum_{k=1}^{500,000}\frac{(-1)^{k-1}}{2k-1} = 3.14159\underline{0}6535897932\underline{40}4626433832\underline{6}9502884197.$$

To one's initial surprise only the underlined digits are wrong. This is explained, *ex post facto*, by setting N equal to one million in the result below:

Theorem 1. *For integer N divisible by 4 the following asymptotic expansion holds*:

$$(2.2)\quad \frac{\pi}{2} - 2\sum_{k=1}^{N/2}\frac{(-1)^{k-1}}{2k-1} \sim \sum_{m=0}^{\infty}\frac{E_{2m}}{N^{2m+1}} = \frac{1}{N} - \frac{1}{N^3} + \frac{5}{N^5} - \frac{61}{N^7} + \cdots$$

where the coefficients are the even Euler numbers 1, -1, 5, -61, 1385, $-50521\ldots$.

The Computational Component. The observation (2.1) arrived in the mail from Roy North. After verifying its truth numerically, it was an easy matter to generate a large number of the "errors" to high precision. We then recognized the sequence of errors in (2.2) as the Euler numbers—with the help of Sloane's *Handbook of Integer Sequences*. The presumption that (2.2) is a form of Euler-Maclaurin summation is now formally verifiable for any fixed N in *Maple*. This allowed us to determine that (2.2) is equivalent to a set of identities between Bernoulli and Euler numbers that could with effort have been established. Secure in the knowledge that (2.2) holds, it is easier, however, to use the Boole Summation formula which applies directly to alternating series and Euler numbers (see [5]).

This is a good example of a phenomenon which really does not become apparent without working to reasonably high precision (who recognizes 2, -2, 10?), and which highlights the role of pattern recognition and hypothesis validation in experimental mathematics. It was an amusing additional exercise to compute π to 5, 000 digits from (2.2). Indeed, with $N = 200,000$ and correcting using the first thousand even Euler numbers, we obtained 5, 263 digits of π (plus 12 guard digits).

2b. The Mathematical Component. The following evaluations are correct to the precision indicated.

Sum 1 (correct to all digits)

$$\sum_{n=1}^{\infty}\frac{o(2^n)}{2^n} = \frac{1}{9}$$

where $o(n)$ counts the <u>odd</u> digits in n: $o(901) = 2$, $o(811) = 2$, $o(406) = 0$.

By comparison

Sum 2 (correct to 30 digits)

$$\sum_{n=1}^{\infty}\frac{e(2^n)}{2^n} = \frac{3166}{3069}$$

where $e(n)$ counts the even digits in n.

Sum 3 (correct to 267 digits)

$$\sum_{n=1}^{\infty}\frac{\lfloor n\tanh\pi\rfloor}{10^n} = \frac{1}{81}$$

where $\lfloor\ \rfloor$ is the greatest integer function: $\lfloor 3.7\rfloor = 3$.

Sum 4 (correct to in excess of 500 million digits)

$$\sum_{n=1}^{\infty}\frac{\lfloor ne^{\sqrt{163\pi/9}}\rfloor}{2^n} = 1280640.$$

Sum 5 (correct to in excess of 42 billion digits)

$$\left(\frac{1}{10^5}\sum_{n=-\infty}^{\infty}e^{-\frac{n^2}{10^{10}}}\right)^2 = \pi.$$

The Computational Component. Analysis of these and other seemingly rational evaluations may be found in [6]. Sum 1 occurred as a problem proposed by Levine, *College Math J.*, **19**, #5 (1989) and Bowman and White, *MAA Monthly*,

96 (1989), 745. Sum 2 relates to a problem of Diamond's in the *MAA Monthly*, **96** (1989), 838. Sums 2, 3, 4 all have transcendental values and are explained by a lovely continued fraction expansion originally studied by Mahler. Computer assisted analysis leads us to a similar more subtle expansion for the generating function of $\lfloor n\alpha+\beta \rfloor$:

$$\sum_{n=0}^{\infty} \lfloor n\alpha+\beta \rfloor x^n.$$

Sum 5 arises from an application of Poisson summation or equivalently as a modular transformation of a theta function. While asymptotically rapid, this series is initially very slow and virtually impossible for high-precision explicit computation.

These evaluations ask the question of how one develops appropriate intuition to be persuaded by say, Sum 1, but not by Sum 2 or Sum 3? They also underline that no level of digit agreement is really conclusive of anything. Ten digits of coincidence is persuasive in some contexts while ten billion is misleading in others. In our experience, symbolic coincidence is much more impressive than undigested numeric coincidence.

3. The Cubic Arithmetic Geometric Mean

The Mathematical Component. For $0 < s < 1$, let $a_0 := 1$ and $b_0 := s$ and define the cubic AGM by

$$a_{n+1} := \frac{a_n + 2b_n}{3}$$

(AG3)

$$b_{n+1} := \sqrt[3]{\frac{(a_n^2 + a_n b_n + b_n^2)b_n}{3}}$$

which converge cubically to a common limit

$$\text{AG}_3(1, s) = \frac{1}{{}_2F_1J(1/3, 2/3; 1; 1-s^3)} \tag{3.1}$$

where the hypergeometric function $F(s) := {}_2F_1(1/3, 2/3; 1; s) = \sum_{n=0}^{\infty} \frac{(3n!)}{(n!)^3 3^{3n}} s^n$. In particular, the hypergeometric function possesses the simple cubic functional equation

$${}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1-x^3\right) = \frac{3}{1+2x}\ {}_2F_1\left(\frac{1}{3}; \frac{2}{3}; 1, \left(\frac{1-x}{1+2x}\right)^3\right).$$

This can be validated symbolically once known! As an example

$$\text{AG}_3(1, 1/100) = \frac{1}{{}_2F_1(1/3, 2/3; 1; 1-100^{-3})}$$

and 4 iterations of (AG3) will compute the hypergeometric function at 0.999999 to 25 significant digits. Any direct computation so near the radius of convergence is doomed.

Continuing, we let

$$L(q) := \sum_{n,m=-\infty}^{\infty} q^{n^2+nm+m^2} \tag{3.2}$$

and

$$M(q) := (3L(q^3) - L(q))/2.$$

Theorem 2. *The functions $L(q)$ and $M(q)$ "parametrize" the cubic AGM in the sense that if $a := L(q)$ and $b := M(q)$ then*

$$L(q^3) = \frac{a+2b}{3}$$

and

$$M(q^3) = \sqrt[3]{\frac{(a^2+ab+b^2)b}{3}}$$

while $\text{AG}_3(1, M(q)/L(q)) = L(q)$.

Thus a step of the iteration has the effect of sending q to q^3. From this, one is led to an easy to state but hard to derive iteration.

Cubic iteration for π. *Let $a_0 := 1/3$, $s_0 := (\sqrt{3}-1)/2$ and set*

$$(1+2s_n)(1+2s_{n-1}^*) = 3 \quad where \quad s^* := \sqrt[3]{1-s^3}$$

$$a_n := (1+2s_n)^2 a_{n-1} - 3^{n-1}[(1+2s_n)^2 - 1],$$

then $1/a_n$ converges cubically to π.

This iteration gives 1, 5, 21, 70, ... digits correct and more than triples accuracy at each step.

The Computational Component. This is the most challenging and most satisfying of our three examples for computer assisted analysis. We began with one of Ramanujan's typically enigmatic entries in Chapter 20 of his notebook, now decoded in [1]. It told us that a "quadratic modular equation" relating to F was

$$(1-u^3)(1-v^3) = (1-uv)^3. \tag{3.3}$$

From this we gleaned that some function R should exist so that $u := R(q)$ and $v := R(q^2)$ would solve (3.3). We formally solved for the coefficients of R and learned nothing. Motivated by the analogy with the classical theory of the AGM iteration [2] we looked at $F(1-R(q)^3)$ which produced

$$F(1-R(q)^3) = 1 + 6q + 6q^3 + 6q^4 + 12q^7 + 6q^9 + 6q^{12} + 12q^{13} + 6q^{16} + 12q^{19} + 12q^{21} + 6q^{25} + \cdots.$$

This was "pay-dirt" since the coefficients were sparse and very regular. Some analysis suggested that they related to the number of representations of the form m^2+3n^2. From this we looked at theta function representations and were rewarded immediately by the apparent identity $F(1-R(q)^3) = L(q)$. Given the truth of this, it was relatively easy to determine that $R(q) = M(q)/L(q)$ with M and L as in (3.2).

It was now clear that the behaviour as q goes to q^3 should be at least as interesting as (3.3). Indeed, motivated by the modular properties of L we observed symbolically that

$$1 = R(q)^3 + \left[\frac{(1-R(q))}{(1+2R(q))}\right]^3. \tag{3.4}$$

At this stage in [4] we resorted rather unsatisfactorily to a classical modular function proof of (3.4) and so to a proof of Theorem 2. Later we returned with Frank Garvan [8] to a search for an elementary proof. This proved successful. By searching for product expansions for M we were lead to an entirely natural computer-guided proof—albeit with human insight along the way.

It is actually possible, as described in [8], to search for, discover and prove *all* modular identities of the type of (3.3) and (3.4) in an entirely automated fashion. Again, this is possible because we have ultimately reduced most of the analytic questions to algebra through the machinery of modular forms.

As a final symbolic challenge we observe that (3.1) may be recast as saying that

$$I(a_n, b_n) = I(a_{n+1}, b_{n+1})$$

where

$$I(a,b) = \int_0^\infty \frac{t\,dt}{\sqrt[3]{(t^3+a^3)(t^3+b^3)^2}}.$$

This invariance should be susceptible to a direct—hopefully experimentally guided—proof.

4. Conclusions

The sort of experiences we have had doing mathematics interactively has persuaded us of several conclusions. It is necessary to develop good context dependent intuition. It is useful to take advantage of the computer to do the easy—many unimaginable hand-calculations are trivial to code. (So trivial, in fact, that one has to resist the temptation to compute mindlessly.) The skill is to recognize when to try speculative variations on a theme and to know when one has actually learned something from them. The mathematical opportunities are virtually unlimited but only in a relatively painless to use high-level and multi-faceted environment.

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Reviews of Mathematical Software

Maple V and *Mathematica*

Reviewed by Paul C. Abbott*

Abstract

A comparison of two popular computer algebra systems (CAS), *Maple* and *Mathematica*, is presented from a users viewpoint. Solved examples highlight the different conventions, environment, and tools that each system provides. Special attention is paid to system design through examples of consistency of function naming, syntax, and the ease with which output from one computation can be entered as input to another.

Introduction

This review assumes a passing knowledge of computer algebra. An introduction to CAS is given in [1–3] and I encourage the reader to refer to the detailed descriptions of *Maple* [4–7] and *Mathematica* [8–10] for more information.

Both *Maple* and *Mathematica* are very large programs, and this review does not even attempt to cover their scope. There have been many reviews of each system individually and some comparative reviews [11–12]. The focus here is on the results of one user trying to solve a set of problems using each system.

Maple and *Mathematica* are under active development, both by their respective manufacturers and by the inclusion of contributed packages from the large and rapidly growing community of CAS users. The capabilities of each package are changing dynamically. Both packages have comprehensive (symbolic) programming languages, and so the

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Vignettes: Obscurer Vices

It would be fascinating to have a study of the after-lunch alcohol content of the American workforce, and of the variations in productivity, work quality, and safety that accompany variations in drinking short of actual drunkenness. Such a study would be expensive and technically difficult, which is one reason it has never been attempted. Another reason is that it would weaken the identification of the alcohol problem with "alcohol abuse and alcoholism" by paying attention to the costs of nonproblem drinking.

—*Mark A. R. Kleiman*, in *Against Excess: Drug Policy for Results* (Basic Books)

Private discourse . . . grew cruder in the decades after World War II. One 1969 study of actual use of language, for example, showed that a group of adults in a leisure setting used *damn* and a four-letter word for excrement more frequently than they did *the* or *and.*

—*John C. Burnham*, in *Bad Habits: Drinking, Smoking, Taking Drugs, Gambling, Sexual Misbehavior, and Swearing in American History* (New York University Press)

third-person singular agreement suffix -s to agree with its subject *Noam*. This is head marking, because the verb is considered the head of the sentence. But in "muchas gracias," the dependent word *muchas*, "many," is marked with a feminine plural suffix *-as* so as to agree with the head word *gracias*, "thanks"; this is dependent marking. Nichols assigns scores to languages depending on which type of morphological marking they favor.

Nichols argues that this feature is of typological importance, that it shows considerable stability, and that its geographical distribution is not random. For example, New World languages tend to mark heads more than dependents, whereas Old World languages tend toward dependent marking. The Pacific region is intermediate between the two. Nichols interprets this distribution as a trace of the earliest expansion of human settlement from Africa and nearby areas to more remote parts of Eurasia and the New World.

Nichols identifies ten major structural features of this kind and investigates their patterns of distribution within a sample of 174 languages. Points of analysis include correlations among features; stability within both language families and geographical areas; evenness of distribution within geographical regions of varying sizes; and areas of maximum diversity. On the basis of such patterns (and other assumptions), Nichols sketches a general picture of linguistic prehistory, consisting of three stages: an initial development of linguistic diversity in the tropical areas of Africa and the nearby parts of Asia; an early expansion from the Old World tropics to Europe, the remainder of Asia, the Pacific, and the New World; and a third, post-glaciation stage in which more complex social groups spread their languages over large areas of the world, thereby removing much of the original linguistic diversity (which remains only in peripheral areas).

In a study involving such a large corpus, it is of course easy to find particular judgments to disagree with. For example, Nichols treats colloquial French as a verb-initial language and assumes that Mandarin Chinese has no prepositional phrases. Though both languages have indeed been analyzed this way, the analyses are certainly debatable.

A more serious problem is that many of the mathematical arguments in the book (and there are quite a number of them) do not inspire confidence. Some examples:

1) In one passage, Nichols says that in measuring the stability of certain features within language families, "two metrics are used" and that "both yield the same hierarchical ranking" of the features' stability. But the two metrics are entirely interdependent: one is the sum of the number of different types in each family for the eight families under consideration, and the other is the mean number of types per family—that is, just the first number divided by 8 (pp. 166–167 and table 52).

2) Data are sometimes inconsistent between tables (as in tables 50 and 53).

3) Nichols makes extensive use of what she calls "Dryer's test"—referring to a procedure for testing hypotheses about linguistic universals proposed by the linguist Matthew Dryer. Though there is no room here for details, her adaptation of Dryer's procedure to measure "significance of divergence" (pp. 187–188) is certainly idiosyncratic and appears to be invalid (as far as may be judged from her laconic description of it).

Of course, these problems do not prove that Nichols's conclusions are wrong: she does make a convincing case that the distribution of typological features among the world's languages is nonrandom and that this distribution may have much to tell us about linguistic—and therefore human—prehistory. Her scenario for the spread of human language must be considered preliminary, but it is not unreasonable. Her book will be a rich source of ideas and techniques for those who wish to pursue this line of investigation further.

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Mathematical Malaises

Pi in the Sky. Counting, Thinking, and Being. JOHN D. BARROW. Clarendon (Oxford University Press), New York, 1992. xii, 317 pp., illus. $25.

The Rock of Gibraltar of most mathematics, indeed of almost all reasoning, is the principle of the excluded middle. We use two-valued logic where statements are either true or false. No middle ground is possible.

> A man of Seville is shaved by the Barber of Seville if and only if the man does not shave himself. Does the Barber shave himself?

If he does he doesn't; if he doesn't he does. The statement can be neither true nor false. The conclusion: the barber cannot exist. The problem is that in a mathematical sense the barber does exist, or at least did by the permissible definitions of the turn of this century. This innocent paradox, recast by Bertrand Russell in only slightly more erudite terms, deeply shook both Russell and the foundations of mathematics. Since it is possible to deduce irrefutably the truth of anything from a contradiction, a single inconsistency in the fabric causes the entire structure to crumble. If Russell's paradox is not resolved then unicorns exist and pigs fly.

The foundations crumbled but the building stood. Mathematicians worked on, largely unimpeded by the most profound crisis imaginable in the philosophy of mathematics. (This is the usual direct impact of philosophy on mathematics.) But inexorably, over this century, the effects of a close examination of the underpinnings of math-

Strange Series and High Precision Fraud

J. M. Borwein and P. B. Borwein

INTRODUCTION. Five of the following twelve series approximations are exact. The remaining seven are not identities but are approximations that are correct to at least 30 digits. One in fact is correct to over 18,000 digits and another to in excess of a billion digits. The reader is invited to separate the true from the bogus. (For answers see the end of the introduction.) Most of these series are easily amenable to high precision calculation in one's favorite high precision environment, such as Maple or MACSYMA, and provide examples of "caveat computat." Things are not always as they appear.

Sum 1

$$\sum_{n=1}^{\infty} \frac{a(2^n)}{2^n} \doteq \frac{1}{99}$$

where $a(n)$ counts the number of odd digits in odd places in the decimal expansion of n. ($a(901) = 2$, $a(210) = 0$, $a(811) = 1$, here the 1st digit is the 1st to the left of the decimal point.)

Sum 2

$$\sum_{n=1}^{\infty} \frac{a(n)}{10^n} \doteq \frac{10}{99}$$

where $a(n)$ is as above.

Sum 3

$$\sum_{n=1}^{\infty} b(n)\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) \doteq \frac{25\pi^2}{297}$$

where $b(n)$ counts the number of odd digits in n ($b(901) = 2$, $b(811) = 2$, ($b(406) = 0$).

Sum 4

$$\sum_{n=1}^{\infty} \frac{c(n)}{2^n} \doteq \frac{511}{8184}$$

where $c(n) := 32c_1(n) - c_2(n)/32$, and $c_1(n)$ counts the number of nines in n, while $c_2(n)$ counts the number of eights in n ($c(8199) = 32 \cdot 2 - 1/32$).

Sum 5

$$1 + \sum_{n=1}^{\infty} \frac{(-1)^{\delta(n)}}{16^{\delta(n)}(D(n))^4} \doteq 2\frac{(e^{\pi/2} - e^{-\pi/2})}{\pi^2}$$

where $\delta(n)$ is the number of ones in the binary expansion of n and $D(n)$ is the product $\prod_i \max\{i\delta_i(n), 1\}$ where $\delta_i(n)$ is the ith binary digit of n ($\delta(1011_2) = 3$, $D(1011_2) = 4 \cdot 2 \cdot 1 = 8$).

Sum 6

$$\sum_{n=1}^{\infty} \frac{e(n)}{n(n+1)} \doteq \frac{10}{99} \log 10$$

where $e(n)$ "reflects" n through the decimal point ($e(123) = .321$, $e(90140) = .04109$).

Sum 7

$$\sum_{n=1}^{\infty} \frac{b(2^n)}{2^n} \doteq \frac{1}{9}$$

where $b(n)$ counts the number of odd digits in n (as in Sum 3).

Sum 8

$$\sum_{n=1}^{\infty} \frac{e(2^n)}{2^n} \doteq \frac{3166}{3069}$$

where $e(n)$ counts the number of even digits in n.

Sum 9

$$\sum_{n=1}^{\infty} \frac{\lfloor n \tanh \pi \rfloor}{10^n} \doteq \frac{1}{81}$$

where $\lfloor\ \rfloor$ is the greatest integer function ($\lfloor 3.7 \rfloor = 3$).

Sum 10

$$\sum_{n=1}^{\infty} \frac{\lfloor n e^{\pi\sqrt{163/9}} \rfloor}{2^n} \doteq 1280640$$

Sum 11

$$\sum_{-\infty}^{\infty} \frac{1}{10^{(n/100)^2}} \doteq 100\sqrt{\frac{\pi}{\log 10}}$$

Sum 12

$$\left(\frac{1}{10^5} \sum_{n=-\infty}^{\infty} e^{-(n^2/10^{10})}\right)^2 \doteq \pi$$

These sums break into four types. Sums 2, 3, 4, 5, and 6 are all specializations of generating functions for digit sums, more-or-less of the type:

$$\prod_{n=0}^{\infty}(1+q^{2^n}) = \sum_{n=0}^{\infty} x^{\delta(n)}q^n \tag{1.1}$$

where $\delta(n)$ counts the number of ones in the binary expansion of n. These are treated in section 2. See also [**14**].

Sums 1 and 7 are related to a problem independently due to E. Levine (*College Math Journal*, Vol. 19, number 5, 1989) and to D. Bowman and T. White (*Amer. Math. Monthly*, Vol. 96 1989, p. 743), which asks if

$$\sum_{n=0}^{\infty}\frac{g(2^n)}{2^n} = \frac{2}{9}$$

where $g(n)$ counts the number of digits ≥ 5 in n. The key to the solution we provide is due to our colleague A. C. Thompson. See section 3.

The sums 8, 9 and 10 revolve around the fact that

$$\sum_{n=0}^{\infty} w^{\lfloor n\alpha\rfloor}q^n$$

has a particularly attractive and rapidly convergent generating function that is related to the continued fraction expansion of α. This is essentially an observation of Mahler's [**11**], though the development we offer in section 4 is quite distinct. See also [**10**], [**3**]. This is closely related to problem #E3353 in the *MAA Monthly* due to H. Diamond [**6**].

The last section deals with series like Sums 11 and 12. There are consequences of the fact that $f(t) := \sum_{n=-\infty}^{\infty} e^{-n^2 t\pi}$ is a modular form and satisfies a simple functional equation linking $f(t)$ and $f(1/t)$.

The fradulent series are: Sum 2 (correct to 99 digits), Sum 4 (correct to 240 digits), Sum 8 (correct to 30 digits), Sum 9 (correct to 267 digits), Sum 10 (correct to at least half a billion digits), Sum 11 (correct to at least 18,000 digits), and Sum 12 (correct to at least 42 billion digits).

GENERATING FUNCTIONS—PART ONE. Many digit sums are generated by the following type of argument.

Example 2.1. Let $b(n)$ count the number of odd digits in n base 10 (as in Sums 3 and 7). Then for $|q| < 1$,

$$\sum_{n=0}^{\infty} x^{b(n)}q^n = \prod_{n=0}^{\infty}\left(1 + xq^{10^n} + q^{2\cdot 10^n} + xq^{3\cdot 10^n} + q^{4\cdot 10^n} + xq^{5\cdot 10^n} + q^{6\cdot 10^n}\right.$$
$$\left. + xq^{7\cdot 10^n} + q^{8\cdot 10^n} + xq^{9\cdot 10^n}\right)$$
$$=: \prod_{n=0}^{\infty} r(x, q^{10^n}). \tag{2.1}$$

To see this, observe that in the expansion of the product each power of q^m arises in exactly one way. This is just the unique expansion of m base 10. The coefficient of q^m is just a product of x's, one for each odd digit in m. If we differentiate (2.1) with respect to x as is legitimate since $b(n) = O(n)$ and the derivatives converge uniformly, we get

$$\frac{\sum_{n=0}^{\infty} b(n) x^{b(n)-1} q^n}{\sum_{n=0}^{\infty} x^{b(n)} q^n} = \sum_{n=0}^{\infty} \frac{q^{10^n} + q^{3\cdot 10^n} + q^{5\cdot 10^n} + q^{7\cdot 10^n} + q^{9\cdot 10^n}}{1 + xq^{10^n} + q^{2\cdot 10^n} + \cdots + q^{8\cdot 10^n} + xq^{9\cdot 10^n}} \quad (2.2)$$

and at $x := 1$

$$\frac{\sum_{n=0}^{\infty} b(n) q^n}{(1-q)^{-1}} = \sum_{n=0}^{\infty} \frac{q^{10^n} + q^{3\cdot 10^n} + \cdots + q^{9\cdot 10^n}}{1 + q^{10^n} + q^{2\cdot 10^n} + \cdots + q^{9\cdot 10^n}} \quad (2.3)$$

$$= \sum_{n=0}^{\infty} \frac{q^{10^n}}{1+q^{10^n}}$$

$$=: \sum_{n=0}^{\infty} R(q^{10^n})$$

where the second last equality follows on factoring each term. It is apparent from this representation for example that

$$\sum_{n=0}^{\infty} b(n) q^n = \frac{1}{1-q}\left(\frac{q^1}{1+q^1} + \frac{q^{10}}{1+q^{10}}\right) + O(q^{100}). \quad (2.4)$$

We need the following observation which we encapsulate as Lemma 2.1.

Lemma 2.1. *Suppose $R(q)$ is a non-negative, measurable function on* $[0,1]$. *If $b > 1$ and*

$$f(q) := \sum_{n=0}^{\infty} R(q^{b^n}) \qquad |q| < 1$$

then

$$\int_0^1 \frac{f(q)}{q}\, dq = \frac{b}{b-1} \int_0^1 \frac{R(q)}{q}\, dq.$$

Proof:

$$\int_0^1 \frac{f(q)}{q}\, dq = \int_0^1 \sum_{n=0}^{\infty} \frac{R(q^{b^n})}{q}\, dq$$

$$= \sum_{n=0}^{\infty} \int_0^1 \frac{R(q^{b^n})}{q}\, dq$$

$$= \sum_{n=0}^{\infty} \int_0^1 \frac{S(q^{b^n}) q^{b^n}}{q}\, dq$$

where $S(q) := R(q)/q$.

Now set $u = q^{b^n}$ and observe that

$$\int_0^1 \frac{f(q)}{q}\, dq = \sum_{n=0}^{\infty} \int_0^1 \frac{S(u)}{b^n}\, du$$

and the lemma is proved. (The interchange of sum and integral is just the monotone convergence theorem.) ■

From (2.3) we have

$$\sum_{n=0}^{\infty} b(n)q^{n-1}(1-q) = \sum_{n=0}^{\infty} \frac{R(q^{10^n})}{q} \tag{2.5}$$

and with Lemma 2.1,

$$\sum_{n=1}^{\infty} b(n)\left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{10}{9}\int_0^1 \frac{1}{1+q}\,dq$$

or

$$\sum_{n=1}^{\infty} \frac{b(n)}{n(n+1)} = \frac{10}{9}\log 2. \tag{2.6}$$

Indeed this process iterates, in the sense that we can keep dividing by q and integrating in (2.5). This yields with some effort the following

Sum 13. For k a positive integer

$$\sum_{n=1}^{\infty} b(n)\left(\frac{1}{n^k} - \frac{1}{(n+1)^k}\right) = \frac{10^k}{10^k - 1}\alpha(k)$$

where, α is the alternating zeta function,

$$\alpha(s) := (1 - 2^{1-s})\zeta(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}.$$

Note that Sum 3 is just the $k := 2$ case of the above, while $k := 1$ gives (2.6).

A direct derivation of Sum 13 valid for non-integer k can be based on the fact that:

$$\alpha(s)\sum_{n=1}^{\infty} a_n n^{-s} = \sum_{n=1}^{\infty} b_n n^{-s}$$

if and only if

$$\sum_{n=1}^{\infty} a_n \frac{x^n}{1+x^n} = \sum_{n=1}^{\infty} b_n x^n.$$

This identity is now coupled with (2.3). See **[18]**.

Example 2.2. The generating function for q, the number of odd digits in odd places (as in Sums 1 and 2), is given by

$$\sum_{n=0}^{\infty} x^{a(n)}q^n = \prod_{n=0}^{\infty} r\left(x, q^{10^{2n}}\right)$$

where

$$\begin{aligned} r(x,q) := &\left(1 + xq + q^2 + xq^3 + q^4 + \cdots + xq^9\right) \\ &\cdot\left(1 + q^{10} + q^{2\cdot 10} + q^{3\cdot 10} + \cdots + q^{9\cdot 10}\right) \end{aligned}$$

and leads, as in (2.3), to the series

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{1-q}\sum_{n=0}^{\infty} \frac{q^{100^n}}{1+q^{100^n}}. \tag{2.7}$$

Sum 2 now appears on taking $q := \frac{1}{10}$ and using the first term of the above expansion. It is apparent that the remainder is positive of size very close to $\frac{1}{9} \cdot 10^{-99}$. This gives the nature of the estimate in Sum 2.

In similar fashion

$$\sum_{n=0}^{\infty} A_k(n) q^n = \frac{1}{1-q} \sum_{n=0}^{\infty} \frac{q^{(10^k)^n}}{1 + q^{(10^k)^n}} \tag{2.8}$$

is the generating function for the number of odd digits in the 1st, $(k+1)$th, $(2k+1)$th places of k. So with $k = 10$, for example

$$\sum_{n=0}^{\infty} \frac{A_{10}(n)}{10^n} = \frac{10}{99} + \varepsilon_n \tag{2.9}$$

where $0 < |\varepsilon_n| < \frac{10}{9} \cdot 10^{-10^{10}}$, and the above approximation is correct to over a billion digits. ■

Example 2.3. The number of times the digit $i > 0$ occurs in n has generating function

$$\sum_{n=0}^{\infty} g(n) q^n = \frac{1}{1-q} \sum_{n=0}^{\infty} \frac{q^{i \cdot 10^n}}{1 + q^{10^n} + \cdots + q^{9 \cdot 10^n}}.$$

So the generating function for $c(n)$ in Sum 4 is just

$$\sum_{n=0}^{\infty} c(n) q^n = \frac{1}{1-q} \sum_{n=0}^{\infty} \frac{32 q^{9 \cdot 10^n} - \dfrac{q^{8 \cdot 10^n}}{32}}{1 + q^{10^n} + \cdots + q^{9 \cdot 10^n}}.$$

At $q := \frac{1}{2}$, the second term vanishes to give

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{c(n)}{2^n} &= \frac{1}{1-q} \left(\frac{32 q^9 - \dfrac{q^{8 \cdot 10^n}}{32}}{1 + \cdots + q^9} \right) + O(q^{800}) \\ &= \frac{511}{8184} + \varepsilon \end{aligned}$$

where $\varepsilon < 10^{-241}$.

Example 2.4. The generating function which reverses digits, as in Sum 6, is

$$\sum_{n=0}^{\infty} x^{e(n)} q^n = \prod_{n=0}^{\infty} \left(1 + x^{1/10^{n+1}} q^{10^n} + \cdots + x^{9/10^{n+1}} q^{9 \cdot 10^n}\right). \tag{2.10}$$

So

$$\sum_{n=0}^{\infty} e(n) q^n = \frac{1}{1-q} \sum_{n=0}^{\infty} \frac{\dfrac{1}{10^{n+1}} q^{10^n} + \cdots + \dfrac{9}{10^{n+1}} q^{9 \cdot 10^n}}{1 + q^{10^n} + \cdots + q^{9 \cdot 10^n}} \tag{2.11}$$

and as in Lemma 2.1

$$\sum_{n=1}^{\infty} \frac{e(n)}{n(n+1)} = \frac{10}{99} \log 10.$$

There are very many analogues of these results. All have variations in different bases. The binary digit counting functions δ has generating function

$$\sum_{n=0}^{\infty} x^{\delta(n)} q^n = \prod_{n=0}^{\infty} (1 + x q^{2^n}) \tag{2.12}$$

and

$$\sum_{n=0}^{\infty} \delta(n) q^n = \frac{1}{1-q} \sum_{n=0}^{\infty} \frac{q^{2^n}}{1+q^{2^n}} \tag{2.13}$$

whence

$$\sum_{n=1}^{\infty} \frac{\delta(n)}{n(n+1)} = 2\log 2. \tag{2.14}$$

(See the Putnam examinations of 1981, 1984 and 1987.) As in Example 2.1 we have Sum 14.

Sum 14. Let $\delta(n)$ denote the sum of the binary digits of n. Then

$$\sum_{n=1}^{\infty} \delta(n)\left(\frac{1}{n^k} - \frac{1}{(n+1)^k}\right) = \left(\frac{2^k}{2^k - 1}\right)\alpha(k)$$

where $\alpha(k)$ is the alternating zeta function.

The sum of the decimal digits of n denoted $s(n)$ has generating function

$$\sum_{n=0}^{\infty} x^{s(n)} q^n = \prod_{n=0}^{\infty} (1 + x q^{10^n} + x^2 q^{2\cdot 10^n} + \cdots + x^9 q^{9\cdot 10^n}) \tag{2.15}$$

from which we deduce that

$$\sum_{n=1}^{\infty} \frac{s(n)}{n(n+1)} = \frac{10}{9}\log 10. \tag{2.16}$$

Loxton and van der Poorten **[10]** and Mahler **[11]** treat transcendence questions for functions, with power series expansions at zero which satisfy functional equations. From these results, one knows that if f, holomorphic at zero and not an algebraic function, satisfies a function equation of the form

$$f(q^m) = f(q) + R(q) \tag{2.17}$$

where m is an integer and R is a rational function, then $f(\alpha)$ is transcendental for algebraic α. From this we deduce that the exact answers in Sum 2, Sum 4 and Sum 8, are transcendental. This can also be deduced easily from Roth's Theorem **[8]**.

GENERATING FUNCTIONS—PART TWO. A second type of digit function arises as follows.

Example 3.1. Let $\delta(n)$ as before, denote the sum of the binary digits of n, and let $\rho(n) := \prod\{S_i\colon i\text{th binary digit of } n \neq 0\}$ and $\rho(0) := 1$, where S_i is a given sequence and the product is taken over those binary digits of n which equal one. Then formally

$$\sum_{n=0}^{\infty} \frac{x^{\delta(n)} q^n}{\rho(n)} = \prod_{n=0}^{\infty}\left(1 + \frac{x}{S_{n+1}} q^{2^n}\right) \tag{3.1}$$

and

$$\sum_{n=0}^{\infty} \frac{x^{\delta(n)}}{\rho(n)} = \prod_{n=0}^{\infty}\left(1 + \frac{x}{S_{n+1}}\right).$$

Example 3.2. Let $\delta(n)$ denote the sum of the binary digits of n, and let

$$D(n) = \prod i$$

where the product is taken over those i where the ith binary digit of n is non-zero (as in Sum 5). So, if $0 < n_1 < n_2 < \cdots < n_k$,

$$D(2^{n_1} + 2^{n_2} + \cdots + 2^{n_k}) = (n_1 + 1)(n_2 + 1) \cdots (n_k + 1).$$

Then as in Example 3.1, starting with

$$F_q(x) := x\prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2}q^{2^{n-1}}\right) = x\prod_{n=0}^{\infty}\left(1 - \frac{x^2}{(n+1)^2}q^{2^n}\right) \tag{3.2}$$

we have, for $|x| < 1$,

$$F_1(x) = \frac{\sin \pi x}{\pi} = \sum_{n=0}^{\infty} \frac{(-1)^{\delta(n)} x^{2\delta(n)+1}}{[D(n)]^2} \tag{3.3}$$

and at $x := \frac{1}{2}$

$$\frac{2}{\pi} = \sum_{n=0}^{\infty} \frac{(-1)^{\delta(n)}}{4^{\delta(n)}[D(n)]^2}. \tag{3.4}$$

Similarly, starting with

$$\frac{(\sin \pi x)(\sinh \pi x)}{\pi^2} = x^2\prod_{n=1}^{\infty}\left(1 - \frac{x^4}{n^4}\right) \tag{3.5}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{\delta(n)} x^{4\delta(n)+2}}{[D(n)]^4},$$

we have, at $x := \frac{1}{2}$,

$$2\left(\frac{e^{\pi/2} - e^{-\pi/2}}{\pi^2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^{\delta(n)}}{16^{\delta(n)}[D(n)]^4}, \tag{3.6}$$

which is Sum 5.

Example 3.3. Let $t(n) := \Sigma i$, where the sum is taken over the non-zero digits on n base 2. So $t(1011_2) = 4 + 0 + 2 + 1 = 7$. Then

$$\prod_{n=0}^{\infty}(1 - x^{n+1}q^{2^n}) = \sum_{n=0}^{\infty}(-1)^{\delta(n)}x^{t(n)}q^n. \tag{3.7}$$

So

$$\sum_{n=0}^{\infty}(-1)^{\delta(n)}x^{t(n)} = \prod_{n=1}^{\infty}(1 - x^n) = \sum_{-\infty}^{\infty}(-1)^n x^{(3n+1)n/2} \tag{3.8}$$

on using Euler's pentagonal number theorem [2] and on integrating, from zero to one,

$$\sum_{n=0}^{\infty} \frac{(-1)^{\delta(n)}}{t(n)+1} = \sum_{-\infty}^{\infty} \frac{2(-1)^n}{3n^2+n+2}. \tag{3.9}$$

4. CONTINUED FRACTION EXPANSIONS. The identities of this section are based on the two functions

$$G_\alpha(z,w) := \sum_{n=1}^{\infty} z^n w^{\lfloor n\alpha \rfloor} \tag{4.1}$$

and

$$F_\alpha(z,w) := \sum_{n=1}^{\infty} z^n \sum_{m=1}^{\lfloor n\alpha \rfloor} w^m \tag{4.2}$$

where α is a non-negative real number and $\lfloor n\alpha \rfloor$ is the integer part of $n\alpha$, while z and w are complex with modulus so as to ensure convergence. The function F_α was studied by Mahler [**11**] and is obviously related to G_α by

$$F_\alpha(z,w) + \frac{w}{1-w} G_\alpha(z,w) = \frac{zw}{(1-z)(1-w)} \tag{4.3}$$

for $|z|, |w| < 1$. Van der Poorten [**10**] comments that Mahler's paper has been largely overlooked. In [**3**] we explore these matters further. Note that for positive z and w, F_α is strictly increasing as a function of α.

For irrational α we will use the infinite continued fraction approximations generated by

$$\begin{aligned} &\text{(a)} \quad p_{n+1} := p_n a_{n+1} + p_{n-1} \qquad p_0 := a_0 = \lfloor \alpha \rfloor, \quad p_{-1} := 1 \\ &\text{(b)} \quad q_{n+1} := q_n a_{n+1} + q_{n-1} \qquad q_0 := 1, \qquad q_{-1} := 0 \end{aligned} \tag{4.4}$$

for $n \geq 0$ where

$$\alpha = [a_0, a_1, \ldots, a_n, a_{n+1}, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

so that each a_i is integral, $a_0 \geq 0$ and $a_n \geq 1$ for $n \geq 1$. Then for $n \geq 0$ p_{2n}/q_{2n} increases to α while p_{2n+1}/q_{2n+1} decreases to α and

$$\frac{1}{q_n(q_n + q_{n+1})} < \left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}}. \tag{4.5}$$

All of this is standard and may be found in [**8**], [**9**], or [**16**]. We will avoid using finite continued fractions which arise only for rational α. Let us write $q_n\alpha - p_n$ as ε_n. By (4.5) and (4.4)

$$|\varepsilon_{n+1}| < \frac{1}{q_n + q_{n+1}} < |\varepsilon_n| < \frac{1}{q_{n+1}} \leq 1.$$

A key lemma is:

Lemma 4.1. *For irrational* $\alpha > 0$ *and* n, N *in* $\mathbf{N}$

(a) $$\lfloor n\alpha + \varepsilon_N \rfloor = \lfloor n\alpha \rfloor \qquad \text{for } n < q_{N+1}$$

(b) $$\lfloor n\alpha + \varepsilon_N n \rfloor = \lfloor n\alpha \rfloor + (-1)^N \quad \text{for } n = q_{N+1}.$$

Proof: Suppose N is even (the odd case is entirely parallel). Then $\varepsilon_N > 0$ and (a) fails when

$$n\alpha + \varepsilon_N > m > n\alpha \quad \text{for some } m \text{ in } \mathbf{N}. \tag{4.6}$$

As $\alpha > p_N/q_N$, we have an integer k with

$$(n + q_N)\varepsilon_N > mq_N - np_N = k > 0.$$

If $k \geq 2$ then $n + q_n > 2/\varepsilon_N > 2q_{N+1}$ and $n > q_{N+1}$.

If $k = 1$ we have

$$p_N q_N - q_N p_N = 0, \qquad p_{N+1} q_N - q_{N+1} p_N = 1,$$

so that the linear Diophantine equation $mq_N - np_N = 1$ has general solution $m = p_{N+1} + sp_N$, $n = q_{N+1} + sq_N$ for s integer. However, $n + q_N > 1/\varepsilon_N > q_{N+1}$ so that s is non-negative. This establishes (a). For $n = q_{N+1}$ we have

$$q_{N+1}\alpha < p_{N+1} < q_{N+1}\alpha + \varepsilon_N < p_{N+1} + 1$$

since $p_{N+1} > q_{N+1}\alpha$ and $0 < \varepsilon_{N+1} + \varepsilon_N < 1$. This yields (b). ∎

Theorem 4.1.

(a) *For rational* $\alpha = p/q$ *(reducible or irreducible)*

$$(1 - z^q w^p) G_\alpha(z, w) = \sum_{j=1}^{q} z^j w^{\lfloor jp/q \rfloor}.$$

(b) *For irrational* α *and* $N > 0$

$$(1 - z^{q_N} w^{p_N}) G_\alpha(z, w) = \sum_{n=1}^{q_n} z^n w^{\lfloor n\alpha \rfloor} + (-1)^N \left(\frac{w-1}{w}\right) z^{q_N} w^{p_N} z^{q_{N+1}} w^{p_{N+1}} + R_N(z, w)$$

with

$$|R_N(z, w)| \leq |1 - w| \frac{|z|^{q_{N+1} + q_N + 1}}{1 - |z|}.$$

Proof:

(a) $$G_\alpha(z, w) = \sum_{k=0}^{\infty} \sum_{j=1}^{q} z^{qk+j} w^{kp + \lfloor j(p/q) \rfloor}$$
$$= \sum_{k=0}^{\infty} (z^q w^p)^k \sum_{j=1}^{q} z^j w^{\lfloor j(p/q) \rfloor},$$

(b) $$(1 - z^{q_N} w^{p_N}) G_\alpha(z, w) - \sum_{n=1}^{q_N} z^n w^{\lfloor n\alpha \rfloor}$$
$$= \sum_{n=1}^{\infty} z^{n+q_N} \{ w^{\lfloor (n+q_N)\alpha \rfloor} - w^{p_N + \lfloor n\alpha \rfloor} \}$$
$$= \sum_{n=1}^{\infty} z^{n+q_N} w^{\lfloor n\alpha \rfloor + p_N} \{ w^{\lfloor n\alpha + \varepsilon_N \rfloor - \lfloor n\alpha \rfloor} - 1 \}.$$

By the proof of Lemma 4.1, the first non-zero term in this last expression is $(-1)^N(w-1)/w)z^{q_N+q_{N+1}}w^{p_N+p_{N+1}}$ while the other terms are dominated by $|z|^n|1-w|$ with $n > q_N + q_{N+1}$. ■

For fixed $\alpha > 0$ we write

$$P_N := \sum_{n=1}^{q_N} z^n w^{\lfloor n\alpha \rfloor}, \qquad Q_N := 1 - z^{q_N} w^{p_N}$$

and observe that Theorem 4.1 shows that

$$G_\alpha - \frac{P_N}{Q_N} = (-1)^N \left(\frac{w-1}{w} \right) \frac{z^{q_N} w^{p_N} z^{q_{N+1}} w^{p_{N+1}}}{Q_N} + O(z^{q_N+q_{N+1}} + 1) \quad (4.7)$$

for α irrational (while $G_\alpha = P_N/Q_N$ for rational α). Thus as a function of z P_N/Q_N is the main diagonal Padé approximation to G_α of order q_N.

Corollary 4.1. *For irrational $\alpha > 0$*

$$G_\alpha(z,w) = \frac{zw^{p_0}}{1-zw^{p_0}} - \frac{1-w}{w} \sum_{n=0}^{\infty} (-1)^n \frac{z^{q_n} w^{p_n} z^{q_{n+1}} w^{p_{n+1}}}{(1-z^{q_n}w^{p_n})(1-z^{q_{n+1}}w^{p_{n+1}})}. \quad (4.8)$$

Proof: Let $A_N := P_{N+1}Q_N - Q_{N+1}P_N$. Then A_N is a polynomial of degree at most $q_{N+1} + q_N$ in z. From (4.7) we see that

$$\frac{P_{N+1}}{Q_{N+1}} - \frac{P_N}{Q_N} = \frac{A_N}{Q_N Q_{N+1}} = (-1)^N \left(\frac{w-1}{w} \right) \left\{ \frac{z^{q_N} w^{p_N} z^{q_{N+1}} w^{p_{N+1}}}{Q_N Q_{N+1}} \right\}.$$

On summing from zero to infinity we produce (4.8). ■

This is derived by Mahler for $\alpha \in (0,1)$ in [**11**].

Corollary 4.2. *For irrational $\alpha > 0$ and for $w \neq 1$*

$$F_\alpha(z,w) = \frac{zw}{(1-z)(1-w)} \frac{1-w^{p_0}}{1-zw^{p_0}} + \sum_{n=0}^{\infty} \frac{(-1)^n z^{q_n} w^{p_n} z^{q_{n+1}} w^{p_{n+1}}}{(1-z^{q_n}w^{p_n})(1-z^{q_{n+1}}w^{p_{n+1}})}. \quad (4.9)$$

In particular, for $w = 1$, the spectrum of α [7] is generated by

$$\sum_{n=1}^{\infty} \lfloor n\alpha \rfloor z^n = \frac{p_0 z}{(1-z)^2} + \sum_{n=0}^{\infty} (-1)^n \frac{z^{q_n} z^{q_{n+1}}}{(1-z^{q_n})(1-z^{q_{n+1}})}. \quad (4.10)$$

Proof: Equation (4.9) follows from (4.8) and (4.3). Equation (4.10) is now obtained by letting w tend to 1. ■

If F_N denotes the truncation of the right-hand side of (4.9)

$$\frac{zw}{(1-z)(1-zw^{p_0})} \left(\frac{1-w^{p_0}}{1-w} \right) + \sum_{n=0}^{N-1} (-1)^n \frac{z^{q_n} w^{p_n} z^{q_{n+1}} w^{p_{n+1}}}{(1-z^{q_n}w^{p_n})(1-z^{q_{n+1}}w^{p_{n+1}})}$$

we observe that (4.7) and (4.3) show that

$$F_N = \frac{\left(\frac{w}{1-w} \right) [zQ_N - (1-z)P_N]}{(1-z)(1-z^{q_N}w^{p_N})} \quad (4.11)$$

and some manipulation shows that, for $q_N > 1$, the numerator may be rewritten as

$$B_N := wz \sum_{n=1}^{q_N} z^n \left(\frac{w^{\lfloor (n+1)\alpha \rfloor} - w^{\lfloor n\alpha \rfloor}}{1-w} \right) + wz\left(\frac{1-w^{p_0}}{1-w}\right)(1 - z^{q_N} w^{p_N}) \quad (4.12)$$

so that B_N is a very simple integer polynomial in w and z (of degree $q_N + 1$ in z), while

$$F_\alpha - F_N = O(z^{q_N + q_{N+1}}).$$

Note that F_N is especially simple for $w := 1$ and $0 < \alpha < 1$.

Example 4.1. (a) Let $\alpha := \pi/2$ in (4.11) or (4.10). As

$$\frac{\pi}{2} = [1, 1, 1, 31, \dots]$$

we have $p_0 = 1$, $p_1 = 2$, $p_2 = 3$, $p_3 = 11$, $p_4 = 344$ and $q_0 = 1$, $q_1 = 1$, $q_2 = 2$, $q_3 = 7$, $q_4 = 219$. Thus

$$\begin{aligned} F_{\pi/2}(z,1) &= \sum_{n=1}^{\infty} \left\lfloor \frac{\pi}{2} n \right\rfloor z^n \\ &= \frac{z}{(1-z)^2} + \frac{z^2}{(1-z)^2} - \frac{z^3}{(1-z)(1-z^2)} \\ &\quad + \frac{z^9}{(1-z^2)(1-z^7)} - \frac{z^{226}}{(1-z^7)(1-z^{219})} + \dots \end{aligned}$$

and the approximation F_4 is also expressible as

$$\frac{z(z^7 + z^6 + 2z^5 + z^4 + 2z^3 + z^2 + 2z + 1)}{(1-z^7)(1-z)}$$

and has an error like z^{226}. In particular

$$\sum_{n=1}^{\infty} \frac{\left\lfloor \frac{\pi}{2} n \right\rfloor}{2^n} \doteq \frac{339}{127}$$

with error less than 10^{-68}.

(b) Sum 9 follows from using (4.10) for $\tanh(\pi) = [0, 1, 267, \dots]$. This produces

$$\sum_{n=1}^{\infty} \lfloor n \tanh \pi \rfloor z^n = \frac{z^2}{(1-z)^2} - \frac{z^{269}}{(1-z)(1-z^{268})} + \cdots.$$

(c) Sum 10 follows similarly from (4.10) with one of our favorite transcendental numbers $\alpha := e^{\pi\sqrt{163/9}} = [640320, 1653264929, \dots]$.

(d) Let $\alpha := \log_{10}(2) = [0, 3, 3, 9, \dots]$. Then (4.11) with $N := 3$, $z := \frac{1}{2}$ and $w := 1$ gives

$$\sum_{n=1}^{\infty} \frac{\lfloor n \log_{10}(2) \rfloor}{2^n} \doteq \frac{146}{1023}$$

to 30 places since $q_0 = 1$, $q_1 = 3$, $q_2 = 10$, $q_3 = 93$. Thus, as the number of even digits in 2^n is $\lfloor n \log_{10}(2) \rfloor + 1$ less the number of odd digits in 2^n, the "false" Sum 8 follows from Sum 7 and this "false" identity. In fact, see below, Sum 8 is transcendental while Sum 7 is rational. ■

Other lovely approximations follow from

$$\log_{10}(6) = [0,1,3,1,1,32,\dots]$$
$$\tanh(1) = [1,3,7,9,11,\dots]$$
$$\frac{e-1}{2} = [0,1,6,10,14,\dots]$$

and other simple transcendental numbers. Thus

$$\sum_{n=1}^{\infty} \frac{\lfloor n\zeta(3)\rfloor}{2^n} \doteq \frac{64}{31}$$

to 30 places.

Example 4.2. Many other related sums can be derived from (4.8) and (4.9). We indicate some classes.

(a) For irrational $\alpha > 0$

$$G_\alpha(1,w) = \sum_{n=1}^{\infty} w^{\lfloor n\alpha\rfloor} = \left(\frac{1-w}{w}\right)F_{1/\alpha}(w,1),$$

and more generally

$$G_\alpha(z,w) = \left(\frac{1-w}{w}\right)F_{1/\alpha}(w,z).$$

This follows either from the elementary identity in [**11**]

$$F_\alpha(z,w) + F_{\alpha^{-1}}(w,z) = \frac{zw}{(1-z)(1-w)} \tag{4.13}$$

or from Theorem 2 in [**13**], when $z = 1$.

(b) Letting $w := -1$ in (4.9) produces a Lambert-like series for $\sum_{\lfloor n\alpha\rfloor \text{ odd}} z^n$. As an example,

$$\sum \left\{\frac{1}{2^n}\middle| \text{length}(2^n) \text{ even}\right\} \doteq \frac{114}{1025}$$

to 30 places.

(c) Observe that

$$\sum_{k=0}^{M} \frac{(-1)^k\binom{M}{R}G_\alpha(z,w^k)}{(1-w)^M} = \sum_{n=1}^{\infty}\left(\frac{1-w^{\lfloor n\alpha\rfloor^M}}{1-w}\right)z^n$$

so that on letting w tend to unity we obtain the approximation

$$\sum_{n=1}^{\infty} \lfloor n\alpha\rfloor^M z^n = \frac{\Delta_N^M(z)}{(1-z)(1-z^{q_N})^M} + O(z^{q_N+q_{N+1}})$$

where Δ_N^M is an integer polynomial in z of degree $MqN+1$. In particular

$$\sum_{n=1}^{\infty} \lfloor n\alpha\rfloor^2 z^n = \sum_{n=0}^{\infty} \frac{z^{q_n+q_{n+1}}}{(1-z^{q_n})^2(1-z^{q_{n+1}})^2}$$
$$\times\{(2p_n + 2p_{n+1} - 1) - z^{q_n}z^{q_{n+1}} - (2p_n - 1)z^{q_{n+1}} - (2p_{n+1} - 1)z^{q_n}\}$$

for $0 < \alpha < 1$, α irrational. Thus

$$\sum_{n=1}^{\infty} \frac{(\text{length}(6^n))^2}{6^n} \doteq \frac{196669}{37303}$$

to 88 places.

(d) Similarly, if w is a primitive Nth root of unity

$$\frac{1}{N} \sum_{k=1}^{N} G_\alpha(z, w^k) \overline{w}^{Mk} = \sum_{\lfloor n\alpha \rfloor \equiv M \pmod{N}} z^n$$

[compare (b)]. Thus

$$\sum_{3 \mid \lfloor n \log_{10} 2 \rfloor} \frac{1}{3^n} \doteq \frac{3554}{7381}$$

to 50 places.

(e) Let $w := e^{i\theta}$ (θ real) in (4.9). We obtain

$$\sum_{n=1}^{\infty} \cos(\lfloor n\alpha \rfloor) z^n = \frac{\sum_{n=1}^{q_N} \cos(\lfloor n\alpha \rfloor \theta) z^n - \sum_{n=1}^{q_N} \cos(p_N - \lfloor n\alpha \rfloor \theta) z^{n+q_N}}{1 - 2z^{q_N} \cos(p_N \theta) + z^{2q_N}} + O(z^{q_N + q_{N+1}}),$$

with a similar expression for sin replacing cos. ■

The rational counterpart to (4.13) is

$$F_{p/q}(z, w) + F_{q/p}(w, z) = \frac{zw}{(1-z)(1-w)} + \frac{z^q w^p}{1 - z^q w^p}, \tag{4.14}$$

for p and q relatively prime.

We consider $F(\alpha) := F_\alpha(z, w)$ as a function of α, and observe that $F(\alpha)$ is continuous at each irrational. Moreover, $\lim_{\alpha \downarrow p/q} F(\alpha) = F(p/q)$. Thus, on using (4.13) and (4.14) $\lim_{\alpha \uparrow p/q} F(\alpha) = F(p/q) - z^q w^p/(1 - z^q w^p)$. In consequence, F is discontinuous at every rational and $F(1) - F(0) = \sum_{0 < p/q < 1} \{F(p/q) - F(\frac{p}{q} -)\}$ so that dF is a "pure jump measure" on the rationals in $[0, 1]$. [This observation was made by H. Diamond.] Explicitly the jumps are expressed as

$$\begin{aligned} J &:= \sum_{s=1}^{\infty} \sum_{\substack{1 \le r \le s \\ (r,s)=1}} \frac{z^s w^r}{1 - z^s w^r} \\ &= \sum_{k=1}^{\infty} \sum_{s=1}^{\infty} z^{sk} \sum_{\substack{(r,s)=1 \\ 1 \le r \le s}} w^{rk}. \end{aligned} \tag{4.15}$$

Now, on setting $n = sk$, this yields

$$\sum_{n=1}^{\infty} z^k \left\{ \sum_{s/n} \sum_{\substack{(r,s)=1 \\ 1 \le r \le s}} w^{(r/s)n} \right\}.$$

Equation (16.2.3) in [8] applies with $F(w) := w^n$ and shows that the bracketed term is just $\sum_{m=1}^{n} w^m$. Hence $J = \sum_{n=1}^{\infty} z^n \sum_{m=1}^{n} w^m = F_1(z, w)$ as claimed. This is valid for $|z| < 1$, $|w| \le 1$. ■

We have also shown, using Theorem 4.1(a) and $\lfloor n\alpha \rfloor = \lfloor n(p_N/q_N) \rfloor$ for $n < q_N$, that for $0 < \alpha < 1$

$$F_N = \begin{cases} F_{p_N/q_N} & N \text{ even} \\ F_{p_N/q_N} - \dfrac{z^{q_N} w^{p_N}}{1 - z^{q_N} w^{p_N}} & N \text{ odd.} \end{cases} \tag{4.16}$$

Clearly $F\colon Q \to Q$. In **[10]**, **[11]** (4.9) is used to obtain transcendence estimates by functional equation methods. For $w := \pm 1$ and $z := 1/b$, $b = 2, 3, 4, \dots$ we can get very accessible estimates for F_α or G_α from Roth's theorem **[2]**, **[9]**, **[15]**.

First, observe that Corollary 4.2 shows F_α is irrational when α is irrational and w, z are rational. It is convenient to introduce

$$s := s(\alpha) = \limsup_{n \to \infty} a_n.$$

Thus s is infinite when α has unbounded continued fraction coefficients. For b and w as above, we have from (4.12)

$$0 < \left| F(\alpha) - \frac{P_N}{Q_N} \right| \le O\left(\frac{1}{b^{q_N + q_{N+1}}} \right) \le O\left(\frac{1}{Q_N^{(1 + q_{N+1}/q_N)}} \right) \tag{4.17}$$

for integers P_N and $Q_N := (b-1)(b^{q_n} - w^{p_N})$. Hence, Roth's theorem shows $F(\alpha)$ is transcendental when

$$\limsup_{n \to \infty} \frac{q_{N+1}}{q_N} > 1,$$

and clearly α is Liouville when $s(\alpha) = \infty$. Since almost all numbers have unbounded coefficients, $F(\alpha)$ is Liouville in almost all cases and F maps Liouville numbers to Liouville numbers as they have s = infinity. When $s(\alpha)$ is finite, we have $q_{N+1} \le s q_N + q_{N-1} \le (s+1) q_N$ eventually and so infinitely often

$$q_{N+1} \ge s q_N + q_{N-1} \ge \frac{s^2 + s + 1}{s + 1} q_N$$

and (4.17) shows $F(\alpha)$ is approximable to order at least $(s+1) + (1/(s+1)) \ge 5/2$. If $s = 1$ then α is equivalent to $(\sqrt{5} + 1)/2$. In every other case $F(\alpha)$ is approximable to order $10/3$. In summary $F(\alpha)$ is never algebraic, indeed never has the expected rate of rational approximation and is usually Liouville (**[2]**, **[8]**, **[15]**). In fact almost all irrationals have only finitely many solutions to

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^2 (\log q)^{1+}}.$$

Example 4.3. (a) Arguing similarly from Example 4.2 we see that for almost all α,

$$\sum_{n=1}^{\infty} \frac{p(\lfloor n\alpha \rfloor)}{b^n}$$

is a Liouville number, for any integer polynomial p.

It is hard to find explicit numbers with unbounded continued fraction coefficients but e and $\tanh(1)$ are two examples:

$$\sum_{n=1}^{\infty} \frac{p(\lfloor ne \rfloor)}{b^n}$$

is Liouville for all p and b.

(b) Correspondingly, $\sum_{n=1}^{\infty} p(\lfloor n\alpha \rfloor)/b^n$ is approximable to order at least

$$\frac{1 + s(\alpha)}{\deg(p)}.$$

■

For irrational $0 < \alpha < 1$, $F_\alpha(z,w)$ may be computed entirely from the continued fraction expansion via

$$F_\alpha(z,w) = \sum_{n=0}^{\infty} (-1)^n \frac{z_n z_{n+1}}{(1 - z_n)(1 - z_{n+1})}$$

where $z_{n+1} := z_n^{a_{n+1}} z_{n-1}$, $z_0 := z$, $z_{-1} := w$. This follows from (4.9) and an easy induction.

We conclude with some remarks about iterates of $F(\alpha) := \sum_{n=1}^{\infty} \lfloor n\alpha \rfloor 2^{-n}$. For $\alpha = p/q$ $(0 < \alpha < 1)$ we have

$$F_\alpha(z,w) = zw \frac{\sum_{n=1}^{q} \left(\left\lfloor (n+1)\frac{p}{q} \right\rfloor - \left\lfloor n\frac{p}{q} \right\rfloor \right) w^{\lfloor n(p/q) \rfloor} z^n}{(1 - z)(1 - z^q w^p)} \tag{4.18}$$

either by direct computation or from (4.11) and (4.16). We now set $z := \frac{1}{2}$, $w := 1$ and observe that

$$F\left(\frac{p}{q}\right) + F\left(1 - \frac{p}{q}\right) = 1 + \frac{1}{2^q - 1}.$$

In particular $F(\frac{1}{2}) = \frac{2}{3}$. Moreover, (4.18) shows that

$$F\left(1 - \frac{1}{q}\right) = 1 - \frac{1}{2^q - 1}.$$

Let $q_0 := 2$ and $q_{n+1} := 1/(2^{q_n} - 1)$ to deduce that

$$F^{(n)}\left(\frac{1}{2}\right) = 1 - \frac{1}{q_{n+1}}$$

and so converges to 1. Similar analysis shows that

$$F\left(\frac{1}{q}\right) = \frac{2}{2^q - 1} < \frac{1}{2^{q-2}},$$

and so that

$$F^{(n)}\left(\frac{1}{3}\right) \to 0, \quad \text{because } F^{(2)}\left(\frac{1}{3}\right) = \frac{18}{127} < \frac{1}{7}.$$

Note that $\alpha \geq \frac{1}{2}$ implies $F^{(n)}(\alpha) \geq F^{(n)}(\frac{1}{2})$ and $\alpha < \frac{1}{2}$ implies $F^{(n+1)}(\alpha) \to 0$ for $0 \leq \alpha < \frac{1}{2}$. For rational α, the entire sequence is rational, otherwise it is entirely transcendental, usually Liouville.

5. RATIONAL DIGIT SUMS. This section is based on the following Lemma whose proof we owe to A. C. Thompson.

Lemma 5.1. *For* $0 < q < 1$ *and integer* $m > 1$

$$q = \sum_{n=1}^{\infty} \frac{\lfloor m^n q \rfloor \pmod{m}}{m^n}. \tag{5.1}$$

Proof: Consider the base m expansion of q

$$q = \sum_{k=1}^{\infty} \frac{a_k}{m^k} \qquad 0 \le a_k < m$$

where when ambiguous we take the terminating expansion. Then

$$m^n q = \sum_{k=1}^{n-1} m^{n-k} a_k + a_n + \theta_n$$

for some θ_n in $[0,1[$. Thus a_n is the remainder of $\lfloor m^n q \rfloor$ modulo m, and (5.1) follows. ■

Let $F(q) := \sum_{n=1}^{\infty} c_n q^n$ be any formal power series.

Theorem 5.1. *For* $0 < q < 1/\limsup_{n\to\infty} |c_n|^{1/n}$,

$$F(q) = \sum_{n=1}^{\infty} \frac{f(n)}{m^n}$$

where

$$f(n) = \sum_{k\ge 1} c_k(\lfloor m^n q^k \rfloor \bmod m).$$

Proof: From Lemma 5.1

$$F(q) = \sum_{k=1}^{\infty} c_k q^k = \sum_{k=1}^{\infty} c_k \sum_{n=1}^{\infty} \frac{\lfloor m^n q^k \rfloor \bmod m}{m^n}$$
$$= \sum_{n=1}^{\infty} \frac{f(n)}{m^n}$$

on exchanging order of summation, as is valid within the radius of convergence of F. ■

Theorem 5.1 can be extended so as to replace m^n by $\prod_{k=1}^{n} r_k$ where r_k are integers ≥ 2, and where the remainder is computed modulo r_n.

If we specialize Theorem 5.1 to the case where $q := 1/b$ and b is an integer divisible by m we may observe that $\lfloor m^n/b^k \rfloor \bmod m$ coincides with the coefficient (mod m) of b^k in the base b expansion of m^n (the $(k+1)^{th}$ digit).

Specializing further so that $m := 2$ and b is even we have

$$F\left(\frac{1}{b}\right) = \sum_{n=1}^{\infty} \frac{f_b(n)}{2^n} \tag{5.2}$$

where

$$f_b(n) := \sum \left\{ c_k \mid 2^n \text{ has } (k+1)^{\text{th}} \text{ digit odd base } b \right\}.$$

Example 5.1. (a) Let $F(q) := q/(1-q)$. Then $f_b(n)$ counts the number of odd digits in 2^n base b. Sum 7 is established on setting $b := 10$.

(b) Sum 1 corresponds to taking $F(q) = q^2/(1-q^2)$ and $q = 1/10$.

(c) Let $F(q) = q/(1-q-q^2)$. Now F is the generating function of the Fibonacci numbers ($F_1 = 1$, $F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$). Again with $q := 1/10$, we obtain for

$$f(n) := \sum \left\{ F_k | \, 2^n \text{ has } (k+1)^{\text{th}} \text{ digit odd} \right\},$$

as in Bowman and White [4], that

$$\sum_{n=1}^{\infty} \frac{f(n)}{2^n} = \frac{10}{89}.$$

The generating function for F_k^2 is $\dfrac{q - q^2}{1 - 2q - 2q^2 + q^3}$ and so for

$$f(n) := \sum \left\{ F_k^2 | \, 2^n \text{ has } (k+1)^{\text{th}} \text{ digit odd} \right\}$$

$$\sum_{n=1}^{\infty} \frac{f(n)}{2^n} = \frac{90}{781}.$$

(d) Let

$$F(q) = \sum_{n=1}^{\infty} q^{n^2} = \frac{\theta_3(q) - 1}{2}.$$

Then

$$\sum_{n=1}^{\infty} \frac{f(n)}{2^n} = \frac{\theta_3\left(\frac{1}{10}\right) - 1}{2}$$

where $f(n)$ counts the number of odd digits of 2^n in square positions (the second, fifth, tenth digits etc.).

(e) If we apply Theorem 5.1 to $F(q) := q/(1-q)$ with $b := 10$ and $m := 5$ we deduce that again

$$\sum_{n=1}^{\infty} \frac{f(n)}{5^n} = \frac{1}{9}$$

where $f(n)$ sums the digits (mod 5) of 5^n base 10 (e.g. $f(3125) = 6$). ∎

6. THETA FUNCTION EXAMPLES. The underlying identity for this section is really just a modular transformation of $\theta_3(q) := \sum_{n=-\infty}^{\infty} q^{n^2}$. (See [2].)

Lemma 6.1. *For $\alpha, \beta > 0$ with $\alpha\beta = 2\pi$*

$$\sqrt{\alpha}\left[\sum_{n=-\infty}^{\infty} e^{-\alpha^2 n^2/2}\right] = \sqrt{\beta}\left[\sum_{n=-\infty}^{\infty} e^{-\beta^2 n^2/2}\right].$$

Example 6.1. From the Lemma, with $s = 2/\beta^2$ so $\alpha^2 = 2\pi^2 s$

$$\sqrt{\pi s} - \sum_{n=-\infty}^{\infty} e^{-n^2/s} = 2\sqrt{\pi s}\, e^{-\pi^2 s} + O(e^{-\pi^2 4s}) \tag{6.1}$$

$$\sim 2\sqrt{\pi s}\, 10^{-(4.2863\ldots)s}.$$

Now with $s := 10^{10}$ we get

$$\left|\sqrt{\pi} - \left(\frac{1}{10^5}\sum_{n=-\infty}^{\infty} e^{-n^2/10^{10}}\right)\right| \leq 10^{-4.2 \cdot 10^{10}}, \tag{6.2}$$

which is Sum 12.

If we set

$$s = \frac{1}{\log 10^{1/N}} = \frac{N}{\log 10}$$

we get

$$\sqrt{\frac{N\pi}{\log 10}} - \sum_{-\infty}^{\infty} \frac{1}{10^{n^2/N}} \sim 2 \cdot \sqrt{\frac{N\pi}{\log 10}}\, 10^{-(1.861\cdots)N} \tag{6.3}$$

and with $N := 10^4$ we get Sum 11.

Similarly we have

$$\sqrt{\frac{q\pi}{\log q}} - \sum_{-\infty}^{\infty} \frac{1}{q^{n^2/q}} \sim 2\sqrt{\frac{q\pi}{\log q}}\, e^{-\pi^2 q/\log q}. \tag{6.4}$$

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ematics and reasoning have in fact changed the way many of us view the world of mathematics.

A hundred years before Russell's paradox ushered in the 20th century, the great French scientist J. L. Lagrange wrote,

> It seems to me that the mine is already almost too deep, and unless we discover new seams we shall sooner or later have to abandon it. Today Physics and Chemistry offer more brilliant and more easily exploited riches; and it seems that the taste of the century has turned entirely in that direction. It is not impossible that the mathematical positions in the Academies will one day become what the University chairs in Arabic are now.

This lament echoes a *fin de siècle* pessimism that has struck mathematics toward the end of each of the last three centuries. Very likely we will indulge in a similar malaise of millennialism over the next few years. If we are as lucky as were our ancestors, this will be followed, in the manner of an economy coming out of a long recession, by a tremendous burst of productivity in which new and unexpected directions will be taken. Paradigms will shift, perhaps as dramatically as they did at the beginning of both this century (in the shadow of the modern atom) and last century (with the advent of rigor à la Cauchy and the later disquieting discovery of non-Euclidean geometries).

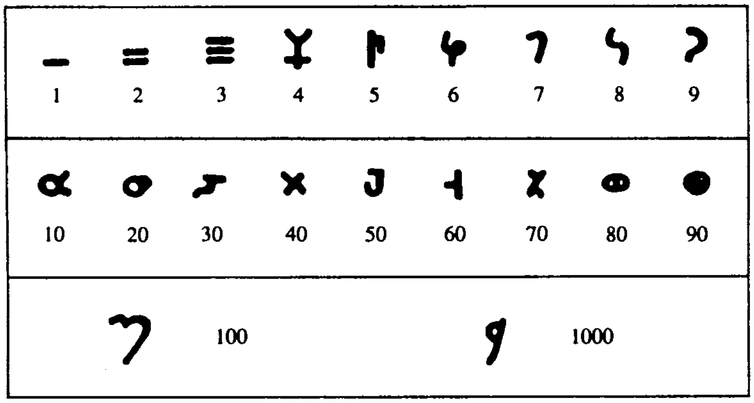
For Lagrange mathematics was prosaically Platonic, intellectual coal to be mined. The lament was not for the passing of mathematics, it was for the passing of mathematicians. Mathematical ore is still there even if no one is digging. It is the cultural loss, or perhaps the loss of a pleasant livelihood, not the lost science, that is found troubling.

Lagrange, comfortable in his Platonic belief in a tangible, physical mathematics and its concomitant discovery and exploitation, might have been much shaken by the crisis induced by Russell and his contemporaries. Others certainly were. Many mathematicians today take a much more formalist, axiomatic, and bloodless approach to their subject. Some take an extreme constructivist position: things that cannot be constructed finitely do not exist. Others take an intuitionist point of view: proofs must eschew the principle of the excluded middle and must be fully (psychologically) analyzable.

The questioning of foundations has led to some of the truly profound insights of the century about the nature of knowledge, uncertainty, randomness and unknowability, the gulf between truth and proof. Out of the brains of logicians like Turing sprang fully formed theoretical computers—with all the power of the physical ones that were still to be built. Thus computers were in fact discovered before they were invented—or perhaps it is the other way around.

This begins to touch the themes of Barrow's richly woven book, which is really a collection of six long, lucid, loosely linked essays in the philosophy, history, and culture of mathematics.

We would not indulge in "millennialism" if we had six fingers on each hand, nor would we tend to encapsulate by centuries. But we would almost certainly still count, and quite probably in very similar fashion, even if in a different base. In a long chapter on the cultural development of counting and numeration, Barrow asserts, "The Indian system of counting has been the most successful intellectual innovation ever made on our planet"—a grand claim that is persuasively defended. The Indian innovation is primarily the number zero. A Wonderland feature, that nothing can be claimed more successful and inevitable than the discovery (invention) of nothing.



"An early form of Indian numerals in Brâhmî script. Our own number symbols are descended from them." [From *Pi in the Sky*]

Most of this book revolves around making the case for and against the various competing philosophies of mathematics. By and large the case against wins each round. It is very hard to embrace any of these philosophies wholeheartedly. We tend to set our own personal demarcations. But for most of us some parts of mathematics *exist*: natural numbers, triangles, perhaps pi. Some of the more exotic and abstract bits just don't have the same claim to a life of their own. The average mathematician is a mosaic: perhaps two parts Platonist to one part formalist, with a taste for constructive proofs when possible. (We challenge the reader to find a working mathematician of any philosophical stripe who would refuse authorship of a classically valid but nonconstructive proof of the celebrated Riemann hypothesis no matter what axiomatics that proof demanded.)

What keeps this book so readable is the texture: the historical anecdotes; the careful biographical sketches of Goedel, Cantor, Brouwer, Hilbert, and others; the excursions into the bizarre world of undecidability; the speculations on the future; the thought-provoking ripostes. (In answer to Roger Penrose, Barrow suggests that the capacity to encode undecidable statements is a precondition for consciousness of a structure.) Throughout Barrow demonstrates a remarkable scope, a fine sense of how mathematics works, and considerable insight into how it may be evolving. Occasional minor technical infelicities do nothing to mar the success of his project.

Barrow writes, "Today it is not unexpected to find the 'computer' or the 'program' as central paradigms in our attempts to interpret the Universe" and observes that "the concept of experimental mathematics has begun to take on a new and more adventurous complexion." This pervasive use of the computer to attempt to interpret mathematics rather than just the universe is surprisingly new. Mathematicians invented computers and then for several decades proceeded largely to ignore them. It is only recently, with the advent of really successful symbolic manipulation of computer algebra packages, that computers have come of mathematical age—or, more accurately, have entered puberty.

This book is not so much about mathematics as specialist subject as it is about mathematics as universal language. Talking meaningfully about mathematics without talking in mathematics is a difficult and underpracticed art. Barrow's book is a very welcome addition to this literature.

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■

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BOOK REVIEWS

EDITED BY C. W. GROETSCH AND K. R. MEYER

The Encyclopedia of Integer Sequences. *By N. J. A. Sloane and Simon Plouffe.* Academic Press, San Diego, CA, 1995. $44.95. xii+587 pp., hardcover. ISBN 0-12-558630-2.

The Encyclopedia of Integer Sequences by Sloane and Plouffe, published by Academic Press, is not a normal book. It contains a lexicographically ordered collection of integer sequences together with references to where these sequences appear in the literature. The idea is that a researcher who encounters a sequence in her or his work, and wishes to quickly find out what is known about the sequence (does it have a name, for example, such as "the Euler numbers" or "the Stirling numbers of the first kind"?), can look it up here.

On the face of it this seems a difficult task to accomplish, because surely there are very many sequences of interest. However, by Pareto's principle (80% of your work is done with 20% of your tools) we would expect that simple sequences would occur often, and thus such a book would be useful.

Indeed, this is the case, and even if the book were no more than the handsomely bound physical collection it is, it would have been worthwhile to create, publish, or buy, because it provides a very cheap and efficient route to answers that will work sometimes: if it doesn't work on a particular problem, no great effort has been expended, while if it *does* work you may save a lot of time.

But the physical book is *not* the whole story. Sloane and Plouffe have also created two "avatars" of the book; these are freely available online computer programs (which we will call `sequences` and `superseeker`) for people to send their sequences to. Because the programs can be accessed by people who do not own the book, we think that Academic Press deserves considerable praise for its enlightened attitude toward the changing shape of publishing.

This is not the first, but is one of the first of a growing list of sophisticated tools which are accessible to even relatively naive users, and which dramatically illustrate a positive use of the Internet. Our dream work environment would provide us with a whole palette of such tools and a simple key to what exists and how to use it. These tools should ideally be fully compatible with your favourite working environment (MATLAB, Axiom, Maple, Mathematica, etc.).

In our opinion the physical book is itself worthwhile not only because it is pleasant to browse in (electrons are so cold, in comparison) but also because of the discussion at the beginning on analysis of sequences. Some of the heuristics discussed in Chapters 1, 2, and 3 (before the table of sequences proper begins) give useful hints about what to do when the computer programs don't work; they also give a nice conceptual model of the inner workings of the programs.

One can turn the tables (so to speak) and use the sequences from the book as a test of each of the subprograms in `sequences` and `superseeker`. Simon Plouffe tells us that each subprogram was considered useful enough to be included if it could identify on the order of 10–100 of the sequences from the book. Further, about 25% of the sequences in the book are obtained from a rational generating function or elementary manipulation thereof (reversion, the undoing of a logarithmic differentiation, etc.). The addition of various other classes such as hypergeometric functions and preprocessing (adding "1" to each term or doubling the terms, etc.) significantly increased the hit rate. It is to be emphasized that not every plausible transformation was included, and much expertise on the part of the authors was needed to choose useful transformations and to avoid "the curse of exponentiality."

Finally, some "off-the-wall" sequences are also included, such as the numbers on the New York subway stops in Figure M5405.

Incidentally, due to a printer's error the table of figures was not included in the book, and as these "silly" sequences are not actually indexed or numbered in the book, one must either use the programs or know that they are contained in Figure M5405 to find them.

We now give some examples of the uses of the book and the programs to demonstrate their utility (and also some limitations).

Publishers are invited to send books for review to Book Reviews Editor, SIAM, 3600 *University City Science Center, Philadelphia, PA* 19104-2688.

1. How to use the programs.

1. Prepare a message with a line of the form

```
lookup 1 1 5 61 1385 50521 2702765
```

in it (obviously change the sequence to the one you want to look up).

2. Send the message to

```
sequences@research.att.com
```

for the simple lookup service. In this case, omit the initial terms, and omit all minus signs. Separate the entries in the sequence with spaces, not commas. This simply looks up sequences in the Encyclopedia. The answer frequently comes back within minutes.

3. Send the message to

```
superseeker@research.att.com
```

for a more "enthusiastic" attempt to identify your sequence. This time, include the initial terms, *and the minus signs.* If possible, give from 10 to 20 terms. This program tries over 100 transformations in an attempt to match the given sequence with ones in the Encyclopedia.

2. Related books and programs.

- *A Handbook of Integer Sequences* by N. J. A. Sloane (1973). This might be considered the "first edition" of the book under review. It contained only 2372 sequences, compared to 5488 in the current volume. As of this writing, there are 6222 basic entries in the dynamic online version, and because of the transformations many more sequences can be identified.
- *A Dictionary of Real Numbers* by Borwein and Borwein (1990).
- ISC—the Inverse Symbolic Calculator, which can be found easily from

```
http://www.cecm.sfu.ca.
```

When you give this program an approximation to a real number, it will do its best to decide what that number "really" is—in essence, this is a greatly expanded online version of *A Dictionary of Real Numbers*, mentioned above. For example, the ISC describes $\int_0^1 dx/\sqrt{1-x^4} = 1.311028777\ldots$ as the "lemniscate number." Simon Plouffe is currently working on this program here at the Centre.

- `gfun` the Generating Function Package by Salvy and Zimmerman [4]. This Maple package from the `share` library contains functions for manipulating sequences, linear recurrences, differential equations, and generating functions of various types. Simon Plouffe and F. Bergeron had some input into this package as well.
- `numapprox[pade]` (formerly `convert/ratpoly`) in Maple. This utility uses clever algorithms to convert power series into Padé approximants.
- Mathematica has facilities for conversion of series into Padé approximants and the like as well.

3. Examples for superseeker and sequences.

3.1. Example 1. We begin with a classical analytic example from *Pi, Euler numbers, and asymptotic expansions* [1].

R. D. North asked for an explanation of the following fact:

$$4\sum_{k=1}^{500{,}000}\frac{(-1)^{k-1}}{2k-1}$$
$$= 3.14159\underline{0}6535897932\underline{4}0462643383\underline{2}69502884197.$$

The number on the right is not π to 40 places. As one would expect, the 6th digit after the decimal point is wrong. The surprise is that *only* the underlined digits are wrong. This is explained in detail in [1]. The discovery of the explanation is quite difficult from this result, but is somewhat easier from the following similar one:

$$\frac{\pi}{2}\approx 2\sum_{k=1}^{50{,}000}\frac{(-1)^{k-1}}{2k-1}$$
$$= 1.5707\underline{8}63267948\underline{9}761923132\underline{1}9163975\underline{205}209\ldots.$$

Here we note that if we add 1, −1, 5, and −61 to the incorrect digits, we get equality (to 40 places) with $\pi/2$. With the help of Sloane and Plouffe (or indeed with the help of Sloane's original *Handbook*, as was actually the case, or the computer programs) we can identify these as the first four nonzero Euler numbers. We conjecture, then, that the error is of the form

$$\frac{\pi}{2} = 2\sum_{k=1}^{50{,}000}\frac{(-1)^{k-1}}{2k-1} + \sum_{k\geq 1}\frac{E_{2k}}{100{,}000^{2k+1}},$$

where E_{2k} is the kth nonzero Euler number. We can test this conjecture by computation, and find by adding the first 80 terms in the error formula above to the sum that we get $\pi/2$ to 500 digits. This does not tell us that our conjecture is true, but at least it encourages us that a proof might be possible. See [1] for the proof.

The point of this example is that recognition of the Euler numbers in this at first required ingenuity (to shift from π to $\pi/2$, because the original problem has *twice* the Euler numbers appearing in it). However, the case has changed: the new programs both recognize twice the Euler numbers.

If we do not give enough terms to `superseeker`, it fails to return anything (the heuristics of the program are not designed for short sequences, which, after all, can represent far too many things to be really useful). If we put in 7 terms, however, it returns the following.

```
Report on [ 2,2,10,122,2770,
101042,5405530]:

Many tests are carried out, but
only potentially useful
information (if any)
is reported here.

TEST: APPLY VARIOUS
TRANSFORMATIONS TO
SEQUENCE AND LOOK IT UP
IN THE ENCYCLOPEDIA AGAIN

        SUCCESS
        (limited to 10 matches):

Transformation T003 gave a
match with sequence A0364
Transformation T004 gave a
match with sequence A0364

List of sequences mentioned:

%I A0364 M4019 N1667
%S A0364 1,1,5,61,1385,50521,
2702765,199360981,
19391512145,2404879675441,
%T A0364 370371188237525,
69348874393137901,
15514534163557086905,
%U A0364 4087072509293123892361
%N A0364 Euler numbers:
expansion of sec $x$.
%R A0364 AS1 810. MOC 21 675 67.

<some cryptic material omitted>

References (if any):

[AS1] = M. Abramowitz and
I. A. Stegun, {Handbook of
Mathematical Functions}, National
Bureau of Standards,
Washington DC, 1964.
[MOC] = {Mathematics of
Computation} (formerly
{Mathematical Tables and
Other Aids to Computation}).

List of transformations used:
T003  sequence divided by the
gcd of its elements
T004  sequence divided by the
gcd of its elements,
from the 2nd term

Abbreviations used in the above
list of transformations:
u[j] = j-th term
of the sequence
v[j] = u[j]/(j-1)!
Sn(z) = ordinary
generating function
En(z) = exponential
generating function
```

The Euler numbers appear as sequence M4019 in the book. (The code here is to the explicit tag in the book; A0364 is an internal absolute code while T003 tags the transformation used.)

3.2. Example 2. The following sequence arose in the analysis of the long-term dynamics of numerical methods. For details on the mathematics of this sequence, see [2], but for now note that this could (broadmindedly) be considered as applied mathematics because RMC was investigating the reliability of numerical methods for solving nonlinear differential equations over long time intervals (the classical theory gives results useful only on compact time intervals, and the presence of exponentially growing terms in the classical error bounds raises questions about the validity of numerical solutions over long time intervals).

Define the function $B(v) = 1 - v + 3v^2/2 - 8v^3/3 + 31v^4/6 - 157v^5/15 + \cdots$. Then multiplying each coefficient by $k!$ we get the following sequence:

$$1, -1, 3, -16, 124, -1256, 15576, \ldots.$$

This (modulo the obviously trivial minus signs) is sequence M3024 in the book, which gives the reference to [3].

The history of the example is perhaps more interesting than the mathematics. The first few terms of a series representing the "modified equation" solved by $u_{n+1} = u_n + hu_n^2$, which arises from forward Euler applied to $\dot{y} = y^2$, were laboriously computed using Maple. Bruno Salvy's `gfun` package was then used to identify the sequence; it succeeded, but on checking it was found that the wrong sequence had been generated in the first place (i.e., there was a bug in my Maple program—RMC). Once the bug was fixed, `gfun` could no longer identify the sequence. Bruno Salvy (who is at INRIA in France) was asked for help, and he remarked (immediately) that he *recognized the sequence*. It turned out that he had a prepublication version of the book under review here and, as stated previously, the sequence is listed in the book! Coincidentally, Gilbert Labelle (from Montréal, the author of reference [3]) was visiting INRIA at this time as well, so it is conceivable that even without the book the sequence would have been recognized, but the book did play a role.

It is worth remarking that the paper by Labelle that was uncovered by this recognition was extremely apt, and *would never have been discovered otherwise* because it is extremely unlikely that RMC would have looked in a combinatorics journal for a result on reliability of numerical methods for dynamical systems.

3.3. Example 3. Consider $a = 1 + \sqrt{2}$, which is a Pisot number because the other root of $a^2 - 2a - 1$ is inside the unit circle. Then a^n is asymptotically an integer, and indeed $a^n + (-1)^n/a^n = 2, 6, 14, 34, 82, 198, \ldots$.

The sequence as such is not in the book (we must divide by 2) but even without division by 2, `sequences` return the following:

```
Matches (at most 7) found
for  2 6 14 34 82 198:

%I A2203 M0360 N0136
%S A2203 2,2,6,14,34,82,198,478,
1154,2786,6726,16238,39202,
94642,228486,
%T A2203 551614,1331714,3215042,
7761798,18738638,45239074,
109216786,263672646
%N A2203 Companion Pell numbers:
   $a(n) = 2a(n-1) + a(n-2)$.
%R A2203 AJM 1 187 1878.
FQ 4 373 66. BPNR 43.
%O A2203 0,1
%C A2203 njas
%K A2203

References (if any):
[AJM] = {American Journal
of Mathematics}.
[BPNR] = P. Ribenboim,
{The Book of Prime Number
Records}, Springer-Verlag,
NY, 2nd ed., 1989.
[FQ] = {The Fibonacci Quarterly}.
```

Instead of mentioning Pisot numbers, the sequence is (correctly) identified as being related to Companion Pell numbers. This connection also would have been unlikely without this compendium.

3.4. Example 4. A problem that recently arose on `sci.math` was the (well-known) problem of finding when triangular numbers are square numbers; the first few quickly lead to the sequence $1, 8, 49, 288, 1681, \ldots$, as can be determined with a few minutes computation. This is sequence M4536 in the book, and references are provided to Dickson's History, to Beiler, and other recreational mathematics books. In some sense this is what the book is for: to give people an *index* into what is "well known" and perhaps to avoid ingenious but ultimately wasted rediscovery.

3.5. Failed examples. The book and programs are not oracles, and cannot perform miracles. For example, if we submit the following sequence, which simply counts the number of terms in a particular arrangement of a perturbation solution of a heat transfer problem (we would like to know how quickly the size of the solution is growing),

$$2, 12, 44, 100, 203, 344, 558,$$

$$824, 1189, 1620, 2176, 2812, \ldots,$$

we get no answer.

Other failures can of course occur. The following example shows what *might* happen, and the potential for misidentification of a sequence. If we submit the sequence [0, 1, 2, 3, 5, 7, 9, 12, 15, 18] to `superseeker`, it returns matches for both M0638 and for M0639, which agree to the first 10 entries. The 11th entry for M0638 is 22, while the 11th entry for M0639 is 23. One of them must be wrong, and this brings home the fact that even if the programs or book say that the sequence you give it is X *only*, that might just be a numerical coincidence.

The user of the book and programs must remember that a match does not prove that the sequence found is the one you are looking for, and it is up to the user to demonstrate that any matches found by the programs or in the book are really appropriate for the problem at hand.

4. About the reviewers. One of us is more pure than applied, while the other is more vice than versa. However, neither of us believes in drawing artificial boundaries between branches of mathematics, and to us the most exciting thing about the book under review is that it helps to erase such boundaries (and indeed narrow the real gaps that exist). RMC is currently visiting the Centre for Experimental and Constructive Mathematics, where JMB is Director. It should be noted that this review is not entirely at "arm's length," because (as previously mentioned) Simon Plouffe has recently joined the CECM.

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Transport Simulation in Microelectronics. *By Alfred Kersch and William J. Morokoff.* Birkhäuser, Basel, 1995. $89.00. 235 pp., cloth. ISBN 3-7643-5168-3.

This book deals with important applications of the Boltzmann equation to the field of microelectronics. Other known examples are studies of upper atmosphere flight, which occur, e.g., in connection with the re-entry of a space shuttle, micromachines, environmental problems, such as understanding and control of the formation, motion, reactions, and evolution of particles of varying composition and shapes, ranging from a diameter of the order of $.001\mu m$ to $50\mu m$, as well as their space-time distribution under gradients of concentration, pressure, temperature, and the action of radiation. Modified versions of the Boltzmann equation appear in other well-known applications, such as neutron transport, plasma physics, and radiative transfer [1].

In these fields of modern technology the concepts and tools introduced by Boltzmann are essential. This would have pleased Boltzmann, who was very much interested in technological advances (he also wrote a paper in which he correctly predicted the superiority of airplanes over dirigible airships) and is the author of the following sentences written in 1902 [2], which nowadays may sound a bit trivial:

However much science prides itself on the ideal character of its goal, looking down somewhat contemptuously on technology and practice, it cannot be denied that it took its rise from a striving for satisfaction of purely practical needs. Besides, the victorious campaign of contemporary natural science would never have been so incomparably brilliant, had not science found in technologists such capable pioneers.

All the problems mentioned above have in common the fact that the mean free path is not negligible with respect to some other characteristic length. In fact, simple considerations indicate that for rarefied gases one cannot rely on the usual Navier-Stokes equations for a compressible fluid and must resort to kinetic theory.

The basic evolution equation in kinetic theory is the Boltzmann equation [1, 3, 4], which governs the time development of the distribution function $f = f(x, v, t)$, i.e., the probability density (in the phase space described by $(x, v) \in \Omega \times \Re^3$, $\Omega \subset \Re^3$) of finding a molecule with position x and velocity v at time t. In the absence of a body force and for the case of a monatomic gas, this equation may be written as follows:

(1)
$$\partial f/\partial t + v \cdot (\partial f/\partial x) = \iint (f' f'_* - f f_*) B(\theta, \mid v - v_* \mid) dv_* d\theta d\phi.$$

Here $B(\theta, \mid v - v_* \mid)$ is a kernel containing the details of the molecular interaction and f', f'_*, f_* are the same as f, except for the fact that the argument v is replaced by v', v'_*, v_*, respectively, v_* being an integration variable (having

Chapter 7

Making Sense of Experimental Mathematics

7.1 Introduction

Philosophers have frequently distinguished mathematics from the physical sciences. While the sciences were constrained to fit themselves via experimentation to the "real" world, mathematicians were allowed more or less free reign within the abstract world of the mind. This picture has served mathematicians well for the past few millennia but the computer has begun to change this. The computer has given us the ability to look at new and unimaginably vast worlds. It has created mathematical worlds that would have remained inaccessible to the unaided human mind, but this access has come at a price. Many of these worlds, at present, can only be known experimentally. The computer has allowed us to fly through the rarefied domains of hyperbolic spaces and examine more than a billion digits of π but experiencing a world and understanding it are two very different phenomena. Like it or not, the world of the mathematician is becoming experimentalized.[1]

The computers of tomorrow promise even stranger worlds to explore. Today, however, most of these explorations into the mathematical wilderness remain isolated illustrations. Heuristic conventions, pictures and diagrams developing

[1]This entire chapter is a reprint (with permission) of "Making Sense of Experimental Mathematics," by J. M. Borwein, P. B. Borwein, R. Girgensohn, and S. Parnes, *Mathematical Intelligencer*, vol. 18 (1996), page 12–18.

in one sub-field often have little content for another. In each sub-field unproven results proliferate but remain conjectures, strongly held beliefs or perhaps mere curiosities passed like folk tales across the Internet. The computer has provided extremely powerful computational and conceptual resources but it is only recently that mathematicians have begun to systematically exploit these abilities. It is our hope that by focusing on experimental mathematics today, we can develop a unifying methodology tomorrow.

7.1.1 Our Goals

The genesis of this article was a simple question: "How can one use the computer in dealing with computationally approachable but otherwise intractable problems in mathematics?" We began our current exploration of *experimental mathematics* by examining a number of very long–standing conjectures and strongly held beliefs regarding decimal and continued fraction expansions of certain elementary constants. These questions are uniformly considered to be hopelessly intractable given present mathematical technology. Unified field theory or cancer's "magic bullet" seem accessible by comparison. But like many of the most tantalizing problems in mathematics their statements are beguilingly simple. Since our experimental approach was unlikely to result in any new discoveries[2], we focused on two aspects of experimentation: systematization and communication.

For our attempted systematization of *experimental mathematics* we were concerned with producing data that were "completely" reliable and insights that could be quantified and effectively communicated. We initially took as our model experimental physics. We were particularly interested in how physicists verified their results and the efforts they took to guarantee the reliability of their data. The question of reliability is undoubtedly central to mathematicians and here we believe we can draw a useful distinction between experimental physics and mathematics. While it is clearly impossible to extract perfect experimental data from nature such is not the case with mathematics. Indeed, reliability of raw mathematical data is far from the most vexing issue.

Let us turn to our second and primary concern: insight. All experimental sciences turn on the intuitions and insights uncovered through modeling and the

[2]We will not discuss the computational difficulties here but there are many non-trivial mathematical and computer–related issues involved in this project.

use of probabilistic, statistical and visual analysis. There is really no other way to proceed, but this process even when applied to mathematics inevitably leads to some considerable loss of exactness.

The communication of insight, whether derived from mathematical experiment or not, is a complex issue. Unlike most experimentalized fields, Mathematics does not have a "vocabulary" tailored to the transmission of condensed data and insight. As in most physics experiments the amount of raw data obtained from mathematical experiments is, in general, too large for anyone to grasp. The collected data needs to be compressed and compartmentalized. To make up for this lack of unifying vocabulary we have borrowed heavily from statistics and data analysis to interpret our results. For now we have used restraint in the presentation of our results in what we hope is an intuitive, friendly and convincing manner. Eventually what will probably be required is a multi-leveled hypertextual presentation of mathematics, allowing mathematicians from diverse fields to quickly examine and interpret the results of others—without demanding the present level of specialist knowledge. [Not only do mathematicians have trouble communicating with lay audiences, but they have significant difficulty talking to each other. There are hundreds of distinct mathematical languages. The myth of a universal language of mathematics is just that. Many subdisciplines simply can not comprehend each other.]

7.1.2 Unifying Themes

We feel that many of these problems can be addressed through the development of a rigorous notion of experimental mathematics. In keeping with the positivist tradition, mathematics is viewed as the most exact of sciences and mathematicians have long taken pride in this. But as mathematics has expanded, many mathematicians have begun to feel constrained by the bonds placed upon us by our collective notion of proof. Mathematics has grown explosively during our century with many of the seminal developments in highly abstract seemingly non-computational areas. This was partly from taste and the power of abstraction but, we would argue, equally much from the lack of an alternative. Many intrinsically more concrete areas were, by 1900, explored to the limits of pre-computer mathematics. Highly computational, even "brute-force" methods were of necessity limited but the computer has changed all that. A re-concretization is now underway. The computer–assisted proofs of the four color theorem are a prime example of computer–dependent methodology and have been highly controversial despite the fact that such proofs are much more likely to be error free than, say, even the revised proof of Fermat's Last Theorem.

Still, these computerized proofs need offer no insight. The Wilf/Zeilberger algorithms for "hypergeometric" summation and integration, if properly implemented, can rigorously prove very large classes of identities. In effect, the algorithms encapsulate parts of mathematics. The question raised is: "How can one make full use of these very powerful ideas?" Doron Zeilberger has expressed his ideas on experimental mathematics in a paper dealing with what he called "semi-rigorous" mathematics. While his ideas as presented are somewhat controversial, many of his ideas have a great deal of merit.

The last problem is perhaps the most surprising. As mathematics has continued to grow there has been a recognition that the age of the mathematical generalist is long over. What has not been so readily acknowledged is just how specialized mathematics has become. As we have already observed, sub-fields of mathematics have become more and more isolated from each other. At some level, this isolation is inherent but it is imperative that communications between fields should be left as wide open as possible.

As fields mature, speciation occurs. The communication of sophisticated proofs will never transcend all boundaries since many boundaries mark true conceptual difficulties. But experimental mathematics, centering on the use of computers in mathematics, would seem to provide a common ground for the transmission of many insights. And this requires a common meta–language[3]. While such a language may develop largely independent of any conscious direction on the part of the mathematical community, some focused effort on the problems of today will result in fewer growing pains tomorrow.

7.2 Experimental Mathematics

7.2.1 Journal of

A professor of psychology was exploring the creative process and as one of his subjects chose a mathematician who was world famous for his ability to solve

[3]This may not be a fanciful dream as the *Computer Algebra Systems* (CAS) of today are beginning to provide just that.

problems. They gave him a problem to work on. He wrote something down and immediately scribbled it out. He wrote something else down and scribbled it out. The professor asked him to leave everything on the page. He explained that he was interested in the process, the wrong answers and the right answers. The mathematician sat down. Wrote something. The psychology professor waited in anticipation but the mathematician announced he could not proceed without erasing his mistakes. While the mathematician in this situation is undoubtedly fairly idiosyncratic in how he attacks problems there is a strongly felt separation between the creative process of mathematics and the published or finished product.

A current focal point for experimental mathematics is the journal called *Experimental Mathematics*. But does it really seek to change the way we do mathematics, or to change the way we write mathematics? We begin by attempting to extract a definition of "experimental" from the Journal's introductory article ([86]) "About this Journal" by David Epstein, Silvio Levy and Rafael de la Llave.

> The word "experimental" is conceived broadly: many mathematical experiments these days are carried out on computers, but others are still the result of pencil-and-paper work, and there are other experimental techniques, like building physical models. ([86] p. 1)

It seems that almost anything can be conceived of as being experimental. Let us try again.

> Experiment has always been, and increasingly is, an important method of mathematical discovery. (Gauss declared that his way of arriving at mathematical truths was "through systematic experimentation".) Yet this tends to be concealed by the tradition of presenting only elegant, well-rounded and rigorous results. ([86] p. 1)

Now we begin to get closer to the truth. Experimentation is still ill defined but is clearly an important part of the mathematical process. It is clearly not new but by implication must be inelegant, lopsided and lax. We, of course, dispute all three of these points and while we do not reply directly to these charges, we hope the reader will be convinced that there need be no compromises made with respect to the quality of the work.

But what is the journal interested in publishing? Their goal seems to be two-fold.

> While we value the theorem-proof method of exposition, and while we do not depart from the established view that a result can only become part of mathematical knowledge once it is supported by a logical proof, we consider it anomalous that an important component of the process of mathematical creation is hidden from public discussion. It is to our loss that most of the mathematical community are almost always unaware of how new results have been discovered. ([86] p. 1)

and

> The early sharing of insights increases the possibility that they will lead to theorems: an interesting conjecture is often formulated by a researcher who lacks the techniques to formalize a proof, while those who have the techniques at their fingertips have been looking elsewhere.

It appears that through the journal *Experimental Mathematics* the editors advocate a not undramatic change in writing style. So what does a paper published in that journal look like? A recent example is “Experimental Evaluation of Euler sums” by D. H. Bailey, J. Borwein and R. Girgensohn ([20]). The authors describe how their interest in Euler sums was roused by a surprising discovery:

> In April 1993, Enrico Au-Yeung, an undergraduate at the University of Waterloo, brought to the attention of one of us the curious fact that
>
> $$\begin{aligned}\sum_{k=1}^{\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{k}\right)^2 k^{-2} &= 4.59987\cdots\\ &\approx \frac{17}{4}\zeta(4)=\frac{17\pi^4}{360}\end{aligned}$$
>
> based on a computation to 500,000 terms. This author’s reaction was to compute the value of this constant to a higher level of precision in order to dispel this conjecture. Surprisingly, a computation to 30 and later to 100 decimal digits still affirmed it. ([20] p. 17)

After Enrico Au-Yeung's serendipitous discovery, D. Bailey, J. Borwein and R. Girgensohn launched a full fledged assault on the problem. This is documented in "Experimental Detection of Euler Sums" (the material below was taken from David Bailey's slides).

Experimental Approach

1. Employ an advanced scheme to compute high-precision (100+ digit) numerical values for various constants in a class.
2. Conjecture the form of terms involved in possible closed-form evaluations.
3. Employ an integer relation finding algorithm to determine if an Euler sum value is given by a rational linear combination of the conjectured terms.
4. Attempt to find rigorous proofs of experimental results.
5. Attempt to generalize proofs for specific cases to general classes of Euler sums.

Table 7.1: Serendipity and experimentation.

This type of serendipitous discovery must go on all the time, but it needs the flash of insight that will place it in a broader context. It is like a gold nugget waiting to be refined—without a context it would remain a curiosity. The authors now proceeded to provide a context by mounting a full-fledged assault on the problem. They systematically applied an integer relation detection algorithm to large classes of sums of the above type, trying to find evaluations of these sums in terms of zeta functions (see Table 7.1 and 7.2 for details). Some of the experimentally discovered evaluations were then proven rigorously, others remain conjectures. While Au-Yeung's insight may fill us with a sense of amazement, the experimenters' approach appears quite natural and systematic.

The editors of *Experimental Mathematics* are advocating a change in the way mathematics is written, placing more emphasis on the mathematical process. Imre Lakatos in his influential though controversial book *Proofs and Refutations*

[137] advocated a similar change from what he called the deductivist style of proof to the heuristic style of proof. In the deductivist style, the definitions are carefully tailored to the proofs. The proofs are frequently elegant and short. But it is difficult to see what process led to the discovery of the theorem and its proof. The heuristic style maintains the mathematical rigor but again the emphasis is more on process. One does not merely give the definition but perhaps includes a comment on why this definition was chosen and not another. This is clearly an important shift if the editors wish to meet their second objective, the sharing of insights.

7.2.2 The Deductivist Style

The major focus of this section is Imre Lakatos's description of the deductivist style in *Proofs and Refutations*. An extreme example of this style is given in the form of a computer generated proof of $(1+1)^n = 2^n$ in Table 7.3.

> Euclidean Methodology has developed a certain obligatory style of presentation. I shall refer to this as "deductivist style." This style starts with a painstakingly stated list of *axioms, lemmas* and/or *definitions*. The axioms and definitions frequently look artificial and mystifyingly complicated. One is never told how these complications arose. The list of axioms and definitions is followed by the carefully worded *theorems*. These are loaded with heavy-going conditions; it seems impossible that anyone should ever have guessed them. The theorem is followed by the *proof*. ([137] p. 142)

This is the essence of what we have called formal understanding. We know that the results are true because we have gone through the crucible of the mathematical process and what remains is the essence of truth. But the insight and thought processes that led to the result are hidden.

> In deductivist style, all propositions are true and all inferences valid. Mathematics is presented as an ever-increasing set of eternal, immutable truths. ([137] p. 142)

> Deductivist style hides the struggle, hides the adventure. The whole story vanishes, the successive tentative formulations of the theorem

Definitions:

$$\zeta(s) = \sum_{k=1}^{\infty} k^{-s}$$

$$s_{\rm h}(m,n) = \sum_{k=1}^{\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right)^m (k+1)^{-n} \quad m \geq 1,\, n \geq 2$$

Some experimentally derived conjectures:

$$\begin{aligned}
s_{\rm h}(3,2) &= \frac{15}{2}\zeta(5) + \zeta(2)\zeta(3) \\
s_{\rm h}(3,3) &= -\frac{33}{16}\zeta(6) + 2\zeta^2(3) \\
s_{\rm h}(3,4) &= \frac{119}{16}\zeta(7) - \frac{33}{4}\zeta(3)\zeta(4) + 2\zeta(2)\zeta(5) \\
s_{\rm h}(3,6) &= \frac{197}{24}\zeta(9) - \frac{33}{4}\zeta(4)\zeta(5) - \frac{37}{8}\zeta(3)\zeta(6) + \zeta^3(3) + 3\zeta(2)\zeta(7) \\
s_{\rm h}(4,2) &= \frac{859}{24}\zeta(6) + 3\zeta^2(3)
\end{aligned}$$

We are given the raw data with which to work, carefully organized to give us a glimpse into the investigators' insights on the problem. Note in the first formula for $s_{\rm h}(3,2)$, $3+2=5$, on the right hand side of the equation we have $\zeta(5)$ and $\zeta(3)\zeta(2)$.

Some proven Euler sums:

$$\begin{aligned}
s_{\rm h}(2,2) &= \frac{3}{2}\zeta(4) + \frac{1}{2}\zeta^2(2) = \frac{11\pi^4}{360} \\
s_{\rm h}(2,4) &= \frac{2}{3}\zeta(6) - \frac{1}{3}\zeta(2)\zeta(4) + \frac{1}{3}\zeta^3(2) - \zeta^2(3) = \frac{37\pi^6}{22680} - \zeta^2(3)
\end{aligned}$$

The proven evaluation for $s_{\rm h}(2,2)$ above implies the truth of Au-Yeung's discovery.

Table 7.2: Some experimental results.

When one first learns to sum infinite series one is taught to sum geometric series

$$1 + \rho + \rho^2 + \cdots + \rho^k + \cdots = \frac{1}{1-\rho}$$

when $|\rho| < 1$. Next one learns to sum telescoping series. For example if $f(i) = \frac{1}{i+1} - \frac{1}{i+2}$, it is not to hard to see that

$$\sum_{i=0}^{n} f(i) = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) \cdots (\frac{1}{n+1} - \frac{1}{n+2}) = 1 - \frac{1}{n+2}$$

and in particular that

$$\sum_{i=0}^{\infty} f(i) = 1.$$

The *Wilf-Zeilberger* algorithms employ "creative telescoping" to show that a sum or integral is zero. The algorithms really provide a meta-insight into a broad range of problems involving identities. Unfortunately the proofs produced by the computer, while understandable by most mathematicians are at the same time uninteresting. On the other hand, the existence of WZ proofs for large classes of objects gives us a global insight into these areas.

Table 7.3: Shrinking or encapsulating mathematics.

> in the course of the proof-procedure are doomed to oblivion while the end result is exalted into sacred infallibility. ([137] p. 142)

Perhaps the most extreme examples of the deductivist style come out of the computer generated proofs guaranteed by Wilf and Zeilberger's algorithmic proof theory. It is important to note here that Wilf and Zeilberger transform the problem of proving identities to the more computer oriented problem of solving a system of linear equations with symbolic coefficients.

These WZ proofs (see Table 7.4) are perhaps the ultimate in the deductivist tradition. At present, knowing the WZ proof of an identity amounts to little more (We will discuss the importance of certificates later.) than knowing that the identity is true. In fact, Doron Zeilberger in [194] has advocated leaving only a QED at the end of the statement, the author's seal that he has had the computer perform the calculations needed to prove the identity. The advantage of this approach is that the result is completely encapsulated. Just as one would not worry about how the computer multiplied two huge integers together or inverted a matrix, one now has results whose proofs are uninteresting.

7.3 Zeilberger and the Encapsulation of Identity

7.3.1 Putting a Price on Reliability

In the last two sections we talked about the importance of communicating insights within the mathematical community. There we focused on the process of mathematical thought but now we want to talk about communicating insights that have not been made rigorous.

We have already briefly talked about Wilf and Zeilberger's algorithmic proof theory and its denial of insight. In this section we will discuss the implications of this theory and D. Zeilberger's philosophy of mathematics as contained in *Theorems for a Price: Tomorrow's Semi-Rigorous Mathematical Culture* ([195]).

It is probably unfortunate but perhaps necessary that the two voices most strongly advocating truly experimental math are also at times the most hyperbolic in their language. We will concentrate mostly on the ideas of Doron Zeilberger but G. J. Chaitin should not and will not be ignored.

We will begin with D. Zeilberger's "Abstract of the future"

> We show in a certain precise sense that the Goldbach conjecture is true with probability larger than 0.99999 and that its complete truth could be determined with a budget of 10 billion. ([195] p. 980)

Once people get over the shock of seeing probabilities assigned to truth in mathematics the usual complaint is that the 10 billion is ridiculous. Computers have been getting better and cheaper for years. What can it mean that "the complete truth could be determined with a budget of 10 billion?" What is clear from the article is that this is an additive measure of the difficulty of completely solving this problem. If we know that the Riemann hypothesis will be proven if we prove lemmas costing 10 billion, 2 billion and 2 trillion dollars respectively, we can tell at a glance not merely what it would "cost" to prove the hypothesis but also

Below is a sample WZ proof of $(1+1)^n = 2^n$ (this proof is a modified version of the output of Doron Zeilberger's original *Maple* program, influenced by the proof in [194]).

Let $F(n,k) = \left(\begin{array}{c} n \\ k \end{array}\right) 2^{-n}$. We have to show that $l(n) = \sum_k F(n,k) = 1$. To do this we will show that $l(n+1) - l(n) = 0$ for every $n \geq 0$ and that $l(0) = 1$. The second half is trivial since for $n = 0$, $F(0,0)$ is equal to 1 and 0 otherwise. The first half is proved by the WZ algorithm.

We construct

$$G(n,k) = \frac{-1}{2^{(n+1)}} \left(\begin{array}{c} n \\ k-1 \end{array}\right) \left(= \frac{-k}{2(n-k+1)} F(n,k)\right),$$

with the motive that

$$WZ = F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k) \quad \text{(check!)}.$$

Summing WZ with respect to k gives

$$\sum_k F(n+1,k) - \sum_k F(n,k) =$$

$$\sum_k (G(n,k+1) - G(n,k)) = 0$$

(by telescoping). We have now established that $l(n+1) - l(n) = 0$ and we are done.

The proof gives little insight into this binomial coefficient identity. However, the algorithms give researchers in other fields direct access to the field of special function identities.

Table 7.4: An uninteresting(?) proof.

where new ideas will be essential in any proof. (This assumes that 2 trillion is a lot of "money.")

The introduction of "cost" leads immediately to consideration of a trend that has over taken the business world and is now intruding rapidly on academia: a focus on productivity and efficiency.

> It is a waste of money to get absolute certainty, unless the conjectured identity in question is known to imply the Riemann Hypothesis ([195] p. 980)

We have taken this quote out of its context (Wilf and Zeilberger's algorithmic proof theory of identities) [195] but even so we think it is indicative of a small but growing group of mathematicians who are asking us to look at not just the benefits of reliability in mathematics but also the associated costs. See for example A. Jaffe and F. Quinn in [118, 119] and G. Chaitin in [66]. Still, we have not dealt with the central question. Why does D. Zeilberger need to introduce probabilistic "truths?" and how might we from a "formalist" perspective not feel this to be a great sacrifice?

7.3.2 It's All About Insight

Why is Zeilberger so willing to give up on absolute truths? The most reasonable answer is that he is pursuing deeper truths. In *Identities in Search of Identities*, Zeilberger advocates an examination of identities for the sake of studying identities. Still as Herb Wilf and others have pointed out it is possible to produce an unlimited number of identities. It is the context, the ability to use and manipulate these identities, that make them interesting. Why then might we think that studying identities for their own sake may lead us down the golden path rather than the garden path?

We are now looking for what might be called meta-mathematical structures. We remove the math from its original context and isolate it, trying to detect new structures. When doing this it is impossible to collect only the relevant information that will lead to the new discovery. One collects objects (theorems, statistics, conjectures, etc.) that have a reasonable degree of similarity and familiarity and then attempts to eliminate the irrelevant or the untrue (counter examples). We are preparing for some form of eliminative induction. There is a built in stage, where objects are subject to censorship. In this context, it is not unreasonable to introduce objects where one is not sure of their truth, since all the objects, whether proved or not, will be subject to the same degree of scrutiny. Moreover, if these probably true objects fall into the class of reliable (i.e., they fit the new conjecture) objects, it may be possible to find a legitimate proof in the new context. Recall that the fast WZ algorithms transform the problem of proving an identity to one of solving a system of linear equations with symbolic coefficients.

> It is very time consuming to solve a system of linear equations with symbolic coefficients. By plugging in specific values for n and other parameters if present, one gets a system with numerical coefficients, which is much faster to handle. Since it is unlikely that a random system of inhomogeneous linear equations with more equations than unknowns can be solved, the solvability of the system for a number of special values of n and the other parameters is a very good indication that the identity is indeed true. ([195] p. 980)

Suppose we can solve the system above for ten different assignments for n and the other parameters but cannot solve the general system. What do we do if we really need this identity? We are in a peculiar position. We have reduced the problem of proving identities involving sums and integrals of proper-hypergeometric terms to the problem of solving a possibly gigantic system of inhomogeneous linear equations with more equations than unknowns. We have an appropriately strong belief that this system has a solution but do not have the resources to uncover this solution.

What can we do with our result? If we agree with G. J. Chaitin, we may want to introduce it as an "axiom."

> I believe that elementary number theory and the rest of mathematics should be pursued more in the spirit of experimental science, and that you should be willing to adopt new principles. I believe that Euclid's statement that an axiom is a self-evident truth is a big mistake[4]. The Schrödinger equation certainly isn't a self-evident truth! And the Riemann Hypothesis isn't self-evident either, but it's very useful. A physicist would say that there is ample experimental evidence for

[4]There is no evidence that Euclid ever made such a statement. However, the statement does have an undeniable emotional appeal.

> the Riemann Hypothesis and would go ahead and take it as a working assumption. ([66] p. 24)

In this case, we have ample experimental evidence for the truth of our identity and we may want to take it as something more than just a working assumption. We may want to introduce it formally into our mathematical system.

7.4 Experiment and "Theory"

We have now examined two views of experimental mathematics but we appear to be no closer to a definition than when we began. However, we are now ready to begin in full our exploration of experiment. In *Advice to a Young Scientist*, P. B. Medawar defines four different kinds of experiment: the Kantian, Baconian, Aristotelian, and the Galilean. Mathematics has always participated deeply in the first three categories but has somehow managed to avoid employing the Galilean model. In developing our notion of experimental mathematics we will try to adhere to this Galilean mode as much as possible.

We will begin with the Kantian experiments. Medawar gives as his example:

> generating "the classical non-Euclidean geometries (hyperbolic, elliptic) by replacing Euclid's axiom of parallels (or something equivalent to it) with alternative forms." ([152] pp. 73–74)

It seems clear that mathematicians will have difficulty escaping from the Kantian fold. Even a Platonist must concede that mathematics is only accessible through the human mind and thus at a basic level all mathematics might be considered a Kantian experiment. We can debate whether Euclidean geometry is but an idealization of the geometry of nature (where a point has no length or breadth and a line has length but no breadth?) or nature an imperfect reflection of "pure" geometrical objects, but in either case the objects of interest lie within the minds eye.

Similarly, we cannot escape the Baconian experiment. In Medawar's words this

> is a contrived as opposed to a natural happening, it "is the consequence of 'trying things out' or even of merely messing about." ([152] p. 69)

Most of the research described as experimental is Baconian in nature and in fact one can argue that all of mathematics proceeds out of Baconian experiments. One tries out a transformation here, an identity there, examines what happens when one weakens this condition or strengthens that one. Even the application of probabilistic arguments in number theory can be seen as a Baconian experiment. The experiments may be well thought out and very likely to succeed but the ultimate criteria of inclusion of the result in the literature is success or failure. If the "messing about" works (e.g., the theorem is proved, the counterexample found) the material is kept; otherwise, it is relegated to the scrap heap.

The Aristotelian experiments are described as demonstrations:

> apply electrodes to a frog's sciatic nerve, and lo, the leg kicks; always precede the presentation of the dog's dinner with the ringing of a bell, and lo, the bell alone will soon make the dog dribble. ([152] p. 71)

The results are tailored to demonstrate the theorems, as opposed to the experiments being used to devise and revise the theorems. This may seem to have little to do with mathematics but it has everything to do with pedagogy. The Aristotelian experiment is equivalent to the concrete examples we employ to help explain our definitions, theorems, or the problems assigned to students so they can see how their newly learned tools will work.

The last and most important is the Galilean experiment:

> (the) Galilean Experiment is a critical experiment—one that discriminates between possibilities and, in doing so, either gives us confidence in the view we are taking or makes us think it in need of correction. ([152])

Ideally one devises an experiment to distinguish between two or more competing hypotheses. In subjects like medicine the questions are in principal more clear cut (the Will Roger's phenomenon or Simpson's paradox complicates matters [5] Does this medicine work (longevity, quality of life, cost effectiveness, etc.)? Is this treatment better than that one? Unfortunately, these questions are extremely difficult to answer and the model Medawar presents here does not correspond

[5] Simpson's paradox notes that two data sets can separately support one conclusion while the union of the data supports the opposite conclusion. Will Roger's phenomenon notes that in a medical study it is possible to transfer a patient from one group to another and improve the statistics of both groups.

with the current view of experimentation. Since the spectacular "failure" (i.e., it worked beautifully but ultimately was supplanted see [138]) of Newtonian physics it has been widely held that no amount of experimental evidence can prove or disprove a theorem about the world around us and it is widely known that in the real world the models one tests are not true. Medawar acknowledges the difficulty of proving a result but has more confidence than modern philosophers in disproving hypotheses. If experiment cannot distinguish between hypotheses or prove theorems, what can it do? What advantages does it have? Is it necessary?

7.5 "Theoretical" Experimentation

While there is an ongoing crisis in mathematics, it is not as severe as the crisis in physics. The untestability of parts of theoretical physics (e.g., string theory) has led to a greater reliance on mathematics for "experimental verification." This may be in part what led Arthur Jaffe and Frank Quinn to advocate what they have named "Theoretical Mathematics"(note that many mathematicians think they have been doing theoretical mathematics for years) but which we like to think of as "theoretical experimentation." There are certainly some differences between our ideas and theirs but we believe they are more of emphasis than substance.

Unlike our initial experiment where we are working with and manipulating floating point numbers, "theoretical experimentation" would deal directly with theorems, conjectures, the consequences of introducing new axioms. ... Note that by placing it in the realm of experimentation, we shift the focus from the more general realm of mathematics, which concerns itself with the transmission of both truth and insight, to the realm of experimentation, which primarily deals with the establishment of and transmission of insight. Although it was originally conceived outside the experimental framework, the central problems Jaffe and Quinn need to deal with are the same. They must attempt to preserve the rigorous core of mathematics, while contributing to an increased understanding of mathematics both formally and intuitively.

As described in Arthur Jaffe and Frank Quinn's *"Theoretical Mathematics": Toward a Cultural Synthesis of Mathematics and Theoretical Physics* it appears to be mainly a call for a loosening of the bonds of rigor. They suggest the creation of a branch of theoretical (experimental) mathematics akin to theoretical physics, where one produces speculative and intuitive works that will later be made reliable through proof. They are concerned about the slow pace of mathematical developments when all the work must be rigorously developed prior to publication. They argue convincingly that a haphazard introduction of conjectorial mathematics will almost undoubtedly result in chaos.

Their solution to the problems involved in the creation of theoretical (experimental) mathematics comes in two parts. They suggest that

> theoretical work should be explicitly acknowledged as theoretical and incomplete; in particular, a major share of credit for the final result must be reserved for the rigorous work that validates it. ([118] p.10)

This is meant to ensure that there are incentives for following up and proving the conjectured results.

To guarantee that work in this theoretical mode does not affect the reliability of mathematics in general, they propose a linguistic shift.

> Within a paper, standard nomenclature should prevail: in theoretical material, a word like "conjecture" should replace "theorem"; a word like "predict" should replace "show" or "construct"; and expressions such as "motivation" or "supporting argument" should replace "proof." Ideally the title and abstract should contain a word like "theoretical", "speculative", or "conjectural". ([118] p.10)

Still, none of the newly suggested nomenclature would be entirely out of place in a current research paper. Speculative comments have always had and will always have a place in mathematics.

This is clearly an exploratory form of mathematics. But is it truly experimental in any but the Baconian sense? The answer will of course lie in its application. If we accept the description at face value, all we have is a lessening of rigor, covered by the introduction of a new linguistic structure. More "mathematics" will be produced but it is not clear that this math will be worth more, or even as much as, the math that would have been done without it.

It is not enough to say that mathematical rigor is strangling mathematical productivity. One needs to argue that by relaxing the strictures temporarily one can achieve more. If we view theoretical (experimental) mathematics as a form of Galilean experimentation then in its idealized form "theoretical" (experimental) mathematics should choose between directions (hypotheses) in mathematics.

Like any experimental result the answers will not be conclusive, but they will need to be strong enough to be worth acting on.

Writing in this mode, a good theoretical paper should do more than just sketch arguments and motivations. Such a paper should be an extension of the survey paper, defining not what has been done in the field but what the author feels can be done, should be done and might be done, as well as documenting what is known, where the bottlenecks are, etc. In general, we sympathize with the desire to create a "theoretical" mathematics but without a formal structure and methodology it seems unlikely to have the focus required to succeed as a separate field.

One final comment seems in order here. "Theoretical" mathematics, as practiced today, seems a vital and growing instititution. Mathematicians now routinely include conjectures and insights with their work (a trend that seems to be growing). This has expanded in haphazard fashion to include algorithms, suggested algorithms and even pseudo algorithms. We would distinguish our vision of "experimental" mathematics from "theoretical" mathematics by an emphasis on the constructive/algorithmic side of mathematics. There are well established ways of dealing with conjectures but the rules for algorithms are less well defined. Unlike most conjectures, algorithms if sufficiently efficacious soon find their way into general use.

While there has been much discussion of setting up standardized data bases to run algorithms on, this has proceeded even more haphazardly. Addressing these issues of reliability would be part of the purview of experimental mathematics. Not only would one get a critical evaluation of these algorithms but by reducing the problems to their algorithmic core, one may facilitate the sharing of insights both within and between disciplines. At its most extreme, a researcher from one discipline may not need to understand anything more than the outline of the algorithm to make important connections between fields.

7.6 A Mathematical Experiment

7.6.1 Experimentation

We now turn to a more concrete example of a mathematical experiment. Our meta-goal in devising this experiment was to investigate the similarities and differences between experiments in mathematics and in the natural sciences, *Definition.* A real number is normal to the base 10 if every block of digits of length k occurs with frequency $1/10^k$.

Example: the Champernowne number

$$0.01234567891011121314\ldots 99100101\ldots$$

is known to be normal base 10.

Except for artificially created examples no numbers have been proven normal in any particular base. If we allow artificial numbers there are no explicit numbers known to be normal in every base[a]

Questions:

- Are all non-rational algebraic numbers normal base 10?
- Do all non-rational algebraic numbers have uniformly distributed digits?

[a]G. J. Chaitin in *Randomness and Complexity in Pure Mathematics*, has a number he calls $\Omega = \sum_{p \text{ halts}} 2^{-|p|}$, the halting probability, which he notes is "sort of a mathematical pun", but is normal to all bases. He does this by identifying integers with binary strings representing Turing machines and summing over the programs that stopped (non-trivially, see [66] p.12).

Table 7.5: Background on normality.

particularly in physics. We therefore resolved to examine a conjecture which could be approached by collecting and investigating a huge amount of data: the conjecture that every non-rational algebraic number is normal in every base (see Table 7.5). It is important to understand that we did not aim to prove or disprove this conjecture; our aim was to find evidence pointing in one or the other direction. We were hoping to gain insight into the nature of the problem from an experimental perspective.

The actual experiment consisted of computing to 10,000 decimal digits the square roots and cube roots of the positive integers smaller than 1000 and then subjecting these data to certain statistical tests (again, see Table 7.5). Under the hypothesis that the digits of these numbers are uniformly distributed (a much weaker hypothesis than normality of these numbers), we expected the probability values of the statistics to be distributed uniformly between 0 and 1. Our first run showed fairly conclusively that the digits were distributed uniformly. In fact, the Anderson-Darling test, which we used to measure how uniformly distributed our probabilities were suggested that the probabilities might have been "too uniform" to be random. We therefore ran the same tests again, only this time for the first 20,000 decimal digits, hoping to detect some non-randomness in the data. The data were not as interesting on the second run.

7.6.2 Verification

It is even more important in mathematics than in the physical sciences that the data under investigation are completely reliable. At first glance it may seem that the increasing reliance of mathematicians on programs such as *Maple* and *Mathematica*, has decreased the need for verification. Computers very rarely make arbitrary mistakes in arithmetic and algebra. But all the systems have known and unknown bugs in their programming. It is therefore imperative that we check our results. So what efforts did we take to verify our findings?

First of all, we had to make sure that the roots we computed were accurate to at least 10,000 (resp. 20,000) digits. We computed these roots using *Maple* as well as *Mathematica*, having them compute the roots to an accuracy of 10,010 digits. We then did two checks on the computed approximation s_n to $\sqrt{n}$. First, we tested that $\sqrt{n} \in [s_n - 10^{-10005}, s_n + 10^{-10005}]$ by checking that $(s_n - 10^{-10005})^2 < n < (s_n + 10^{-10005})^2$. Second, we tested that the 10,000th through 10,005th digits were not all zeroes or nines. This ensures that we actually computed the first 10,000 digits of the decimal expansion of $\sqrt{n}$. (We note that *Maple* initially did not give us an accuracy of 10,000 digits for all of the cube roots, so that we had to increase the precision.)

We then had to make sure that we computed the statistics and probability values accurately—or at least to a reasonable precision, since we used asymptotic formulas anyway. We did this by implementing them both in *Maple* and in *Mathematica* and comparing the results. We detected no significant discrepancy.

We claim that these measures reasonably ensure the reliability of our experimental results.

We looked at the first 10,000 digits after the decimal point of the $\sqrt{n}$ where $n < 1000$ is not a perfect square and of $\sqrt[3]{n}$ where $n < 1000$ is not a perfect cube.
Tests used:

- χ^2—to check that each digit occurs 1/10 of the time (discrete uniform distribution base 10).
- Discrete Cramér-von Mises—to check that all groups of 4 consecutive digits occurs 1/10,000 of the time (discrete uniform distribution base 10,000).
- Anderson-Stephens —to check that the power spectrum of the sequence matches that of white noise (periodicity).
- Anderson-Darling—continuous uniform distribution.

Important point: In order for us to claim we have generated any evidence at all either for or against we have made two fairly strong assumptions.

- The first 10,000 digits are representative of the remaining digits.
- These digits behave as far as our statistical tests go like independent random variables.

In fact, for the first and second 10,000 digits our final conclusions are identical. The second assumption is problematic. Since we have beautiful algorithms to calculate these numbers, by most reasonable definitions of independent and random, these digits are neither.

Table 7.6: Data and statistics.

7.6.3 Interpretation

Our experimental results support the conjecture that every non-rational algebraic number is normal; more precisely, we have found no evidence against this conjecture. In this section we will describe how we looked at and interpreted the experimental data to arrive at this conclusion. We include only a few examples of how we looked at the data here. In fact, we have only looked at certain aspects of normality and randomness in decimal expansions. Thus our results may be interpreted more narrowly to support the hypothesis that algebraic numbers are normal base 10. A full description will be found in [51].

Our main goal here is to give a quick visual summary that is at once convincing and data rich. These employ some of the most basic tools of visual data analysis and should probably form part of the basic vocabulary of an experimental mathematician. Note that traditionally one would run a test such as the Anderson-Darling test (which we have done) for the continuous uniform distribution and associate a particular probability with each of our sets of probability, but unless the probability values are extremely high or low it is difficult to interpret these statistics.

Experimentally, we want to test graphically the hypothesis of normality and randomness (or non-periodicity) for our numbers. Because the statistics themselves do not fall into the nicest of distributions, we have chosen to plot only the associated probabilities. We include two different types of graphs here. A quantile-quantile plot is used to examine the distribution of our data and scatter plots are used to check for correlations between statistics.

The first is a quantile-quantile plot of the chi square base 10 probability values versus a a discrete uniform distribution. For this graph we have placed the probabilities obtained from our square roots and plotted them against a perfectly uniform distribution. Finding nothing here is equivalent to seeing that the graph is a straight line with slope one. This is a crude but effective way of seeing the data. The disadvantage is that the data are really plotted along a one dimensional curve and as such it may be impossible to see more subtle patterns.

The other graphs are examples of scatter plots. The first scatter plot shows that nothing interesting is occurring. We are again looking at probability values this time derived from the discrete Cramer-von Mises (CVM) test base 10,000. For each cube root we have plotted the point (f_i, s_i), where f_i is the CVM base 10,000 probability associated with the first 2500 digits of the cube root of i and s_i is the probability associated with the next 2500 digits. A look at the graph reveals that we have now plotted our data on a two dimensional surface and there is a lot more “structure” to be seen. Still, it is not hard to convince oneself that there is little or no relationship between the probabilities of the first 2500 digits and the second 2500 digits.

The last graph is similar to the second. Here we have plotted the probabilities associated with the Anderson-Stephens statistic of the first 10,000 digits versus the first 20,000 digits. We expect to find a correlation between these tests since there is a 10,000 digit overlap. In fact, although the effect is slight, one can definitely see the thinning out of points from the upper left hand corner and lower right hand corner.

7.7 Conclusion

All the versions of experimental mathematics that we have dealt with so far have two characteristics: their main interest is in expanding our mathematical knowledge as rapidly as possible and none of them stray too far from the mainstream. In many cases this urgency leads to a temporary relaxation of rigor, a relaxation that is well documented and hopefully can be cleaned up afterwards. In other cases it may be intrinsic to the mathematics they wish to explore. When a field has been as wildly successful as mathematics has been in the past few centuries there is a reluctance to change. We have hoped to convince some of the readers that these changes are revolutionary only in the same sense that the earth revolves around the sun.

We conclude with a definition of experimental mathematics.

Experimental Mathematics is that branch of mathematics that concerns itself ultimately with the codification and transmission of insights within the mathematical community through the use of experimental (in either the Galilean, Baconian, Aristotelian or Kantian sense) exploration of conjectures and more informal beliefs and a careful analysis of the data acquired in this pursuit.

Results discovered experimentally will, in general, lack some of the rigor associated with mathematics but will provide general insights into mathematical problems to guide further exploration, either experimental or traditional. We have restricted our definition of experimental mathematics to methodological pursuits that in some way mimic Medawar’s views of Gallilean experimentation. However, our emphasis on insight also calls for the judicious use of examples

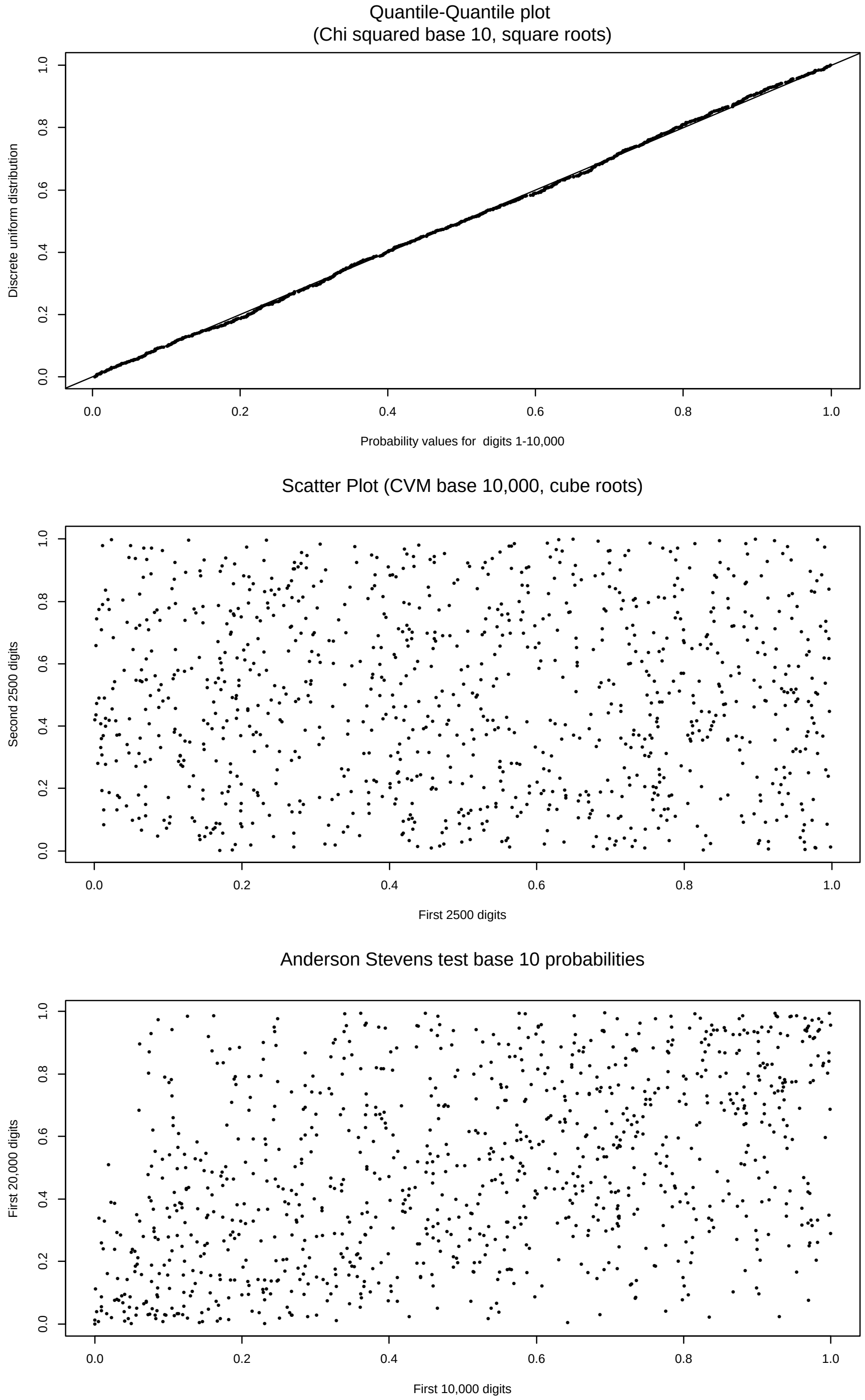


Figure 7.1: Graphical statistics of our experiments.

(Aristotelian experimentation).

If the mathematical community as a whole was less splintered, we would probably remove the word "codification" from the definition. That is to say that a great deal of time will need to be spent on developing a language for the expression of experimental results. Since there are real communications problems between fields and since the questions to be explored will be difficult, it seems imperative that experimental investigators make every effort to organize their insights and present their data in a manner that will be as widely accessible as possible[6].

With respect to reliability and rigor, the main tools here are already in place. We need to stress systematization of our exploration. As in our experimental project on normality, it is important to clearly define what has been looked at, how things have been examined, and what confidence the reader should have in the data. Although mathematicians may not like to admit it, ease of use will have to be a primary consideration if experimental results are to be of widescale use. As such, visualization and hypertextual presentations of material will become increasingly important in the future. We began by stealing some of the basic tools of scientific analysis and laying claim to them. As the needs of the community become more apparent one would expect these tools and others to evolve into a form better suited to the particular needs of the mathematical community. Someday, who knows, first year graduate students may be signing up for Experimental Methods in Mathematics I.

[6]It is clear that mechanisms are developing for transmitting insights within fields, even if this is only through personal communications.

The Quest for Pi

D. H. BAILEY, J. M. BORWEIN, P. B. BORWEIN, AND S. PLOUFFE

This article gives a brief history of the analysis and computation of the mathematical constant $\pi = 3.14159\ldots$, including a number of formulas that have been used to compute π through the ages. Some exciting recent developments are then discussed in some detail, including the recent computation of π to over six billion decimal digits using high-order convergent algorithms, and a newly discovered scheme that permits arbitrary individual hexadecimal digits of π to be computed.

For further details of the history of π up to about 1970, the reader is referred to Petr Beckmann's readable and entertaining book [3]. A listing of milestones in the history of the computation of π is given in Tables 1 and 2, which we believe to be more complete than other readily accessible sources.

The Ancients

In one of the earliest accounts (about 2000 B.C.) of π, the Babylonians used the approximation $3\frac{1}{8} = 3.125$. At this same time or earlier, according to an account in an ancient Egyptian document, Egyptians were assuming that a circle with diameter nine has the same area as a square of side eight, which implies $\pi = 256/81 = 3.1604\ldots$. Others of antiquity were content to use the simple approximation 3, as evidenced by the following passage from the Old Testament:

> Also, he made a molten sea of ten cubits from brim to brim, round in compass, and five cubits the height thereof; and a line of thirty cubits did compass it round about. (I Kings 7:23; see also 2 Chron. 4:2)

The first rigorous mathematical calculation of the value of π was due to Archimedes of Syracuse (~250 B.C.), who used a geometrical scheme based on inscribed and circumscribed polygons to obtain the bounds $3\frac{10}{71} < \pi < 3\frac{1}{7}$, or in other words, $3.1408\ldots < \pi < 3.1428\ldots$ [11]. No one was able to improve on Archimedes's method for many centuries, although a number of persons used this general method to obtain more accurate approximations. For example, the astronomer Ptolemy, who lived in Alexandria in A.D. 150, used the value $3\frac{17}{120} = 3.141666\ldots$, and the fifth-century Chinese mathematician Tsu Chung-Chih used a variation of Archimedes's method to compute π correct to seven digits, a level not attained in Europe until the 1500s.

The Age of Newton

As in other fields of science and mathematics, progress in the quest for π in medieval times occurred mainly in the Islamic world. Al-Kashi of Samarkand computed π to 14 places in about 1430.

In the 1600s, with the discovery of calculus by Newton

TABLE 1. History of π Calculations (Pre-20th-Century)			
Babylonians	2000? B.C.E.	1	3.125 ($3\frac{1}{8}$)
Egyptians	2000? B.C.E.	0	3.16045 [$4(\frac{8}{9})^2$]
China	1200? B.C.E.	0	3
Bible (1 Kings 7:23)	550? B.C.E.	0	3
Archimedes	250? B.C.E.	3	3.1418 (ave.)
Hon Han Shu	A.D. 130	0	3.1622 (= $\sqrt{10}$?)
Ptolemy	150	3	3.14166
Chung Hing	250?	0	3.16227 ($\sqrt{10}$)
Wang Fau	250?	0	3.15555 ($\frac{142}{45}$)
Liu Hui	263	5	3.14159
Siddhanta	380	4	3.1416
Tsu Ch'ung Chi	480?	7	3.1415926
Aryabhata	499	4	3.14156
Brahmagupta	640?	0	3.162277 (= $\sqrt{10}$)
Al-Khowarizmi	800	4	3.1416
Fibonacci	1220	3	3.141818
Al-Kashi	1429	14	
Otho	1573	6	3.1415929
Viète	1593	9	3.1415926536 (ave.)
Romanus	1593	15	
Van Ceulen	1596	20	
Van Ceulen	1615	35	
Newton	1665	16	
Sharp	1699	71	
Seki	1700?	10	
Kamata	1730?	25	
Machin	1706	100	
De Lagny	1719	127	(112 correct)
Takebe	1723	41	
Matsunaga	1739	50	
Vega	1794	140	
Rutherford	1824	208	(152 correct)
Strassnitzky and Dase	1844	200	
Clausen	1847	248	
Lehmann	1853	261	
Rutherford	1853	440	
Shanks	1874	707	(527 correct)

TABLE 2. History of π Calculations (20th Century)		
Ferguson	1946	620
Ferguson	Jan. 1947	710
Ferguson and Wrench	Sep. 1947	808
Smith and Wrench	1949	1,120
Reitwiesner, *et al.* (ENIAC)	1949	2,037
Nicholson and Jeenel	1954	3,092
Felton	1957	7,480
Genuys	Jan. 1958	10,000
Felton	May 1958	10,021
Guilloud	1959	16,167
Shanks and Wrench	1961	100,265
Guilloud and Filliatre	1966	250,000
Guilloud and Dichampt	1967	500,000
Guilloud and Bouyer	1973	1,001,250
Miyoshi and Kanada	1981	2,000,036
Guilloud	1982	2,000,050
Tamura	1982	2,097,144
Tamura and Kanada	1982	4,194,288
Tamura and Kanada	1982	8,388,576
Kanada, Yoshino, and Tamura	1982	16,777,206
Ushiro and Kanada	Oct. 1983	10,013,395
Gosper	1985	17,526,200
Bailey	Jan. 1986	29,360,111
Kanada and Tamura	Sep. 1986	33,554,414
Kanada and Tamura	Oct. 1986	67,108,839
Kanada, Tamura, Kubo, *et al.*	Jan. 1987	134,217,700
Kanada and Tamura	Jan. 1988	201,326,551
Chudnovskys	May 1989	480,000,000
Chudnovskys	June 1989	525,229,270
Kanada and Tamura	July 1989	536,870,898
Kanada and Tamura	Nov. 1989	1,073,741,799
Chudnovskys	Aug. 1989	1,011,196,691
Chudnovskys	Aug. 1991	2,260,000,000
Chudnovskys	May 1994	4,044,000,000
Takahashi and Kanada	June 1995	3,221,225,466
Kanada	Aug. 1995	4,294,967,286
Kanada	Oct. 1995	6,442,450,938

and Leibniz, a number of substantially new formulas for π were discovered. One of them can be easily derived by recalling that

$$\tan^{-1} x = \int_0^x \frac{dt}{1+t^2} = \int_0^x (1 - t^2 + t^4 - t^6 + \cdots)\, dt$$
$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots.$$

Substituting $x = 1$ gives the well-known Gregory–Leibniz formula

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots.$$

Regrettably, this series converges so slowly that hundreds of terms would be required to compute the numerical value of π to even two digits accuracy. However, by employing the trigonometric identity

$$\frac{\pi}{4} = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)$$

(which follows from the addition formula for the tangent function), one obtains

$$\frac{\pi}{4} = \left(\frac{1}{2} - \frac{1}{3\cdot 2^3} + \frac{1}{5\cdot 2^5} - \frac{1}{7\cdot 2^7} + \cdots\right) + \left(\frac{1}{3} - \frac{1}{3\cdot 3^3} + \frac{1}{5\cdot 3^5} - \frac{1}{7\cdot 3^7} + \cdots\right),$$

which converges much more rapidly. An even faster formula, due to Machin, can be obtained by employing the identity

$$\frac{\pi}{4} = 4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right)$$

in a similar way. Shanks used this scheme to compute π to 707 decimal digits accuracy in 1873. Alas, it was later found that this computation was in error after the 527th decimal place.

Newton discovered a similar series for the arcsine function:

$$\sin^{-1}x = x + \frac{1 \cdot x^3}{2 \cdot 3} + \frac{1 \cdot 3 \cdot x^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5 \cdot x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots.$$

π can be computed from this formula by noting that $\pi/6 = \sin^{-1}(1/2)$. An even faster formula of this type is

$$\pi = \frac{3\sqrt{3}}{4} + 24\left(\frac{1}{3 \cdot 2^3} - \frac{1}{5 \cdot 2^5} - \frac{1}{7 \cdot 2^7} - \frac{1}{9 \cdot 2^9} - \cdots\right).$$

Newton himself used this particular formula to compute π. He published only 15 digits, but later sheepishly admitted, "I am ashamed to tell you how many figures I carried these computations, having no other business at the time."

In the 1700s, the mathematician Euler, arguably the most prolific mathematician in history, discovered a number of new formulas for π. Among these are

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots,$$
$$\frac{\pi^4}{90} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \cdots.$$

A related, more rapidly convergent series is

$$\frac{\pi^2}{6} = 3 \sum_{m=1}^{\infty} \frac{1}{m^2\binom{2m}{m}}.$$

These formulas, despite their important theoretical implications, aren't very efficient for computing π.

One motivation for computations of π during this time was to see if the decimal expansion of π repeats, thus disclosing that π is the ratio of two integers (although hardly anyone in modern times seriously believed this). The question was settled in the late 1700s, when Lambert and Legendre proved that π is irrational. Some still wondered whether π might be the root of some algebraic equation with integer coefficients (although, as before, few really believed that it was). This question was finally settled in 1882 when Lindemann proved that π is transcendental. Lindemann's proof also settled once and for all, in the negative, the ancient Greek question of whether the circle could be squared with rule and compass. This is because constructible numbers are necessarily algebraic.

In the annals of π, the march of the nineteenth-century progress sometimes faltered. Three years prior to the turn of the century, one Edwin J. Goodman, M.D. introduced into the Indiana House of Representatives a "new Mathematical truth" to enrich the state, which would profit from the royalties ensuing from this discovery. Section two of his bill included the passage

> disclosing the fourth important fact that the ratio of the diameter and circumference is as five-fourths to four;

Thus, one of Goodman's new mathematical "truths" is that $\pi = \frac{16}{5} = 3.2$. The Indiana House passed the bill unanimously on Feb. 5, 1897. It then passed a Senate committee and would have been enacted into law had it not been for the last-minute intervention of Prof. C. A. Waldo of Purdue University, who happened to hear some of the deliberation while on other business.

The Twentieth Century

With the development of computer technology in the 1950s, π was computed to thousands and then millions of digits, in both decimal and binary bases (see, for example, [17]). These computations were facilitated by the discovery of some advanced algorithms for performing the required high-precision arithmetic operations on a computer. For example, in 1965, it was found that the newly discovered fast Fourier transform (FFT) could be used to perform high-precision multiplications much more rapidly than conventional schemes. These methods dramatically lowered the computer time required for computing π and other mathematical constants to high precision. See [1], [7], and [8].

In spite of these advances, until the 1970s all computer evaluations for π still employed classical formulas, usually a variation of Machin's formula. Some new infinite series formulas were discovered by the Indian mathematician Ramanujan around 1910, but these were not well known until quite recently when his writings were widely published. One of these is the remarkable formula

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103 + 26{,}390k)}{(k!)^4 396^{4k}}.$$

Each term of this series produces an additional eight correct digits in the result. Gosper used this formula to compute 17 million digits of π in 1985.

Although Ramanujan's series is considerably more efficient than the classical formulas, it shares with them the property that the number of terms one must compute increases linearly with the number of digits desired in the result. In other words, if one wishes to compute π to twice as many digits, then one must evaluate twice as many terms of the series.

In 1976, Eugene Salamin [16] and Richard Brent [8] independently discovered a new algorithm for π, which is based on the arithmetic–geometric mean and some ideas originally due to Gauss in the 1800s (although, for some reason, Gauss never saw the connection to computing π). This algorithm produces approximations that converge to π much more rapidly than any classical formula. The Salamin–Brent algorithm may be stated as follows. Set $a_0 = 1$, $b_0 = 1/\sqrt{2}$, and $s_0 = 1/2$. For $k = 1, 2, 3, \ldots$ compute

$$a_k = \frac{a_{k-1} + b_{k-1}}{2},$$
$$b_k = \sqrt{a_{k-1}b_{k-1}},$$
$$c_k = a_k^2 - b_k^2,$$
$$s_k = s_{k-1} - 2^k c_k,$$
$$p_k = \frac{2a_k^2}{s_k}.$$

Then p_k converges *quadratically* to π. This means that each iteration of this algorithm approximately *doubles* the number of correct digits. To be specific, successive iterations produce 1, 4, 9, 20, 42, 85, 173, 347, and 697 correct digits of π. Twenty-five iterations are sufficient to compute π to over 45 million decimal digit accuracy. However, each of these iterations must be performed using a level of numeric precision that is at least as high as that desired for the final result.

The Salamin–Brent algorithm requires the extraction of square roots to high precision, operations not required, for example, in Machin's formula. High-precision square roots can be efficiently computed by means of a Newton iteration scheme that employs only multiplications, plus some other operations of minor cost, using a level of numeric precision that doubles with each iteration. The total cost of computing a square root in this manner is only about three times the cost of performing a single full-precision multiplication. Thus, algorithms such as the Salamin–Brent scheme can be implemented very rapidly on a computer.

Beginning in 1985, two of the present authors (Jonathan and Peter Borwein) discovered some additional algorithms of this type [5–7]. One is as follows. Set $a_0 = 1/3$ and $s_0 = (\sqrt{3} - 1)/2$. Iterate

$$r_{k+1} = \frac{3}{1 + 2(1 - s_k^3)^{1/3}},$$
$$s_{k+1} = \frac{r_{k+1} - 1}{2},$$
$$a_{k+1} = r_{k+1}^2 a_k - 3^k (r_{k+1}^2 - 1).$$

Then $1/a_k$ converges *cubically* to π—each iteration approximately triples the number of correct digits.

A quartic algorithm is as follows: Set $a_0 = 6 - 4\sqrt{2}$ and $y_0 = \sqrt{2} - 1$. Iterate

$$y_{k+1} = \frac{1 - (1 - y_k^4)^{1/4}}{1 + (1 - y_k^4)^{1/4}},$$
$$a_{k+1} = a_k(1 + y_{k+1})^4 - 2^{2k+3} y_{k+1}(1 + y_{k+1} + y_{k+1}^2).$$

Then $1/a_k$ converges *quartically* to π. This particular algorithm, together with the Salamin–Brent scheme, has been employed by Yasumasa Kanada of the University of Tokyo in several computations of π over the past 10 years or so. In the latest of these computations, Kanada computed over 6.4 billion decimal digits on a Hitachi supercomputer. This is presently the world's record in this arena.

More recently, it has been further shown that there are algorithms that generate mth-order convergent approximations to π for any m. An example of a nonic (ninth-order) algorithm is the following: Set $a_0 = 1/3$, $r_0 = (\sqrt{3} - 1)/2$, and $s_0 = (1 - r_0^3)^{1/3}$. Iterate

$$t = 1 + 2r_k,$$
$$u = [9r_k(1 + r_k + r_k^2)]^{1/3},$$
$$v = t^2 + tu + u^2,$$
$$m = \frac{27(1 + s_k + s_k^2)}{v},$$
$$a_{k+1} = ma_k + 3^{2k-1}(1 - m),$$
$$s_{k+1} = \frac{(1 - r_k)^3}{(t + 2u)v},$$
$$r_{k+1} = (1 - s_k^3)^{1/3}.$$

Then $1/a_k$ converges *nonically* to π. It should be noted, however, that these higher-order algorithms do not appear to be faster as computational schemes than, say, the Salamin–Brent or the Borwein quartic algorithms. Although fewer iterations are required to achieve a given level of precision in the higher-order schemes, each iteration is more expensive.

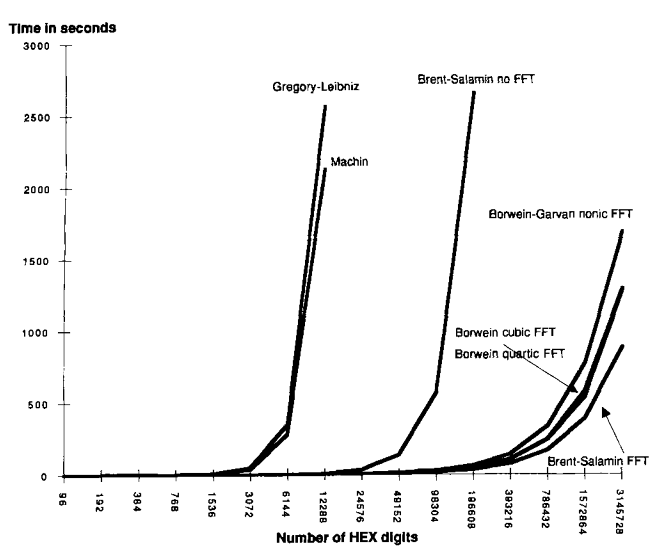
A comparison of actual computer run times for various π algorithms is shown in Figure 1. These run times are for computing π in binary to various precision levels on an IBM RS6000/590 workstation. The abscissa of this plot is in hexadecimal digits—multiply these numbers by 4 to obtain equivalent binary digits, or by $\log_{10}(16) = 1.20412\ldots$ to obtain equivalent decimal digits. Other implementations on other systems may give somewhat different results—for example, in Kanada's recent computation of π to over six billion digits, the quartic algorithm ran somewhat faster than the Salamin–Brent algorithm (116 hours versus 131 hours). But the overall picture from such comparisons is unmistakable: the modern schemes run many times faster than the classical schemes, especially when implemented using FFT-based arithmetic.

David and Gregory Chudnovsky of Columbia University have also done some very high-precision computations of π in recent years, alternating with Kanada for the world's record. Their most recent computation (1994) produced over four billion digits of π [9]. They did not employ a high-order convergent algorithm, such as the Salamin–Brent or Borwein algorithms, but instead utilized the following infinite series (which is in the spirit of Ramanujan's series above):

$$\frac{1}{\pi} = 12 \sum_{k=0}^{\infty} \frac{(-1)^k\,(6k)!(13{,}591{,}409 + 545{,}140{,}134k)}{(3k)!\,(k!)^3\,640{,}320^{3k+3/2}}$$

Each term of this series produces an additional 14 correct digits. The Chudnovskys implemented this formula with a very clever scheme that enabled them to utilize the results

FIGURE 1



of a certain level of precision to extend the calculation to even higher precision. Their program was run on a home-brewed supercomputer that they have assembled using private funds. An interesting personal glimpse of the Chudnovsky brothers is given in [14].

Computing Individual Digits of π

At several junctures in the history of π, it was widely believed that virtually everything of interest with regard to this constant had been discovered and, in particular, that no fundamentally new formulas for π lay undiscovered. This sentiment was even suggested in the closing chapters of Beckmann's 1971 book on the history of π [3], p. 172. Ironically, the Salamin–Brent algorithm was discovered only 5 years later.

A more recent reminder that we have not come to the end of humanity's quest for knowledge about π came with the discovery of the Rabinowitz–Wagon "spigot" algorithm for π in 1990 [15]. In this scheme, successive digits of π (in any desired base) can be computed with a relatively simple recursive algorithm based on the previously generated digits. Multiple-precision computation software is not required; therefore, this scheme can be easily implemented on a personal computer.

Note, however, that this algorithm, like all of the other schemes mentioned above, still has the property that in order to compute the dth digit of π, one must first (or simultaneously) compute each of the preceding digits. In other words, there is no "shortcut" to computing the dth digit with these formulas. Indeed, it has been widely assumed in the field (although never proven) that the computational complexity of computing the dth digit is not significantly less than that of computing all of the digits up to and including the dth digit. This may still be true, although it is probably very hard to prove. Another common feature of the previously known π algorithms is that they all appear to require substantial amounts of computer memory, amounts that typically grow linearly with the number of digits generated.

Thus, it was with no small surprise that a novel scheme was recently discovered for computing individual hexadecimal digits of π [2]. In particular, this algorithm (1) produces the dth hexadecimal (base 16) digit of π directly, without the need of computing any previous digits, (2) is quite simple to implement on a computer, (3) does not require multiple-precision arithmetic software, (4) requires very little memory, and (5) has a computational cost that grows only slightly faster than the index d. For example, the one millionth hexadecimal digit of π can be computed in only a minute or two on a current RISC workstation or high-end personal computer. This algorithm is not fundamentally faster than other known schemes for computing all digits up to some position d, but its elegance and simplicity are, nonetheless, of considerable interest.

This scheme is based on the following remarkable new formula for π:

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i}\left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6}\right).$$

The proof of this formula is not very difficult. First, note that for any $k < 8$,

$$\int_0^{1/\sqrt{2}} \frac{x^{k-1}}{1-x^8}\,dx = \int_0^{1/\sqrt{2}} \sum_{i=0}^{\infty} x^{k-1+8i}\,dx = \frac{1}{2^{k/2}} \sum_{i=0}^{\infty} \frac{1}{16^i(8i+k)}.$$

Thus, we can write

$$\sum_{i=0}^{\infty} \frac{1}{16^i}\left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6}\right) = \int_0^{1/\sqrt{2}} \frac{4\sqrt{2} - 8x^3 - 4\sqrt{2}x^4 - 8x^5}{1-x^8}\,dx,$$

which on substituting $y := \sqrt{2}x$ becomes

$$\int_0^1 \frac{16y - 16}{y^4 - 2y^3 + 4y - 4}\,dy = \int_0^1 \frac{4y}{y^2-2}\,dy - \int_0^1 \frac{4y-8}{y^2-2y+2}\,dy = \pi,$$

reflecting a partial fraction decomposition of the integral on the left-hand side.

However, this derivation is dishonest, in the sense that the actual route of discovery was much different. This formula was actually discovered not by formal reasoning, but instead by numerical searches on a computer using the "PSLQ" integer-relation-finding algorithm [10]. Only afterward was a proof found.

A similar formula for π^2 (which also was first discovered using the PSLQ algorithm) is as follows:

$$\pi^2 = \sum_{i=0}^{\infty} \frac{1}{16^i}\left(\frac{16}{(8i+1)^2} - \frac{16}{(8i+2)^2} - \frac{8}{(8i+3)^2} - \frac{16}{(8i+4)^2} - \frac{4}{(8i+5)^2} - \frac{4}{(8i+6)^2} + \frac{2}{(8i+7)^2}\right).$$

Formulas of this type for a few other mathematical constants are given in [2].

Computing individual hexadecimal digits of π using the above formula crucially relies on what is known as the binary algorithm for exponentiation, wherein one evaluates x^n by successive squaring and multiplication. This reduces the number of multiplications required to less than 2 $\log_2(n)$. According to Knuth, this technique dates back at least to 200 B.C. [13]. In our application, we need to obtain the exponentiation result modulo a positive integer c. This can be efficiently done with the following variant of the binary exponentiation algorithm, wherein the result of each multiplication is reduced modulo c:

To compute $r = b^n \bmod c$, first set t to be the largest power of $2 \le n$, and set $r = 1$. Then

A: if $n \ge t$ then $r \leftarrow br \bmod c$; $n \leftarrow n - t$; endif
$t \leftarrow t/2$
if $t \ge 1$ then $r \leftarrow r^2 \bmod c$; go to A; endif

Upon exit from this algorithm, r has the desired value. Here "mod" is used in the binary operator sense, namely as the binary function defined by $x \bmod y := x - [x/y]y$. Note that the above algorithm is entirely performed with positive integers that do not exceed c^2 in size. As an example, when computing 3^{49} mod 400 by this scheme, the variable r assumes the values 1, 9, 27, 329, 241, 81, 161, 83. Indeed $3^{49} =$ 239299329230617529590083, so that 83 is the correct result.

Consider now the first of the four sums in the formula above for π:

$$S_1 = \sum_{k=0}^{\infty} \frac{1}{16^k(8k+1)}.$$

First observe that the hexadecimal digits of S_1 beginning at position $d + 1$ can be obtained from the fractional part of $16^d S_1$. Then we can write

$$\begin{aligned} \operatorname{frac}(16^d S_1) &= \sum_{k=0}^{\infty} \frac{16^{d-k}}{8k+1} \bmod 1 \\ &= \sum_{k=0}^{d} \frac{16^{d-k} \bmod 8k+1}{8k+1} \bmod 1 \\ &+ \sum_{k=d+1}^{\infty} \frac{16^{d-k}}{8k+1} \bmod 1. \end{aligned}$$

For each term of the first summation, the binary exponentiation scheme can be used to rapidly evaluate the numerator. In a computer implementation, this can be done using either integer or 64-bit floating-point arithmetic. Then floating-point arithmetic can be used to perform the division and add the quotient to the sum mod 1. The second summation, where the exponent of 16 is negative, may be evaluated as written using floating-point arithmetic. It is only necessary to compute a few terms of this second summation, just enough to ensure that the remaining terms sum to less than the "epsilon" of the floating-point arithmetic being used. The final result, a fraction between 0 and 1, is then converted to base 16, yielding the $(d + 1)$th hexadecimal digit, plus several additional digits. Full details of this scheme, including some numerical considerations, as well as analogous formulas for a number of other basic mathematical constants, can be found in [2]. Sample implementations of this scheme in both Fortran and C are available from the web site *http://www.cecm.sfu.ca/personal/pborwein/*.

As the reader can see, there is nothing very sophisticated about either this new formula for π, its proof, or the scheme just described to compute hexadecimal digits of π using it. In fact, this same scheme can be used to compute binary (or hexadecimal) digits of log(2) based on the formula

$$\log(2) = \sum_{k=1}^{\infty} \frac{1}{k2^k},$$

which has been known for centuries. Thus, it is astonishing that these methods have lain undiscovered all this time. Why shouldn't Euler, for example, have discovered them? The only advantage that today's researchers have in this regard is advanced computer technology. Table 3 gives some hexadecimal digits of π computed using the above scheme.

One question that immediately arises is whether or not there is a formula of this type and an associated computational scheme to compute individual *decimal* digits of π. Alas, no decimal scheme for π is known at this time, although there is for certain constants such as log(9/10)—see [2]. On the other hand, there is not yet any proof that a decimal scheme for π cannot exist. This question is currently being actively pursued. Based on some numerical searches using the PSLQ algorithm, it appears that there are no simple formulas for π of the above form with 10 in the place of 16. This, of course, does not rule out the possibility of completely different formulas that nonetheless permit rapid computation of individual decimal digits of π.

TABLE 3. Hexadecimal Digits of π

Position	Hex digits beginning at this position
10^6	26C65E52CB4593
10^7	17AF5863EFED8D
10^8	ECB840E21926EC
10^9	85895585A0428B
10^{10}	921C73C6838FB2

Fabrice Bellard tells us that he recently completed the computation of the 100 billion'th hexadecimal digit by this method, this gives:

9C381872D27596F81D0E. . .

Why?

A value of π to 40 digits would be more than enough to compute the circumference of the Milky Way galaxy to an error less than the size of a proton. There are certain scientific calculations that require intermediate calculations to be performed to significantly higher precision than required for the final results, but it is doubtful that anyone will ever need more than a few hundred digits of π for such purposes. Values of π to a few thousand digits are sometimes employed in explorations of mathematical questions using a computer, but we are not aware of any significant applications beyond this level.

One motivation for computing digits of π is that these calculations are excellent tests of the integrity of computer hardware and software. This is because if even a single error occurs during a computation, almost certainly the final result will be in error. On the other hand, if two independent computations of digits of π agree, then most likely both computers performed billions or even trillions of operations flawlessly. For example, in 1986, a π-calculating program detected some obscure hardware problems in one of the original Cray-2 supercomputers [1].

The challenge of computing π has also stimulated research into advanced computational techniques. For example, some new techniques for efficiently computing linear convolutions and fast Fourier transforms, which have applications in many areas of science and engineering, had their origins in efforts to accelerate computations of π.

Beyond immediate practicality, decimal and binary expansions of π have long been of interest to mathematicians, who have still not been able to resolve the question of whether the expansion of π is normal [18]. In particular, it is widely suspected that the decimal expansions of π, e, $\sqrt{2}$, $\sqrt{10}$, and many other mathematical constants all have the property that the limiting frequency of any digit is one-tenth, and the limiting frequency of any n-long string

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Jonathan Borwein (http://www.cecm.sfu.ca/~jborwein) and Peter Borwein (http://www.cecm.sfu.ca/~pborwein) have provided fuller biographic information in a recent article on Experimental Mathematics in *The Intelligencer.* They direct the Centre for Experimental and Constructive Mathematics at which Simon Plouffe works.

of decimal digits is 10^{-n} (and similarly for binary expansions). Such a guaranteed property could, for instance, be the basis of a reliable pseudo-random-number generator for scientific calculations. Unfortunately, this assertion has not been proven in even one instance. Thus, there is a continuing interest in performing statistical analyses on the expansions of these numbers to see if there is any irregularity that would make them look unlike random sequences. So far, such studies of high-precision values of π have not disclosed any irregularities. Along this line, new formulas and schemes for computing digits of π are of interest because they may suggest new approaches to the normality question.

Finally, there is a more fundamental motivation for computing π, the challenge, like that of a lofty mountain or a major sporting event: "it is there." π is easily the most famous of the basic constants of mathematics. Every technical civilization has to master π, and we wonder if it may be equally inevitable that someone feels the challenge to raise the precision of its computation.

The constant π has repeatedly surprised humanity with new and unanticipated results. If anything, the discoveries of this century have been even more startling, with respect to the previous state of knowledge, than those of past centuries. We guess from this that even more surprises lurk in the depths of undiscovered knowledge regarding this famous constant.

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CHALLENGES IN MATHEMATICAL COMPUTING

Almost all interesting mathematical algorithmic questions relate to NP-hard questions. Such computation is prone to explode exponentially. The authors anticipate the greatest benefit will come from mathematical platforms that allow for computer-assisted insight generation, not from solutions of grand-challenge problems.

Some say that pure mathematicians invented digital computers and then proceeded to ignore them for the better part of half a century. In the past two decades, this situation has changed with a vengeance.

Major *symbolic mathematics* and *computer algebra* packages (see the sidebar), most notably Maple and Mathematica, have reached a remarkable degree of sophistication over the last 15 years. (We should also allude to counterparts such as Axiom, Macsyma, Reduce, MuPad; Matlab; and other more specialized packages such as GAP, Magma, or Cayley [for group theoretic computation], Pari [for number theory], KnotPlot [for knot theory], SnapPea [for hyperbolic 3-manifolds], and SPlus [for statistics].) This sophistication has relied on a confluence of algorithmic breakthroughs, dramatically increased processor power, almost limitless storage capacity, and, most recently, network communication, excellent online databases, and Web-distributed (often Java-based) computational tools. Examples include the mathematics front end to the Los Alamos Preprint ArXiv (http://front.math.ucdavis.edu), mathematical reviews on the Web (http://e-math.ams.org/mathscinet), Neil Sloane's encyclopedia of integer sequences (www.research.att.com/personal/njas/sequences/eisonline.html), our own inverse symbolic calculator (www.cecm.sfu.ca/projects/ISC/ISCmain.html), and integer relation finders (www.cecm.sfu.ca/projects/IntegerRelations).

The relatively seamless integration of all these components arguably represents the key challenge for 21st-century computational mathematics. It's hard to think of mathematical problems where a dramatic increase in computational speed and scale would enable a presently intractable line of research. It's easy to give examples where it would not—consider Clement W.H. Lam's 1991 proof (www.cecm.sfu.ca/organics/papers/lam/index.html) of the nonexistence of a finite projective plane of order 10 (a hunt for a configuration of $n^2 + n^1 + 1$ points and lines). It involved thousands of hours of computation. Lam's estimate is that the next case ($n = 18$) susceptible to his methods would take millions of years on any conceivable architecture. Although a certain class of mathematical enquiries is susceptible to massively parallel Web-based computation (for example, discovering Mersenne primes of the form $2^n - 1$), these tend not to be problems central to mathematics.

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Computational excursions in contemporary mathematics

Many researchers have made significant in-roads into some rather difficult—previously viewed as intractable—problems such as exact integration of elementary functions. Some of the most important mathematical algorithms of the 20th century include

- the fast Fourier transform,
- lattice basis reduction methods and related integer relation algorithms,
- the Risch algorithm for indefinite integration,
- the Gröbner basis computation for solving algebraic equations, and
- the Wilf/Zeilberger algorithms for hypergeometric summation and integration, which rigorously prove very large classes of identities.

All these are—or soon will be—centrally incorporated into symbolic mathematics or computer algebra packages. In fact, the first two were counted among the 10 algorithms with "the greatest influence on the development and practice of science and engineering in the 20th century."[1] Of course, many of the others, such as the sorting algorithms, are fundamental to the needs of contemporary mathematics.

Such packages can now substantially deal with large parts of the standard mathematics curriculum—and can out-perform most of our undergraduates to boot. They provide extraordinary opportunities for research that most mathematicians are only beginning to appreciate and digest. They also provide access to sophisticated mathematics to a very broad cross-section of scientists and engineers.

The emergence of such packages—and their integration into mathematical parole—represents the most significant part of a paradigm shift in how mathematics is done. Certainly these packages have already become a central research tool in many subareas of mathematics, both from an exploratory and a formal point of view—it is acceptable now to see a line in a proof that begins "by a large calculation in Maple, we see …" The first objective of symbolic algebra packages was to do as much exact mathematics as possible. A second, increasingly important objective is to do it very fast and to deal in an arbitrary-precision environment with the more standard algorithms of mathematical analysis. Roughly, users would like to be able to incorporate the usual methods of numerical analysis into an exact environment or at least into an arbitrary-precision environment.

The problems are obvious and hard. For example, how do we do arbitrary precision numerical quadrature? When do we switch methods with precision required or with different analytic properties of the integrand? How do we deal with branch cuts of analytic functions? How do we deal consistently with log? More ambitiously, how do we do a similar analysis for differential equations? Ultimately, can we certify that a given numeric or symbolic computation is indeed a proof or even just correct? The goal is to marry the algorithms of analysis with symbolic and exact computation and to do this with as little loss of speed as possible. Sometimes this means we must first go back and speed up the core algebraic calculations.

Within this context, a number of very interesting problems concerning the visualization of mathematics arise. How do we actually "see" what we are doing? Some say that Cartesian graphing was the most important invention of the last millennium. Certainly it changed how we think about mathematics—the subsequent development of differential calculus rested on it. More subtle and complicated graphics, like those of fractals, enable a previously impossible kind of exploration. There are many issues to work out at the interface of mathematics, pedagogy, and even psychology that are important to get right. An instructive example is the growing reliance of numerical analysts on graphic representation of large sparse matrices—the pictures show structure while numerical output is little help. (An example is JavaView [www-sfb288.math.tu-berlin.de/vgp/javaview/index.html] for 3D geometry on the Web.)

Some Significant Mathematical Packages

Axiom: www.axiomtek.com
Derive: www.derive.com
KnotPlot: www.pims.math.ca/knotplot
Macsyma: http://wombat.doc.ic.ac.uk/foldoc/foldoc.cgi? MACSYMA
Maple: www.maplesoft.com
Mathematica: www.wolfram.com
Matlab: www.mathworks.com
MuPad: www.mupad.com
Pari: www.cs.sunysb.edu/~algorith/implement/pari/implement.shtml
Reduce: www.uni-koeln.de/REDUCE
SnapPea: www.ptf.com/ptf/products/UNIX/current/0465.0.html
SPlus: www.insightful.com

The twin successes of the symbolic algebra packages have been their mathematical generality and ease of use. These packages deal most successfully with algebraic problems whereas many (perhaps most) serious applications require analytic objects such as definite integrals, series, and differential equations. All the elementary notions of analysis, such as continuity and differentiability, need precise computational meaning. The first challenge to meeting this need involves mathematical algorithmic developments to allow the handling of a variety of these only partially handled problems—including the analysis of functions given by programs. Many of these relate to the difficult mathematical problems involved in automatic simplification of complicated analytic formulae and recognition of when two very different such expressions represent the same object. There is also an intrinsic need to mix numeric and symbolic (exact and inexact) methods. Human mathematicians often criticize programs for making dumb errors, but often these errors (such as oversimplifying expressions, leaving out hypotheses, or dividing by zero) are precisely how we start when we do it ourselves. As Jacques Hadamard noted almost a century ago, "The object of mathematical rigor is to sanction and legitimize the conquests of intuition."

The Reimann hypothesis

The question that a pure mathematician is most likely to sell his soul to solve is the so-called Riemann hypothesis, first described in 1859. The bounty on its solution now exceeds $1 million. At the Clay Mathematics Institute's Web site (www.claymath.org/prize_problems/rules.htm), the problem is described in the following form:

> Some numbers have the special property that they cannot be expressed as the product of two smaller numbers, e.g. 2, 3, 5, 7, etc. Such numbers are called prime numbers, and they play an important role, both in pure mathematics and its applications. The distribution of such prime numbers among all natural numbers does not follow any regular pattern, however the German mathematician G.F.B. Riemann (1826-1866) observed that the frequency of prime numbers is closely related to the behavior of an elaborate function $\zeta(s)$ called the Riemann Zeta function. The Riemann hypothesis asserts that all interesting solutions of the equation $\zeta(s) = 0$ lie on a straight line. This has been checked for the first 1,500,000,000 solutions. A proof that it is true for every interesting solution would shed light on many of the mysteries surrounding the distribution of prime numbers.

A little more precisely, the Riemann hypothesis is usually formulated as

> All the zeros in the right half of the complex plane of the analytic continuation of
>
> $$\zeta(s) := \sum_{n=0}^{\infty} \frac{1}{n^s}$$
>
> lie on the vertical line $\Re(s) = 1/2$.

(One of the most famous results in elementary mathematics is Euler's evaluation of $\zeta(2) = \pi^2/6$.)

Without doubt this is one of the "grand challenge" problems of mathematics and for good reason. Large tracts of mathematics fall into place if the Riemann hypothesis is true: while the proof methods may be tremendously significant, the truth of the Riemann hypothesis is central—its falseness would be disquieting. Most mathematicians believe the Riemann hypothesis to be true, although there are notable dissenters. John Littlewood, one of the last century's great analytic number theorists, has hypothesized its falseness.[2] Of course, finding just one nontrivial zero off the line $\Re(s) = 1/2$, should it exist, is worth $1 million, and this might provide additional motivation to extend this particular mountain's climb. (Perhaps the prize is only for a proof, not a disproof—certainly a proof is more interesting.) The fact that more than the first billion zeros are known, by computation, to satisfy the Riemann hypothesis might be considered "strong numerical evidence." However, it is far from overwhelming—there are subtle phenomena in this branch of mathematics that only manifest themselves far outside present computer range.

One reason to extend such computations—which are neither easy nor obvious and rely on some fairly subtle mathematics—is the hope that someone will uncover delicate phenomena that give insight for a proof. Greatly more ambitious is the possibility that, in the very long run, it will be possible to machine-generate a proof—even for problems as difficult as this one.

P vs. *NP*

Of the seven $1 million "Millennium Prize" problems on the Clay Web site, the one that is most germane to this discussion is the so-called $P \neq NP$ problem. Again, from the site:

It is Saturday evening and you arrive at a big party. Feeling shy, you wonder whether you already know anyone in the room. Your host proposes that you must certainly know Rose, the lady in the corner next to the dessert tray. In a fraction of a second you are able to cast a glance and verify that your host is correct. However, in the absence of such a suggestion, you are obliged to make a tour of the whole room, checking out each person one by one, to see if there is anyone you recognize. This is an example of the general phenomenon that generating a solution to a problem often takes far longer than verifying that a given solution is correct. Similarly, if someone tells you that the number 13,717,421 can be written as the product of two smaller numbers, you might not know whether to believe him, but if he tells you that it can be factored as $3{,}607 \times 3{,}803$, you can easily check that it is true using a hand calculator. One of the outstanding problems in logic and computer science is determining whether questions exist whose answer can be quickly checked (for example by computer), but which require a much longer time to solve from scratch (without knowing the answer). There certainly seem to be many such questions. But so far no one has proved that any of them really does require a long time to solve; it may be that we simply have not yet discovered how to solve them quickly. Stephen Cook formulated the P versus NP problem in 1971.

Although in many instances you could question the practical distinction between polynomial and nonpolynomial algorithms, this problem is central to our current understanding of computing. Roughly, it conjectures that many of the problems we currently find computationally difficult must perforce be that way. It is a question about methods, not about actual computations, but it underlies many of the challenge problems we can imagine posing. A question that requests us to "compute such and such a sized incidence of this or that phenomena" always risks having the answer "it's just not possible" because $P \neq NP$.

Two specific challenges

With the caveat that although factoring is difficult, it is not generally assumed to be in the class of NP-hard problems, let us offer two challenges that are far-fetched but not inconceivable goals for the next few decades.

Design an algorithm that can reliably factor a random thousand-digit integer

Even with a huge effort, current algorithms get stuck at about 150 digits. (See www.rsasecurity.com/rsalabs/challenges/factoring/index.html for a list of current factoring challenges.) And there is a $100,000 cash prize offered for any reliable 10-million-digit prime (www.mersenne.org/prime.htm).

Primality checking is currently easier than factoring, and there are some very fast and powerful probabilistic primality tests—much faster than those providing certificates. Given that any computation has potential errors due to subtle (or even not-so-subtle) programming bugs, compiler errors, software errors, or undetected hardware integrity errors, it may be pointless to distinguish between these two types of primality tests. Many would take their chances with a $(1 - 10^{-100})$ probability statistic over a proof any day.

Learn how to factor large numbers, and most current security systems are crackable.

These questions are intimately related to the Riemann hypothesis, although not obviously so to the nonafficionado. They are also critical to issues of Internet security—learn how to factor large numbers, and most current security systems are crackable.

Find the minima in the merit factor problem up to size 100

There are many old problems that lend themselves to extensive numerical exploration. For example, in signal processing there is the *merit factor problem*, which is due to Marcel Golay with closely related versions due to Littlewood and Paul Erdös. Its pedigree is long, but not as long as the Riemann hypothesis (see http://athene.nat.uni-magdeburg.de/~mertens for recent records and references).

We can formulate it as follows. Suppose $(a_0 := 1, a_1, ..., a_n)$ is a sequence of length $n + 1$, where each a_i is either 1 or –1. If

$$c_k = \sum_{j=0}^{n-k} a_j a_{j+k} \tag{1}$$

then the problem is, for each fixed n, to minimize:

$$\sum_{k=-n}^{n} c_k^2 \; . \tag{2}$$

We can find exact minima up to about $n = 50$. The search space of sequences at size 50 is 2^{50}, which is about today's limit for a very large-scale calculation. In fact, the records use a branch-and-bound algorithm that more or less grows like 1.8^n. This is marginally better than the naive 2^n of a completely exhaustive search, but it is still painfully exponential.

The best hope for a solution is better algorithms. The problem is widely acknowledged as a very hard problem in combinatorial optimization, but it isn't known to be in one of the recognized hard classes like NP. The next best hope is radically different computers, perhaps quantum computers. And there is always a remote chance that analysis will lead to a mathematical solution.

The next best hope is radically different computers, perhaps quantum computers.

A concrete example

Let's examine some of the mathematical challenges in a specific problem Donald Knuth recently proposed. He asked solvers to evaluate the following sum:[3]

$$\sum_{k=1}^{\infty}\left(\frac{k^k}{k!e^k}-\frac{1}{\sqrt{2\pi k}}\right). \tag{3}$$

We answer Knuth's question in the following steps.

1. A very rapid Maple computation yielded –0.08406950872765600... as the first 16 digits of the sum.
2. The inverse symbolic calculator has a "smart lookup" feature—alternatively, we could use a sufficiently robust integer relation finder—that replied that this was probably $-2/3-\zeta(1/2)/\sqrt{2\pi}$.
3. Checking this to 50 digits provided ample experimental confirmation. Thus, within minutes we knew the answer.
4. So why did these numerical and symbolic numbers match? A clue was provided by the surprising speed with which Maple computed the slowly convergent infinite sum. The package clearly knew something the user did not. Peering under the covers revealed that it used the *LambertW* function, W, which is the inverse of $w = z \exp(z)$. (A search for "Lambert W function" on MathSciNet provided nine references—all since 1997 when the function appears named for the first time in Maple and Mathematica.)
5. The presence of ζ(1/2) and standard Euler-MacLaurin techniques, using Stirling's formula (as might be anticipated from the

   $$\sum_{k=1}^{\infty}\left(\frac{1}{\sqrt{2\pi k}}-\frac{1}{\sqrt{2}}\frac{(1/2)_{k-1}}{(k-1)!}\right)=\frac{\zeta(1/2)}{\sqrt{2\pi}} \tag{4}$$

   where the binomial coefficients are those in the series for

   $$\frac{1}{\sqrt{2-2z}} .$$

   Now Equation 4 is a formula Maple can prove.
6. However, we still need to show

   $$\sum_{k=1}^{\infty}\left(\frac{k^k}{k!e^k}-\frac{1}{\sqrt{2}}\frac{(1/2)_{k-1}}{(k-1)!}\right)=-\frac{2}{3} . \tag{5}$$

7. Guided by the presence of W and its series

   $$\sum_{k=1}^{\infty}\frac{(-k)^{k-1}z^k}{k!} ,$$

   an appeal to Abel's limit theorem lets us deduce the need to evaluate

   $$\lim_{z\to 1}\left(\frac{d}{dz}W(-z/e)+\frac{1}{\sqrt{2-2z}}\right)=\frac{2}{3} . \tag{6}$$

   Again, Maple can establish Equation 6.

Of course, this all took a fair amount of human mediation and insight.

In 1996, discussing the philosophy and practice of experimental mathematics, we wrote[4]

> As mathematics has continued to grow there has been a recognition that the age of the mathe-

matical generalist is long over. What has not been so readily acknowledged is just how specialized mathematics has become. As we have already observed, subfields of mathematics have become more and more isolated from each other. At some level, this isolation is inherent but it is imperative that communications between fields should be left as wide open as possible. As fields mature, speciation occurs. The communication of sophisticated proofs will never transcend all boundaries since many boundaries mark true conceptual difficulties. But experimental mathematics, centering on the use of computers in mathematics, would seem to provide a common ground for the transmission of many insights.

This common ground continues to increase and extends throughout the sciences and engineering.

The corresponding need is to retain the robustness and unusually long-livedness of the rigorous mathematical literature. Doron Zeilberger's proposed *Abstract of the Future* challenges this in many ways: "We show in a certain precise sense that the Goldbach conjecture (where every even number is the sum of two primes) is true with probability larger than 0.99999 and that its complete truth could be determined with a budget of 10 billion."[4]

He goes on to suggest that only the Riemann hypothesis merits paying really big bucks for certainty. Relatedly, Greg Chaitin argued that we should introduce the Riemann hypothesis as an axiom: "I believe that elementary number theory and the rest of mathematics should be pursued more in the spirit of experimental science, and that you should be willing to adopt new principles. I believe that Euclid's statement that an axiom is a self-evident truth is a big mistake. The Schrödinger equation certainly isn't a self-evident truth! And the Riemann hypothesis isn't self-evident either, but it's very useful. A physicist would say that there is ample experimental evidence for the Riemann hypothesis and would go ahead and take it as a working assumption."[4]

How do we reconcile these somewhat combative challenges with the inarguable power of the deductive method? How do we continue to produce rigorous mathematics when more research will be performed in large computational environments where we might or might not be able to determine what the system has done or why? This is often described as "relying on proof by 'Von Neumann says'."

At another level we see the core challenge for mathematical computing to be the construction of workspaces that largely or completely automate the diverse steps illustrated in Knuth's and similar problems.

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The Best Teacher I Ever Had Was ...

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My first response when approached to write on this subject was "but I remember an anecdote here or there nothing more." On my next plane trip I nonetheless made a mental inventory of my most memorable teachers – gifted and otherwise. As an undergraduate in the late sixties: Peter Fraser, a mathematical physicist at Western Ontario, whose flawless lectures persuaded my second year mechanics class that no effort was required on their part, until a failing grade on the final proved otherwise. My to-be-nameless third year number theory teacher, a non-researcher whose stolid indifference to scholarship could not destroy a for-me magical subject.

As a graduate student in Oxford I was entranced by the distinguished (mathematical and linguistic) philosopher Michael Dummett who taught me about Frege and Heyting, expected me to prove the "Independence of the Continuum Hypothesis" in viva, and lectured seemingly sans notes with a lucidity that provided "camera ready copy". His curiously two-tone hair (bright yellow at the front, silver gray at the back) was limned by my sudden realization half way through a course that each cycle across the proscenium had regular nicotine producing features. Inhale from the cigarette in the elegant ivory holder; exhale, expostulate and excogitate, run fingers through yellow–gray mane; repeat for duration of hour. And Brian Birch the (then and now) leading British number theorist whose elliptic argot left me temporarily reeling. Phrases something like "kill the integral and shove the function to the left" I eventually decoded as a good prescription for contour integration. Equally, "the zeta function associated with an elliptic curve is exactly what you thought it would be" proved more useful with hindsight than on first meeting.

At that stage, I decided I had taken my brief somewhat too narrowly. Teachers should include my academic mentors. With some sense of relief I added Michael Edelstein at Dalhousie, a Polish–Israeli functional analyst who by example and engagement converted a generation of young analysts into full blooded independent researchers; Dick Duffin, a distinguished mathematician and engineer who was my colleague at Carnegie-Mellon in the early eighties. Dick, who had trained Nash (a future Economics laureate), Raul Bott (himself the supervisor of future Fields medalists) and Hans Weinberger among others, is a man absolutely without pretensions who forcibly taught me that great depth of insight and what Louis Wolpert has nicely termed "a passion for science" had no need of flashy or pretentious packages.

In diverse ways from these teachers and many more (some superb exponents of indifferent subjects, some indifferent exponents of intoxicating subjects, and all other combinations) I learned the myriad skills of a successful academic who must be a teacher, researcher, editor, reviewer, administrator, and all too often in loco parentis. That said, one person emerged from my archival hunt – my mathematician father David Borwein, FRSE, head of Pure Mathematics at Western from 1967 to 1989, President of the Canadian Mathematical Society from 1984-1986 and my frequent co–author.[1]

Both my brother and I ultimately became academic mathematicians and not surprisingly have from time to time mulled over what factors lead us to take up the same vocation. I started University determined to be a historian. Neither of us was in any sense "hot–housed". In my undergraduate career I had precisely one lecture from my father; otherwise he assiduously scheduled classes so as to avoid our meeting. The only exception being a 1957 bet with his colleagues in St. Andrews –for a large quantity of of cheese – that he could teach his six year old son to solve two–by–two simultaneous linear equations by making it into a game. In still recently post–war Britain I was so taught and while conning neither reason nor rationale I loved playing this mysterious game and taught my best friend also to play.

[1]Ten joint papers, nine of which followed my father's somewhat mis-described "retirement", and still counting.

From then until I was a third year undergraduate David's (Dad's) role in my education was restrained. I was offered very little overt enrichment. Nor, in the politically heated and drug laden late sixties would I have brooked much intrusion. But what I did infuse in confrontational discussions at the dinner table over Johnson and later Nixon, and more quietly as we began jointly to solve problems posed in the *American Mathematical Monthly*[2] was the timbre of a to–the–manor–born academic; a man who nonetheless cared deeply about the external world; a man with a subtle and inexhaustible sense of humor; a man who would happily stay up all night polishing a proof or hunting for the resolution to an obdurately untameable mistake. Above all a man who demonstrated with every fibre that he was doing just what he wanted to be doing, that fads were fads but that scientific knowledge would not ever be entirely deconstructed. And so by 1971 when I graduated from UWO and went somewhat uncertainly as a Rhodes scholar to Oxford, he had helped me become inescapably a mathematician despite James Sinclair's (Trudeau's father in law) offer that if I studied 'PPE' (Politics, Philosophy and Economics) in Oxford he would give me a cement factory to manage on my return!

My father's mentorship did not end in 1971. He offered easy and low risk access to a knowing guide. "Dad, I don't follow much at conference talks." Answer "Neither do I." Consequence: an early realization of the real purpose of mathematics meetings as places to meet, mix and to discover what one wished subsequently to learn. "David, I'm taking a job in the States and so should resign my position at Dalhousie." Answer "Has any one asked you to resign?" Consequence: two years later I moved back to Dalhousie for another decade, easily and I trust to the benefit of both the institution and myself.

Thus was I taught about the *erkensis und praxis* of mathematics: that strange blend of arts and science, of austere Platonic edifice and fallible human creation. I learned of the quiet satisfactions of an intellectual life; but not of a life lived in a vacuum. I was taught to make yet one more revision to a paper and to savour the polish and finish it provided. And finally, my overarching memory is of my father, at frequent parties arranged by my more outgoing mother, playing generously if not exhuberantly the

[2]The MAA Monthly, the most widely circulated mathematics journal, has for generations published a section of problems, both solved and unsolved, on which researchers, novice and adept, expend considerable effort.

role of host. By mid–evening his eyes would slightly glaze and a stream of cocktail napkins would issue forth covered with formulae and expressions in his careful and concise italic script.

I have been unusually privileged. I have worked intensively with both brother and father as equals and have known my father as an intellectual peer for more than a quarter century. In every sense he has been 'the best teacher I ever had."

Experimental Mathematics: Recent Developments and Future Outlook

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1 Introduction

While extensive usage of high-performance computing has been a staple of other scientific and engineering disciplines for some time, research mathematics is one discipline that has heretofore not yet benefited to the same degree. Now, however, with sophisticated mathematical computing tools and environments widely available on desktop computers, a growing number of remarkable new mathematical results are being discovered partly or entirely with the aid of these tools. With currently planned improvements in these tools, together with substantial increases expected in raw computing power, due both to Moore's Law and the expected implementation of these environments on parallel supercomputers, we can expect even more remarkable developments in the years ahead.

This article briefly discusses the nature of mathematical experiment. It then presents a few instances primarily of our own recent computer-aided mathematical discoveries, and sketches the outlook for the future. Additional examples in diverse fields and broader citations to the literature may be found in [16] and its references.

2 Preliminaries

The crucial role of high performance computing is now acknowledged throughout the physical, biological and engineering sciences. Numerical experimentation, using increasingly large-scale, three-dimensional simulation programs, is now a staple of fields such as aeronautical and electrical engineering, and research scientists heavily utilize computing technology to collect and analyze data, and to explore the implications of various physical theories.

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However, "pure" mathematics (and closely allied areas such as theoretical physics) only recently has begun to capitalize on this new technology. This is ironic, because the basic theoretical underpinnings of modern computer technology were set out decades ago by mathematicians such as Alan Turing and John Von Neumann. But only in the past decade, with the emergence of powerful mathematical computing tools and environments, together with the growing availability of very fast desktop computers and highly parallel supercomputers, as well as the pervasive presence of the Internet, has this technology reached the level where the research mathematician can enjoy the same degree of intelligent assistance that has graced other technical fields for some time.

This new approach is often termed *experimental mathematics*, namely the utilization of advanced computing technology to explore mathematical structures, test conjectures and suggest generalizations. And there is now a thriving journal of *Experimental Mathematics*. In one sense, there is nothing new in this approach — mathematicians have used it for centuries. Gauss once confessed, "I have the result, but I do not yet know how to get it." [2]. Hadamard declared, "The object of mathematical rigor is to sanction and legitimize the conquests of intuition, and there was never any other object for it." [34]. In recent times Milnor has stated this philosophy very clearly:

> If I can give an abstract proof of something, I'm reasonably happy. But if I can get a concrete, computational proof and actually produce numbers I'm much happier. I'm rather an addict of doing things on computer, because that gives you an explicit criterion of what's going on. I have a visual way of thinking, and I'm happy if I can see a picture of what I'm working with. [35]

What is really meant by an *experiment* in the context of mathematics? In *Advice to a Young Scientist*, Peter Medawar [31] identifies four forms of experiment:

1. The *Kantian* experiment is one such as generating "the classical non-Euclidean geometries (hyperbolic, elliptic) by replacing Euclid's axiom of parallels (or something equivalent to it) with alternative forms."
2. The *Baconian* experiment is a contrived as opposed to a natural happening, it "is the consequence of 'trying things out' or even of merely messing about."
3. The *Aristotelian* experiment is a demonstration: "apply electrodes to a frog's sciatic nerve, and lo, the leg kicks; always precede the presentation of the dog's dinner with the ringing of a bell, and lo, the bell alone will soon make the dog dribble."
4. The *Galilean* experiment is "a critical experiment – one that discriminates between possibilities and, in doing so, either gives us confidence in the view we are taking or makes us think it in need of correction."

The first three are certainly common in mathematics. However, as discussed in detail in [15], the Galilean experiment is the only one of the four forms which can make experimental mathematics a truly serious enterprise.

3 Tools of the Trade

The most obvious development in mathematical computing technology has been the growing availability of powerful symbolic computing tools. Back in the 1970s, when the first symbolic computing tools became available, their limitations were quite evident — in many cases, these programs were unable to handle operations that could be done by hand. In the intervening years these programs, notably the commercial products such as Maple and Mathematica, have greatly improved. While numerous deficiencies remain, they nonetheless routinely and correctly dispatch many operations that are well beyond the level that a human could perform with reasonable effort.

Another recent development that has been key to a number of new discoveries is the emergence of practical integer relation detection algorithms. Let $x = (x_1, x_2, \cdots, x_n)$ be a vector of real or complex numbers. x is said to possess an integer relation if there exist integers a_i, not all zero, such that $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$. By an *integer relation algorithm*, we mean a practical computational scheme that can recover the vector of integers a_i, if it exists, or can produce bounds within which no integer relation exists. The problem of finding integer relations was studied by numerous mathematicians, including Euclid and Euler. The first general integer relation algorithm was discovered in 1977 by Ferguson and Forcade [24]. There is a close connection between integer relation detection and finding small vectors in an integer lattice, and thus one common solution to the integer relation problem is to apply the Lenstra-Lenstra-Lovasz (LLL) lattice reduction algorithm [30]. At the present time, the most effective scheme for integer relation detection is Ferguson's "PSLQ" algorithm [23,6].

Integer relation detection, as well as a number of other techniques used in modern experimental mathematics, relies heavily on very high precision arithmetic. The most advanced tools for performing high precision arithmetic utilize fast Fourier transforms (FFTs) for multiplication operations. Armed with one of these programs, a researcher can often effortlessly evaluate mathematical constants and functions to precision levels in the many thousands of decimal digits. The software products Maple and Mathematica include relatively complete and well-integrated multiple precision arithmetic facilities, although until very recently they did not utilize FFTs, or other accelerated multiplication techniques. One may also use any of several freeware multiprecision software packages [3,22] and for many purposes tools such as Matlab, MuPAD or more specialized packages like Pari-GP are excellent.

High precision arithmetic, when intelligently used with integer relation detection programs, allows researchers to discover heretofore unknown mathematical identities. It should be emphasized that these numerically discovered "identities" are only approximately established. Nevertheless, in the cases we are aware of, the results have been numerically verified to hundreds and in some cases thousands of decimal digits beyond levels that could reasonably be dismissed as numerical artifacts. Thus while these "identities" are not firmly established in a formal sense, they are supported by very compelling numerical evidence. After all, which is more compelling, a formal proof that in its full exposition requires hundreds of difficult pages of reasoning, fully understood by only two or three colleagues, or the numerical verification of a conjecture to 100,000 decimal digit accuracy, subsequently validated by numerous subsidiary computations? In the same way, these tools are often even more useful as a way of *excluding* the possibility of hoped for relationships, as in equation (1) below.

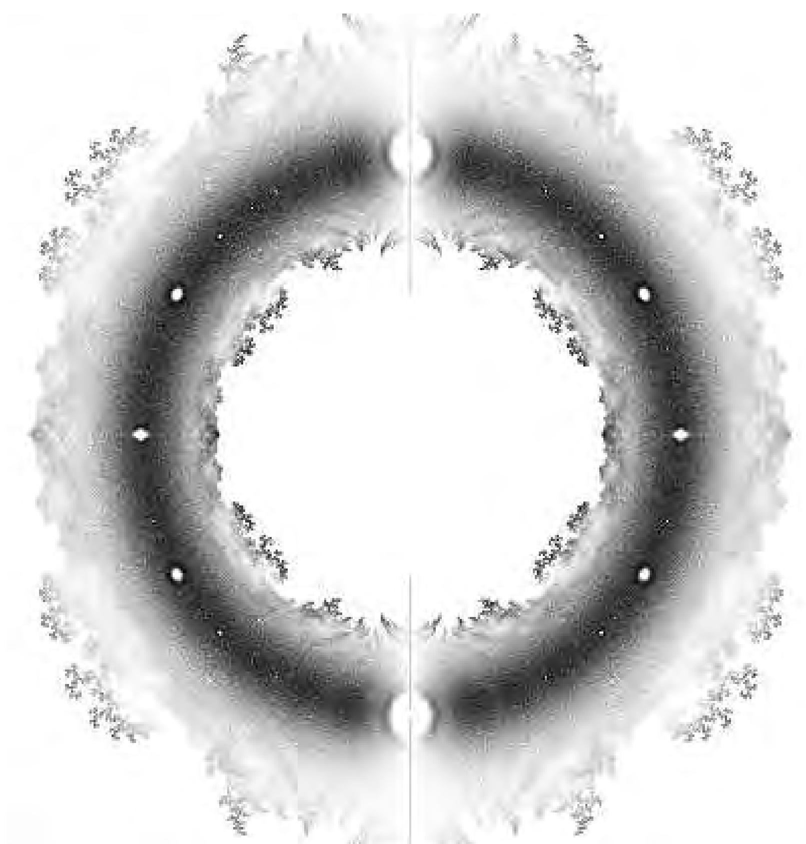

FIGURE 1(A-D): -1/1 POLYNOMIALS (TO BE SET IN COLOR)

We would be remiss not to mention the growing power of visualization especially when married to high performance computation. The pictures in FIGURE 1 represents the zeroes of all polynomials with ± 1 coefficients of degree at most 18. One of the most striking features of the picture, its fractal nature excepted, is the appearance of different sized "holes" at what transpire to be roots of unity. This observation which would be very hard to make other than pictorially led to a detailed and rigorous analysis of the phenomenon and more [17,27]. They were lead to this analysis by the interface which was built for Andrew Odlyzko's seminal online paper [32].

One additional tool that has been utilized in a growing number of studies is Sloane and Plouffe's *Encyclopedia of Integer Sequences* [36]. As the title indicates, it identifies many integer sequences based on the first few terms. A very powerful on-line version is also available and is a fine example of the changing research paradigm. Another wonderful resource is Stephen Finch's "Favorite Mathematical Constants," which contains a wealth of frequently updated information, links and references on 125 constants, [25], such as the *hard hexagon constant* $\kappa \approx 1.395485972$ for which Zimmermann obtained a minimal polynomial of degree 24 in 1996.[1]

In the following, we illustrate this – both new and old – approach to mathematical research using a handful of examples with which we are personally familiar. We will then sketch some future directions in this emerging methodology. We have focussed on the research of our own circle of direct collaborators. We do so for resaons of fmailiarity and because we believe it is representative of broad changes in the way mathematics is being done rather than to claim primacy for our own skills or expertise.

4 A New Formula for Pi

Through the centuries mathematicians have assumed that there is no shortcut to determining just the n-th digit of π. Thus it came as no small surprise when such a scheme was recently discovered [5]. In particular, this simple algorithm allows one to calculate the n-th hexadecimal (or binary) digit of π without computing any of the first $n-1$ digits, without the need for multiple-precision arithmetic software, and requiring only a very small amount of memory. The one millionth hex digit of π can be computed in this manner on a current-generation personal computer in only about 30 seconds run time.

This scheme is based on the following remarkable formula, whose formal proof involves nothing more sophisticated than freshman calculus:

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left[\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right]$$

This formula was found using months of PSLQ computations, after corresponding but simpler n-th digit formulas were identified for several

[1] See `http://www.mathsoft.com/asolve/constant/square/square.html`.

other constants, including log(2). This is likely the first instance in history that a significant new formula for π was discovered by a computer.

Similar base-2 formulas are given in [5,21] for a number of other mathematical constants. In [20] some base-3 formulas were obtained, including the identity

$$\pi^2 = \frac{2}{27}\sum_{k=0}^{\infty}\frac{1}{729^k}\left[\frac{243}{(12k+1)^2} - \frac{405}{(12k+2)^2} - \frac{81}{(12k+4)^2} - \frac{27}{(12k+5)^2} - \frac{72}{(12k+6)^2} - \frac{9}{(12k+7)^2} - \frac{9}{(12k+8)^2} - \frac{5}{(12k+10)^2} + \frac{1}{(12k+11)^2}\right]$$

In [8], it is shown that the question of whether π, $\log 2$ and certain other constants are normal can be reduced to a plausible conjecture regarding dynamical iterations of the form $x_0 = 0$,

$$x_n = (bx_{n-1} + r_n) \bmod 1$$

where b is an integer and $r_n = p(n)/q(n)$ is the ratio of two nonzero polynomials with $\deg(p) < \deg(q)$. The conjecture is that these iterates either have a finite set of attractors or else are equidistributed in the unit interval. In particular, it is shown that the question of whether π is normal base 16 (and hence base 2) can be reduced to the assertion that the dynamical iteration

$$x_n = \left(16x_{n-1} + \frac{120n^2 - 89n + 16}{512n^4 - 1024n^3 + 712n^2 - 206n + 21}\right) \bmod 1$$

is equidistributed in $[0, 1)$. There are also connections between the question of normality for certain constants and the theory of linear congruential pseudorandom number generators. All of these results derive from the discovery of the individual digit-calculating formulas mentioned above. For details, see [8].

5 Identities for the Riemann Zeta Function

Another application of computer technology in mathematics is to determine whether or not a given constant α, whose value can be computed to high precision, is algebraic of some degree n or less. This can be done by first computing the vector $x = (1, \alpha, \alpha^2, \cdots, \alpha^n)$ to high precision and then applying an integer relation algorithm. If a relation is found for x, then this relation vector is precisely the set of integer coefficients of a polynomial satisfied by α. Even if no relation is found, integer relation detection programs can produce bounds within which no relation can exist. In fact, exclusions of this type are solidly established by integer relation calculations, whereas "identities" discovered in this fashion are only approximately established, as noted above.

Consider, for example, the following identities, with that for $\zeta(3)$ due to Apéry [10,14]:

$$\zeta(2) = 3\sum_{k=1}^{\infty} \frac{1}{k^2\binom{2k}{k}}$$
$$\zeta(3) = \frac{5}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3\binom{2k}{k}}$$
$$\zeta(4) = \frac{36}{17}\sum_{k=1}^{\infty} \frac{1}{k^4\binom{2k}{k}}$$

where $\zeta(n) = \sum_k k^{-n}$ is the Riemann zeta function at n. These results have led many to hope that

$$Z_5 = \zeta(5) / \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^5\binom{2k}{k}} \tag{1}$$

might also be a simple rational or algebraic number. However, computations using PSLQ established, for instance, that if Z_5 satisfies a polynomial of degree 25 or less, then the Euclidean norm of the coefficients must exceed 2×10^{37}. Given these results, there is no "easy" identity, and researchers are licensed to investigate the possibility of multi-term identities for $\zeta(5)$. One recently discovered [14], using a PSLQ computation, was the polylogarithmic identity

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^5\binom{2k}{k}} = 2\zeta(5) + 80\sum_{k=1}^{\infty}\left[\frac{1}{(2k)^5} - \frac{L}{(2k)^4}\right]\rho^{2k}$$
$$- \frac{4}{3}L^5 + \frac{8}{3}L^3\zeta(2) + 4L^2\zeta(3)$$

where $L = \log(\rho)$ and $\rho = (\sqrt{5}-1)/2$. This illuastrates neatly that one can only find a closed form if one knows where to look.

Other earlier evaluations involving the central binomial coefficient suggested general formulas [12], which were pursued by a combination of PSLQ and heavy-duty symbolic manipulation. This led, most unexpectedly, to the identity

$$\sum_{k=1}^{\infty} \zeta(4k+3)z^{4k} = \sum_{k=1}^{\infty} \frac{1}{k^3(1-z^4/k^4)}$$
$$= \frac{5}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3\binom{2k}{k}(1-z^4/k^4)} \prod_{m=1}^{k-1} \frac{1+4z^4/m^4}{1-z^4/m^4}.$$

Experimental analysis of the first ten terms showed that the rightmost above series necessarily had the form

$$\frac{5}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}P_k(z)}{k^3\binom{2k}{k}(1-z^4/k^4)}$$

where

$$P_k(z) = \prod_{j=1}^{k-1} \frac{1 + 4z^4/j^4}{1 - z^4/j^4}.$$

Also discovered in this process was the intriguing *equivalent* combinatorial identity

$$\binom{2n}{n} = \sum_{k=1}^{\infty} \frac{2n^2 \prod_{i=1}^{n-1}(4k^4 + i^4)}{k^2 \prod_{i=1, i\neq k}^{n}(k^4 - i^4)}.$$

This evaluation was discovered as the result of an serendipitous error in an input to Maple[2]— the computational equivalent of discovering penicillin after a mistake in a Petri dish.

With the recent proof of this last conjectured identity, by Almkvist and Granville [1], the above identities have now been rigorously established. But other numerically discovered "identities" of this type appear well beyond the reach of current formal proof methods. For example, in 1999 British physicist David Broadhurst used a PSLQ program to recover an explicit expression for $\zeta(20)$ involving 118 terms. The problem required 5,000 digit arithmetic and over six hours computer run time. The complete solution is given in [6].

6 Identification of Multiple Sum Constants

Numerous identities were experimentally discovered in some recent research on multiple sum constants. After computing high-precision numerical values of these constants, a PSLQ program was used to determine if a given constant satisfied an identity of a conjectured form. These efforts produced empirical evaluations and suggested general results [4]. Later, elegant proofs were found for many of these specific and general results [13], using a combination of human intuition and computer-aided symbolic manipulation. Three examples of experimentally discovered re-

[2] Typing 'infty' for 'infinity' revealed that the program had an algorithm when a formal variable was entered.

sults that were subsequently proven are:

$$\sum_{k=1}^{\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{k}\right)^2 (k+1)^{-4} = \frac{37}{22680}\pi^6 - \zeta^2(3)$$

$$\sum_{k=1}^{\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{k}\right)^3 (k+1)^{-6} = \zeta^3(3) + \frac{197}{24}\zeta(9) + \frac{1}{2}\pi^2\zeta(7) - \frac{11}{120}\pi^4\zeta(5) - \frac{37}{7560}\pi^6\zeta(3)$$

$$\sum_{k=1}^{\infty}\left(1-\frac{1}{2}+\cdots+(-1)^{k+1}\frac{1}{k}\right)^2 (k+1)^{-3} = 4\,\mathrm{Li}_5(\tfrac{1}{2}) - \frac{1}{30}\ln^5(2) - \frac{17}{32}\zeta(5) - \frac{11}{720}\pi^4\ln(2) + \frac{7}{4}\zeta(3)\ln^2(2) + \frac{1}{18}\pi^2\ln^3(2) - \frac{1}{8}\pi^2\zeta(3)$$

where again $\zeta(n) = \sum_{j=1}^{\infty} j^{-n}$ is a value of the Riemann zeta function, and $\mathrm{Li}_n(x) = \sum_{j=1}^{\infty} x^j j^{-n}$ denotes the classical polylogarithm function.

More generally, one may define *multi-dimensional Euler sums* (or *multiple zeta values*) by

$$\zeta\begin{pmatrix} s_1, & s_2 & \cdots & s_r \\ \sigma_1, & \sigma_2 & \cdots & \sigma_r \end{pmatrix} := \sum_{k_1>k_2>\cdots>k_r>0} \frac{\sigma_1^{k_1}}{k_1^{s_1}}\frac{\sigma_2^{k_2}}{k_2^{s_2}}\cdots\frac{\sigma_r^{k_r}}{k_r^{s_r}}$$

where $\sigma_j = \pm 1$ are signs and $s_j > 0$ are integers. When all the signs are positive, one has a multiple zeta value. The integer r is the sum's depth and $s_1 + s_2 + \cdots + s_r$ is the weight. These sums have connections with diverse fields such as knot theory, quantum field theory and combinatorics. Constants of this form with alternating signs appear in problems such as computation of the magnetic moment of the electron.

Multi-dimensional Euler sums satisfy many striking identities. The discovery of the more recondite identities was facilitated by the development of Hölder convolution algorithms that permit very high precision numerical values to be rapidly computed. See [13] and a computational interface at `www.cecm.sfu.ca/projects/ezface+/`. One beautiful general identity discovered by Zagier [37] in the course of similar research is

$$\zeta(3,1,3,1,\ldots,3,1) = \frac{1}{2n+1}\zeta(2,2,\ldots,2) = \frac{2\pi^{4n}}{(4n+2)!}$$

where there are n instances of '$(3,1)$' and '2' in the arguments to $\zeta(\cdot)$. This has now been proven in [13] and the proof, while entirely conventional, was obtained by guided experimentation. A related conjecture for which overwhelming evidence but no hint of a proof exists is the "identity"

$$8^n \zeta\begin{pmatrix} 2, & 1, & 2, & 1, \cdots, & 2, & 1 \\ -1, & 1, & -1, & 1, \cdots, & -1, & 1 \end{pmatrix} = \zeta(2,1,2,1,\ldots,2,1).$$

Along this line, Broadhurst conjectured, based on low-degree numerical results, that the dimension of the space of Euler sums with weight w is the Fibonacci number $F_{w+1} = F_w + F_{w-1}$, with $F_1 = F_2 = 1$. In testing this conjecture, complete reductions of all Euler sums to a basis of size F_{w+1} were obtained with PSLQ at weights $w \leq 9$. At weights $w = 10$ and $w = 11$ the conjecture was stringently tested by application of PSLQ in more than 600 cases. At weight $w = 11$ such tests involve solving integer relations of size $n = F_{12} + 1 = 145$. In a typical case, each of the 145 constants was computed to more than 5,000 digit accuracy, and a working precision level of 5,000 digits was employed in an advanced "multi-pair" PSLQ program. In these problems the ratios of adjacent coefficients in the recovered integer vector usually have special values, such as $11! = 39916800$. These facts, combined with confidence ratios typically on the order of 10^{-300} in the detected relations, render remote the chance that these identities are spurious numerical artifacts, and lend substantial support to this conjecture [6].

7 Mathematical Computing Meets Parallel Computing

The potential future power of highly parallel computing technology has been underscored in some recent results. Not surprisingly, many of these computations involve the constant π, underscoring the enduring interest in this most famous of mathematical constants. In 1997 Fabrice Bellard of INRIA used a more efficient formula, similar to the one mentioned in section three, programmed on a network of workstations, to compute 150 binary digits of π starting at the *trillionth* position. Not to be outdone, 17-year-old Colin Percival of Simon Fraser University in Canada organized a computation of 80 binary digits of π beginning at the five trillionth position, using a network of 25 laboratory computers. He an many others are presently computing binary digits at the quadrillionth position on the web [33]. As we write, the most recent computational result was Yasumasa Kanada's calculation (September 1999) of the first 206 billion decimal digits of π. This spectacular computation was made on a Hitachi parallel supercomputer with 128 processors, in little over a day, and employed the Salamin-Brent algorithm [10], with a quartically convergent algorithm from [10] as an independent check.

Several large-scale parallel integer relation detection computations have also been performed in the past year or two. One arose from the discovery by Broadhurst that

$$\alpha^{630} - 1 = \frac{(\alpha^{315}-1)(\alpha^{210}-1)(\alpha^{126}-1)^2(\alpha^{90}-1)(\alpha^3-1)^3(\alpha^2-1)^5(\alpha-1)^3}{(\alpha^{35}-1)(\alpha^{15}-1)^2(\alpha^{14}-1)^2(\alpha^5-1)^6\alpha^{68}}$$

where $\alpha = 1.176280818\ldots$ is the largest real root of Lehmer's polynomial [29]

$$0 = 1 + \alpha - \alpha^3 - \alpha^4 - \alpha^5 - \alpha^6 - \alpha^7 + \alpha^9 + \alpha^{10}.$$

The above cyclotomic relation was first discovered by a PSLQ computation, and only subsequently proven. Broadhurst then conjectured that there might be integers a, b_j, c_k such that

$$a\,\zeta(17) = \sum_{j=0}^{8} b_j\,\pi^{2j}(\log\alpha)^{17-2j} + \sum_{k\in D(\mathcal{S})} c_k\,\mathrm{Li}_{17}(\alpha^{-k})$$

where the 115 indices k are drawn from the set, $D(\mathcal{S})$, of positive integers that divide at least one element of

$$\mathcal{S} = \{29, 47, 50, 52, 56, 57, 64, 74, 75, 76, 78, 84, 86, 92, 96, 98, 108, 110, 118, 124, 130, 132, 138, 144, 154, 160, 165, 175, 182, 186, 195, 204, 212, 240, 246, 270, 286, 360, 630\}.$$

Indeed, such a relation was found, using a parallel multi-pair PSLQ program running on a SGI/Cray T3E computer system at Lawrence Berkeley Laboratory. The run employed 50,000 decimal digit arithmetic and required approximately 44 hours on 32 processors. The resulting integer coefficients are as large as 10^{292}, but the "identity" nonetheless was confirmed to 13,000 digits beyond the level of numerical artifact [7].

8 Connections to Quantum Field Theory

In another surprising recent development, David Broadhurst has found, using these methods, that there is an intimate connection between Euler sums and constants resulting from evaluation of Feynman diagrams in quantum field theory [18,19]. In particular, the renormalization procedure (which removes infinities from the perturbation expansion) involves multiple zeta values. As before, a fruitful theory has emerged, including a large number of both specific and general results [13].

Some recent quantum field theory results are even more remarkable. Broadhurst has now shown [20], using PSLQ computations, that in each of ten cases with unit or zero mass, the finite part the scalar 3-loop tetrahedral vacuum Feynman diagram reduces to 4-letter "words" that represent iterated integrals in an alphabet of seven "letters" comprising the one-forms $\Omega := dx/x$ and $\omega_k := dx/(\lambda^{-k} - x)$, where $\lambda := (1 + \sqrt{-3})/2$ is the primitive sixth root of unity, and k runs from 0 to 5. A 4-letter word is a 4-dimensional iterated integral, such as

$$U := \zeta(\Omega^2\omega_3\omega_0) = \int_0^1 \frac{dx_1}{x_1}\int_0^{x_1}\frac{dx_2}{x_2}\int_0^{x_2}\frac{dx_3}{(-1-x_3)}\int_0^{x_3}\frac{dx_4}{(1-x_4)} = \sum_{j>k>0}\frac{(-1)^{j+k}}{j^3 k}.$$

There are 7^4 such four-letter words. Only two of these are primitive terms occurring in the 3-loop Feynman diagrams: U, above, and

$$V := \mathrm{Real}[\zeta(\Omega^2\omega_3\omega_1)] = \sum_{j>k>0} \frac{(-1)^j \cos(2\pi k/3)}{j^3 k}.$$

The remaining terms in the diagrams reduce to products of constants found in Feynman diagrams with fewer loops. These ten cases as shown in Figure 1. In these diagrams, dots indicate particles with nonzero rest mass. The formulas that have been found, using PSLQ, for the corresponding constants are given in Table 2. In the table the constant $C = \sum_{k>0} \sin(\pi k/3)/k^2$.

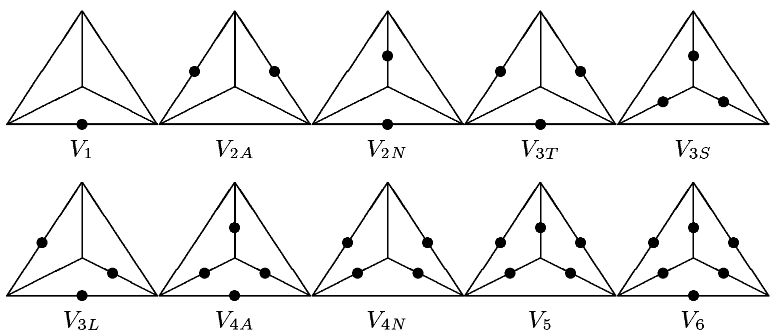

Fig. 1. The ten tetrahedral cases

V_1	$= 6\zeta(3) + 3\zeta(4)$
V_{2A}	$= 6\zeta(3) - 5\zeta(4)$
V_{2N}	$= 6\zeta(3) - \frac{13}{2}\zeta(4) - 8U$
V_{3T}	$= 6\zeta(3) - 9\zeta(4)$
V_{3S}	$= 6\zeta(3) - \frac{11}{2}\zeta(4) - 4C^2$
V_{3L}	$= 6\zeta(3) - \frac{15}{4}\zeta(4) - 6C^2$
V_{4A}	$= 6\zeta(3) - \frac{77}{12}\zeta(4) - 6C^2$
V_{4N}	$= 6\zeta(3) - 14\zeta(4) - 16U$
V_5	$= 6\zeta(3) - \frac{469}{27}\zeta(4) + \frac{8}{3}C^2 - 16V$
V_6	$= 6\zeta(3) - 13\zeta(4) - 8U - 4C^2$

Table 1. Formulas found by PSLQ for the ten cases of Figure 1

9 A Note of Caution

In spite of the remarkable successes of this methodology, some caution is in order. First of all, the fact that an identity is established to high precision is *not* a guarantee that it is indeed true. One example is

$$\sum_{n=1}^{\infty} \frac{[n \tanh \pi]}{10^n} \approx \frac{1}{81}$$

which holds to 267 digits, yet is not an exact identity, failing in the 268'th place. Several other such bogus "identities" are exhibited and explained in [11].

More generally speaking, caution must be exercised when extrapolating results true for small n to all n. For example,

$$\begin{aligned}
\int_0^\infty \frac{\sin(x)}{x}\,dx &= \frac{\pi}{2} \\
\int_0^\infty \frac{\sin(x)}{x}\frac{\sin(x/3)}{x/3}\,dx &= \frac{\pi}{2} \\
&\cdots \\
\int_0^\infty \frac{\sin(x)}{x}\frac{\sin(x/3)}{x/3}\cdots\frac{\sin(x/13)}{x/13}\,dx &= \frac{\pi}{2}
\end{aligned}$$

yet

$$\int_0^\infty \frac{\sin(x)}{x}\frac{\sin(x/3)}{x/3}\cdots\frac{\sin(x/15)}{x/15}\,dx = \frac{467807924713440738696537864469}{935615849440640907310521750000}\pi.$$

When this fact was recently observed by a researcher using a mathematical software package, he concluded that there must be a "bug" in the software. Not so. What is happening here is that

$$\int_0^\infty \frac{\sin(x)}{x}\frac{\sin(x/h_1)}{x/h_1}\cdots\frac{\sin(x/h_n)}{x/h_n}\,dx = \frac{\pi}{2}$$

only so long as $1/h_1 + 1/h_2 + \cdots + 1/h_n < 1$. In the above example, $1/3 + 1/5 + \cdots + 1/13 < 1$, but with the addition of $1/15$, the sum exceeds 1 and the identity no longer holds [9]. Changing the h_n lets this pattern persist indefinitely but still fail in the large.

10 Future Outlook

Computer mathematics software is now becoming a staple of university departments and government research laboratories. Many university departments now offer courses where the usage of one of these software packages is an integral part of the course. But further expansion of these facilities into high schools has been inhibited by a number of factors, including the fairly high cost of such software, the lack of appropriate computer equipment, difficulties in standardizing such coursework at a regional or national level, a paucity of good texts incorporating such tools into a realistic curriculum, lack of trained teachers and many other demands on their time.

But computer hardware continues its downward spiral in cost and its upward spiral in power. It thus appears that within a very few years, moderately powerful symbolic computation facilities can be incorporated into relatively inexpensive hand calculators, at which point it will be much easier to successfully integrate these tools into high school curricula. Thus it seems that we are poised to see a new generation of students coming into university mathematics and science programs who are completely comfortable using such tools. This development is bound to have a profound impact on the future teaching, learning and doing of mathematics.

A likely and fortunate spin-off of this development is that the commercial software vendors who produce these products will likely enjoy a broader financial base, from which they can afford to further enhance their products geared at serious researchers. Future enhancements are likely to include more efficient algorithms, more extensive capabilities mixing numerics and symbolics, more advanced visualization facilities, and software optimized for emerging symmetric multiprocessor and highly parallel, distributed memory computer systems. When combined with expected increases in raw computing power due to Moore's Law — improvements which almost certainly will continue unabated for at least ten years and probably much longer — we conclude that enormously more powerful computer mathematics systems will be available in the future.

We only now are beginning to experience and comprehend the potential impact of computer mathematics tools on mathematical research. In ten more years, a new generation of computer-literate mathematicians, armed with significantly improved software on prodigiously powerful computer systems, are bound to make discoveries in mathematics that we can only dream of at the present time. Will computer mathematics eventually replace, in near entirety, the solely human form of research, typified by Andrew Wiles' recent proof of Fermat's Last Theorem? Will computer mathematics systems eventually achieve such intelligence that they discover deep new mathematical results, largely or entirely without human assistance? Will new computer-based mathematical discovery techniques enable mathematicians to explore the realm, proved to exist by Gödel, Chaitin and others, that is fundamentally beyond the limits of formal reasoning?

11 Conclusion

We have shown a small but we hope convincing selection of what the present allows and what the future holds in store. We have hardly mentioned the growing ubiquity of web based computation, or of pervasive access to massive data bases, both public domain and commercial. Neither have we raised the human/computer interface or intellectual property issues and the myriad other not-purely-technical issues these raise.

Whatever the outcome of these developments, we are still persuaded that mathematics is and will remain a uniquely human undertaking. One could even argue that these developments confirm the fundamentally human nature of mathematics. Indeed, Reuben Hersh's arguments [26] for a humanist philosophy of mathematics, as paraphrased below, become more convincing in our setting:

1. *Mathematics is human.* It is part of and fits into human culture. It does not match Frege's concept of an abstract, timeless, tenseless, objective reality.
2. *Mathematical knowledge is fallible.* As in science, mathematics can advance by making mistakes and then correcting or even re-correcting them. The "fallibilism" of mathematics is brilliantly argued in Lakatos' *Proofs and Refutations* [28].
3. *There are different versions of proof or rigor.* Standards of rigor can vary depending on time, place, and other things. The use of computers in formal proofs, exemplified by the computer-assisted proof of the four color theorem in 1977, is just one example of an emerging nontraditional standard of rigor.
4. *Empirical evidence, numerical experimentation and probabilistic proof all can help us decide what to believe in mathematics.* Aristotelian logic isn't necessarily always the best way of deciding.
5. *Mathematical objects are a special variety of a social-cultural-historical object.* Contrary to the assertions of certain post-modern detractors, mathematics cannot be dismissed as merely a new form of literature or religion. Nevertheless, many mathematical objects can be seen as shared ideas, like Moby Dick in literature, or the Immaculate Conception in religion.

Certainly the recognition that "quasi-intuitive" analogies can be used to gain insight in mathematics can assist in the learning of mathematics. And honest mathematicians will acknowledge their role in discovery as well.

We look forward to what the future will bring.

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Knowledge and Community in Mathematics

Jonathan Borwein and
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The Opinion column offers mathematicians the opportunity to write about any issue of interest to the international mathematical community. Disagreement and controversy are welcome. The views and opinions expressed here, however, are exclusively those of the author, and neither the publisher nor the editor-in-chief endorses or accepts responsibility for them. An Opinion should be submitted to the editor-in-chief, Chandler Davis.

Mathematical Knowledge—As We Knew It

Each society has its regime of truth, its "general politics" of truth: that is, the types of discourse which it accepts and makes function as true; the mechanisms and instances which enable one to distinguish true and false statements, the means by which each is sanctioned; the techniques and procedures accorded value in the acquisition of truth; the status of those who are charged with saying what counts as truth.[1] *(Michel Foucault)*

Henri Lebesgue once remarked that "a mathematician, in so far as he is a mathematician, need not preoccupy himself with philosophy." He went on to add that this was "an opinion, moreover, which has been expressed by many philosophers."[2] The idea that mathematicians can *do* mathematics without a precise philosophical understanding of what they are doing is, by observation, mercifully true. However, while a neglect of philosophical issues does not impede mathematical discussion, discussion *about* mathematics quickly becomes embroiled in philosophy, and perforce encompasses the question of the nature of mathematical knowledge. Within this discussion, some attention has been paid to the resonance between the failure of twentieth-century efforts to enunciate a comprehensive, absolute foundation for mathematics and the postmodern deconstruction of meaning and its corresponding banishment of encompassing philosophical perspectives from the *centre fixe*.

Of note in this commentary is the contribution of Vladimir Tasić. In his book, *Mathematics and the Roots of Postmodern Thought*, he comments on the broad range of ideas about the interrelationship between language, meaning, and society that are commonly considered to fall under the umbrella of postmodernism. Stating that "attempts to make sense of this elusive concept threaten to outnumber attempts to square the circle," he focuses his attention on two relatively well-developed aspects of postmodern theory: "poststructuralism" and "deconstruction."[3] He argues that the development of these theories, in the works of Derrida and others, resonates with the debates surrounding foundationism which preoccupied the philosophy of mathematics in the early stages of the last century and may even have been partly informed by those debates. Our present purpose is not to revisit the connections between the foundationist debates and the advent of postmodern thought, but rather to describe and discuss some of the ways in which epistemological relativism and other postmodern perspectives are manifest in the changing ways in which mathematicians do mathematics and express mathematical knowledge. The analysis is not intended to be a lament; but it does contain an element of warning. It is central to our purpose that the erosion of universally fixed perspectives of acceptable practice in both mathematical activity and its publication be acknowledged as presenting significant challenges to the mathematical community.

Absolutism and Typographic Mathematics

I believe that mathematical reality lies outside us, that our function is to discover or observe it, and that the theorems which we prove, and which we describe grandiloquently as our "creations," are simply the notes of our observations.[4] (G. H. Hardy)

[1]Michel Foucault, "Truth and Power," *Power/Knowledge: Selected Interviews and Other Writings 1972–1977*, edited by Colin Gordon.

[2]Freeman Dyson, "Mathematics in the Physical Sciences," *Scientific American* 211, no. 9 (1964):130.

[3]Vladimir Tasić, *Mathematics and the Roots of Postmodern Thought* (Oxford: Oxford University Press, 2001), 5.

We follow the example of Paul Ernest and others and cast under the banner of *absolutism* descriptions of mathematical knowledge that exclude any element of uncertainty or subjectivity.[5] The quote from Hardy is frequently cited as capturing the essence of Mathematical Platonism, a philosophical perspective that accepts any reasonable methodology and places a minimum amount of responsibility on the shoulders of the mathematician. An undigested Platonism is commonly viewed to be the default perspective of the research mathematician, and, in locating mathematical reality outside human thought, ultimately holds the mathematician responsible only for discovery, observations, and explanations, not creations.

Absolutism also encompasses the logico-formalist schools as well as intuitionism and constructivism—in short, any perspective which strictly defines what constitutes mathematical knowledge or how mathematical knowledge is created or uncovered. Few would oppose the assertion that an absolutist perspective, predominately in the *de facto* Platonist sense, has been the dominant epistemology amongst working mathematicians since antiquity. Perhaps not as evident are the strong connections between epistemological perspective, community structure, and the technologies which support both mathematical activity and mathematical discourse. The media culture of typographic mathematics is defined by centres of publication and a system of community elites which determines what, and by extension *who*, is published. The abiding ethic calls upon mathematicians to respect academic credentialism and the systems of publication which further refine community hierarchies. Community protocols exalt the published, peer-reviewed article as the highest form of mathematical discourse.

The centralized nature of publication and distribution both sustains and is sustained by the community's hierarchies of knowledge management. Publishing houses, the peer review process, editorial boards, and the subscription-based distribution system require a measure of central control. The centralized protocols of typographic discourse resonate strongly with absolutist notions of mathematical knowledge. The emphasis on an encompassing mathematical truth supports and is supported by a hierarchical community structure possessed of well-defined methods of knowledge validation and publication. These norms support a system of community elites to which ascension is granted through a successful history with community publication media, most importantly the refereed article.

The interrelationships between community practice, structure, and epistemology are deep-rooted. Rigid epistemologies require centralized protocols of knowledge validation, and these protocols are only sustainable in media environments which embrace centralized modes of publication and distribution. As an aside, we emphasize that this is not meant as an indictment of publishers as bestowers of possibly unmerited authority—though the present disjunct between digitally "published" *eprints* which are read and typographically published *reprints* which are cited is quite striking. Rather, it is a description of a time-honoured and robust definition of *merit* in a typographical publishing environment. In the latter part of the twentieth century, a critique of absolutist notions of mathematical knowledge emerged in the form of the experimental mathematics methodology and the social constructivist perspective.

In the next section, we consider how evolving notions of mathematical knowledge and new media are combining to change not only the way mathematicians do and publish mathematics, but also the nature of the mathematical community.

Towards Mathematical Fallibilism

This new approach to mathematics—the utilization of advanced computing technology in mathematical research—is often called experimental mathematics. The computer provides the mathematician with a laboratory in which he or she can perform experiments: analyzing examples, testing out new ideas, or searching for patterns.[6] (David Bailey and Jonathan Borwein)

The experimental methodology embraces digital computation as a means of discovery and verification. Described in detail in two recently published volumes, *Mathematics by Experiment: Plausible Reasoning in the 21st Century* and *Experimentation in Mathematics: Computational Paths to Discovery*, the methodology as outlined by the authors (joined by Roland Girgensohn in the later work) accepts, as part of the experimental process, standards of certainty in mathematical knowledge which are more akin to the empirical sciences than they are to mathematics. As an experimental tool, the computer can provide strong, but typically not conclusive, evidence regarding the validity of an assertion. While with appropriate validity checking, confidence levels can in many cases be made arbitrarily high, it is notable that the concept of a "confidence level" has traditionally been a property of statistically oriented fields. It is important to note that the authors are not calling for a new standard of certainty in mathematical knowledge but rather the appropriate use of a methodology which may produce, as a product of its methods, definably uncertain transitional knowledge.

What the authors do advocate is closer attention to and acceptance of degrees of certainty in mathematical knowledge. This recommendation is made on the basis of argued assertions such as:

1. Almost certain mathematical knowledge is valid if treated appropriately;
2. In some cases "almost certain" is as good as it gets;
3. In some cases an almost certain computationally derived assertion is at least as strong as a complex formal assertion.

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[4]G. H. Hardy, *A Mathematician's Apology* (London: Cambridge University Press, 1967), 21.

[5]Paul Ernest, *Social Constructivism As a Philosophy of Mathematics* (Albany: State University of New York Press, 1998), 13.

[6]J. M. Borwein and D. H. Bailey, *Mathematics by Experiment: Plausible Reasoning in the 21st Century*, A. K. Peters Ltd, 2003. ISBN: 1-56881-211-6, 2-3.

The first assertion is addressed by the methodology itself, and in *Mathematics by Experiment*, the authors discuss in detail and by way of example the appropriate treatment of "almost certain" knowledge. The second assertion is a recognition of the limitations imposed by Gödel's Incompleteness Theorem, not to mention human frailty. The third is more challenging, for it addresses the idea that *certainty* is in part a function of the community's knowledge validation protocols. By way of example, the authors write,

> *... perhaps only 200 people alive can, given enough time, digest all of Andrew Wiles' extraordinarily sophisticated proof of Fermat's Last Theorem. If there is even a one percent chance that each has overlooked the same subtle error (and they may be psychologically predisposed so to do, given the numerous earlier results that Wiles' result relies on), then we must conclude that computational results are in many cases actually more secure than the proof of Fermat's Last Theorem.*[7]

Three mathematical examples

Our first and pithiest example answers a question set by Donald Knuth,[8] who asked for a closed form evaluation of the expression below.

Example 1: Evaluate

$$\sum_{k=1}^{\infty}\left\{\frac{k^k}{k!\,e^k}-\frac{1}{\sqrt{2\pi k}}\right\} = -0.084069508727655996461\ldots.$$

It is currently easy to compute 20 or 200 digits of this sum. Using the "smart lookup" facility in the *Inverse Symbolic Calculator*[9] rapidly returns

$$0.0840695087276559964 \approx \frac{2}{3}+\frac{\zeta(1/2)}{\sqrt{2\pi}}.$$

We thus have a prediction which Maple 9.5 on a laptop confirms to 100 places in under 6 seconds and to 500 in 40 seconds. Arguably we are done. □

The second example originates with a multiple integral which arises in Gaussian and spherical models of ferromagnetism and in the theory of random walks. This leads to an impressive closed form evaluation due to G. N. Watson:

Example 2:

$$\widehat{W}_3 = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{1}{3-\cos(x)-\cos(y)-\cos(z)}\,dx\,dy\,dz$$
$$= \frac{(\sqrt{3}-1}{96}\,\Gamma^2\left(\frac{1}{24}\right)\Gamma^2\left(\frac{11}{24}\right).$$

The most self-contained derivation of this very subtle Green's function result is recent and is due to Joyce and Zucker.[10] Computational confirmation to very high precision is, however, easy.

Further experimental analysis involved writing $\widehat{W}_3$ as a product of only Γ-values. This form of the answer is then susceptible to integer relation techniques. To high precision, an *Integer Relation* algorithm returns:

```
0= -1.* log[w3] + -1.* log[gamma[1/24]]
  + 4.*log[gamma[3/24]]
+ -8.*log[gamma[5/24]]
  + 1.* log[gamma[7/24]] + 14.*log[gamma[9/24]]
+ -6.*log[gamma[11/24]] + -9.*log[gamma[13/24]]
+ 18.*log[gamma[15/24]]
  + -2.*log[gamma[17/24]]-7.*log[gamma[19/24]]
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Proving this discovery is achieved by comparing the outcome with Watson's result and establishing the implicit Γ-representation of $(\sqrt{3}-1)^2/96$.

Similar searches suggest there is no similar four-dimensional closed form for $\widehat{W}_4$. Fortunately, a one-variable integral representation is at hand in $\widehat{W}_4 := \int_0^{\infty}\exp(-4t)I_0^4(t)dt$, where I_0 is the Bessel integral of the first kind. The high cost of four-dimensional numeric integration is thus avoided. A numerical search for identities then involves the careful computation of $\exp(-t)\,I_0(t)$, using

$$\exp(-t)\,I_0(t) = \exp(-t)\sum_{n=0}^{\infty}\frac{t^{2n}}{2^{2n}(n!)^2}$$

for t up to roughly $1.2\cdot d$, where d is the number of significant digits needed, and

$$\exp(-t)\,I_0(t) = \frac{1}{\sqrt{2\pi t}}\sum_{n=0}^{N}\frac{\Pi_{k=1}^{n}(2k-1)^2}{(8t)^n n!}$$

for larger t, where the limit N of the second summation is chosen to be the first index n such that the summand is less than 10^{-d}. (This is an asymptotic expansion, so taking more terms than N may increase, not decrease the error.)

Bailey and Borwein found that $\widehat{W}_4$ is not expressible as a product of powers of $\Gamma(k/120)$ (for $0 < k < 120$) with coefficients of less than 12 digits. This result does not, of course, rule out the possibility of a larger relation, but it does cast experimental doubt that such a relation exists—more than enough to stop one from looking. □

The third example emphasizes the growing role of visual discovery.

Example 3: Recent *continued fraction* work by Borwein and Crandall illustrates the methodology's embracing of computer-aided visualization as a means of discovery. They

[7]Borwein and Bailey, p. 10.

[8]Posed as *MAA Problem* 10832, November 2002. Solution details are given on pages 15–17 of Borwein, Bailey, and Girgensohn.

[9]At www.cecm.sfu.ca/projects/ISC/ISCmain.html

[10]See pages 117–121 of J. M. Borwein, D. H. Bailey, and R. Girgensohn, *Experimentation in Mathematics: Computational Paths to Discovery*, A.K. Peters Ltd, 2003. ISBN: 1-56881-136-5.

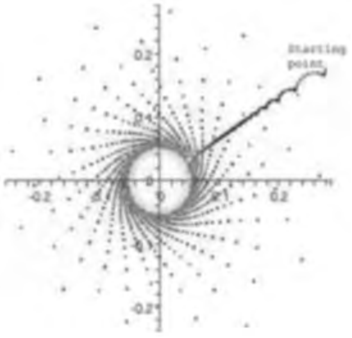

Fig. 1. The starting point depends on the choice of unit vectors, *a* and *b*.

investigated the *dynamical system* defined by: $t_0 := t_1 := 1$ and

$$t_n \hookleftarrow \frac{1}{n}\, t_{n-1} + \omega_{n-1}\left(1 - \frac{1}{n}\right) t_{n-2},$$

where $\omega_n = a^2, b^2$ are distinct unit vectors, for n even, odd, respectively—that occur in the original continued fraction. Treated as a black box, all that can be verified numerically is that $t_n \to 0$ slowly. Pictorially one *learns* more, as illustrated by Figure 1.

Figure 2 illustrates the fine structure that appears when the system is scaled by $\sqrt{n}$ and odd and even iterates are coloured distinctly.

With a lot of work, everything in these pictures is now explained. Indeed from these four cases one is compelled to conjecture that the attractor is finite of cardinality N exactly when the input, a or b, is an Nth root of unity; otherwise it is a circle. Which conjecture one then repeatedly may test. □

The idea that what is accepted as mathematical knowledge is, *to some degree*, dependent upon a community's methods of knowledge acceptance is an idea that is central to the *social constructivist* school of mathematical philosophy.

The social constructivist thesis is that mathematics is a social construction, a cultural product, fallible like any other branch of knowledge.[11] (Paul Ernest)

Associated most notably with the writing of Paul Ernest, an English mathematician and Professor in the Philosophy of Mathematics Education, social constructivism seeks to define mathematical knowledge and epistemology through the social structure and interactions of the mathematical community and society as a whole. In *Social Constructivism As a Philosophy of Mathematics*, Ernest carefully traces the intellectual pedigree for his thesis, a pedigree that encompasses the writings of Wittgenstein, Lakatos, Davis, and Hersh among others.[12]

For our purpose, it is useful to note that the philosophical aspects of the experimental methodology combined with the social constructivist perspective provide a pragmatic alternative to Platonism—an alternative which furthermore avoids the Platonist pitfalls. The apparent paradox in suggesting that the dominant community view of mathematics—Platonism—is at odds with a social constructivist accounting is at least partially countered by the observation that we and our critics have inhabited quite distinct communities. The impact of one on the other was well described by Dewey a century ago:

Old ideas give way slowly; for they are more than abstract logical forms and categories. They are habits, predispositions, deeply engrained attitudes of aversion and preference. . . . Old questions are solved by disappearing, evaporating, while new questions corresponding to the changed attitude of endeavor and preference take their place. Doubtless the greatest dissolvent in contemporary thought of old questions, the greatest precipitant of new methods, new intentions, new problems, is the one effected by the scientific revolution that found its climax in the "Origin of Species."[13] (John Dewey)

New mathematics, new media, and new community protocols

With a proclivity towards centralized modes of knowledge validation, absolutist epistemologies are supported by well-defined community structures and publication protocols. In contrast, both the experimental methodology and social constructivist perspective resonate with a more fluid community structure in which communities, along with their implicit and explicit hierarchies, form and dissolve in response to the establishment of common purposes. The experimental methodology, with its embracing of computational methods, de-emphasizes individual accomplishment by encouraging collaboration not only between mathematicians but between mathematicians and researchers from various branches of computer science.

Conceiving of mathematical knowledge as a function of the social structure and interactions of mathematical communities, the social constructivist perspective is inherently accepting of a realignment of community authority away from easily identified elites and in the direction of those who can most effectively harness the potential for collaboration and publication afforded by new media. The capacity for mass publication no longer resides exclusively in the hands of publishing houses; any workstation equipped with a LaTeX compiler and the appropriate interpreters is all that is needed. The changes that are occurring in the ways we do mathematics, the ways we publish mathematical research, and the nature of the mathematical community leave little opportunity for resistance

[11] Ernest, p. 39ff.
[12] Ernest, p. 39ff.
[13] Quoted from *The Influence of Darwin on Philosophy*, 1910.

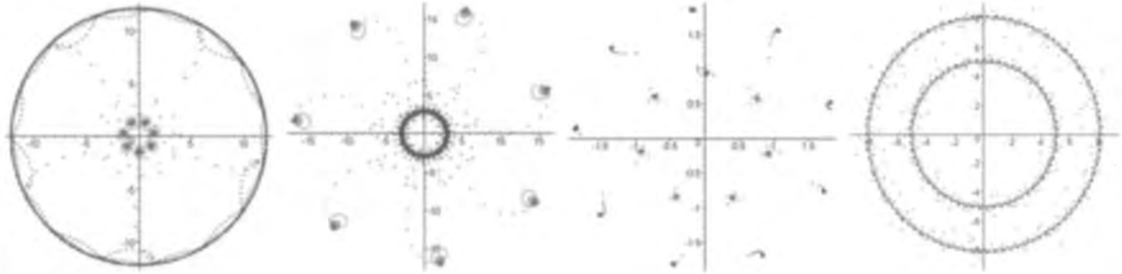

Fig. 2. The attractors for various $|a| = |b| = 1$.

or nostalgia. From a purely pragmatic perspective, the community has little choice but to accept a broader definition of valid mathematical knowledge and valid mathematical publication. In fact, in the transition between publishing protocols based upon mechanical typesetting to protocols supported by digital media, we are already witnessing the beginnings of a realignment of elites and hierarchies and a corresponding re-evaluation of the mathematical skill-set. Before considering more carefully the changes that are occurring in mathematics, we turn our attention to some perhaps immutable aspects of mathematical knowledge.

Some Societal Aspects of Mathematical Knowledge

The question of the ultimate foundations and the ultimate meaning of mathematics remains open: we do not know in what direction it will find its final solution or even whether a final objective answer can be expected at all. "Mathematizing" may well be a creative activity of man, like language or music, of primary originality, whose historical decisions defy complete objective rationalisation.[14] (Hermann Weyl)

Membership in a community implies mutual identification with other members which is manifest in an assumption of some level of shared language, knowledge, attitudes, and practices. Deeply woven into the sensibilities of mathematical research communities, and to varying degrees the sensibilities of society as a whole, are some assumptions about the role of mathematical knowledge in a society and what constitutes essential mathematical knowledge. These assumptions are part of the mythology of mathematical communities and the larger society, and it is reasonable to assume that they will not be readily surrendered in the face of evolving ideas about the epistemology of mathematics or changes in the methods of practicing and publishing mathematics.

Mathematics as fundamental knowledge

Mathematics is the tool specially suited for dealing with abstract concepts of any kind and there is no limit to its power in this field.[15] (Paul Dirac)

In the epistemological universe, mathematics is conceived as a large mass about which orbit many other bodies of knowledge and whose gravity exerts influence throughout. The medieval recognition of the centrality of mathematics was reflected in the quadrivium, which ascribed to the sciences of number—arithmetic, geometry, astronomy, and music—four out of the seven designated liberal arts. Today, mathematics is viewed by many as an impenetrable, but essential, subject that is at the foundation of much of the knowledge that informs our understanding of the scientific universe and human affairs. We are somehow reassured by the idea of a Federal Reserve Chairman who purportedly solves differential equations in his spare time.

The high value that society places on an understanding of basic mathematics is reflected in UNESCO's specification of numeracy, along with literacy and essential life skills, as a fundamental educational objective. This place of privilege bestows upon the mathematical research community some unique responsibilities. Among them, the articulation of mathematical ideas to research, business, and public policy communities whose prime objective is not the furthering of mathematical knowledge. As well, as concerns are raised in many jurisdictions about poor performance in mathematics at the grade-school level, research communities are asked to participate in the general discussion about mathematical education.

The mathematical canon

I will be glad if I have succeeded in impressing the idea that it is not only pleasant to read at times the works of the old mathematical authors, but this may occasionally be of use for the actual advancement of science.[16] (Constantin Carathéodory)

The mathematical community is the custodian of an extensive collection of core knowledge to a larger degree than any other basic discipline with the arguable exception of the combined fields of rhetoric and literature. Preserved largely by the high degree of harmonization of grade-school and undergraduate university curricula, this mathematical canon is at once a touchstone of shared experience of com-

[14]Cited in: *Obituary: David Hilbert 1862–1943*, RSBIOS, 4, 1944, pp. 547–553.
[15]Dirac writing in the preface to *The Principles of Quantum Mechanics* (Oxford, 1930).
[16]Speaking to an MAA meeting in 1936.

munity members and an imposing barrier to anyone who might seek to participate in the discourse of the community without having some understanding of the various relationships between the topics of core knowledge. While the exact definition of the canon is far from precise, to varying degrees of mastery it certainly includes Euclidean geometry, differential equations, elementary algebra, number theory, combinatorics, and probability. It is worth noting parenthetically that while mathematical notation can act as a barrier to mathematical discourse, its universality helps promote the universality of the canon.

At the level of individual works and specific problems, mathematicians display a high degree of respect for historical antecedent. Mathematics has advanced largely through the careful aggregation of a mathematical literature whose reliability has been established by a slow but thorough process of formal and informal scrutiny. Unlike the other sciences, mathematical works and problems need not be recent to be pertinent. Tom Hales's recent computer-assisted solution of Kepler's problem makes this point and many others. Kepler's conjecture—that the densest way to stack spheres is in a pyramid—is perhaps the oldest problem in discrete geometry. It is also the most interesting recent example of computer-assisted proof. The publication of Hales's result in the *Annals of Mathematics*, with an "only 99% checked" disclaimer, has triggered varied reactions.[17]

The mathematical aesthetic

The mathematician's patterns, like the painter's or the poet's, must be beautiful; the ideas, like the colors or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in this world for ugly mathematics.[18] (G. H. Hardy)

Another distinguishing preoccupation of the mathematical community is the notion of a mathematical aesthetic. It is commonly held that good mathematics reflects this aesthetic and that a developed sense of the mathematical aesthetic is an attribute of a good mathematician. The following exemplifies the *"infinity in the palm of your hand"* encapsulation of complexity which is one aspect of the aesthetic sense in mathematics.

$$1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \ \ldots\ = \int_0^1 \frac{1}{x^x}\,dx$$

Discovered in 1697 by Johannes Bernoulli, this formula has been dubbed the *Sophomore's Dream* in recognition of the surprising similarities it reveals between a series and its integral equivalent. Its proof is not too simple and not too hard, and the formula offers the mix of surprise and simplicity that seems central to the mathematical aesthetic. By contrast several of the recent very long proofs are neither simple nor beautiful.

To see a World in a Grain of Sand; and a Heaven in a Wild Flower; Hold Infinity in the palm of your hand; And Eternity in an hour. (William Blake)

Freedom and Discipline

In this section, we make some observations about the tension between conformity and diversity which is present in the protocols of both typographically and digitally oriented communities.

The only avenue towards wisdom is by freedom in the presence of knowledge. But the only avenue towards knowledge is by discipline in the acquirement of ordered fact.[19] (Alfred North Whitehead)

Included in the introduction to his essay *The Rhythmic Claims of Freedom and Discipline*, Whitehead's comments about the importance of the give and take between freedom and discipline in education can be extended to more general domains. In the discourse of mathematical research, tendencies towards freedom and discipline, decentralization and centralization, the organic and the ordered, coexist in both typographic and digital environments. While it may be true that typographic norms are characterized by centralized nodes of publication and authority and the community order that they impose, an examination of the mathematical landscape in the mid-twentieth century reveals strong tendencies towards decentralization occurring independently of the influence of digital media. Mutually reinforcing trends, including an increase in the number of PhD's, an increase in the number of journals and published articles, and the application of advanced mathematical methods to fields outside the domain of the traditional mathematical sciences combined to challenge the tendency to maintain centralized community structures. The result was, and continues to be, a replication of a centralized community structure in increasingly specialized domains of interest. In mathematics more than in any other field of research, the knowledge explosion has led to increased specialization, with new fields giving birth to new journals and the organizational structures which support them.

While the structures and protocols which describe the digital mathematical community are still taking shape, it would be inaccurate to suggest that the tendency of digital media to promote freedom and decentralized norms of knowledge-sharing is unmatched by tendencies to impose control and order. If the natively centralized norms of typographic mathematics manifest decentralization as *knowledge fragmentation*, we are presently observing tendencies emerging from digital mathematics communities to find order and control in the *knowledge atomization* that results from the codification of mathematical knowledge at the level of micro-ontologies. The World Wide Web Con-

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[17]See "In Math, Computers Don't Lie. Or Do They?", *The New York Times*, April 6, 2004.

[18]G. H. Hardy, *A Mathematician's Apology* (London: Cambridge University Press, 1967), 21.

[19]Alfred North Whitehead, *The Aims of Education* (New York: The Free Press, 1957), 30.

sortium (W3C) *MathML* initiative and the European Union's *OpenMath* project are complementary efforts to construct a comprehensive, fine-grained codification of mathematical knowledge that binds semantics to notation and the context in which the notation is used.[20] The tongue-in-cheek indictment of typographic subject specialization as producing experts who learn more and more about less and less until achieving complete knowledge of nothing-at-all becomes, under the digital norms, the increasingly detailed description of increasingly restricted concepts until one arrives at a complete description of nothing-at-all. Ontologies become micro-ontologies and risk becoming *"nontologies."* If typographic modes of knowledge validation and publication are collapsing under the weight of subject specialization, the digital ideal of a comprehensive meta-mathematical descriptive and semantic framework which embraces all mathematics may also prove to be overreaching.

Some Implications

Communication of mathematical research and scholarship is undergoing profound change as new technology creates new ways to disseminate and access the literature. More than technology is changing, however, the culture and practices of those who create, disseminate, and archive the mathematical literature are changing as well. For the sake of present and future mathematicians, we should shape those changes to make them suit the needs of the discipline.[21] (International Math Union Committee on Electronic Information and Communication)

. . . to suggest that the normal processes of scholarship work well on the whole and in the long run is in no way contradictory to the view that the processes of selection and sifting which are essential to the scholarly process are filled with error and sometimes prejudice.[22] (Kenneth Arrow)

Our present idea of a mathematical research community is built on the foundation of the protocols and hierarchies which define the practices of typographic mathematics. At this point, how the combined effects of digital media will affect the nature of the community remains an open question; however, some trends are emerging:

1. **Changing modes of collaboration:** With the facilitation of collaboration afforded by digital networks, individual authorship is increasingly ceding place to joint authorship. It is possible that forms of community authorship, such as are common in the Open Source programming community, may find a place in mathematical research. Michael Kohlhase and Romeo Anghelache have proposed a version-based content management system for mathematical communities which would permit multiple users to make joint contributions to a common research effort.[23] The system facilitates collaboration by attaching version control to electronic document management. Such systems, should they be adopted, challenge not only the notion of authorship but also the idea of what constitutes a valid form of publication.
2. **The ascendancy of gray literature:** Under typographic norms, mathematical research has traditionally been conducted with reference to journals and through informal consultation with colleagues. Digital media, with its non-discriminating capacity for facilitating instantaneous publication, has placed a wide range of sources at the disposal of the research mathematician. Ranging from Computer Algebra System routines to Home Pages and conference programmes, these sources all provide information that may support mathematical research. In particular, it is possible that a published paper may not be the most appropriate form of publication to emerge from a multi-user content management such as proposed by Kohlhase and Anghelache. It may be that the contributors deem it more appropriate to let the result of their efforts stand with its organic development exposed through a history of its versions.
3. **Changing modes of knowledge authentication:** The refereeing process, already under overload-induced stress, depends upon a highly controlled publication process. In the distributed publication environment afforded by digital media, new methods of knowledge authentication will necessarily emerge. By necessity, the idea of authentication based on the ethics of referees will be replaced by authentication based on various types of valuation parameters. Services that track citations are currently being used for this purpose by the Web document servers *CiteSeer* and *citebase*, among others.[24] Certainly the ability to compute informedly with formulae in a preprint can dramatically reduce the reader's or referee's concern about whether the result is reliable. More than we typically admit or teach our students, mathematicians work without proof if they feel secure in the correctness of their thought processes.
4. **Shifts in epistemology:** The increasing acceptance of the experimental methodology and social constructivist

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[20]For background on these projects, see: www.w3.org/Math/ and www.openmath.org, respectively.

[21]The IMU's Committee on Electronic Information and Communication (CEIC) reports to the IMU on matters concerning the digital publication of mathematics. See www.ceic.math.ca/Publications/Recommendations/3_best_practices.shtml

[22]E. Roy Weintraub and Ted Gayer, "Equilibrium Proofmaking," *Journal of the History of Economic Thought*, 23 (Dec. 2001), 421–442. This provides a remarkably detailed analysis of the genesis and publication of the Arrow-Debreu theorem.

[23]Michael Kohlhase and Romeo Anghelache, "Towards Collaborative Content Management and Version Control for Structured Mathematical Knowledge," *Lecture Notes in Computer Science no. 2594: Mathematical Knowledge Management: Proceedings of The Second International Conference*, Andrea Asperti, Bruno Buchberger, and James C. Davenport editors, (Berlin: Springer-Verlag, 2003) 45.

[24]citeseer.ist.psu.edu and citebase.eprints.org, respectively.

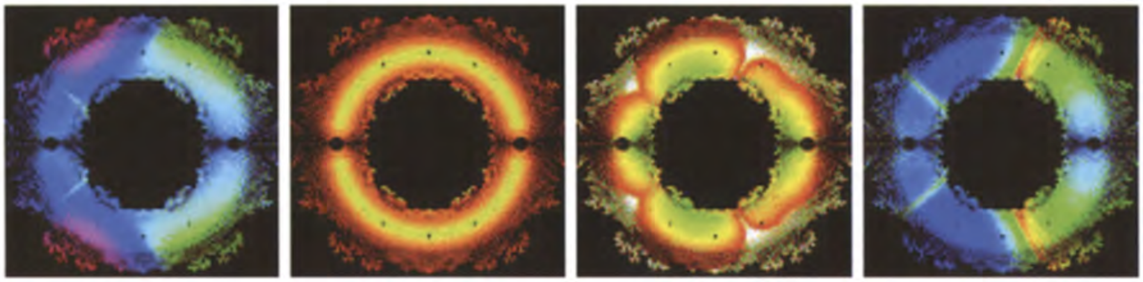

Fig. 3. What you draw *is* what you see. ***Roots of polynomials with coefficients* 1 *or* −1 *up to degree* 18. *The coloration is determined by a normalized sensitivity of the coefficients of the polynomials to slight variations around the values of the zeros, with red indicating low sensitivity and violet indicating high sensitivity. The bands visible in the last picture are unexplained, but believed to be real—not an artifact.***

perspective is leading to a broader definition of valid knowledge and valid forms of knowledge representation. The rapidly expanding capacity of computers to facilitate visualization and perform symbolic computations is placing increased emphasis on visual arguments and interactive interfaces, thereby making practicable the call by Philip Davis and others a quarter-century ago to admit visual proofs more fully into our canon.

The price of metaphor is eternal vigilance (Arturo Rosenblueth & Norbert Wiener)

For example, experimentation with various ways of representing stability of computation led to the four images in Figure 3. They rely on perturbing some quantity and recomputing the image, then coloring to reflect the change. Some features are ubiquitous while some, like the bands, only show up in certain settings. Nonetheless, they are thought not to be an artifact of roundoff or other error but to be a real yet unexplained phenomenon.

5. **Re-evaluation of valued skills and knowledge:** Complementing a reassessment of assumptions about mathematical knowledge, there will be a corresponding reassessment of core mathematical knowledge and methods. Mathematical creativity may evolve to depend less upon the type of virtuosity which characterized twentieth-century mathematicians and more upon an ability to use a variety of approaches and draw together and synthesize materials from a range of sources. This is as much a transfer of attitudes as a transfer of skill sets; the experimental method presupposes an experimental mind-set.
6. **Increased community dynamism:** Relative to computer- and network-mediated research, the static social entities which intermesh with the typographic research environment extend the timeline for research and publication and support stability in inter-personal relationships. Collaborations, when they arise, are often career-long, if not life-long, in their duration. The highly productive friendship between G. H. Hardy and J. E. Littlewood provides a perhaps extreme example. While long-term collaborations are not excluded, the form of collaboration supported by digital media tends to admit a much more fluid community dynamic. Collaborations and coalitions will form as needed and dissolve just as quickly. The four authors of *The SIAM 100-digit Challenge: A Study In High-accuracy Numerical Computing*[25] never met while solving Nick Trefethen's 2002 ten challenge problems which form the basis for their lovely book.

At the extreme end of the scale, distributed computing can facilitate virtually anonymous collaboration. In 2000, Colin Percival used the *Bailey-Borwein-Plouffe* algorithm and connected 1,734 machines from 56 countries to determine the quadrillionth bits of π. Accessing an equivalent of more than 250 cpu years, this calculation (along with *Toy Story Two* and other recent movies) ranks as one of the largest computations ever. The computation was based on the computer-discovered identity

$$\pi = \sum_{k=0}^{\infty} \left(\frac{1}{16}\right)^k \left\{\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right\},$$

which allows binary digits to be computed independently.[26]

A Temporary Epilogue

The plural of "anecdote" is not "evidence."[27] (Alan L. Leshner)

These trends are presently combining to shape a new community ethic. Under the dictates of typographic norms, ethical behaviour in mathematical research involves adhering to well-established protocols of research and publi-

[25]Folkmar Bornemann, Dirk Laurie, Stan Wagon, Jörg Waldvogel, SIAM 2004.

[26]See Borwein and Bailey, Chapter 3.

[27]The publisher of *Science* speaking at the Canadian Federal Science and Technology Forum, Oct 2, 2002.

cation. While the balance of personal freedom against community order which defines the ethic of digitally oriented mathematical research communities may never be as firm or as enforceable by community protocols, some principles are emerging. The CEIC's statement of *best current practices for mathematicians* provides a snapshot of the developing consensus on this question. Stating that *"those who write, disseminate, and store mathematical literature should act in ways that serve the interests of mathematics, first and foremost,"* the recommendations advocate that mathematicians take full advantage of digital media by publishing structured documents which are appropriately linked and marked-up with meta-data.[28] Researchers are also advised to maintain personal homepages with links to their articles and to submit their work to preprint and archive servers.

Acknowledging the complexity of the issue, the final CEIC recommendation concerns the question of copyright: it makes no attempt to recommend a set course of action, but rather simply advises mathematicians to be aware of copyright law and custom and consider carefully the options. Extending back to Britain's first copyright law, *The Statute of Anne*, enacted in 1710, the idea of copyright is historically bound to typographic publication and the protocols of typographic society. Digital copyright law is an emerging field; it is presently unclear how copyright, and the economic models of knowledge distribution that depend upon it, will adapt to the emerging digital publishing environment. The relatively liberal epistemology offered by the *experimental method* and the *social constructivist perspective* and the potential for distributed research and publication afforded by digital media will reshape the protocols and hierarchies of mathematical research communities. Along with long-held beliefs about what constitutes mathematical knowledge and how it is validated and published, at stake are our personal assumptions about the nature of mathematical communities and mathematical knowledge.[29]

While the norms of typographic mathematics are not without faults and weaknesses, we are familiar with them to the point that they instill in us a form of faith; a faith that if we play along, on balance we will be granted fair access to opportunity. As the centralized protocols of typographic mathematics give way to the weakly defined protocols of digital mathematics, it may seem that we are ceding a system that provided a way to agree upon mathematical truth for an environment undermined by relativism that will mix verifiably true statements with statements that guarantee only the probability of truth and an environment which furthermore is bereft of reliable systems for assessing the validity of publications. The simultaneous weakening of community authority structures as typographic elites are rendered increasingly irrelevant by digital publishing protocols may make it seem as though the social imperatives that bind the mathematical community have been weakened. Any sense of loss is the mathematician's version of postmodern malaise; we hope and predict that, as the community incorporates these changes, the malaise will be short-lived. That incorporation is taking place, there can be no doubt. In higher education, we now assume that our students can access and share information via the Web, and we require that they learn how to use reliably vast mathematical software packages whose internal algorithms are not necessarily accessible to them even in principle.

One reason that, in the mathematical case, the "unbearable lightness" may prove to be bearable after all is that while fundamental assumptions about mathematical knowledge may be reinterpreted, they will survive. In particular, the idea of mathematical knowledge as being central to the advancement of science and human affairs, the idea of a mathematical canon and its components, and the idea of a mathematical aesthetic will each find expression in the context of the emerging epistemology and protocols of research and publication. In closing, we note that to the extent that there may be an opportunity to shape the epistemology, protocols, and fundamental assumptions that guide the mathematical research communities of the future, that opportunity is most effectively seized upon during these initial stages of digital mathematical research and publishing.

Whether we scientists are inspired, bored, or infuriated by philosophy, all our theorizing and experimentation depends on particular philosophical background assumptions. This hidden influence is an acute embarrassment to many researchers, and it is therefore not often acknowledged. Such fundamental notions as reality, space, time, and causality—notions found at the core of the scientific enterprise—all rely on particular metaphysical assumptions about the world.[30] (Christof Koch)

The assumptions that we have sought to address in this article are those that define how mathematical reality is investigated, created, and shared by mathematicians working within the social context of the mathematical community and its many sub-communities. We have maintained that those assumptions are strongly guided by technology and epistemology, and furthermore that technological and epistemological change are revealing the assumptions to be more fragile than, until recently, we might have reasonably assumed.

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[28]CEIC Recommendations. See: http://www.ceic.math.ca

[29]As one of our referees has noted, "The law is clearly 25 years behind info-technology." He continues, "What is at stake here is not only intellectual property but the whole system of priorities, fees, royalties, accolades, recognition of accomplishments, jobs."

[30]In "Thinking About the Conscious Mind," a review of John R. Searle's *Mind. A Brief Introduction*, Oxford University Press, 2004.

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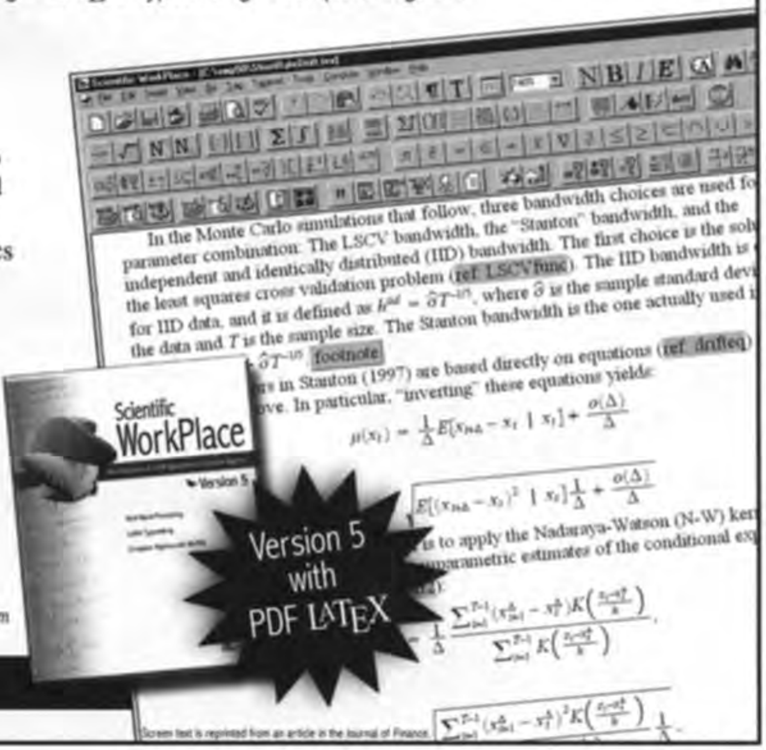

these cases. We must weigh the apparent security purchased by requiring predicative definitions against the burden of having to abandon in many cases what we, as mathematicians, consider natural definitions.

2. It is unclear exactly what objects we are committed to when we are committed to Peano Arithmetic. There are plenty of problems in number theory whose proofs use analytic means, for instance. Does commitment to Peano Arithmetic entail commitment to whatever objects are needed for these proofs? More generally, does commitment to a mathematical theory mean commitment to any objects needed for solving problems of that theory? If so, then Gödel's incompleteness theorems suggest that it is open what objects commitment to Peano Arithmetic entails.
3. As Feferman admits, it is unclear how to account predicatively for some mathematics used in currently accepted scientific practice, for instance, in quantum mechanics. In addition, I think that Feferman would not want to make the stronger claim that *all future* scientifically applicable mathematics will be accountable for by predicative means. However, the claim that *currently* scientifically applicable mathematics can be accounted for predicatively seems too time-bound to play an important role in a foundation of mathematics. Though it is impossible to predict all future scientific advances, it is reasonable to aim at a foundation of mathematics that has the potential to support these advances. Whether or not predicativity is such a foundation should be studied critically.
4. Whether the use of impredicative sets, and the uncountable more generally, is needed for ordinary finite mathematics, depends on whether by "ordinary" we mean "current." If so, then this is subject to the same worry I raised for (3). It also depends on where we draw the line on what counts as finite mathematics. If, for instance, Goldbach's conjecture counts as finite mathematics, then we have a statement of finite mathematics for which it is completely open whether it can be proved predicatively or not.

In emphasizing the degree to which concerns about predicativism shape this book, I should not overemphasize it. There is much besides predicativism in this book, as I have tried to indicate. In fact, Feferman advises that we not read his predicativism too strongly. In the preface, he describes his interest in predicativity as concerned with seeing how far in mathematics we can get without resorting to the higher infinite, whose justification he thinks can only be platonic. It may turn out that uncountable sets are needed for doing valuable mathematics, such as solving currently unsolved problems. In that case, Feferman writes, we "should look to see where it is necessary to use them and what we can say about what it is we know when we do use them" (p. ix).

Nevertheless, Feferman's committed anti-platonism is a crucial influence on the book. For mathematics right now, Feferman thinks, "a little bit goes a long way," as one of the essay titles puts it. The full universe of sets admitted by the platonist is unnecessary, he thinks, for doing the mathematics for which we must currently account. Time will tell if future developments will support that view, or whether, like Brouwer's view, it will require the alteration or outright rejection of too much mathematics to be viable. Feferman's book shows that, far from being over, work on the foundations of mathematics is vibrant and continuing, perched deliciously but precariously between mathematics and philosophy.

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The SIAM 100-Digit Challenge: A Study in High-Accuracy Numerical Computing

by Folkmar Bornemann, Dirk Laurie, Stan Wagon, and Jörg Waldvogel

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REVIEWED BY JONATHAN M. BORWEIN

Lists, challenges, and competitions have a long and primarily lustrous history in mathematics. This is the story of a recent highly successful challenge. The book under review makes it clear that with the continued advance of computing power and accessibility, the view that "real mathematicians don't compute" has little traction, especially for a newer generation of mathematicians who may readily take advantage of the maturation of computational packages such as *Maple*, *Mathematica*, and MATLAB.

Numerical Analysis Then and Now

George Phillips has accurately called Archimedes the first numerical analyst [2, pp. 165–169]. In the process of obtaining his famous estimate $3 + 10/71 < \pi < 3 + 1/7$, he had to master notions of recursion without computers, interval analysis without zero or positional arithmetic, and trigonometry without any of our modern analytic scaffolding. . . . Two millennia later, the same estimate can be obtained by a computer algebra system [3].

Example 1. A modern computer algebra system can tell one that

$$0 < \int_0^1 \frac{(1-x)^4 x^4}{1+x^2}\, dx = \frac{22}{7} - \pi, \qquad (1.1)$$

since the integral may be interpreted as the area under a positive curve.

This leaves us no wiser as to why! If, however, we ask the same system to compute the indefinite integral, we are likely to be told that

$$\int_0^t \cdot = \frac{1}{7}t^7 - \frac{2}{3}t^6 + t^5 - \frac{4}{3}t^3 + 4t - 4\arctan(t).$$

Then (1.1) is now rigorously established by differentiation and an appeal to Newton's Fundamental theorem of calculus. □

While there were many fine arithmeticians over the next 1500 years, this anecdote from Georges Ifrah reminds us that mathematical culture in Europe had not sustained Archimedes's level up to the Renaissance.

> *A wealthy (15th-century) German merchant, seeking to provide his son with a good business education, consulted a learned man as to which European institution offered the best training. "If you only want him to be able to cope with addition and subtraction," the expert replied, "then any French or German university will do. But if you are intent on your son going on to multiplication and division—assuming that he has sufficient gifts—then you will have to send him to Italy.*[1]

By the 19th century, Archimedes had finally been outstripped both as a theorist and as an (applied) numerical analyst, see [7].

> *In 1831, Fourier's posthumous work on equations showed 33 figures of solution, got with enormous labour. Thinking this a good opportunity to illustrate the superiority of the method of W. G. Horner, not yet known in France, and not much known in England, I proposed to one of my classes, in 1841, to beat Fourier on this point, as a Christmas exercise. I received several answers, agreeing with each other, to 50 places of decimals. In 1848, I repeated the proposal, requesting that 50 places might be exceeded: I obtained answers of 75, 65, 63, 58, 57, and 52 places.* (Augustus De Morgan[2])

De Morgan seems to have been one of the first to mistrust William Shanks's epic computations of Pi—to 527, 607, and 727 places [2, pp. 147–161], noting there were too few sevens. But the error was only confirmed three quarters of a century later in 1944 by Ferguson with the help of a calculator in the last pre-computer calculations of π—though until around 1950 a "computer" was still a person and ENIAC was an "Electronic Numerical Integrator and Calculator" [2, pp. 277–281] on which Metropolis and Reitwiesner computed Pi to 2037 places in 1948 and confirmed that there were the expected number of sevens.

Reitwiesner, then working at the Ballistics Research Laboratory, Aberdeen Proving Ground in Maryland, starts his article [2, pp. 277–281] with

> *Early in June, 1949, Professor* JOHN VON NEUMANN *expressed an interest in the possibility that the ENIAC might sometime be employed to determine the value of π and e to many decimal places with a view toward obtaining a statistical measure of the randomness of distribution of the digits.*

The paper notes that *e* appears to be *too* random—this is now proven—and ends by respecting an oft-neglected "best-practice":

> *Values of the auxiliary numbers* arccot 5 *and* arccot 239 *to 2035D . . . have been deposited in the library of Brown University and the UMT file of MTAC.*

The 20th century's "Top Ten"

The digital computer, of course, greatly stimulated both the appreciation of and the need for algorithms and for algorithmic analysis. At the beginning of this century, Sullivan and Dongarra could write, "Great algorithms are the poetry of computation," when they compiled a list of the 10 algorithms having "the greatest influence on the development and practice of science and engineering in the 20th century".[3] Chronologically ordered, they are:

- #1. 1946: **The Metropolis Algorithm for Monte Carlo**. Through the use of random processes, this algorithm offers an efficient way to stumble toward answers to problems that are too complicated to solve exactly.
- #2. 1947: **Simplex Method for Linear Programming.** An elegant solution to a common problem in planning and decision making.
- #3. 1950: **Krylov Subspace Iteration Method.** A technique for rapidly solving the linear equations that abound in scientific computation.
- #4. 1951: **The Decompositional Approach to Matrix Computations.** A suite of techniques for numerical linear algebra.
- #5. 1957: **The Fortran Optimizing Compiler.** Turns high-level code into efficient computer-readable code.
- #6. 1959: **QR Algorithm for Computing Eigenvalues.** Another crucial matrix operation made swift and practical.

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[1]From page 577 of *The Universal History of Numbers: From Prehistory to the Invention of the Computer*, translated from French, John Wiley, 2000.

[2]Quoted by Adrian Rice in "What Makes a Great Mathematics Teacher?" on page 542 of The *American Mathematical Monthly*, June–July 1999.

[3]From "Random Samples," *Science* page 799, February 4, 2000. The full article appeared in the January/February 2000 issue of *Computing in Science & Engineering.*

#7. 1962: **Quicksort Algorithms for Sorting.** For the efficient handling of large databases.

#8. 1965: **Fast Fourier Transform.** Perhaps the most ubiquitous algorithm in use today, it breaks down waveforms (like sound) into periodic components.

#9. 1977: **Integer Relation Detection.** A fast method for spotting simple equations satisfied by collections of seemingly unrelated numbers.

#10. 1987: **Fast Multipole Method.** A breakthrough in dealing with the complexity of n-body calculations, applied in problems ranging from celestial mechanics to protein folding.

I observe that eight of these ten winners appeared in the first two decades of serious computing, and that Newton's method was apparently ruled ineligible for consideration.[4] Most of the ten are multiply embedded in every major mathematical computing package.

Just as layers of software, hardware, and middleware have stabilized, so have their roles in scientific, and especially mathematical, computing. When I first taught the simplex method thirty years ago, the texts concentrated on "Y2K"-like tricks for limiting storage demands. Now serious users and researchers will often happily run large-scale problems in MATLAB and other broad-spectrum packages, or rely on *NAG* library routines embedded in *Maple*.

While such out-sourcing or commoditization of scientific computation and numerical analysis is not without its drawbacks, I think the analogy with automobile driving in 1905 and 2005 is apt. We are now in possession of mature—not to be confused with "error-free"—technologies. We can be fairly comfortable that *Mathematica* is sensibly handling round-off or cancelation error, using reasonable termination criteria and the like. Below the hood, *Maple* is optimizing polynomial computations using tools like Horner's rule, running multiple algorithms when there is no clear best choice, and switching to reduced complexity (Karatsuba or FFT-based) multiplication when accuracy so demands. Wouldn't it be nice, though, if all vendors allowed as much peering under the bonnet as *Maple* does!

Example 2. The number of *additive partitions* of n, $p(n)$, is generated by

$$P(q) = 1 + \sum_{n\geq 1} p(n)q^n = \prod_{n\geq 1} (1 - q^n)^{-1}. \tag{1.2}$$

Thus $p(5) = 7$, because

$$5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1$$
$$= 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1,$$

as we ignore "0" and permutations. Additive partitions are less tractable than multiplicative ones, for there is no analogue of unique prime factorization nor the corresponding structure. Partitions provide a wonderful example of why Keith Devlin calls mathematics "the science of patterns."

Formula (1.2) is easily seen by expanding $(1 - q^n)^{-1}$ and comparing coefficients. A modern computational temperament leads to

Question: How hard is $p(n)$ to compute—in 1900 (for MacMahon the "father of combinatorial analysis") or in 2000 (for *Maple* or *Mathematica*)?

Answer: The computation of $p(200) = 3972999029388$ took MacMahon months and intelligence. Now, however, we can use the most naïve approach: Computing 200 terms of the series for the inverse product in (1.2) instantly produces the result, using either *Mathematica* or *Maple*. Obtaining the result $p(500) = 2300165032574323995027$ is not much more difficult, using the *Maple* code

```
N : =500; coeff(series(1/product
             (1-q^n,n=1..N+1),q,N+1),q,N);
```

Euler's Pentagonal number theorem

Fifteen years ago computing $P(q)$ in *Maple*, was very slow, while taking the series for the reciprocal $Q(q) = \Pi_{n\geq 1}(1 - q^n)$ was quite manageable! Why? Clearly the series for Q must have special properties. Indeed it is *lacunary*:

$$Q(q) = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + q^{26} - q^{35} - q^{40} + q^{51} + q^{57} - q^{70} - q^{77} + q^{92} + O(q^{100}). \tag{1.3}$$

This lacunarity is now recognized automatically by *Maple*, so the platform works much better, but we are much less likely to discover Euler's gem:

$$\prod_{n=1}^{\infty} (1 - q^n) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2}.$$

If we do not immediately recognize these *pentagonal numbers*, then Sloane's online *Encyclopedia of Integer Sequences*[5] immediately comes to the rescue, with abundant references to boot.

This sort of mathematical computation is still in its reasonably early days, but the impact is palpable—and no more so than in the contest and book under review.

About the Contest

For a generation Nick Trefethen has been at the vanguard of developments in scientific computation, both through his own research, on topics such as pseudo-spectra, and through much thoughtful and vigorous activity in the community. In a 1992 essay "The Definition of Numerical Analysis"[6] Trefethen engagingly demolishes the conventional definition of Numerical Analysis as "the science of rounding errors." He explores how this hyperbolic view emerged, and finishes by writing,

> *I believe that the existence of finite algorithms for certain problems, together with other historical forces, has*

[4]It would be interesting to construct a list of the ten most influential earlier algorithms.

[5]A fine model for of 21st-century databases, it is available at www.research.att.com/~njas/sequences

[6]*SIAM News*, November 1992.

distracted us for decades from a balanced view of numerical analysis. Rounding errors and instability are important, and numerical analysts will always be the experts in these subjects and at pains to ensure that the unwary are not tripped up by them. But our central mission is to compute quantities that are typically uncomputable, from an analytical point of view, and to do it with lightning speed. For guidance to the future we should study not Gaussian elimination and its beguiling stability properties, but the diabolically fast conjugate gradient iteration, or Greengard and Rokhlin's O(N) multipole algorithm for particle simulations, or the exponential convergence of spectral methods for solving certain PDEs, or the convergence in O(N) iterations achieved by multigrid methods for many kinds of problems, or even Borwein and Borwein's[7] *magical AGM iteration for determining 1,000,000 digits of π in the blink of an eye. That is the* heart *of numerical analysis.*

In the January 2002 issue of *SIAM News*, Nick Trefethen, by then of Oxford University, presented ten diverse problems used in teaching *modern* graduate numerical analysis students at Oxford University, the answer to each being a certain real number. Readers were challenged to compute ten digits of each answer, with a $100 prize to be awarded to the best entrant. Trefethen wrote, "If anyone gets 50 digits in total, I will be impressed."

And he was. A total of 94 teams, representing 25 different nations, submitted results. Twenty of these teams received a full 100 points (10 correct digits for each problem). They included the late John Boersma, working with Fred Simons and others; Gaston Gonnet (a Maple founder) and Robert Israel; a team containing Carl Devore; and the authors of the book under review variously working alone and with others. These results were much better than expected, but an originally anonymous donor, William J. Browning, provided funds for a $100 award to each of the twenty perfect teams. The present author, David Bailey,[8] and Greg Fee entered, but failed to qualify for an award.[9]

The ten challenge problems

The purpose of computing is insight, not numbers. (Richard Hamming[10])

The ten problems are:

#1. What is $\lim_{\epsilon\to 0}\int_\epsilon^1 x^{-1}\cos(x^{-1}\log x)dx$?

#2. A photon moving at speed 1 in the x-y plane starts at $t = 0$ at $(x,y) = (1/2, 1/10)$ heading due east. Around every integer lattice point (i, j) in the plane, a circular mirror of radius 1/3 has been erected. How far from the origin is the photon at $t = 10$?

#3. The infinite matrix A with entries $a_{11} = 1$, $a_{12} = 1/2$, $a_{21} = 1/3$, $a_{13} = 1/4$, $a_{22} = 1/5$, $a_{31} = 1/6$, etc., is a bounded operator on ℓ^2. What is $\|A\|$?

#4. What is the global minimum of the function $\exp(\sin(50x)) + \sin(60e^y) + \sin(70 \sin x) + \sin(\sin(80y)) - \sin(10(x + y)) + (x^2 + y^2)/4$?

#5. Let $f(z) = 1/\Gamma(z)$, where $\Gamma(z)$ is the gamma function, and let $p(z)$ be the cubic polynomial that best approximates $f(z)$ on the unit disk in the supremum norm $\|\cdot\|_\infty$. What is $\|f - p\|_\infty$?

#6. A flea starts at (0,0) on the infinite 2-D integer lattice and executes a biased random walk: At each step it hops north or south with probability 1/4, east with probability $1/4 + \epsilon$, and west with probability $1/4 - \epsilon$. The probability that the flea returns to (0,0) sometime during its wanderings is 1/2. What is ϵ?

#7. Let A be the 20000×20000 matrix whose entries are zero everywhere except for the primes $2, 3, 5, 7, \cdots, 224737$ along the main diagonal and the number 1 in all the positions a_{ij} with $|i - j| = 1, 2, 4, 8, \cdots, 16384$. What is the (1,1) entry of A^{-1}?

#8. A square plate $[-1,1] \times [-1,1]$ is at temperature $u = 0$. At time $t = 0$ the temperature is increased to $u = 5$ along one of the four sides while being held at $u = 0$ along the other three sides, and heat then flows into the plate according to $u_t = \Delta u$. When does the temperature reach $u = 1$ at the center of the plate?

#9. The integral $I(\alpha) = \int_0^2 [2 + \sin(10\alpha)]x^\alpha \sin(\alpha/(2 - x))\,dx$ depends on the parameter α. What is the value $\alpha \in [0,5]$ at which $I(\alpha)$ achieves its maximum?

#10. A particle at the center of a 10×1 rectangle undergoes Brownian motion (i.e., 2-D random walk with infinitesimal step lengths) till it hits the boundary. What is the probability that it hits at one of the ends rather than at one of the sides?

Answers correct to 40 digits to the problems are available at http://web.comlab.ox.ac.uk/oucl/work/nick.trefethen/hundred.html

Quite full details on the contest and the now substantial related literature are beautifully recorded on Bornemann's Web site

http://www-m8.ma.tum.de/m3/bornemann/challenge book/

which accompanies *The SIAM 100-digit Challenge: A Study In High-accuracy Numerical Computing*, which, for brevity, I shall call *The Challenge.*

About the Book and Its Authors

Success in solving these problems requires a broad knowledge of mathematics and numerical analysis, together with

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[7] As in many cases, this eponym is inaccurate, if flattering: it really should be Gauss-Brent-Salamin.

[8] Bailey wrote the introduction to the book under review.

[9] We took Nick at his word and turned in 85 digits! We thought that would be a good enough entry and returned to other activities.

[10] In *Numerical Methods for Scientists and Engineers*, 1962.

significant computational effort, to obtain solutions and ensure correctness of the results. The strengths and limitations of *Maple, Mathematica*, MATLAB (The *3Ms*), and other software tools such as PARI or GAP, are strikingly revealed in these ventures. Almost all of the solvers relied in large part on one or more of these three packages, and while most solvers attempted to confirm their results, there was no explicit requirement for proofs to be provided. In December 2002, Keller wrote:

> *To the Editor:*
> *Recently, SIAM News published an interesting article by Nick Trefethen (July/August 2002, page 1) presenting the answers to a set of problems he had proposed previously (January/February 2002, page 1). The answers were computed digits, and the clever methods of computation were described.*
> *I found it surprising that no proof of the correctness of the answers was given. Omitting such proofs is the accepted procedure in scientific computing. However, in a contest for calculating precise digits, one might have hoped for more.*
> *Joseph B. Keller, Stanford University*

In my view Keller's request for proofs as opposed to compelling evidence of correctness is, in this context, somewhat unreasonable, and even in the long term counterproductive [3, 4]. Nonetheless, the authors of *The Challenge* have made a complete and cogent response to Keller and much much more. The interest generated by the contest has with merit extended to *The Challenge*, which has already received reviews in places such as *Science*, where mathematics is not often seen.

Different readers, depending on temperament, tools, and training, will find the same problem more or less interesting and more or less challenging. The book is arranged so the ten problems can be read independently. In all cases multiple solution techniques are given; background, mathematics, implementation details—variously in each of the *3Ms* or otherwise—and extensions are discussed, all in a highly readable and engaging way.

Each problem has its own chapter with its own lead author. The four authors, Folkmar Bornemann, Dirk Laurie, Stan Wagon, and Jörg Waldvogel, come from four countries on three continents and did not know each other as they worked on the book, though Dirk did visit Jörge and Stan visited Folkmar as they were finishing their manuscript. This illustrates the growing power of the collaboration, networking, and the grid—both human and computational.

Some high spots

As we saw, Joseph Keller raised the question of proof. On careful reading of the book, one may discover proofs of correctness for all problems except for #1, #3, and #5. For problem #5, one difficulty is to develop a robust interval implementation for both complex number computation and, more importantly, for the *Gamma function*. While error bounds for #1 may be out of reach, an analytic solution to #3 seems to this reviewer tantalizingly close.

The authors ultimately provided 10,000-digit solutions to nine of the problems. They say that this improved their knowledge on several fronts as well as being "cool." When using Integer Relation Methods, ultrahigh precision computations are often needed [3]. One (and only one) problem remains totally intractable[11]—at press time, getting more than 300 digits for #3 was impossible.

Some surprises

According to the authors,[12] they were surprised by the following, listed by problem:

#1. The best algorithm for 10,000 digits was the trusty *trapezoidal rule*—a not uncommon personal experience of mine.

#2. Using *interval arithmetic* with starting intervals of size smaller than 10^{-5000}, one can still find the position of the particle at time 2000 (not just time ten), which makes a fine exercise for very high-precision interval computation.

#4. Interval analysis algorithms can handle similar problems in higher dimensions. As a foretaste of future graphic tools, one can solve this problem using current *adaptive 3-D plotting* routines which can catch all the bumps. As an optimizer by background, this was the first problem my group solved using a damped Newton method.

#5. While almost all canned optimization algorithms failed, *differential evolution*, a relatively new type of evolutionary algorithm, worked quite well.

#6. This problem has an almost-closed form in terms of elliptic integrals and leads to a study of random walks on hypercubic lattices, and Watson integrals [3, 4, 5].

#9. The maximum parameter is expressible in terms of a *MeijerG function*. While this was not common knowledge among the contestants, *Mathematica* and *Maple* both will figure this out. This is another measure of the changing environment. It is usually a good idea—and not at all immoral—to data-mine[13] and find out what your favourite one of the *3Ms* knows about your current object of interest. For example, Maple tells one that:

---

[11]If only by the authors' new gold standard of 10,000 digits.

[12]Stan Wagon, private communication.

[13]By its own count, Wal-Mart has 460 terabytes of data stored on Teradata mainframes, made by NCR, at its Bentonville headquarters. To put that in perspective, the Internet has less than half as much data . . . ," Constance Hays, "What Wal-Mart Knows About Customers' Habits," *New York Times*, Nov. 14, 2004. Mathematicians also need databases.

```
  The Meijer G function is defined by the inverse
                           Laplace transform
  MeijerG([as,bs],[cs,ds],z)
               /
        1      |  GAMMA(1-as+y) GAMMA(cs-y)   y
  = ---------- |  ---------------------------- z dy
    2 Pi I   0   GAMMA(bs-y) GAMMA(1-ds+y)
               |
               /
                L
where
  as = [a1, · · · ,am], GAMMA(1-as+y) = GAMMA(1-a1+y) · · · GAMMA(1-am+y)
  bs = [b1, · · · ,bn], GAMMA(bs-y) = GAMMA(b1-y)        · · · GAMMA(bn-y)
  cs = [c1, · · · ,cp], GAMMA(cs-y) = GAMMA(c1-y)        · · · GAMMA(cp-y)
  ds = [d1, · · · dq],  GAMMA(1-ds+y) = GAMMA(1-d1+y) · · · GAMMA(1-dq+y)
```

Another excellent example of how packages are changing mathematics is the *Lambert W function* [4], whose properties and development are very nicely described in a recent article by Brian Hayes [8], *Why W?*

Two big surprises

I finish this section by discussing in more detail the two problems whose resolution most surprised the authors.

The essay on Problem #7, whose principal author was Bornemann, is titled: "Too Large to be Easy, Too Small to Be Hard." Not so long ago a $20{,}000 \times 20{,}000$ matrix was large enough to be hard. Using both *congruential* and *p-adic* methods, Dumas, Turner, and Wan obtained a fully *symbolic* answer, a rational with a 97,000-digit numerator and like denominator. Wan has reduced the time to obtain this to about 15 minutes on one machine, from using many days on many machines. While *p-adic* analysis is susceptible to parallelism, it is less easily attacked than are congruential methods; the need for better parallel algorithms lurks below the surface of much modern computational mathematics.

The surprise here, though, is not that the solution is rational, but that it can be explicitly constructed. The chapter, like the others, offers an interesting menu of numeric and exact solution strategies. Of course, in any numeric approach *ill-conditioning* rears its ugly head, while *sparsity* and other core topics come into play.

My personal favourite, for reasons that may be apparent, is:

Problem #10: "Hitting the Ends." Bornemann starts the chapter by exploring *Monte-Carlo methods*, which are shown to be impracticable. He then reformulates the problem *deterministically* as the value at the center of a 10×1 rectangle of an appropriate harmonic measure of the ends, arising from a 5-point discretization of Laplace's equation with Dirichlet boundary conditions. This is then solved by a well-chosen *sparse Cholesky* solver. At this point a reliable numerical value of $3.837587979 \cdot 10^{-7}$ is obtained. And the posed problem is solved numerically to the requisite 10 places.

But this is only the warm-up. We proceed to develop two analytic solutions, the first using *separation of variables* on the underlying PDE on a general $2a \times 2b$ rectangle. We learn that

$$(3.4)\qquad p(a,b) = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\operatorname{sech}\left(\frac{\pi(2n+1)}{2}\rho\right)$$

where $\rho := a/b$. A second method using *conformal mappings* yields

$$(3.5)\qquad \operatorname{arccot}\rho = p(a,b)\,\frac{\pi}{2} + \arg K(e^{ip(a,b)\pi}),$$

where K is the *complete elliptic integral* of the first kind. It will not be apparent to a reader unfamiliar with inversion of elliptic integrals that (3.4) and (3.5) encode the same solution; but they must, as the solution is unique in (0,1); each can now be used to solve for $p = 10$ to arbitrary precision.

Bornemann finally shows that, for far from simple reasons, the answer is

$$(3.6)\qquad p = \frac{2}{\pi}\arcsin(k_{100}),$$

where

$$k_{100} := \left((3-2\sqrt{2})\,(2+\sqrt{5})\,(-3+\sqrt{10})\,(-\sqrt{2}+\sqrt[4]{5})^2\right)^2$$

a simple composition of one arcsin and a few square roots. No one anticipated a closed form like this.

Let me show how to finish up. An apt equation is [5, (3.2.29)] showing that

$$(3.7)\qquad \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\operatorname{sech}\left(\frac{\pi(2n+1)}{2}\rho\right) = \frac{1}{2}\arcsin k,$$

exactly when $k = k_{\rho^2}$ is parametrized by *theta functions* in terms of the so-called *nome*, $q = \exp(-\pi\rho)$, as Jacobi discovered. We have

$$(3.8)\qquad k_{\rho^2} = \frac{\theta_2^2(q)}{\theta_3^2(q)} = \frac{\sum_{n=-\infty}^{\infty} q^{(n+1/2)^2}}{\sum_{n=-\infty}^{\infty} q^{n^2}}$$

Comparing (3.7) and (3.4), we see that the solution is

$k_{100} = 6.02806910155971082882540712292\ldots \cdot 10^{-7},$

as asserted in (3.6). The explicit form now follows from classical nineteenth-century theory as discussed in [1, 5]. In fact k_{210} is the singular value sent by Ramanujan to Hardy in his famous letter of introduction [2, 5]—if only Trefethen had asked for a $\sqrt{210} \times 1$ box, or even better a $\sqrt{15} \times \sqrt{14}$ one!

Alternatively, armed only with the knowledge that the singular values are always algebraic, we may finish with an *au courant* proof: numerically obtain the minimal polynomial from a high-precision computation with (3.8), and recover the surds [4].

Example 3. *Maple* allows the following

```
> Digits:=100:with(PolynomialTools):
> k:=s->evalf(EllipticModulus(exp(-Pi*sqrt(s)))):
> p:=latex(MinimalPolynomial(k(100),12)):
> 'Error',fsolve(p)[1]-evalf(k(100)); galois(p);
                Error, 4 10^-106
"8T9", {"D(4)[x]2", "E(8):2"}, "+", 16, {"4 5)(6 7)", "(4 8)(1 5)(2 6)(3 7)",
      "(1 8)(2 3)(4 5)(6 7)", "(2 8)(1 3)(4 6)(5 7)"}
```

which finds the minimal polynomial for k_{100}, checks it to 100 places, tells us the *galois group*, and returns a latex expression 'p' which sets as:

$$p(\_X) = 1 - 1658904\ \_X - 3317540\ \_X^2 + 1657944\ \_X^3 + 6637254\ \_X^4 + 1657944\ \_X^5 - 3317540\ \_X^6 - 1658904\ \_X^7 + \_X^8,$$

and is *self-reciprocal*: it satisfies $p(x) = x^8 p(1/x)$. This suggests taking a square root, and we discover that $y = \sqrt{k_{100}}$ satisfies

$$1 - 1288y + 20y^2 - 1288y^3 - 26y^4 + 1288y^5 + 20y^6 + 1288y^7 + y^8.$$

Now life is good. The prime factors of 100 are 2 and 5, prompting

```
subs(_X=z,
    [op(((factor(p,{sqrt(2),sqrt(5)})))))])
```

This yields four quadratic terms, the desired one being

$$q = z^2 + 322\, z - 228\, z\sqrt{2} + 144\, z\sqrt{5} - 102\, z\sqrt{2}\sqrt{5} + 323 - 228\,\sqrt{2} + 144\sqrt{5} - 102\sqrt{2}\sqrt{5}.$$

For security,

```
w:=solve(q)[2]: evalf[1000](k(100)-w^2);
```

gives a 1000-digit error check of $2.20226255 \cdot 10^{-998}$.

We leave it to the reader to find, using one of the 3Ms, the more beautiful form of k_{100} given above in (3.6). □

Considering also the many techniques and types of mathematics used, we have a wonderful advertisement for multi-field, multi-person, multi-computer, multi-package collaboration.

Concrete Constructive Mathematics

Elsewhere Kronecker said "In mathematics, I recognize true scientific value only in concrete mathematical truths, or to put it more pointedly, only in mathematical formulas." . . . I would rather say "computations" than "formulas," but my view is essentially the same. (Harold M. Edwards [6, p. 1])

Edwards comments elsewhere in his recent *Essays on Constructive Mathematics* that his own preference for constructivism was forged by experience of computing in the fifties, when computing power was, as he notes, "trivial by today's standards." My own similar attitudes were cemented primarily by the ability in the early days of personal computers to decode—with the help of *APL*—exactly the sort of work by Ramanujan which finished #10.

The SIAM 100-Digit Challenge: A Study In High-accuracy Numerical Computing is a wonderful and well-written book full of living mathematics by lively mathematicians. It shows how far we have come computationally and hints tantalizingly at what lies ahead. Anyone who has been interested enough to finish this review, and had not yet read the book, is strongly urged to buy and plunge in—computer in hand—to this fine advertisement for constructive mathematics 21st-century style. I would equally strongly suggest a cross-word solving style—pick a few problems from the list given, and try them before peeking at the answers and extensions given in *The Challenge*. Later, use it to illustrate a course or just for a refresher; and be pleasantly reminded that challenging problems rarely have only one path to solution and usually reward study.

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Conversations on Mind, Matter, and Mathematics

by Jean-Pierre Changeux & Alain Connes
edited and translated by M. B. DeBevoise

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REVIEWED BY JEAN PETITOT

What exactly is the type of reality of mathematical ideal entities? This problem remains largely an open question. Any ontology of abstract entities will encounter certain antinomies which have been well known for centuries if not millennia. These antinomies have led the various schools of contemporary epistemology increasingly to deny any reality to mathematical ideal objects, structures, constructions, proofs, and to justify this denial philosophically, thus rejecting the spontaneous naïve Platonism of most professional mathematicians. But they throw out the baby with the bath water. Contrary to such figures as Poincaré, Husserl, Weyl, Borel, Lebesgue, Veronese, Enriques, Cavaillès, Lautman, Gonseth, or the late Gödel, the dominant epistemology of mathematics is no longer an epistemology of mathematical content. For quite serious and precise philosophical reasons, it refuses to take into account what the great majority of creative brilliant mathematicians consider to be the true nature of mathematical knowledge. And yet, to quote the subtitle of Hao Wang's (1985) book *Beyond Analytic Philosophy*, one might well ask whether the imperative of any valid epistemology should not be "doing justice to what we know."

The remarkable debate *Conversations on Mind, Matter, and Mathematics* between Alain Connes and Jean-Pierre Changeux, both scientific minds of the very first rank and professors at the Collège de France in Paris, takes up the old question of the reality of mathematical idealities in a rather new and refreshing perspective. To be sure, since it is designed to be accessible to a wide audience, the debate is not framed in technical terms; the arguments often employ a broad brush and are not always sufficiently developed. Nevertheless, thanks to the exceptional standing of the protagonists, the debate manages to be compelling and relevant.

Jean-Pierre Changeux's Neural Materialism

Let me begin by summarizing some of Jean-Pierre Changeux's arguments.

Because mathematics is a human and cognitive activity, it is natural first to analyze it in psychological and neurocognitive terms. Psychologism, which formalists and logicists have decried since the time of Frege and Husserl, develops the reductionist thesis that mathematical objects and the logical idealities that formulate them can be reduced—as far as their reality is concerned—to mental states and processes. Depending on whether or not mental representations are themselves conceived as reducible to the underlying neural activity, this psychologism is either a materialist reductionism or a mentalist functionalism.

J-P. Changeux defends a variant of materialist reductionism. His aim is twofold: first, to inquire into the nature of mathematics, but also, at a more strategic level, to put mathematics in its place, so to speak. He has never concealed his opposition to Cartesian or Leibnizian rationalisms that have made mathematics the "queen" of the sciences. In his view, mathematics must abdicate its overly arrogant sovereignty, stop laying claim to universal validity and absolute truth, and accept the humbler role assigned to it by Bacon and Diderot—that of "servant" to the natural sciences (p. 7). And what better way to make mathematics surrender its prestigious seniority than to demonstrate scientifically that its claims to absolute truth have no more rational basis than do those of religious faith?

Pursuing his mission with great conviction, Changeux revisits all the traditional touchstones of the empiricist, materialist, and nominalist critiques of Platonist idealism in mathematics. He cites an impressive mass of scientific data along the way, including results from neurobiology and cognitive psychology in which he has played a leading role. It is this aspect of his approach which commands attention.

1. The empiricist and constructivist theses hold that mathematical objects are "creatures of reason" whose reality is purely cerebral (p. 11). They are representations, that is, mental objects that exist materially in the brain, and "corresponding to physical [i.e., neural] states" (p. 14).

> Mental representations—memory objects—are coded in the brain as forms in the Gestalt sense, and stored in the neurons and synapses, despite significant variability in synaptic efficacy (p. 128).

Their object-contents are reflexively analyzable and their properties can be clarified axiomatically. But that is possible only because, as mental representations, they are endowed with a material reality (pp. 11–15). What's more, the axiomatic method of analysis is itself a "cerebral process" (p. 30).

2. One might try to salvage an autonomy for the formal logical and mathematical levels by admitting, in line with

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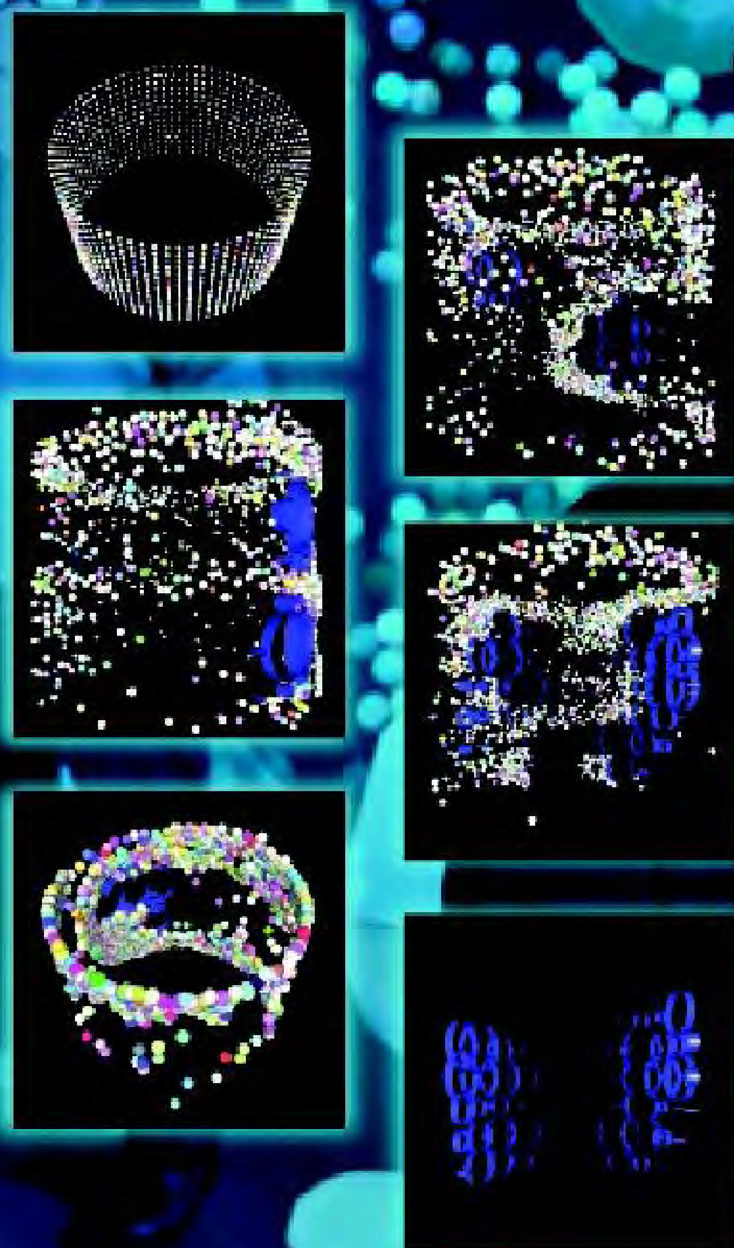

Simulating structure formation in the cosmos (see page 438)

Experimental Mathematics: Examples, Methods and Implications

David H. Bailey and Jonathan M. Borwein

> The object of mathematical rigor is to sanction and legitimize the conquests of intuition, and there was never any other object for it.
>
> —Jacques Hadamard[1]

> If mathematics describes an objective world just like physics, there is no reason why inductive methods should not be applied in mathematics just the same as in physics.
>
> —Kurt Gödel[2]

Introduction

Recent years have seen the flowering of "experimental" mathematics, namely the utilization of modern computer technology as an active tool in mathematical research. This development is not limited to a handful of researchers nor to a handful of universities, nor is it limited to one particular field of mathematics. Instead, it involves hundreds of individuals, at many different institutions, who have turned to the remarkable new computational tools now available to assist in their research, whether it be in number theory, algebra, analysis, geometry, or even topology. These tools are being used to work out specific examples, generate plots, perform various algebraic and calculus manipulations, test conjectures, and explore routes to formal proof. Using computer tools to test conjectures is by itself a major timesaver for mathematicians, as it permits them to quickly rule out false notions.

Clearly one of the major factors here is the development of robust symbolic mathematics software. Leading the way are the Maple and Mathematica products, which in the latest editions are far more expansive, robust, and user-friendly than when they first appeared twenty to twenty-five years ago. But numerous other tools, some of which emerged only in the past few years, are also playing key roles. These include: (1) the Magma computational algebra package, developed at the University of Sydney in Australia; (2) Neil Sloane's online integer sequence recognition tool, available at `http://www.research.att.com/njas/sequences`; (3) the inverse symbolic calculator (an online numeric constant recognition facility), available at `http://www.cecm.sfu.ca/projects/ISC`; (4) the electronic geometry site at `http://www.eg-models.de`; and numerous others. See

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[1] *Quoted at length in E. Borel,* Leçons sur la theorie des fonctions, *1928.*

[2] *Kurt Gödel,* Collected Works, Vol. III, *1951.*

http://www.experimentalmath.info for a more complete list, with links to their respective websites.

We must of course also give credit to the computer industry. In 1965 Gordon Moore, before he served as CEO of Intel, observed:

> The complexity for minimum component costs has increased at a rate of roughly a factor of two per year.... Certainly over the short term this rate can be expected to continue, if not to increase. Over the longer term, the rate of increase is a bit more uncertain, although there is no reason to believe it will not remain nearly constant for at least 10 years. [29]

Nearly forty years later, we observe a record of sustained exponential progress that has no peer in the history of technology. Hardware progress alone has transformed mathematical computations that were once impossible into simple operations that can be done on any laptop.

Many papers have now been published in the experimental mathematics arena, and a full-fledged journal, appropriately titled *Experimental Mathematics*, has been in operation for twelve years. Even older is the AMS journal *Mathematics of Computation*, which has been publishing articles in the general area of computational mathematics since 1960 (since 1943 if you count its predecessor). Just as significant are the hundreds of other recent articles that mention computations but which otherwise are considered entirely mainstream work. All of this represents a major shift from when the present authors began their research careers, when the view that "real mathematicians don't compute" was widely held in the field.

In this article, we will summarize some of the discoveries and research results of recent years, by ourselves and by others, together with a brief description of some of the key methods employed. We will then attempt to ascertain at a more fundamental level what these developments mean for the larger world of mathematical research.

Integer Relation Detection

One of the key techniques used in experimental mathematics is integer relation detection, which in effect searches for linear relationships satisfied by a set of numerical values. To be precise, given a real or complex vector $(x_1, x_2, \cdots, x_n)$, an integer relation algorithm is a computational scheme that either finds the n integers (a_i), not all zero, such that $a_1x_1 + a_2x_2 + \cdots a_nx_n = 0$ (to within available numerical accuracy) or else establishes that there is no such integer vector within a ball of radius A about the origin, where the metric is the Euclidean norm: $A = (a_1^2 + a_2^2 + \cdots + a_n^2)^{1/2}$. Integer relation computations require very high precision in the input vector x to obtain numerically meaningful results—at least dn-digit precision, where $d = \log_{10} A$. This is the principal reason for the interest in very high-precision arithmetic in experimental mathematics. In one recent integer relation detection computation, 50,000-digit arithmetic was required to obtain the result [9].

At the present time, the best-known integer relation algorithm is the PSLQ algorithm [26] of mathematician-sculptor Helaman Ferguson, who, together with his wife, Claire, received the 2002 Communications Award of the Joint Policy Board for Mathematics (AMS-MAA-SIAM). Simple formulations of the PSLQ algorithm and several variants are given in [10]. The PSLQ algorithm, together with related lattice reduction schemes such as LLL, was recently named one of ten "algorithms of the century" by the publication *Computing in Science and Engineering* [4]. PSLQ or a variant is implemented in current releases of most computer algebra systems.

Arbitrary Digit Calculation Formulas

The best-known application of PSLQ in experimental mathematics is the 1995 discovery, by means of a PSLQ computation, of the "BBP" formula for π:

(1)
$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right).$$

This formula permits one to directly calculate binary or hexadecimal digits beginning at the n-th digit, without needing to calculate any of the first $n-1$ digits [8], using a simple scheme that requires very little memory and no multiple-precision arithmetic software.

It is easiest to see how this individual digit-calculating scheme works by illustrating it for a similar formula, known at least since Euler, for $\log 2$:

$$\log 2 = \sum_{n=1}^{\infty} \frac{1}{n2^n}.$$

Note that the binary expansion of $\log 2$ beginning after the first d binary digits is simply $\{2^d \log 2\}$, where by $\{\cdot\}$ we mean fractional part. We can write

(2)
$$\begin{aligned} \{2^d \log 2\} &= \left\{ \sum_{n=1}^{\infty} \frac{2^{d-n}}{n} \right\} = \left\{ \sum_{n=1}^{d} \frac{2^{d-n}}{n} \right\} + \left\{ \sum_{n=d+1}^{\infty} \frac{2^{d-n}}{n} \right\} \\ &= \left\{ \sum_{n=1}^{d} \frac{2^{d-n} \bmod n}{n} \right\} + \left\{ \sum_{n=d+1}^{\infty} \frac{2^{d-n}}{n} \right\}, \end{aligned}$$

where we insert "mod n" in the numerator of the first term of (2), since we are interested only in the fractional part after division by n. Now the expression $2^{d-n} \bmod n$ may be evaluated very rapidly by means of the binary algorithm for exponentiation, where each multiplication is reduced

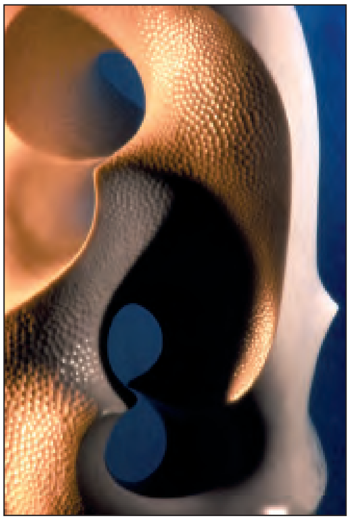

Figure 1. Ferguson's "Figure Eight Knot Complement" sculpture.

modulo n. The entire scheme indicated by formula (2) can be implemented on a computer using ordinary 64-bit or 128-bit arithmetic; high-precision arithmetic software is not required. The resulting floating-point value, when expressed in binary format, gives the first few digits of the binary expansion of $\log 2$ beginning at position $d+1$. Similar calculations applied to each of the four terms in formula (1) yield a similar result for π. The largest computation of this type to date is binary digits of π beginning at the quadrillionth (10^{15}-th) binary digit, performed by an international network of computers organized by Colin Percival.

The BBP formula for π has even found a practical application: it is now employed in the g95 Fortran compiler as part of transcendental function evaluation software.

Since 1995 numerous other formulas of this type have been found and proven using a similar experimental approach. Several examples include:

(3)
$$\pi\sqrt{3} = \frac{9}{32}\sum_{k=0}^{\infty}\frac{1}{64^k}\left(\frac{16}{6k+1}-\frac{8}{6k+2}-\frac{2}{6k+4}-\frac{1}{6k+5}\right),$$

(4)
$$\pi^2 = \frac{1}{8}\sum_{k=0}^{\infty}\frac{1}{64^k}\left[\frac{144}{(6k+1)^2}-\frac{216}{(6k+2)^2}-\frac{72}{(6k+3)^2}-\frac{54}{(6k+4)^2}+\frac{9}{(6k+5)^2}\right],$$

(5)
$$\pi^2 = \frac{2}{27}\sum_{k=0}^{\infty}\frac{1}{729^k}\left[\frac{243}{(12k+1)^2}-\frac{405}{(12k+2)^2}-\frac{81}{(12k+4)^2}-\frac{27}{(12k+5)^2}\right.$$
$$\left.-\frac{72}{(12k+6)^2}-\frac{9}{(12k+7)^2}-\frac{9}{(12k+8)^2}-\frac{5}{(12k+10)^2}+\frac{1}{(12k+11)^2}\right],$$

(6)
$$\sqrt{3}\arctan\left(\frac{\sqrt{3}}{7}\right) = \sum_{k=0}^{\infty}\frac{1}{27^k}\left(\frac{3}{3k+1}+\frac{1}{3k+2}\right),$$

(7)
$$\frac{25}{2}\log\left[\frac{781}{256}\left(\frac{57-5\sqrt{5}}{57+5\sqrt{5}}\right)^{\sqrt{5}}\right] = \sum_{k=0}^{\infty}\frac{1}{5^{5k}}\left(\frac{5}{5k+2}+\frac{1}{5k+3}\right).$$

Formulas (3) and (4) permit arbitrary-position binary digits to be calculated for $\pi\sqrt{3}$ and π^2. Formulas (5) and (6) permit the same for ternary (base-3) expansions of π^2 and $\sqrt{3}\arctan(\sqrt{3}/7)$. Formula (7) permits the same for the base-5 expansion of the curious constant shown. A compendium of known BBP-type formulas, with references, is available at [5].

One interesting twist here is that the hyperbolic volume of one of Ferguson's sculptures (the "Figure Eight Knot Complement";[3] see Figure 1), which is given by

$$V = 2\sqrt{3}\sum_{n=1}^{\infty}\frac{1}{n\binom{2n}{n}}\sum_{k=n}^{2n-1}\frac{1}{k}$$
$$= 2.029883212819307250042405108549\ldots,$$

has been identified in terms of a BBP-type formula by application of Ferguson's own PSLQ algorithm. In particular, British physicist David Broadhurst found in 1998, using a PSLQ program, that

$$V = \frac{\sqrt{3}}{9}\sum_{n=0}^{\infty}\frac{(-1)^n}{27^n}$$
$$\times\left[\frac{18}{(6n+1)^2}-\frac{18}{(6n+2)^2}-\frac{24}{(6n+3)^2}-\frac{6}{(6n+4)^2}+\frac{2}{(6n+5)^2}\right].$$

This result is proven in [15, Chap. 2, Prob. 34].

Does Pi Have a Nonbinary BBP Formula?

Since the discovery of the BBP formula for π in 1995, numerous researchers have investigated, by means of computational searches, whether there is a similar formula for calculating arbitrary digits of π in other number bases (such as base 10). Alas, these searches have not been fruitful.

Recently, one of the present authors (JMB), together with David Borwein (Jon's father) and William Galway, established that there is no degree-1 BBP-type formula for π for bases other than powers of two (although this does not rule out some other scheme for calculating individual digits). We will sketch this result here. Full details and some related results can be found in [20].

In the following, $\Re(z)$ and $\Im(z)$ denote the real and imaginary parts of z, respectively. The integer $b>1$ is not a *proper power* if it cannot be written as c^m for any integers c and $m>1$. We will use the notation $\mathrm{ord_p(z)}$ to denote the p-adic order of the rational $z\in Q$. In particular, $\mathrm{ord_p(p)}=1$ for prime p, while $\mathrm{ord_p(q)}=0$ for primes $q\neq p$, and $\mathrm{ord_p(wz)}=\mathrm{ord_p(w)}+\mathrm{ord_p(z)}$. The notation $\nu_b(p)$ will mean the order of the integer b in the multiplicative group of the integers modulo p. We will say that p is a *primitive prime factor* of b^m-1 if m is the least integer such that $p|(b^m-1)$. Thus p is a primitive prime factor of b^m-1 provided $\nu_b(p)=m$. Given the Gaussian integer $z\in Q[i]$ and the rational prime $p\equiv 1 \pmod 4$, let $\theta_p(z)$ denote $\mathrm{ord}_{\mathfrak{p}}(z)-\mathrm{ord}_{\overline{\mathfrak{p}}}(z)$, where $\mathfrak{p}$ and $\overline{\mathfrak{p}}$ are the two conjugate Gaussian primes dividing p and where we require $0<\Im(\mathfrak{p})<\Re(\mathfrak{p})$ to make the definition of θ_p unambiguous. Note that

(8)
$$\theta_p(wz) = \theta_p(w)+\theta_p(z).$$

Given $\kappa\in R$, with $2\le b\in Z$ and b not a proper power, we say that κ has a Z-linear or Q-linear

[3] *Reproduced by permission of the sculptor.*

Machin-type BBP arctangent formula to the base b if and only if κ can be written as a Z-linear or Q-linear combination (respectively) of generators of the form

$$\arctan\left(\frac{1}{b^m}\right) = \Im \log\left(1+\frac{i}{b^m}\right)$$
$$= b^m \sum_{k=0}^{\infty} \frac{(-1)^k}{b^{2mk}(2k+1)}. \quad (9)$$

We shall also use the following result, first proved by Bang in 1886:

Theorem 1. *The only cases where $b^m - 1$ has no primitive prime factor(s) are when $b = 2$, $m = 6$, $b^m - 1 = 3^2 \cdot 7$ or when $b = 2^N - 1, N \in Z$, $m = 2$, $b^m - 1 = 2^{N+1}(2^{N-1} - 1)$.*

We can now state the main result:

Theorem 2. *Given $b > 2$ and not a proper power, there is no Q-linear Machin-type BBP arctangent formula for π.*

Proof: It follows immediately from the definition of a Q-linear Machin-type BBP arctangent formula that any such formula has the form

$$\pi = \frac{1}{n}\sum_{m=1}^{M} n_m \Im \log(b^m - i), \quad (10)$$

where $n > 0 \in Z$, $n_m \in Z$, and $M \ge 1$, $n_M \ne 0$. This implies that

$$\prod_{m=1}^{M} (b^m - i)^{n_m} \in e^{ni\pi} Q^{\times} = Q^{\times}. \quad (11)$$

For any $b > 2$ and not a proper power, it follows from Bang's Theorem that $b^{4M} - 1$ has a primitive prime factor, say p. Furthermore, p must be odd, since $p = 2$ can only be a *primitive* prime factor of $b^m - 1$ when b is odd and $m = 1$. Since p is a primitive prime factor, it does not divide $b^{2M} - 1$, and so p must divide $b^{2M} + 1 = (b^M + i)(b^M - i)$. We cannot have both $p | b^M + i$ and $p | b^M - i$, since this would give the contradiction that $p | (b^M + i) - (b^M - i) = 2i$. It follows that $p \equiv 1 \pmod 4$ and that p factors as $p = \mathfrak{p}\bar{\mathfrak{p}}$ over $Z[i]$, with exactly one of $\mathfrak{p}$, $\bar{\mathfrak{p}}$ dividing $b^M - i$. Referring to the definition of θ, we see that we must have $\theta_p(b^M - i) \ne 0$. Furthermore, for any $m < M$, neither $\mathfrak{p}$ nor $\bar{\mathfrak{p}}$ can divide $b^m - i$, since this would imply $p \mid b^{4m} - 1$, $4m < 4M$, contradicting the fact that p is a primitive prime factor of $b^{4M} - 1$. So for $m < M$, we have $\theta_p(b^m - i) = 0$. Referring to equation (10) and using equation (8) and the fact that $n_M \ne 0$, we get the contradiction

$$0 \ne n_M \theta_p(b^M - i)$$
$$= \sum_{m=1}^{M} n_m \theta_p(b^m - i) = \theta_p(Q^{\times}) = 0. \quad (12)$$

Thus our assumption that there was a b-ary Machin-type BBP arctangent formula for π must be false.

Normality Implications of the BBP Formulas

One interesting (and unanticipated) discovery is that the existence of these computer-discovered BBP-type formulas has implications for the age-old question of normality for several basic mathematical constants, including π and $\log 2$. What's more, this line of research has recently led to a full-fledged proof of normality for an uncountably infinite class of explicit real numbers.

Given a positive integer b, we will define a real number α to be b-normal if every m-long string of base-b digits appears in the base-b expansion of α with limiting frequency b^{-m}. In spite of the apparently stringent nature of this requirement, it is well known from measure theory that almost all real numbers are b-normal, for all bases b. Nonetheless, there are very few explicit examples of b-normal numbers, other than the likes of *Champernowne's constant* $0.123456789101112131415\ldots$. In particular, although computations suggest that virtually all of the well-known irrational constants of mathematics (such as $\pi, e, \gamma, \log 2, \sqrt{2}$, etc.) are normal to various number bases, there is not a single proof—not for any of these constants, not for any number base.

Recently one of the present authors (DHB) and Richard Crandall established the following result.

Let $p(x)$ and $q(x)$ be integer-coefficient polynomials, with $\deg p < \deg q$, and $q(x)$ having no zeroes for positive integer arguments. By an *equidistributed* sequence in the unit interval we mean a sequence (x_n) such that for every subinterval (a, b), the fraction $\#[x_n \in (a, b)]/n$ tends to $b - a$ in the limit. The result is as follows:

Theorem 3. *A constant α satisfying the BBP-type formula*

$$\alpha = \sum_{n=1}^{\infty} \frac{p(n)}{b^n q(n)}$$

is b-normal if and only if the associated sequence defined by $x_0 = 0$ and, for $n \ge 1$, $x_n = \{b x_{n-1} + p(n)/q(n)\}$ (where $\{\cdot\}$ denotes fractional part as before), is equidistributed in the unit interval.

For example, $\log 2$ is 2-normal if and only if the simple sequence defined by $x_0 = 0$ and $\{x_n = 2x_{n-1} + 1/n\}$ is equidistributed in the unit interval. For π, the associated sequence is $x_0 = 0$ and

$$x_n = \left\{16 x_{n-1} + \frac{120n^2 - 89n + 16}{512n^4 - 1024n^3 + 712n^2 - 206n + 21}\right\}.$$

Full details of this result are given in [11] [15, Section 3.8].

It is difficult to know at the present time whether this result will lead to a full-fledged proof of normality for, say, π or $\log 2$. However, this approach has yielded a solid normality proof for another class of reals: Given $r \in [0,1)$, let r_n be the n-th binary digit of r. Then for each r in the unit interval, the constant

$$\alpha_r = \sum_{n=1}^{\infty} \frac{1}{3^n 2^{3^n + r_n}} \tag{13}$$

is 2-normal and transcendental [12]. What's more, it can be shown that whenever $r \neq s$, then $\alpha_r \neq \alpha_s$. Thus (13) defines an uncountably infinite class of distinct 2-normal, transcendental real numbers. A similar conclusion applies when 2 and 3 in (13) are replaced by any pair of relatively prime integers greater than 1.

Here we will sketch a proof of normality for one particular instance of these constants, namely $\alpha_0 = \sum_{n \geq 1} 1/(3^n 2^{3^n})$. Its associated sequence can be seen to be $x_0 = 0$ and $x_n = \{2x_{n-1} + c_n\}$, where $c_n = 1/n$ if n is a power of 3, and zero otherwise. This associated sequence is a very good approximation to the sequence $(\{2^n \alpha_0\})$ of shifted binary fractions of α_0. In fact, $|\{2^n \alpha_0\} - x_n| < 1/(2n)$. The first few terms of the associated sequence are

$$0,\ 0,\ 0,\ \tfrac{1}{3},\ \tfrac{2}{3},\ \tfrac{1}{3},\ \tfrac{2}{3},\ \tfrac{1}{3},\ \tfrac{2}{3},$$

$$\tfrac{4}{9},\ \tfrac{8}{9},\ \tfrac{7}{9},\ \tfrac{5}{9},\ \tfrac{1}{9},\ \tfrac{2}{9},\ \tfrac{4}{9},\ \tfrac{8}{9},\ \tfrac{7}{9},\ \tfrac{5}{9},\ \tfrac{1}{9},\ \tfrac{2}{9},\ \tfrac{4}{9},\ \tfrac{8}{9},\ \tfrac{7}{9},\ \tfrac{5}{9},\ \tfrac{1}{9},\ \tfrac{2}{9},$$

$$\tfrac{13}{27},\ \tfrac{26}{27},\ \tfrac{25}{27},\ \tfrac{23}{27},\ \tfrac{19}{27},\ \tfrac{11}{27},\ \tfrac{22}{27},\ \tfrac{17}{27},\ \tfrac{7}{27},\ \tfrac{14}{27},\ \tfrac{1}{27},\ \tfrac{2}{27},\ \tfrac{4}{27},\ \tfrac{8}{27},\ \tfrac{16}{27},\ \tfrac{5}{27},\ \tfrac{10}{27},\ \tfrac{20}{27},$$

$$\tfrac{13}{27},\ \tfrac{26}{27},\ \tfrac{25}{27},\ \tfrac{23}{27},\ \tfrac{19}{27},\ \tfrac{11}{27},\ \tfrac{22}{27},\ \tfrac{17}{27},\ \tfrac{7}{27},\ \tfrac{14}{27},\ \tfrac{1}{27},\ \tfrac{2}{27},\ \tfrac{4}{27},\ \tfrac{8}{27},\ \tfrac{16}{27},\ \tfrac{5}{27},\ \tfrac{10}{27},\ \tfrac{20}{27},$$

$$\tfrac{13}{27},\ \tfrac{26}{27},\ \tfrac{25}{27},\ \tfrac{23}{27},\ \tfrac{19}{27},\ \tfrac{11}{27},\ \tfrac{22}{27},\ \tfrac{17}{27},\ \tfrac{7}{27},\ \tfrac{14}{27},\ \tfrac{1}{27},\ \tfrac{2}{27},\ \tfrac{4}{27},\ \tfrac{8}{27},\ \tfrac{16}{27},\ \tfrac{5}{27},\ \tfrac{10}{27},\ \tfrac{20}{27},$$

and so forth. The clear pattern is that of triply repeated segments, each of length $2 \cdot 3^m$, where the numerators range over all integers relatively prime to and less than 3^{m+1}.

Note the very even manner in which this sequence fills the unit interval. Given any subinterval (c,d) of the unit interval, it can be seen that this sequence visits this subinterval no more than $3n(d-c)+3$ times, among the first n elements, provided that $n > 1/(d-c)$. It can then be shown that the sequence $(\{2^j \alpha\})$ visits (c,d) no more than $8n(d-c)$ times, among the first n elements of this sequence, so long as n is at least $1/(d-c)^2$. The 2-normality of α_0 then follows from a result given in [28, p. 77]. Further details on these results are given in [15, Sec. 4.3], [6], [12].

Euler's Multi-Zeta Sums

In April 1993, Enrico Au-Yeung, an undergraduate at the University of Waterloo, brought to the attention of one of us (JMB) the curious result

$$\sum_{k=1}^{\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right)^2 k^{-2} = 4.59987\ldots \approx \frac{17}{4}\zeta(4) = \frac{17\pi^4}{360} \tag{14}$$

where $\zeta(s) = \sum_{n \geq 1} n^{-s}$ is the Riemann zeta function. Au-Yeung had computed the sum in (14) to 500,000 terms, giving an accuracy of five or six decimal digits. Suspecting that his discovery was merely a modest numerical coincidence, Borwein sought to compute the sum to a higher level of precision. Using Fourier analysis and *Parseval's equation*, he wrote

$$\frac{1}{2\pi} \int_0^{\pi} (\pi - t)^2 \log^2(2 \sin \frac{t}{2})\, dt = \sum_{n=1}^{\infty} \frac{(\sum_{k=1}^{n} \frac{1}{k})^2}{(n+1)^2}. \tag{15}$$

The series on the right of (15) permits one to evaluate (14), while the integral on the left can be computed using the numerical quadrature facility of Mathematica or Maple. When he did this, Borwein was surprised to find that the conjectured identity (14) holds to more than 30 digits. We should add here that by good fortune, $17/360 = 0.047222\ldots$ has period one and thus can plausibly be recognized from its first six digits, so that Au-Yeung's numerical discovery was not entirely far-fetched.

Borwein was not aware at the time that (14) follows directly from a 1991 result due to De Doelder and had even arisen in 1952 as a problem in the *American Mathematical Monthly*. What's more, it turns out that Euler considered some related summations. Perhaps it was just as well that Borwein was not aware of these earlier results—and indeed of a large, quite deep and varied literature [21]—because pursuit of this and similar questions had led to a line of research that continues to the present day.

First define the *multi-zeta* constant

$$\zeta(s_1, s_2, \cdots, s_k) := \sum_{n_1 > n_2 > \cdots > n_k > 0} \prod_{j=1}^{k} n_j^{-|s_j|} \sigma_j^{-n_j},$$

where the $s_1, s_2, \ldots, s_k$ are nonzero integers and the $\sigma_j := \mathrm{signum}(s_j)$. Such constants can be considered as generalizations of the Riemann zeta function at integer arguments in higher dimensions.

The analytic evaluation of such sums has relied on fast methods for computing their numerical values. One scheme, based on *Hölder Convolution*, is discussed in [22] and implemented in EZFace+, an online tool available at `http://www.cecm.sfu.ca/projects/ezface+`. We will illustrate its application to one specific case, namely the analytic identification of the sum

$$S_{2,3} = \sum_{k=1}^{\infty} \left(1 - \frac{1}{2} + \cdots + (-1)^{k+1}\frac{1}{k}\right)^2 (k+1)^{-3}. \tag{16}$$

Expanding the squared term in (16), we have

$$(17) \quad \sum_{\substack{0<i,j<k \\ k>0}} \frac{(-1)^{i+j+1}}{ijk^3} = -2\,\zeta(3,-1,-1) + \zeta(3,2).$$

Evaluating this in EZFace+, we quickly obtain

$$S_{2,3} = 0.15616693338117691588103590968798819368577670984030387295752935449707503744029579145520565370935814757 8\ldots.$$

Given this numerical value, PSLQ or some other integer-relation-finding tool can be used to see if this constant satisfies a rational linear relation of certain constants. Our experience with these evaluations has suggested that likely terms would include: π^5, $\pi^4\log(2)$, $\pi^3\log^2(2)$, $\pi^2\log^3(2)$, $\pi\log^4(2)$, $\log^5(2)$, $\pi^2\zeta(3)$, $\pi\log(2)\zeta(3)$, $\log^2(2)\zeta(3)$, $\zeta(5)$, $\mathrm{Li}_5(1/2)$. The result is quickly found to be:

$$S_{2,3} = 4\,\mathrm{Li}_5\left(\frac{1}{2}\right) - \frac{1}{30}\log^5(2) - \frac{17}{32}\zeta(5) - \frac{11}{720}\pi^4\log(2) + \frac{7}{4}\zeta(3)\log^2(2) + \frac{1}{18}\pi^2\log^3(2) - \frac{1}{8}\pi^2\zeta(3).$$

This result has been proven in various ways, both analytic and algebraic. Indeed, all evaluations of sums of the form $\zeta(\pm a_1, \pm a_2, \cdots, \pm a_m)$ with *weight* $w := \sum_k a_m$, for $k < 8$, as in (17) are established.

One general result that is reasonably easily obtained is the following, true for all n:

$$(18) \quad \zeta(\{3\}_n) = \zeta(\{2,1\}_n).$$

On the other hand, a general proof of

$$(19) \quad \zeta(\{2,1\}_n) \stackrel{?}{=} 2^{3n}\,\zeta(\{-2,1\}_n)$$

remains elusive. There has been abundant evidence amassed to support the conjectured identity (19) since it was discovered experimentally in 1996. The first eighty-five instances of (19) were recently affirmed in calculations by Petr Lisoněk to 1000 decimal place accuracy. Lisonek also checked the case $n = 163$, a calculation that required ten hours run time on a 2004-era computer. The only proof known of (18) is a change of variables in a multiple integral representation that sheds no light on (19) (see [21]).

Evaluation of Integrals

This same general strategy of obtaining a high-precision numerical value, then attempting by means of PSLQ or other numeric-constant recognition facilities to identify the result as an analytic expression, has recently been applied with significant success to the age-old problem of evaluating definite integrals. Obviously Maple and Mathematica have some rather effective integration facilities, not only for obtaining analytic results directly, but also for obtaining high-precision numeric values. However, these products do have limitations, and their numeric integration facilities are typically limited to 100 digits or so, beyond which they tend to require an unreasonable amount of run time.

Fortunately, some new methods for numerical integration have been developed that appear to be effective for a broad range of one-dimensional integrals, typically producing up to 1000 digit accuracy in just a few seconds' (or at most a few minutes') run time on a 2004-era personal computer, and that are also well suited for parallel processing [13], [14], [16, p. 312]. These schemes are based on the *Euler-Maclaurin summation* formula [3, p. 180], which can be stated as follows: Let $m \geq 0$ and $n \geq 1$ be integers, and define $h = (b-a)/n$ and $x_j = a + jh$ for $0 \leq j \leq n$. Further assume that the function $f(x)$ is at least $(2m+2)$-times continuously differentiable on $[a,b]$. Then

$$(20) \quad \int_a^b f(x)\,dx = h\sum_{j=0}^{n} f(x_j) - \frac{h}{2}(f(a)+f(b)) - \sum_{i=1}^{m}\frac{h^{2i}B_{2i}}{(2i)!}\left(f^{(2i-1)}(b) - f^{(2i-1)}(a)\right) - E(h),$$

where B_{2i} denote the Bernoulli numbers, and

$$E(h) = \frac{h^{2m+2}(b-a)B_{2m+2}f^{2m+2}(\xi)}{(2m+2)!}$$

for some $\xi \in (a,b)$. In the circumstance where the function $f(x)$ and all of its derivatives are zero at the endpoints a and b (as in a smooth, bell-shaped function), the second and third terms of the Euler-Maclaurin formula (20) are zero, and we conclude that the error $E(h)$ goes to zero more rapidly than any power of h.

This principle is utilized by transforming the integral of some C^∞ function $f(x)$ on the interval $[-1,1]$ to an integral on $(-\infty,\infty)$ using the change of variable $x = g(t)$. Here $g(x)$ is some monotonic, infinitely differentiable function with the property that $g(x) \to 1$ as $x \to \infty$ and $g(x) \to -1$ as $x \to -\infty$, and also with the property that $g'(x)$ and all higher derivatives rapidly approach zero for large positive and negative arguments. In this case we can write, for $h > 0$,

$$\int_{-1}^{1} f(x)\,dx = \int_{-\infty}^{\infty} f(g(t))g'(t)\,dt = h\sum_{j=-\infty}^{\infty} w_j f(x_j) + E(h),$$

where $x_j = g(hj)$ and $w_j = g'(hj)$ are abscissas and weights that can be precomputed. If $g'(t)$ and its derivatives tend to zero sufficiently rapidly for large t, positive and negative, then even in cases where $f(x)$ has a vertical derivative or an integrable singularity at one or both endpoints, the resulting integrand $f(g(t))g'(t)$ is, in many cases, a smooth bell-shaped function for which the Euler-Maclaurin formula applies. In these cases, the error $E(h)$ in this approximation decreases faster than any power of h.

Three suitable g functions are $g_1(t) = \tanh t$, $g_2(t) = \operatorname{erf} t$, and $g_3(t) = \tanh(\pi/2 \cdot \sinh t)$. Among these three, $g_3(t)$ appears to be the most effective for typical experimental math applications. For many integrals, *"tanh-sinh" quadrature*, as the resulting scheme is known, achieves quadratic convergence: reducing the interval h in half roughly doubles the number of correct digits in the quadrature result. This is another case where we have more heuristic than proven knowledge.

As one example, recently the present authors, together with Greg Fee of Simon Fraser University in Canada, were inspired by a recent problem in the *American Mathematical Monthly* [2]. They found by using a tanh-sinh quadrature program, together with a PSLQ integer relation detection program, that if $C(a)$ is defined by

$$C(a) = \int_0^1 \frac{\arctan(\sqrt{x^2+a^2})\,dx}{\sqrt{x^2+a^2}(x^2+1)},$$

then

$$\begin{aligned} C(0) &= \pi \log 2/8 + G/2, \\ C(1) &= \pi/4 - \pi\sqrt{2}/2 + 3\arctan(\sqrt{2})/\sqrt{2}, \\ C(\sqrt{2}) &= 5\pi^2/96. \end{aligned}$$

Here $G = \sum_{k\geq 0}(-1)^k/(2k+1)^2$ is *Catalan's constant*—the simplest number whose irrationality is not established but for which abundant numerical evidence exists. These experimental results then led to the following general result, rigorously established, among others:

$$\begin{aligned} &\int_0^\infty \frac{\arctan(\sqrt{x^2+a^2})\,dx}{\sqrt{x^2+a^2}(x^2+1)} \\ &\quad = \frac{\pi}{2\sqrt{a^2-1}}\left[2\arctan(\sqrt{a^2-1}) - \arctan(\sqrt{a^4-1})\right]. \end{aligned}$$

As a second example, recently the present authors empirically determined that

$$\begin{aligned} &\frac{2}{\sqrt{3}}\int_0^1 \frac{\log^6(x)\arctan[x\sqrt{3}/(x-2)]}{x+1}\,dx = \frac{1}{81648}\,[-229635L_3(8) \\ &+ 29852550L_3(7)\log 3 - 1632960L_3(6)\pi^2 + 27760320L_3(5)\zeta(3) \\ &- 275184L_3(4)\pi^4 + 36288000L_3(3)\zeta(5) - 30008L_3(2)\pi^6 \\ &- 57030120L_3(1)\zeta(7)\,], \end{aligned}$$

where $L_3(s) = \sum_{n=1}^\infty [1/(3n-2)^s - 1/(3n-1)^s]$. Based on these experimental results, general results of this type have been conjectured but not yet rigorously established.

A third example is the following:

$$(21) \qquad \frac{24}{7\sqrt{7}}\int_{\pi/3}^{\pi/2} \log\left|\frac{\tan t + \sqrt{7}}{\tan t - \sqrt{7}}\right| dt \stackrel{?}{=} L_{-7}(2)$$

where

$$\begin{aligned} L_{-7}(s) = \sum_{n=0}^\infty \bigg[&\frac{1}{(7n+1)^s} + \frac{1}{(7n+2)^s} - \frac{1}{(7n+3)^s} \\ &+ \frac{1}{(7n+4)^s} - \frac{1}{(7n+5)^s} - \frac{1}{(7n+6)^s}\bigg]. \end{aligned}$$

The "identity" (21) has been verified to over 5000 decimal digit accuracy, but a proof is not yet known. It arises from the volume of an ideal tetrahedron in hyperbolic space, [15, pp. 90–1]. For algebraic topology reasons, it is known that the ratio of the left-hand to the right-hand side of (21) is rational.

A related experimental result, verified to 1000 digit accuracy, is

$$\begin{aligned} 0 \stackrel{?}{=} &-2J_2 - 2J_3 - 2J_4 + 2J_{10} + 2J_{11} + 3J_{12} + 3J_{13} + J_{14} - J_{15} \\ &-J_{16} - J_{17} - J_{18} - J_{19} + J_{20} + J_{21} - J_{22} - J_{23} + 2J_{25}, \end{aligned}$$

where J_n is the integral in (21), with limits $n\pi/60$ and $(n+1)\pi/60$.

The above examples are ordinary one-dimensional integrals. Two-dimensional integrals are also of interest. Along this line we present a more recreational example discovered experimentally by James Klein—and confirmed by *Monte Carlo* simulation. It is that the expected distance between two random points on different sides of a unit square is

$$\begin{aligned} &\frac{2}{3}\int_0^1\int_0^1 \sqrt{x^2+y^2}\,dx\,dy + \frac{1}{3}\int_0^1\int_0^1 \sqrt{1+(u-v)^2}\,du\,dv \\ &\quad = \frac{1}{9}\sqrt{2} + \frac{5}{9}\log(\sqrt{2}+1) + \frac{2}{9}, \end{aligned}$$

and the expected distance between two random points on different sides of a unit cube is

$$\begin{aligned} &\frac{4}{5}\int_0^1\int_0^1\int_0^1\int_0^1 \sqrt{x^2+y^2+(z-w)^2}\,dw\,dx\,dy\,dz \\ &+\frac{1}{5}\int_0^1\int_0^1\int_0^1\int_0^1 \sqrt{1+(y-u)^2+(z-w)^2}\,du\,dw\,dy\,dz \\ &\quad = \frac{4}{75} + \frac{17}{75}\sqrt{2} - \frac{2}{25}\sqrt{3} - \frac{7}{75}\pi \\ &\qquad + \frac{7}{25}\log\left(1+\sqrt{2}\right) + \frac{7}{25}\log\left(7+4\sqrt{3}\right). \end{aligned}$$

See [7] for details and some additional examples. It is not known whether similar closed forms exist for higher-dimensional cubes.

Ramanujan's AGM Continued Fraction

Given $a, b, \eta > 0$, define

$$R_\eta(a,b) = \cfrac{a}{\eta + \cfrac{b^2}{\eta + \cfrac{4a^2}{\eta + \cfrac{9b^2}{\eta + \ddots}}}}.$$

This continued fraction arises in Ramanujan's *Notebooks*. He discovered the beautiful fact that

$$\frac{R_\eta(a,b) + R_\eta(b,a)}{2} = R_\eta\left(\frac{a+b}{2}, \sqrt{ab}\right).$$

The authors wished to record this in [15] and wished to computationally check the identity. A first attempt to numerically compute $R_1(1,1)$ directly failed miserably, and with some effort only three reliable digits were obtained: 0.693…. With hindsight, the slowest convergence of the fraction occurs in the mathematically simplest case, namely when $a = b$. Indeed $R_1(1,1) = \log 2$, as the first primitive numerics had tantalizingly suggested.

Attempting a direct computation of $R_1(2,2)$ using a depth of 20000 gives us two digits. Thus we must seek more sophisticated methods. From formula (1.11.70) of [16] we see that for $0 < b < a$,

$$\mathcal{R}_1(a,b) = \frac{\pi}{2}\sum_{n\in\mathbb{Z}} \frac{aK(k)}{K^2(k) + a^2n^2\pi^2}\operatorname{sech}\left(n\pi\frac{K(k')}{K(k)}\right), \quad (22)$$

where $k = b/a = \theta_2^2/\theta_3^2, k' = \sqrt{1-k^2}$. Here θ_2, θ_3 are Jacobian theta functions and K is a complete elliptic integral of the first kind.

Writing the previous equation as a Riemann sum, we have

$$\begin{aligned}\mathcal{R}(a) := \mathcal{R}_1(a,a) &= \int_0^\infty \frac{\operatorname{sech}(\pi x/(2a))}{1+x^2}dx \\ &= 2a\sum_{k=1}^\infty \frac{(-1)^{k+1}}{1+(2k-1)a},\end{aligned} \quad (23)$$

where the final equality follows from the Cauchy-Lindelof Theorem. This sum may also be written as $\mathcal{R}(a) = \frac{2a}{1+a}F\left(\frac{1}{2a}+\frac{1}{2}, 1; \frac{1}{2a}+\frac{3}{2}; -1\right)$. The latter form can be used in Maple or Mathematica to determine

$$\mathcal{R}(2) = 0.97499098879872209671990033452 9\ldots.$$

This constant, as written, is a bit difficult to recognize, but if one first divides by $\sqrt{2}$, one can obtain, using the *Inverse Symbolic Calculator*, an online tool available at the URL `http://www.cecm.sfu.ca/projects/ISC/ISCmain.html`, that the quotient is $\pi/2 - \log(1+\sqrt{2})$. Thus we conclude, experimentally, that

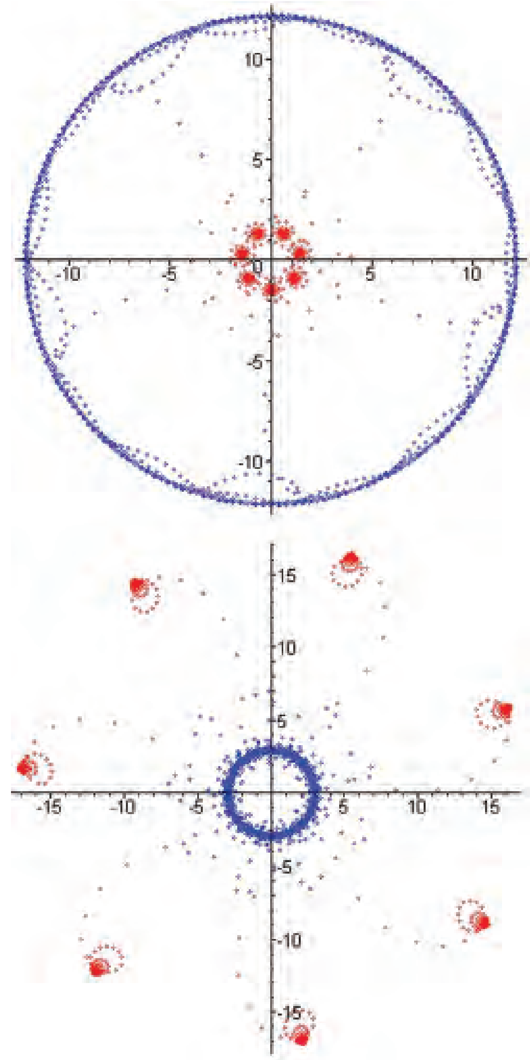

Figure 2. Dynamics and attractors of various iterations.

$$\mathcal{R}(2) = \sqrt{2}[\pi/2 - \log(1+\sqrt{2})].$$

Indeed, it follows (see [19]) that

$$\mathcal{R}(a) = 2\int_0^1 \frac{t^{1/a}}{1+t^2}dt.$$

Note that $\mathcal{R}(1) = \log 2$. No nontrivial closed form is known for $\mathcal{R}(a,b)$ with $a \neq b$, although

$$\mathcal{R}_1\left(\frac{1}{4\pi}\beta\left(\frac{1}{4},\frac{1}{4}\right), \frac{\sqrt{2}}{8\pi}\beta\left(\frac{1}{4},\frac{1}{4}\right)\right) = \frac{1}{2}\sum_{n\in\mathbb{Z}}\frac{\operatorname{sech}(n\pi)}{1+n^2}$$

is close to closed. Here β denotes the classical *Beta function*. It would be pleasant to find a direct proof of (23). Further details are to be found in [19], [17], [16].

Study of these Ramanujan continued fractions has been facilitated by examining the closely related dynamical system $t_0 = 1, t_1 = 1$, and

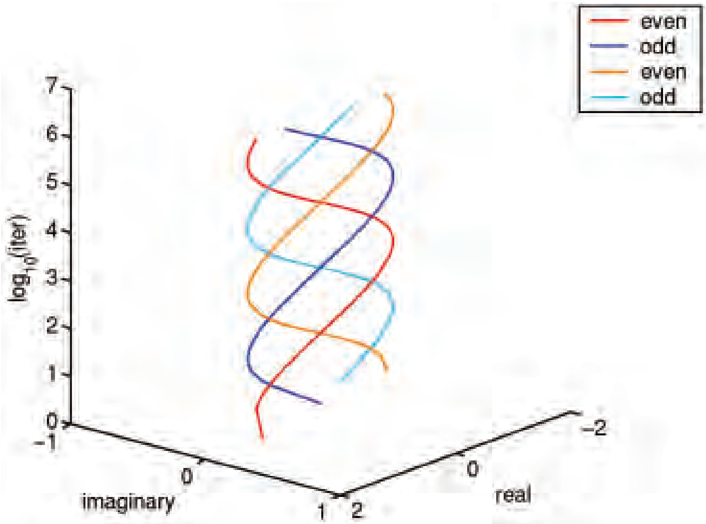

Figure 3. The subtle fourfold serpent.

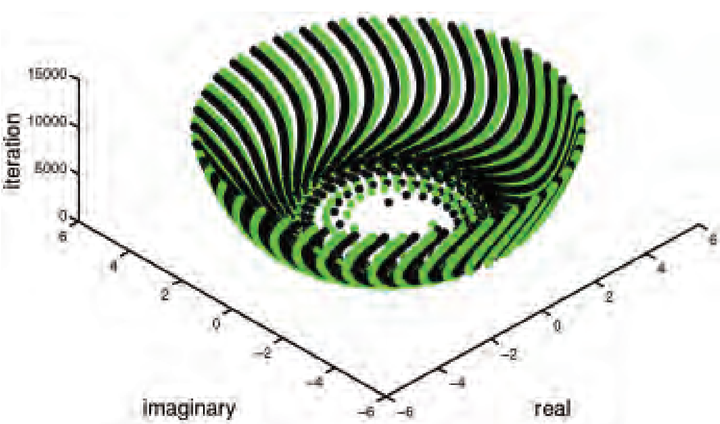

Figure 4. A period three dynamical system (odd and even iterates).

$$(24) \qquad t_n := t_n(a,b) = \frac{1}{n} + \omega_{n-1}\left(1 - \frac{1}{n}\right) t_{n-2},$$

where $\omega_n = a^2$ or b^2 (from the Ramanujan continued fraction definition), depending on whether n is even or odd.

If one studies this based only on numerical values, nothing is evident; one only sees that $t_n \to 0$ fairly slowly. However, if we look at this iteration pictorially, we learn significantly more. In particular, if we plot these iterates in the complex plane and then scale by $\sqrt{n}$ and color the iterations blue or red depending on odd or even n, then some remarkable fine structures appear; see Figure 2. With assistance of such plots, the behavior of these iterates (and the Ramanujan continued fractions) is now quite well understood. These studies have ventured into matrix theory, real analysis, and even the theory of martingales from probability theory [19], [17], [18], [23].

There are some exceptional cases. *Jacobsen-Masson theory* [17], [18] shows that the even/odd fractions for $\mathcal{R}_1(i,i)$ behave "chaotically"; neither converge. Indeed, when $a = b = i$, $(t_n(i,i))$ exhibit a fourfold quasi-oscillation, as n runs through values mod 4. Plotted versus n, the (real) sequence $t_n(\mathbf{i})$ exhibits the serpentine oscillation of four separate "necklaces". The detailed asymptotic is

$$t_n(i,i) = \sqrt{\frac{2}{\pi}\cosh\frac{\pi}{2}}\,\frac{1}{\sqrt{n}}\left(1 + O\left(\frac{1}{n}\right)\right) \times \begin{cases} (-1)^{n/2}\cos(\theta - \log(2n)/2) & n \text{ is even} \\ (-1)^{(n+1)/2}\sin(\theta - \log(2n)/2) & n \text{ odd} \end{cases}$$

where $\theta := \arg\Gamma((1+i)/2)$.

Analysis is easy given the following striking hypergeometric parametrization of (24) when $a = b \neq 0$ (see [18]), which was both *experimentally discovered* and is *computer provable*:

$$(25) \qquad t_n(a,a) = \frac{1}{2}F_n(a) + \frac{1}{2}F_n(-a),$$

where

$$F_n(a) := -\frac{a^n 2^{1-\omega}}{\omega\,\beta(n+\omega,-\omega)}\,{}_2F_1\left(\omega,\omega;n+1+\omega;\frac{1}{2}\right).$$

Here

$$\beta(n+1+\omega,-\omega) := \frac{\Gamma(n+1)}{\Gamma(n+1+\omega)\Gamma(-\omega)}, \quad \text{and} \quad \omega := \frac{1-1/a}{2}.$$

Indeed, once (25) was discovered by a combination of insight and methodical computer experiment, its proof became highly representative of the changing paradigm: both sides satisfy the same recursion and the same initial conditions. This can be checked in Maple, and if one looks inside the computation, one learns which *confluent hypergeometric identities* are needed for an explicit human proof.

As noted, study of $\mathcal{R}$ devolved to *hard but compelling* conjectures on complex dynamics, with many interesting *proven* and *unproven* generalizations. In [23] consideration is made of continued fractions like

$$S_1(a) = \cfrac{1^2 a_1^2}{1 + \cfrac{2^2 a_2^2}{1 + \cfrac{3^2 a_3^2}{1 + \ddots}}}$$

for *any* sequence $a \equiv (a_n)_{n=1}^{\infty}$ and convergence properties obtained for deterministic and random sequences (a_n). For the deterministic case the best results obtained are for periodic sequences, satisfying $a_j = a_{j+c}$ for all j and some finite c. The dynamics are considerably more varied, as illustrated in Figure 4.

Coincidence and Fraud

Coincidences do occur, and such examples drive home the need for reasonable caution in this enterprise. For example, the approximations

$$\pi \;\approx\; \frac{3}{\sqrt{163}}\log(640320), \quad \pi \;\approx\; \sqrt{2}\,\frac{9801}{4412}$$

occur for deep number theoretic reasons: the first good to fifteen places, the second to eight. By contrast

$$e^{\pi} - \pi \;=\; \mathbf{19.999099979}189475768\ldots,$$

most probably for no good reason. This seemed more bizarre on an eight-digit calculator. Likewise, as spotted by Pierre Lanchon recently,

$$e \;=\; \overline{\mathbf{10.10110111111000010}}101000101100\ldots$$

while

$$\pi \;=\; 11.0010\overline{\mathbf{0100001111110110101}}01000\ldots$$

have 19 bits agreeing in base two—with one reading right to left. More extended coincidences are almost always contrived, as illustrated by the following:

$$\sum_{n=1}^{\infty}\frac{[n\tanh(\pi/2)]}{10^n} \approx \frac{1}{81}, \quad \sum_{n=1}^{\infty}\frac{[n\tanh(\pi)]}{10^n} \approx \frac{1}{81}.$$

The first holds to **12** decimal places, while the second holds to **268** places. This phenomenon can be understood by examining the continued fraction expansion of the constants $\tanh(\pi/2)$ and $\tanh(\pi)$: the integer **11** appears as the third entry of the first, while **267** appears as the third entry of the second.

Bill Gosper, commenting on the extraordinary effectiveness of continued-fraction expansions to "see" what is happening in such problems, declared, "It looks like you are cheating God somehow."

A fine illustration is the unremarkable decimal $\alpha = 1.4331274267223117583\ldots$ whose continued fraction begins $[1, 2, 3, 4, 5, 6, 7, 8, 9\ldots]$ and so most probably is a ratio of Bessel functions. Indeed, $I_0(2)/I_1(2)$ was what generated the decimal. Similarly, π and e are quite different as continued fractions, less so as decimals.

A more sobering example of high-precision "fraud" is the integral

$$\pi_2 := \int_0^{\infty}\cos(2x)\prod_{n=1}^{\infty}\cos\left(\frac{x}{n}\right)dx. \tag{26}$$

The computation of a high-precision numerical value for this integral is rather challenging, due in part to the oscillatory behavior of $\prod_{n\ge1}\cos(x/n)$ (see Figure 2), but mostly due to the difficulty of computing high-precision evaluations of the integrand function. Note that evaluating thousands of terms of the infinite product would produce only a few correct digits. Thus it is necessary to rewrite the integrand function in a form more suitable for computation. This can be done by writing

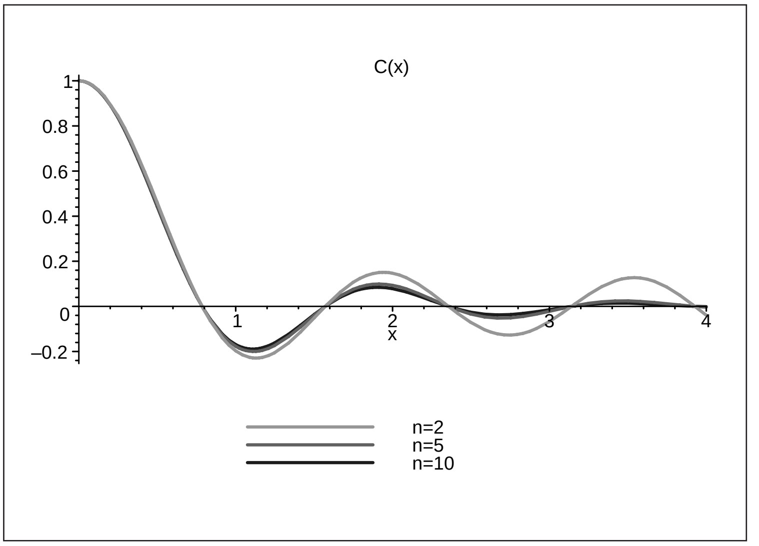

Figure 5. First few terms of $\prod_{n\ge1}\cos(x/k)$.

$$f(x) = \cos(2x)\left[\prod_{1}^{m}\cos(x/k)\right]\exp(f_m(x)), \tag{27}$$

where we choose $m > x$, and where

$$f_m(x) = \sum_{k=m+1}^{\infty}\log\cos\left(\frac{x}{k}\right). \tag{28}$$

The log cos evaluation can be expanded in a Taylor series [1, p. 75], as follows:

$$\log\cos\left(\frac{x}{k}\right) \;=\; \sum_{j=1}^{\infty}\frac{(-1)^j 2^{2j-1}(2^{2j}-1)B_{2j}}{j(2j)!}\left(\frac{x}{k}\right)^{2j},$$

where B_{2j} are *Bernoulli numbers*. Note that since $k > m > x$ in (28), this series converges. We can now write

$$\begin{aligned} f_m(x) &= \sum_{k=m+1}^{\infty}\sum_{j=1}^{\infty}\frac{(-1)^j 2^{2j-1}(2^{2j}-1)B_{2j}}{j(2j)!}\left(\frac{x}{k}\right)^{2j} \\ &= -\sum_{j=1}^{\infty}\frac{(2^{2j}-1)\zeta(2j)}{j\pi^{2j}}\left[\sum_{k=m+1}^{\infty}\frac{1}{k^{2j}}\right]x^{2j} \\ &= -\sum_{j=1}^{\infty}\frac{(2^{2j}-1)\zeta(2j)}{j\pi^{2j}}\left[\zeta(2j)-\sum_{k=1}^{m}\frac{1}{k^{2j}}\right]x^{2j}. \end{aligned}$$

This can now be written in a compact form for computation as

$$f_m(x) = -\sum_{j=1}^{\infty}a_j b_{j,m}x^{2j}, \tag{29}$$

where

$$\begin{aligned} a_j &= \frac{(2^{2j}-1)\zeta(2j)}{j\pi^{2j}}, \\ b_{j,m} &= \zeta(2j) - \sum_{k=1}^{m}1/k^{2j}. \end{aligned} \tag{30}$$

Figure 6. Advanced Collaborative Environment in Vancouver.

Computation of these b coefficients must be done to a much higher precision than that desired for the quadrature result, since two very nearly equal quantities are subtracted here.

The integral can now be computed using, for example, the tanh-sinh quadrature scheme. The first 60 digits of the result are the following:

$$0.392699081698724154807830422909937860524645434187231595926812\ldots.$$

At first glance, this appears to be $\pi/8$. But a careful comparison with a high-precision value of $\pi/8$, namely

$$0.392699081698724154807830422909937860524646174921888227621868\ldots,$$

reveals that they are *not* equal: the two values differ by approximately 7.407×10^{-43}. Indeed, these two values are provably distinct. The reason is governed by the fact that $\sum_{n=1}^{55} 1/(2n+1) > 2 > \sum_{n=1}^{54} 1/(2n+1)$. See [16, Chap. 2] for additional details.

A related example is the following. Recall the *sinc* function

$$\operatorname{sinc}(x) := \frac{\sin x}{x}.$$

Consider the seven highly oscillatory integrals below.

$$I_1 := \int_0^\infty \operatorname{sinc}(x)\,dx = \frac{\pi}{2},$$
$$I_2 := \int_0^\infty \operatorname{sinc}(x)\operatorname{sinc}\left(\frac{x}{3}\right)dx = \frac{\pi}{2},$$
$$I_3 := \int_0^\infty \operatorname{sinc}(x)\operatorname{sinc}\left(\frac{x}{3}\right)\operatorname{sinc}\left(\frac{x}{5}\right)dx = \frac{\pi}{2},$$
$$\ddots$$
$$I_6 := \int_0^\infty \operatorname{sinc}(x)\operatorname{sinc}\left(\frac{x}{3}\right)\cdots\operatorname{sinc}\left(\frac{x}{11}\right)dx = \frac{\pi}{2},$$
$$I_7 := \int_0^\infty \operatorname{sinc}(x)\operatorname{sinc}\left(\frac{x}{3}\right)\cdots\operatorname{sinc}\left(\frac{x}{13}\right)dx = \frac{\pi}{2}.$$

However,

$$I_8 := \int_0^\infty \operatorname{sinc}(x)\operatorname{sinc}\left(\frac{x}{3}\right)\cdots\operatorname{sinc}\left(\frac{x}{15}\right)dx = \frac{467807924713440738696537864469}{935615849440640907310521750000}\pi \approx 0.499999999992646\pi.$$

When this was first found by a researcher using a well-known computer algebra package, both he and the software vendor concluded there was a "bug" in the software. Not so! It is easy to see that the limit of these integrals is $2\pi_1$, where

$$\pi_1 := \int_0^\infty \cos(x)\prod_{n=1}^{\infty}\cos\left(\frac{x}{n}\right)dx. \qquad (31)$$

This can be seen via *Parseval's theorem*, which links the integral

$$I_N := \int_0^\infty \operatorname{sinc}(a_1x)\operatorname{sinc}(a_2x)\cdots\operatorname{sinc}(a_Nx)\,dx$$

with the volume of the polyhedron P_N given by

$$P_N := \{x : |\sum_{k=2}^{N} a_kx_k| \le a_1, |x_k| \le 1, 2 \le k \le N\},$$

where $x := (x_2, x_3, \cdots, x_N)$. If we let

$$C_N := \{(x_2, x_3, \cdots, x_N) : -1 \le x_k \le 1, 2 \le k \le N\},$$

then

$$I_N = \frac{\pi}{2a_1}\frac{\operatorname{Vol}(P_N)}{\operatorname{Vol}(C_N)}.$$

Thus, the value drops precisely when the constraint $\sum_{k=2}^{N} a_kx_k \le a_1$ becomes *active* and bites the hypercube C_N. That occurs when $\sum_{k=2}^{N} a_k > a_1$. In the above, $\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{13} < 1$, but on addition of the term $\frac{1}{15}$, the sum exceeds 1, the volume drops, and $I_N = \frac{\pi}{2}$ no longer holds. A similar analysis applies to π_2. Moreover, it is fortunate that we began with π_1 or the falsehood of the identity analogous to that displayed above would have been much harder to see.

Further Directions and Implications

In spite of the examples of the previous section, it must be acknowledged that computations can in many cases provide very compelling evidence for mathematical assertions. As a single example, recently Yasumasa Kanada of Japan calculated π to over one trillion decimal digits (and also to over one trillion hexadecimal digits). Given that such computations—which take many hours on large, state-of-the-art supercomputers—are prone to many types of error, including hardware failures, system software problems, and especially programming bugs, how can one be confident in such results?

In Kanada's case, he first used two different arctangent-based formulas to evaluate π to over one trillion hexadecimal digits. Both calculations agreed that the hex expansion beginning at position 1,000,000,000,001 is B4466E8D21 5388C4E014. He then applied a variant of the BBP formula for π, mentioned in Section 3, to calculate these hex digits directly. The result agreed exactly. Needless to say, it is exceedingly unlikely that three different computations, each using a completely distinct computational approach, would all perfectly agree on these digits unless all three are correct.

Another, much more common, example is the usage of probabilistic primality testing schemes. Damgard, Landrock, and Pomerance showed in 1993 that if an integer n has k bits, then the probability that it is prime, provided it passes the most commonly used probabilistic test, is greater than $1 - k^2 4^{2-\sqrt{k}}$, and for certain k is even higher [25]. For instance, if n has 500 bits, then this probability is greater than $1 - 1/4^{28m}$. Thus a 500-bit integer that passes this test even once is prime with prohibitively safe odds: the chance of a false declaration of primality is less than one part in Avogadro's number (6×10^{23}). If it passes the test for four pseudorandomly chosen integers a, then the chance of false declaration of primality is less than one part in a googol (10^{100}). Such probabilities are many orders of magnitude more remote than the chance that an undetected hardware or software error has occurred in the computation. Such methods thus draw into question the distinction between a probabilistic test and a "provable" test.

Another interesting question is whether these experimental methods may be capable of discovering facts that are fundamentally beyond the reach of formal proof methods, which, due to Gödel's result, we know must exist; see also [24].

One interesting example, which has arisen in our work, is the following. We mentioned in Section 3 the fact that the question of the 2-normality of π reduces to the question of whether the chaotic iteration $x_0 = 0$ and

$$x_n = \left\{16x_{n-1} + \frac{120n^2 - 89n + 16}{512n^4 - 1024n^3 + 712n^2 - 206n + 21}\right\},$$

where $\{\cdot\}$ denotes fractional part, are equidistributed in the unit interval.

It turns out that if one defines the sequence $y_n = \lfloor 16x_n \rfloor$ (in other words, one records which of the 16 subintervals of $(0, 1)$, numbered 0 through 15, x_n lies in), that the sequence (y_n), when interpreted as a hexadecimal string, appears to precisely generate the hexadecimal digit expansion of π. We have checked this to 1,000,000 hex digits and have found no discrepancies. It is known that (y_n) is a very good approximation to the hex digits of π, in the sense that the expected value of the number of errors is finite [15, Section 4.3] [11]. Thus one can argue, by the second Borel-Cantelli lemma, that in a heuristic sense the probability that there

Figure 7. Polyhedra in an immersive environment.

is any error among the remaining digits after the first million is less than 1.465×10^{-8} [15, Section 4.3]. Additional computations could be used to lower this probability even more.

Although few would bet against such odds, these computations do not constitute a rigorous proof that the sequence (y_n) is identical to the hexadecimal expansion of π. Perhaps someday someone will be able to prove this observation rigorously. On the other hand, maybe not—maybe this observation is in some sense an "accident" of mathematics, for which no proof will ever be found. Perhaps numerical validation is all we can ever achieve here.

Conclusion

We are only now beginning to digest some very old ideas:

> Leibniz's idea is very simple and very profound. It's in section VI of the *Discours [de métaphysique]*. It's the observation that the concept of law becomes vacuous if arbitrarily high mathematical complexity is permitted, for then there is always a law. Conversely, if the law has to be extremely complicated, then the data is irregular, lawless, random, unstructured, patternless, and also incompressible and irreducible. A theory has to be simpler than the data that it explains, otherwise it doesn't explain anything. —Gregory Chaitin [24]

Chaitin argues convincingly that there are many mathematical truths which are logically and computationally irreducible—they have *no good reason* in the traditional rationalist sense. This in turn adds force to the desire for evidence even when proof may not be possible. Computer experiments can provide precisely the sort of evidence that is required.

Although computer technology had its roots in mathematics, the field is a relative latecomer to the application of computer technology, compared, say, with physics and chemistry. But now this is changing, as an army of young mathematicians, many of whom have been trained in the usage of sophisticated computer math tools from their high school years, begin their research careers. Further advances in software, including compelling new mathematical visualization environments (see Figures 6 and 7), will have their impact. And the remarkable trend towards greater miniaturization (and corresponding higher power and lower cost) in computer technology, as tracked by Moore's Law, is pretty well assured to continue for at least another ten years, according to Gordon Moore himself and other industry analysts. As Richard Feynman noted back in 1959, "There's plenty of room at the bottom" [27]. It will be interesting to see what the future will bring.

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JONATHAN M. BORWEIN

THE EXPERIMENTAL MATHEMATICIAN: THE PLEASURE OF DISCOVERY AND THE ROLE OF PROOF

> '...where almost one quarter hour was spent, each beholding the other with admiration before one word was spoken: at last Mr. Briggs began "My Lord, I have undertaken this long journey purposely to see your person, and to know by what wit or ingenuity you first came to think of this most excellent help unto Astronomy, viz. the Logarithms: but my Lord, being by you found out, I wonder nobody else found it out before, when now being known it appears so easy." '[1]

ABSTRACT. The emergence of powerful mathematical computing environments, the growing availability of correspondingly powerful (multi-processor) computers and the pervasive presence of the internet allow for mathematicians, students and teachers, to proceed heuristically and 'quasi-inductively'. We may increasingly use symbolic and numeric computation, visualization tools, simulation and data mining. The unique features of our discipline make this both more problematic and more challenging. For example, there is still no truly satisfactory way of displaying mathematical notation on the web; and we care more about the reliability of our literature than does any other science. The traditional role of proof in mathematics is arguably under siege – for reasons both good and bad.

AMS Classifications: 00A30, 00A35, 97C50

KEY WORDS: aesthetics, constructivism, experimental mathematics, humanist philosophy, insight, integer relations, proof

1. EXPERIMENTAL MATH: AN INTRODUCTION

> "There is a story told of the mathematician Claude Chevalley (1909–1984), who, as a true Bourbaki, was extremely opposed to the use of images in geometric reasoning.
>
> He is said to have been giving a very abstract and algebraic lecture when he got stuck. After a moment of pondering, he turned to the blackboard, and, trying to hide what he was doing, drew a little diagram, looked at it for a moment, then quickly erased it, and turned back to the audience and proceeded with the lecture...

> ...The computer offers those less expert, and less stubborn than Chevalley, access to the kinds of images that could only be imagined in the heads of the most gifted mathematicians, ..." (Nathalie Sinclair[2])

For my coauthors and I, *Experimental Mathematics* (Borwein and Bailey, 2003) connotes the use of the computer for some or all of:

1. Gaining insight and intuition.
2. Discovering new patterns and relationships.
3. Graphing to expose math principles.
4. Testing and especially falsifying conjectures.
5. Exploring a possible result to see if it *merits* formal proof.
6. Suggesting approaches for formal proof.
7. Computing replacing lengthy hand derivations.
8. Confirming analytically derived results.

This process is studied very nicely by Nathalie Sinclair in the context of pre-service teacher training.[3] Limned by examples, I shall also raise questions such as:

What constitutes secure mathematical knowledge? When is computation convincing? Are humans less fallible? What tools are available? What methodologies? What about the 'law of the small numbers'? How is mathematics actually done? How should it be? Who cares for certainty? What is the role of proof?

And I shall offer some personal conclusions from more than twenty years of intensive exploitation of the computer as an adjunct to mathematical discovery.

1.1. *The Centre for Experimental Math*

About 12 years ago I was offered the signal opportunity to found the *Centre for Experimental and Constructive Mathematics* (CECM) at Simon Fraser University. On its web-site (www.cecm.sfu.ca) I wrote

> "At CECM we are interested in developing methods for exploiting mathematical computation as a tool in the development of mathematical intuition, in hypotheses building, in the generation of symbolically assisted proofs, and in the construction of a flexible computer environment in which researchers and research students can undertake such research. That is, in doing 'Experimental Mathematics.'"

The decision to build CECM was based on: (i) more than a decade's personal experience, largely since the advent of the personal computer, of the value of computing as an adjunct to mathematical insight and correctness; (ii) on a growing conviction that the future of mathematics would rely much more on collaboration and intelligent computation; (iii) that such developments needed to be enshrined in, and were equally valuable for, mathematical education; and (iv) that experimental mathematics is *fun*.

A decade or more later, my colleagues and I are even more convinced of the value of our venture – and the 'mathematical universe is unfolding' much as we anticipated. Our efforts and philosophy are described in some detail in the recent books (Borwein and Bailey, 2003; Borwein et al., 2004) and in the survey articles (Borwein et al., 1996; Borwein and Carless, 1999; Bailey and Borwein, 2000; Borwein and Borwein, 2001). More technical accounts of some of our tools and successes are detailed in (Borwein and Bradley (1997) and Borwein and Lisonĕk (2000). About 10 years ago the term 'experimental mathematics' was often treated as an oxymoron. Now there is a highly visible and high quality journal of the same name. About 15 years ago, most self-respecting research pure mathematicians would not admit to using computers as an adjunct to research. Now they will talk about the topic whether or not they have any expertise. The centrality of information technology to our era and the growing need for concrete implementable answers suggests why we had attached the word 'Constructive' to CECM – and it motivated my recent move to Dalhousie to establish a new *Distributed Research Institute and Virtual Environment*, D-DRIVE (www.cs.dal.ca/ddrive).

While some things have happened much more slowly than we guessed (e.g., good character recognition for mathematics, any substantial impact on classroom parole) others have happened much more rapidly (e.g., the explosion of the world wide web[4], the quality of graphics and animations, the speed and power of computers). Crudely, the tools with broad societal or economic value arrive rapidly, those interesting primarily in our niche do not.

Research mathematicians for the most part neither think deeply about nor are terribly concerned with either pedagogy or the philosophy of mathematics. Nonetheless, aesthetic and philosophical notions have always permeated (pure and applied) mathematics. And the top researchers have always been driven by an aesthetic imperative:

> "We all believe that mathematics is an art. The author of a book, the lecturer in a classroom tries to convey the structural beauty of mathematics to his readers, to his listeners. In this attempt, he must always fail. Mathematics is logical to be sure,

> each conclusion is drawn from previously derived statements. Yet the whole of it, the real piece of art, is not linear; worse than that, its perception should be instantaneous. We have all experienced on some rare occasions the feeling of elation in realizing that we have enabled our listeners to see at a moment's glance the whole architecture and all its ramifications." (Emil Artin, 1898–1962)[5]

Elsewhere, I have similarly argued for aesthetics before utility (Borwein, 2004, in press). The opportunities to tie research and teaching to aesthetics are almost boundless – at all levels of the curriculum.[6] This is in part due to the increasing power and sophistication of visualization, geometry, algebra and other mathematical software. That said, in my online lectures and resources,[7] and in many of the references one will find numerous examples of the utility of experimental mathematics.

In this article, my primary concern is to explore the relationship between proof (deduction) and experiment (induction). I borrow quite shamelessly from my earlier writings.

There is a disconcerting pressure at all levels of the curriculum to derogate the role of proof. This is in part motivated by the aridity of some traditional teaching (e.g., of Euclid), by the alternatives now being offered by good software, by the difficulty of teaching and learning the tools of the traditional trade, and perhaps by laziness.

My own attitude is perhaps best summed up by a cartoon in a book on learning to program in APL (a very high level language). The blurb above reads *Remember 10 minutes of computation is worth 10 hours of thought*. The blurb below reads *Remember 10 minutes of thought is worth 10 hours of computation*. Just as the unlived life is not much worth examining, proof and rigour should be in the service of things worth proving. And equally foolish, but pervasive, is encouraging students to 'discover' fatuous generalizations of uninteresting facts. As an antidote, In Section 2, I start by discussing and illustrating a few of George Polya's views. Before doing so, I review the structure of this article.

Section 2 discusses some of George Polya's view on heuristic mathematics, while Section 3 visits opinions of various eminent mathematicians. Section 4 discusses my own view and their genesis. Section 5 contains a set of mathematical examples amplifying the prior discussion. Sections 6 and 7 provide two fuller examples of computer discovery, and in Section 8 I return to more philosophical matters – in particular, a discussion of proof versus truth and the nature of secure mathematical knowledge.

2. POLYA ON PICTURE-WRITING

> "[I]ntuition comes to us much earlier and with much less outside influence than formal arguments which we cannot really understand unless we have reached a relatively high level of logical experience and sophistication." (Geroge Polya)[8]

Polya, in his engaging eponymous 1956 *American Mathematical Monthly* article on picture writing, provided three provoking examples of converting pictorial representations of problems into generating function solutions:

1. *In how many ways can you make change for a dollar?*
   This leads to the (US currency) generating function

$$\sum_{k=1}^{\infty} P_k x^k = \frac{1}{(1-x^1)(1-x^5)(1-x^{10})(1-x^{25})(1-x^{50})},$$

   which one can easily expand using a *Mathematica* command,
   Series $[1/((1-x) \times (1-x\hat{\ }5) \times (1-x\hat{\ }10) \times (1-x\hat{\ }25) \times (1-x\hat{\ }50)),$ $\{x, 0, 100\}]$
   to obtain $P_{100} = 292$ (243 for Canadian currency, which lacks a 50 cent piece but has a dollar coin). Polya's diagram is shown in Figure 1.[9]
   To see why we use geometric series and consider the so-called *ordinary generating function*

$$\frac{1}{1-x^{10}} = 1 + x^{10} + x^{20} + x^{30} + \cdots$$

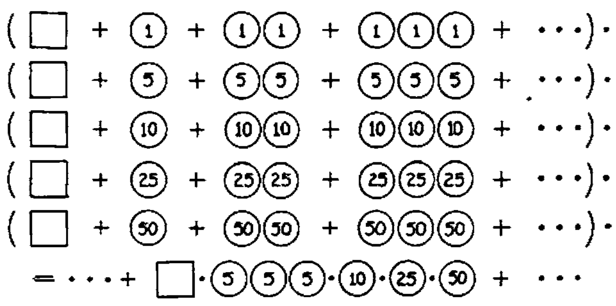

Figure 1. Polya's illustration of the change solution.

for dimes and

$$\frac{1}{1-x^{25}} = 1 + x^{25} + x^{50} + x^{75} + \cdots$$

for quarters, etc. If we multiply these two together and compare coefficients, we get

$$\frac{1}{1-x^{10}} \times \frac{1}{1-x^{25}} = 1 + x^{10} + x^{20} + x^{25} + x^{30} + x^{35} + x^{40} + x^{45} + 2x^{50} + x^{55} + 2x^{60} + \cdots$$

and can argue that the coefficient of x^{60} on the right is precisely the number of ways of making 60 cents out of identical dimes and quarters.

This is easy to check with a handful of change or a calculator and the more general question with more denominations is handled similarly. I leave it to the reader to decide whether it is easier to decode the generating function from the picture or vice versa. In any event, symbolic and graphic experiment can provide abundant and mutual reinforcement and assistance in concept formation.

2. *Dissect a polygon with n sides into $n-2$ triangles by $n-3$ diagonals and compute D_n, the number of different dissections of this kind.*

This leads to the fact that the generating function for $D_3 = 1$, $D_4 = 2$, $D_5 = 5$, $D_6 = 14$, $D_7 = 42$, ...

$$D(x) = \sum_{k=1}^{\infty} D_k x^k$$

satisfies

$$D(x) = x[1 + D(x)]^2,$$

whose solution is therefore

$$D(x) = \frac{1 - 2x - \sqrt{1-4x}}{x}$$

and D_{n+2} turns out to be the n-th Catalan number $\binom{2n}{n}/(n+1)$.

3. *Compute T_n, the number of different (rooted) trees with n knots.* The generating function of the T_n becomes a remarkable result due to Cayley:

$$T(x) = \sum_{k=1}^{\infty} T_k x^k = x \prod_{k=1}^{\infty} (1 - x^k)^{-T_k}, \qquad (1)$$

where remarkably the product and the sum share their coefficients. This produces a recursion for T_n in terms of $T_1, T_2, \ldots, T_{n-1}$, which starts: $T_1 = 1$, $T_2 = 1$, $T_3 = 2$, $T_4 = 4$, $T_5 = 9$, $T_6 = 20, \ldots$

In each case, Polya's main message is that one can usefully draw pictures of the component elements – (a) in pennies, nickels dimes and quarters (plus loonies in Canada and half dollars in the US), (b) in triangles and (c) in the simplest trees (e.g., those with the fewest branches).

> "In the first place, the beginner must be convinced that proofs deserve to be studied, that they have a purpose, that they are interesting." (George Polya)[10]

While by 'beginner' George Polya largely intended young school students, I suggest that this is equally true of anyone engaging for the first time with an unfamiliar topic in mathematics.

3. GAUSS, HADAMARD AND HARDY'S VIEWS

Three of my personal mathematical heroes, very different men from different times, all testify interestingly on these points and on the nature of mathematics.

3.1. *Carl Friedrich Gauss*

Carl Friedrich Gauss (1777–1855) wrote in his diary[11]

> "I have the result, but I do not yet know how to get it."

Ironically I have been unable to find the precise origin of this quote.

One of Gauss's greatest discoveries, in 1799, was the relationship between the lemniscate sine function and the arithmetic-geometric mean iteration. This was based on a purely computational observation. The young Gauss wrote in his diary that the result "*will surely open up a whole new field of analysis.*"

He was right, as it prised open the whole vista of 19th century elliptic and modular function theory. Gauss's specific discovery, based on tables of integrals provided by Stirling (1692–1770), was that the reciprocal of the integral

$$v\frac{2}{\pi}\int_0^1 \frac{dt}{\sqrt{1-t^4}}$$

agreed numerically with the limit of the rapidly convergent iteration given by $a_0 := 1$, $b_0 := \sqrt{2}$ and computing

$$a_{n+1} := \frac{a_n + b_n}{2}, \quad b_{n+1} := \sqrt{a_n b_n}.$$

The sequences a_n, b_n have a common limit 1.1981402347355922074...

Which object, the integral or the iteration, is more familiar, which is more elegant – then and now? Aesthetic criteria change: 'closed forms' have yielded centre stage to 'recursion', much as biological and computational metaphors (even 'biology envy') have replaced Newtonian mental images with Richard Dawkin's 'blind watchmaker'.

This experience of 'having the result' is reflective of much research mathematics. Proof and rigour play the role described next by Hadamard. Likewise, the back-handed complement given by Briggs to Napier underscores that is often harder to discover than to explain or digest the new discovery.

3.2. *Jacques Hadamard*

A constructivist, experimental and aesthetic driven rationale for mathematics could hardly do better than to start with:

> "The object of mathematical rigor is to sanction and legitimize the conquests of intuition, and there was never any other object for it." (J. Hadamard[12])

Jacques Hadamard (1865–1963) was perhaps the greatest mathematician to think deeply and seriously about cognition in mathematics[13]. He is quoted as saying "... *in arithmetic, until the seventh* *grade, I was last or nearly last''* which should give encouragement to many young students.

Hadamard was both the author of ''The psychology of invention in the mathematical field'' (1945), a book that still rewards close inspection, and co-prover of the *Prime Number Theorem* (1896):

> ''The number of primes less than n tends to ∞ as does $n/\log n$.''

This was one of the culminating results of 19th century mathematics and one that relied on much preliminary computation and experimentation.

One rationale for experimental mathematics and for heuristic computations is that one generally does not know during the course of research how it will pan out. Nonetheless, one must frequently prove all the pieces along the way as assurance that the project remains on course. The methods of experimental mathematics, alluded to below, allow one to maintain the necessary level of assurance without nailing down all the lemmas. At the end of the day, one can decide if the result merits proof. It may not be the answer one sought, or it may just not be interesting enough.

3.3. *Hardy's Apology*

Correspondingly, G. H. Hardy (1877–1947), the leading British analyst of the first half of the 20th century was also a stylish author who wrote compellingly in defense of pure mathematics. He noted that

> ''All physicists and a good many quite respectable mathematicians are contemptuous about proof.''

in his apologia, ''A Mathematician's Apology''. The Apology is a spirited defense of beauty over utility:

> ''Beauty is the first test. There is no permanent place in the world for ugly mathematics.''

That said, his comment that

> ''Real mathematics. . .is almost wholly 'useless'.''

has been over-played and is now to my mind very dated, given the importance of cryptography and other pieces of algebra and number theory devolving from very pure study. But he does acknowledge that

> "If the theory of numbers could be employed for any practical and obviously honourable purpose, ..."

even Gauss would be persuaded.

The Apology is one of Amazon's best sellers. And the existence of Amazon, or Google, means that I can be less than thorough with my bibliographic details without derailing a reader who wishes to find the source.

Hardy, on page 15 of his tribute to Ramanujan entitled *Ramanujan, Twelve Lectures ...,* gives the so-called 'Skewes number' as a "*striking example of a false conjecture*". The integral

$$\mathrm{li}\ x = \int_0^x \frac{dt}{\log t}$$

is a very good approximation to $\pi(x)$, the number of primes not exceeding x. Thus, li $10^8 = 5, 762, 209.375\ldots$ while $\pi(10^8) = 5,$ 761,455.

It was conjectured that

$$\mathrm{li}\ x > \pi(x)$$

holds for all x and indeed it so for many x. Skewes in 1933 showed the first explicit crossing at $10^{10^{10^{34}}}$. This has by now been now reduced to a relatively tiny number, a mere 10^{1167}, still vastly beyond direct computational reach or even insight.

Such examples show forcibly the limits on numeric experimentation, at least of a naive variety. Many will be familiar with the 'Law of large numbers' in statistics. Here we see what some number theorists call the 'Law of small numbers': *all small numbers are special*, many are primes and direct experience is a poor guide. And sadly or happily depending on one's attitude even 10^{1166} may be a small number. In more generality one never knows when the initial cases of a seemingly rock solid pattern are misleading. Consider the classic sequence counting the maximal number of regions obtained by joining n points around a circle by straight lines:

$$1, 2, 4, 8, 16, \mathbf{31}, \mathbf{57}, \ldots$$

(see entry A000127 in Sloane's Encyclopedia).

4. RESEARCH GOALS AND MOTIVATIONS

As a computational and experimental pure mathematician my main goal is: *insight*. Insight demands speed and increasingly parallelism as described in Borwein and Borwein (2001). Extraordinary speed and enough space are prerequisite for rapid verification and for validation and falsification ('proofs *and* refutations'). One can not have an 'aha' when the 'a' and 'ha' come minutes or hours apart.

What is 'easy' changes as computers and mathematical software grow more powerful. We see an exciting merging of disciplines, levels and collaborators. We are more and more able to marry theory & practice, history & philosophy, proofs & experiments; to match elegance and balance to utility and economy; and to inform all mathematical modalities computationally – analytic, algebraic, geometric & topological.

This has lead us to articulate an *Experimental M***a***thodology*,[14] as a philosophy (Borwein et al., 1996; Borwein and Bailey, 2003) and in practice (Borwein and Corless, 1999), based on: (i) meshing computation and mathematics (intuition is often acquired not natural, notwithstanding the truth of Polya's observations above); (ii) visualization (even three is a lot of dimensions). Nowadays we can exploit pictures, sounds and other haptic stimuli; and on (iii) 'caging' and 'monster-barring' (Imre Lakatos' and my terms for how one rules out exceptions and refines hypotheses). Two particularly useful components are:

- *Graphic checks*. comparing $y-y^2$ and y^2-y^4 to $-y^2 \ln(y)$ for $0 < y < 1$ pictorially (as in Figure 2) is a much more rapid way to divine which is larger than traditional analytic methods. It is clear that in the later case they cross, it is futile to try to prove one majorizes the other. In the first case, evidence is provided to motivate a proof.
- *Randomized checks*. of equations, linear algebra, or primality can provide enormously secure knowledge or counter-examples when deterministic methods are doomed.

All of these are relevant at every level of learning and research. My own methodology depends heavily on: (i) (*High Precision*) computation of object(s) for subsequent examination; (ii) *Pattern Recognition* of *Real Numbers* (e.g., using CECM's Inverse Calculator and

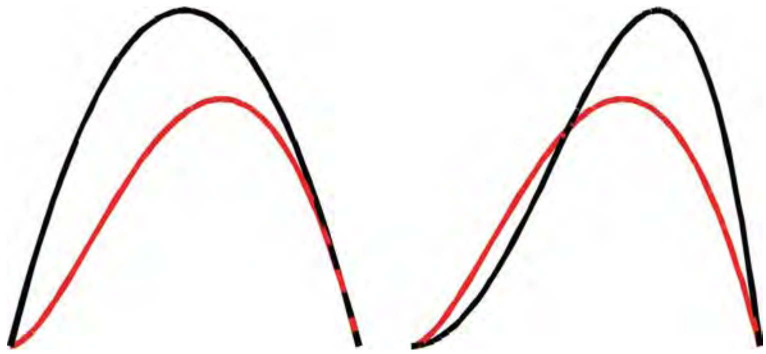

Figure 2. Graphical comparison of $y-y^2$ and y^2-y^4 to $-y^2\ln(y)$ (red).

'RevEng'.[15]) or *Sequences* (e.g., using Salvy & Zimmermann's 'gfun' or Sloane and Plouffe's Online Encyclopedia); and (iii) extensive use of *Integer Relation Methods*: *PSLQ* & *LLL* and FFT.[16] Exclusion bounds are especially useful and such methods provide a great test bed for 'Experimental Mathematics'. All these tools are accessible through the listed CECM websites and those at `www.expmath.info`. To make more sense of this it is helpful to discuss the nature of experiment.

4.1. *Four Kinds of Experiment*

Peter Medawar usefully distinguishes four forms of scientific experiment.

1. The Kantian example: Generating "the classical non-Euclidean geometries (hyperbolic, elliptic) by replacing Euclid's axiom of parallels (or something equivalent to it) with alternative forms."
2. The Baconian experiment is a contrived as opposed to a natural happening, it "is the consequence of 'trying things out' or even of merely messing about."
3. Aristotelian demonstrations: "apply electrodes to a frog's sciatic nerve, and lo, the leg kicks; always precede the presentation of the dog's dinner with the ringing of a bell, and lo, the bell alone will soon make the dog dribble."
4. The most important is Galilean: "a critical experiment – one that discriminates between possibilities and, in doing so, either gives us

confidence in the view we are taking or makes us think it in need of correction.''

The first three forms are common in mathematics, the fourth is not. It is also the only one of the four forms which has the promise to make Experimental Mathematics into a serious replicable scientific enterprise.[17]

5. FURTHER MATHEMATICAL EXAMPLES

The following suite of examples aims to make the case that modern computational tools can assist both by encapsulating concepts and by unpacking them as needs may be.

5.1. *Two Things About* $\sqrt{2}\ldots$

Remarkably one can still find new insights in the oldest areas:

5.1.1. *Irrationality*

We present graphically, Tom Apostol's lovely new geometric proof[18] of the irrationality of $\sqrt{2}$. Earlier variants have been presented, but I like very much that this was published in the present millennium.

PROOF. To say $\sqrt{2}$ is rational is to draw a right-angled isoceles triangle with integer sides. Consider the *smallest* right-angled isoceles

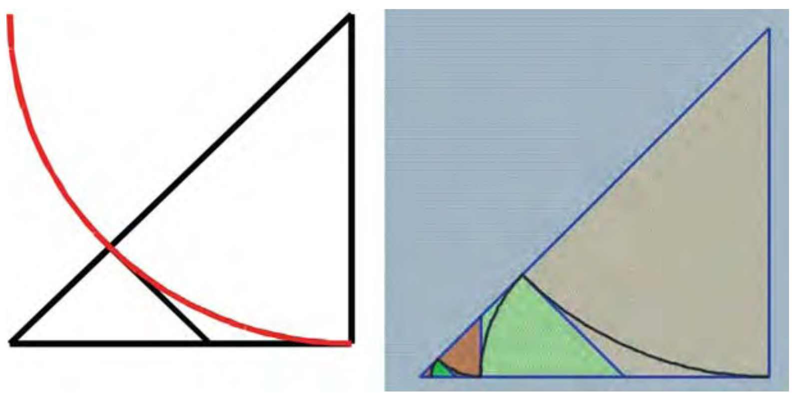

Figure 3. Root two is irrational (static and dynamic pictures).

triangle with integer sides – that is with shortest hypotenuse. Circumscribe a circle of radius the vertical side and construct the tangent on the hypotenuse, as in the picture in Figure 3. Repeating the process once more produces an even smaller such triangle in the same orientation as the initial one.

The *smaller* right-angled isoceles triangle again has integer sides... **QED**

This can be beautifully illustrated in a dynamic geometry package such as *Geometer's SketchPad, Cabri* or *Cinderella*, as used here. We can continue to draw smaller and smaller integer-sided similar triangles until the area palpably drops below 1/2. But I give it here to emphasize the ineffably human component of the best proofs.

A more elaborate picture can be drawn to illustrate the irrationality of $\sqrt{n}$ for $n = 3,5,6,\ldots$

5.1.2. *Rationality*

$\sqrt{2}$ also makes things rational:

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{(\sqrt{2}\cdot\sqrt{2})} = \sqrt{2}^{2} = 2.$$

Hence by *the principle of the excluded middle*

$$\text{Either } \sqrt{2}^{\sqrt{2}} \in \mathbb{Q} \text{ or } \sqrt{2}^{\sqrt{2}} \notin \mathbb{Q}.$$

In either case we can deduce that there are irrational numbers α and β with α^{β} rational. But how do we know which ones? This is not an adequate proof for an Intuitionist or a Constructivist. We may build a whole mathematical philosophy project around this. Compare the assertion that

$$\alpha := \sqrt{2} \quad \text{and} \quad \beta := 2\ln_2(3) \text{ yield } \alpha^{\beta} = 3$$

as *Maple* confirms. This illustrates nicely that verification is often easier than discovery (similarly the fact multiplication is easier than

factorization is at the base of secure encryption schemes for e-commerce).
There are eight possible (ir)rational triples:

$$\alpha^{\beta} = \gamma$$

and finding examples of all cases is now a fine student project.

5.2. *Exploring Integrals and Products*

Even *Maple* 'knows' $\pi \neq 22/7$ since

$$0 < \int_0^1 \frac{(1-x)^4 x^4}{1+x^2} dx = \frac{22}{7} - \pi,$$

though it would be prudent to ask 'why' it can perform the evaluation and 'whether' to trust it?

In this case, asking a computer algebra system to evaluate the indefinite integral

$$\int_0^t \frac{(1-x^4)x^4}{(1+x^2)} dx = \frac{1}{7}t^7 - \frac{2}{3}t^6 + t^5 - \frac{4}{3}t^3 + 4t - 4\ \arctan(t)$$

and differentiation proves the formula completely – after an appeal to the Fundamental theorem of calculus.

The picture in Figure 4 illustrates Archimedes' inequality in Nathalie Sinclair's *Colour calculator* micro-world in which the digits have been coloured modulo 10. This reveals simple patterns in 22/7, more complex in 223/71 and randomness in π. Many new approaches

Figure 4. A colour calculator.

to teaching about fractions are made possible by the use of such a visual representation.

In contrast, *Maple* struggles with the following *sophomore's dream*:[19]

$$\int_0^1 \frac{1}{x^x}\,dx = \sum_{n=1}^{\infty} \frac{1}{n^n},$$

and students asked to confirm this, typically mistake numerical validation for symbolic proof.

Similarly

$$\prod_{n=2}^{\infty} \frac{n^3-1}{n^3+1} = \frac{2}{3} \tag{2}$$

is rational, while the seemingly simpler ($n = 2$) case

$$\prod_{n=2}^{\infty} \frac{n^2-1}{n^2+1} = \frac{\pi}{\sinh(\pi)} \tag{3}$$

is irrational, indeed transcendental. Our Inverse Symbolic Calculator can identify the right-hand side of (3) from it numeric value 0.272029054..., and the current versions *Maple* can 'do' both products, but the student learns little or nothing from this unless the software can also recreate the steps of a validation – thereby unpacking the identity. For example, (2) may be rewritten as a lovely telescoping product, and an attempt to evaluate the finite product

$$\prod_{n=2}^{N} \frac{n^2-1}{n^2+1} \tag{4}$$

leads to a formula involving the *Gamma function*, about which Maple's Help files are quite helpful, and the student can be led to an informative proof on taking the limit in (4) after learning a few basic properties of $\Gamma(x)$. Explicitly, with 'val: = proc(f) f = value(f) end proc;'

```
> P2:=N->Product((n^2-1)/(n^2+1),n=2..N):

> val(P2(infinity));
                    infinity
                   ,--------'
                      | |     2
                      | |    n  - 1       Pi
                      | |    ------ = --------
                      | |     2       sinh(Pi)
                     n = 2   n  + 1

> val(P2(N));
    N
 ,--------'
   | |     2
   | |    n  - 1   GAMMA(N) GAMMA(N + 2) GAMMA(2 - I) GAMMA(2 + I)
   | |    ------ = ----------------------------------------------
   | |     2          2 GAMMA(N + (1 - I)) GAMMA(N + (1 + I))
  n = 2   n  + 1
> simplify(%);
    N
 ,--------'
   | |     2                             2
   | |    n  - 1               I GAMMA(N)  N (N + 1) Pi
   | |    ------ = ----------------------------------------------------
   | |     2       GAMMA(N + (1 - I)) GAMMA(N + (1 + I)) sin((2 + I)Pi)
  n = 2   n  + 1

> evalc(%);
    N
 ,--------'
   | |     2                         2
   | |    n  - 1             GAMMA(N)  N (N + 1) Pi
   | |    ------ = ---------------------------------------------
   | |     2       GAMMA(N + (1 - I)) GAMMA(N + (1 + I)) sinh(Pi)
  n = 2   n  + 1
```

5.3 *Self-Similarity in Pascal's Triangle*

In any event, in each case so far computing adds reality, making concrete the abstract, and making some hard things simple. This is strikingly the case in *Pascal's Triangle*: `www.cecm.sfu.ca/interfaces/` which affords an emphatic example where deep fractal structure is exhibited in the elementary binomial coefficients

$$1, 1, 2, 1, 1, 3, 3, 1, 1, 4, 6, 4, 1, 1, 5, 10, 10, 5, 1$$

becomes the parity sequence

$$1, 1, 0, 1, 1, 1, 1, 1, 1, 0, 0, 0, 1, 1, 1, 0, 0, 1, 1$$

and leads to the picture in Figure 5, in which odd elements of the triangle are coloured purple. Thus, as in the $\sqrt{2}$ example notions of *self-similarity* and invariance of scale can be introduced quite early and naturally in the curriculum.

One can also explore what happens if the coefficients are colored modulo three or four – four is nicer. Many other recursive sequences exhibit similar fractal behaviour.[20]

5.4. *Berlinski on Mathematical Experiment*

David Berlinski[21] writes

> "The computer has in turn changed the very nature of mathematical experience, suggesting for the first time that mathematics, like physics, may yet become an empirical discipline, a place where things are discovered because they are seen."

As all sciences rely more on 'dry experiments', via computer simulation, the boundary between physics (e.g., string theory) and mathematics (e.g., by experiment) is delightfully blurred. An early exciting example is provided by gravitational boosting.

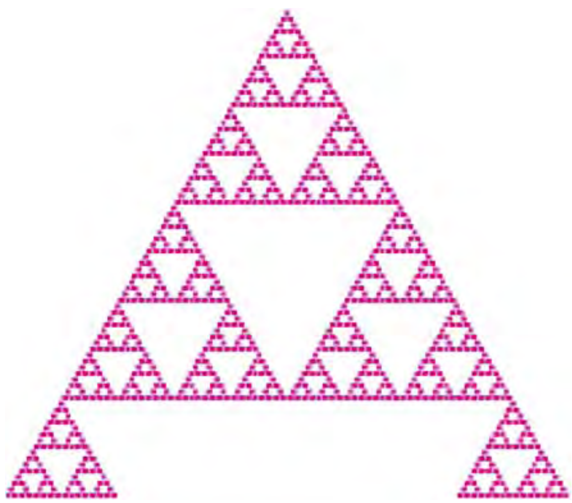

Figure 5. Drawing Pascal's triangle modulo two.

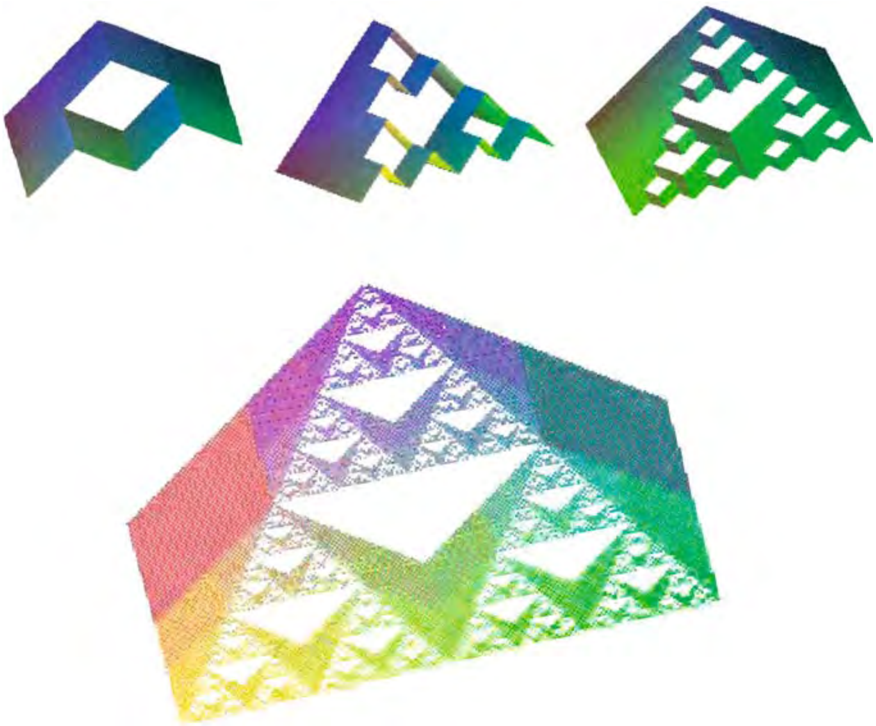

Figure 6. First, second, third and seventh iterates of a Sierpinski triangle.

Gravitational boosting. "The Voyager Neptune Planetary Guide" (JPL Publication 89–24) has an excellent description of Michael Minovitch' computational and unexpected discovery of *gravitational boosting* (otherwise known as slingshot magic) at the Jet Propulsion Laboratory in 1961.

The article starts by quoting Arthur C. Clarke "Any sufficiently advanced technology is indistinguishable from magic." Until Minovitch discovered that the so-called *Hohmann transfer ellipses* were not the minimum energy way of getting to the outer planets, "most planetary mission designers considered the gravity field of a target planet to be somewhat of a nuisance, to be cancelled out, usually by onboard Rocket thrust." For example, without a gravitational boost from the orbits of Saturn, Jupiter and Uranus, the Earth-to-Neptune Voyager mission (achieved in 1989 in little more than a decade) would have taken more than 30 years! We should still be waiting.

5.5. *Making Fractal Postcards*

And yet, as we have seen, not all impressive discoveries require a computer. Elaine Simmt and Brent Davis describe lovely constructions made by repeated regular paper folding and cutting – but no removal of paper – that result in beautiful fractal, self-similar, "pop-up" cards.[22] Nonetheless, in Figure 6, we show various iterates of a pop-up Sierpinski triangle built in software by turning those paper cutting and folding rules into an algorithm. Note the similarity to the triangle in Figure 7. Any regular rule produces a fine card. The pictures should allow the reader to start folding.

Recursive *Maple* code is given below.

```
sierpinski := proc ( n: nonnegint)
             local p1, p2, q1, q2, r1, r2, plotout;
    p1:= [1.,0.,0.]; q1:= [-1.,0.,0.]; r1:= [0.,0.,1.];
    p2:= [1.,1.,0.]; q2:= [-1.,1.,0.]; r2:= [0.,1.,1.];
    plotout:= polys(n, p1, p2, r1, r2, q1, q2);
  return PLOT3D( plotout, SCALING(CONSTRAINED),
  AXESSTYLE(NONE), STYLE(PATCHNOGRID), ORIENTATION(90,45) );
end:

polys:= proc( n::nonnegint, p1, p3, r1, r3, q1, q3 )
            local p2, q2, r2, s1, s2, s3, t1, t2, t3, u2, u3;
  if n=0 then return POLYGONS([p1,p3,r3,r1], [q1,q3,r3,r1]) fi;
    p2:= (p1+p3)/2; q2:= (q1+q3)/2; r2:= (r1+r3)/2;
    s1:= (p1+r1)/2; s2:= (p2+r2)/2; s3:= (p3+r3)/2;
    t1:= (q1+r1)/2; t2:= (q2+r2)/2; t3:= (q3+r3)/2;
    u2:= (p2+q2)/2; u3:= (p3+q3)/2;
  return polys(n-1, p2, p3, s2, s3, u2, u3),
         polys(n-1, s1, s2, r1, r2, t1, t2),
         polys(n-1, u2, u3, t2, t3, q2, q3),
         POLYGONS([p1,p2,s2,s1], [q1,q2,t2,t1]); end:
```

And, as in Figure 7, art can be an additional source of mathematical inspiration.

5.6. *Seeing Patterns in Partitions*

The number of *additive partitions* of n, $p(n)$, is *generated* by

$$1 + \sum_{n \geq 1} p(n) q^n = \frac{1}{\prod_{n \geq 1} (1 - q^n)}. \tag{5}$$

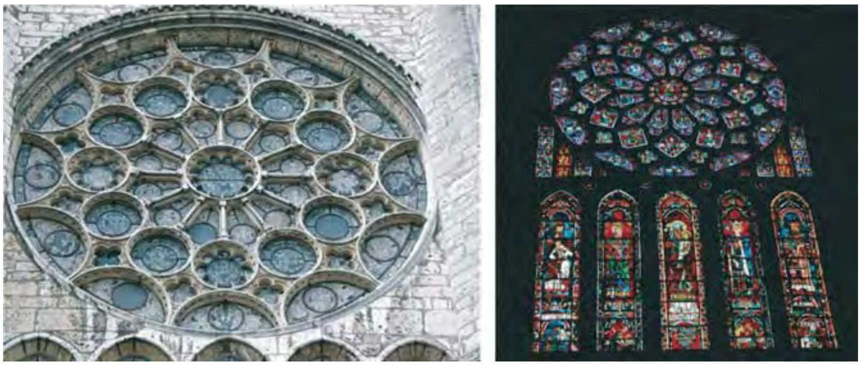

Figure 7. Self similarity at Chartres.

Thus, $p(5) = 7$ since

$$\begin{aligned}5 &= 4+1 = 3+2 = 3+1+1 = 2+2+1\\ &= 2+1+1+1 = 1+1+1+1+1.\end{aligned}$$

Developing (5) is a nice introduction to enumeration via generating functions of the type discussed in Polya's change example.

Additive partitions are harder to handle than multiplicative factorizations, but again they may be introduced in the elementary school curriculum with questions like: *How many 'trains' of a given length can be built with Cuisenaire rods?*

A more modern computationally driven question is *How hard is* $p(n)$ *to compute?*

In 1900, it took the father of combinatorics, Major Percy MacMahon (1854–1929), months to compute $p(200)$ using recursions developed from (5). By 2000, *Maple* would produce $p(200)$ in seconds if one simply demands the 200th term of the Taylor series. A few years earlier it required one to be careful to compute the series for $\prod_{n\geq 1}(1-q^n)$ first and then to compute the series for the reciprocal of that series! This seemingly baroque event is occasioned by *Euler's pentagonal number theorem*

$$\prod_{n\geq 1}(1-q^n) = \sum_{n=-\infty}^{\infty}(-1)^n q^{(3n+1)n/2}$$

The reason is that, if one takes the series for (5) directly, the software has to deal with 200 terms on the bottom. But if one takes the series for $\prod_{n\geq 1}(1-q^n)$, the software has only to handle the 23 non-zero terms in series in the pentagonal number theorem. This expost facto algorithmic analysis can be used to facilitate independent student discovery of the pentagonal number theorem, and like results.

If introspection fails, we can find the *pentagonal numbers* occurring above in *Sloane* and Plouffe's on-line 'Encyclopedia of Integer Sequences' www.research.att.com/personal/njas/sequences/eisonline.html.

Ramanujan used MacMahon's table of $p(n)$ to intuit remarkable and deep congruences such as

$$p(5n+4) \equiv 0 \mod 5$$
$$p(7n+5) \equiv 0 \mod 7$$

and

$$p(11n+6) \equiv 0 \mod 11,$$

from relatively limited data like

$$\begin{aligned} P(q) = {} & 1+q+2q^2+3q^3+\underline{5}q^{\mathbf{4}}+\overline{7}q^{\mathbf{5}}+11q^6+15q^7+22q^8+\underline{30}q^{\mathbf{9}} \\ & +42q^{10}+56q^{11}+\overline{77}q^{\mathbf{12}}+\underline{101}q^{13}+\underline{135}q^{14}+176q^{15}+231q^{16} \\ & +297q^{17}+385q^{18}+\overline{490}q^{\mathbf{19}}+627q^{20}b+792q^{21}+1002q^{22} \\ & +\cdots+p(200)q^{200}\ldots \end{aligned} \tag{6}$$

The exponents and coefficients for the cases $5n + 4$ and $7n + 5$ are highlighted in formula (6). Of course, it is much easier to heuristically confirm than to discover these patterns.

Here we see very fine examples of *Mathematics: the science of patterns* as is the title of Keith Devlin's 1997 book. And much more may similarly be done.

The difficulty of estimating the size of $p(n)$ analytically – so as to avoid enormous or unattainable computational effort – led to some marvellous mathematical advances by researchers including Hardy and Ramanujan, and Rademacher. The corresponding ease of computation may now act as a retardant to mathematical insight. New mathematics is discovered only when prevailing tools run totally out of steam. This raises a caveat against mindless computing: will a student or researcher discover structure when it is easy to compute without needing to think about it? Today, she may thoughtlessly compute $p(500)$ which a generation ago took much, much pain and insight.

Ramanujan typically found results not proofs and sometimes went badly wrong for that reason. So will we all. Thus, we are brought full face to the challenge, such software should be used, but algorithms must be taught and an appropriate appreciation for and facility with proof developed.

For example, even very extended evidence may be misleading. Indeed.

5.7. *Distinguishing Coincidence and Fraud*

Coincidences do occur. The approximations

$$\pi \approx \frac{3}{\sqrt{163}} \log(640320)$$

and

$$\pi \approx \sqrt{2}\,\frac{9801}{4412}$$

occur for deep number theoretic reasons – the first good to 15 places, the second to eight. By contrast

$$e^{\pi} - \pi = \mathbf{19.999099979}189475768\ldots$$

most probably for no good reason. This seemed more bizarre on an eight digit calculator. Likewise, as spotted by Pierre Lanchon recently, in base-two

$$e = \overline{\mathbf{10.10110111111000010}}10100010110001010\ldots$$

$$\pi = 11.0010\overline{\mathbf{0100001111110110101}}0100010001\ldots$$

have 19 bits agreeing – with one read right to left.

More extended coincidences are almost always contrived, as in the following due to Kurt Mahler early last century. Below '[x]' denotes the integer part of x. Consider:

$$\sum_{n=1}^{\infty} \frac{[n \ \tanh(\pi)]}{10^n} \stackrel{?}{=} \frac{1}{81}$$

is valid to **268** places; while

$$\sum_{n=1}^{\infty} \frac{[n \ \tanh(\frac{\pi}{2})]}{0^n} \stackrel{?}{=} \frac{1}{81}$$

is valid to **12** places. Both are actually transcendental numbers.

Correspondingly, the *simple continued fractions* for $\tanh(\pi)$ and $\tanh(\frac{\pi}{2})$ are respectively,

$$[0, 1, \mathbf{267}, 4, 14, 1, 2, 1, 2, 2, 1, 2, 3, 8, 3, 1, \ldots]$$

and

$$[0, 1, \mathbf{11}, 14, 4, 1, 1, 1, 3, 1, 295, 4, 4, 1, 5, 17, 7, \ldots].$$

This is, as they say, no coincidence! While the reasons (Borwein and Bailey, 2003) are too advanced to explain here, it is easy to conduct experiments to discover what happens when $\tanh(\pi)$ is replaced by another irrational number, say log(2).

It also affords a great example of fundamental objects that are hard to compute by hand (high precision sums or continued fractions) but easy even on a small computer or calculator. Indeed, I would claim that continued fractions fell out of the undergraduate curriculum precisely because they are too hard to work with by hand. And, of course the main message, is again that computation without insight is mind numbing and destroys learning.

6. COMPUTER DISCOVERY OF BITS OF π

Bailey, P. Borwein and Plouffe (1996) discovered a series for π (and corresponding ones for some other *polylogarithmic constants*) which somewhat disconcertingly allows one to compute hexadecimal digits of π *without* computing prior digits. The algorithm needs very little memory and no multiple precision. The running time grows only slightly faster than linearly in the order of the digit being computed. Until that point it was broadly considered impossible to compute digits of such a number without computing most of the preceding ones.

The key, found as described above, is

$$\pi = \sum_{k=0}^{\infty} \left(\frac{1}{16}\right)^k \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right).$$

Knowing an algorithm would follow they spent several months hunting by computer using integer relation methods (Bailey and Borwein, 2000; Borwein and Lisoněk, 2000; Dongarra and Sullivan, 2000) for such a formula. Once found, it is easy to prove in *Mathematica*, in *Maple* or by hand – and provides a very nice calculus exercise. This discovery was a most successful case of **REVERSE MATHEMATICAL ENGINEERING**.

The algorithm is entirely practicable, God reaches her hand deep into π: in September 1997 Fabrice Bellard (INRIA) used a variant of this formula to compute 152 binary digits of π, starting at the *trillionth position* (10^{12}). This took 12 days on 20 work-stations working in parallel over the Internet.

In August 1998 Colin Percival (SFU, age 17) finished a similar naturally or "embarrassingly parallel" computation of the *five trillionth bit* (using 25 machines at about 10 times the speed of Bellard). In *hexadecimal notation* he obtained

$$\underline{0}7E45733CC790B5B5979.$$

The corresponding binary digits of π starting at the 40 trillionth place are

$$\underline{0}0000111110011111.$$

By September 2000, the quadrillionth bit had been found to be '0' (using 250 cpu years on 1734 machines from 56 countries). Starting at the 999, 999, 999, 999, 997th bit of π one has

$$111\underline{0}00110001000010110101100000110.$$

Why should we believe this calculation? One good reason is that it was done twice starting at different digits, in which case the algorithm performs entirely different computations. For example, computing 40 hexadecimal digits commencing at the trillionth and trillion-less-tenth place, respectively, should produce 30 shared hex-digits. The probability of those coinciding by chance, at least heuristically, is about

$$\frac{1}{16^{30}} \approx \frac{1}{10^{36.23\ldots}},$$

a stunning small probability. Moreover, since many different machines were engaged no one machine error plays a significant role. I like Hersh – as we shall see later – would be hard pressed to find complex proofs affording such a level of certainty.

In the final mathematical section we attempt to capture all of the opportunities in one more fleshed-out, albeit more advanced, example.

7. A SYMBOLIC-NUMERIC EXAMPLE

I illustrate more elaborately some of the continuing and engaging mathematical challenges with a specific problem, proposed in the *American Mathematical Monthly* (November, 2000), originally discussed in Borwein and Borwein (2001) and Borwein and Bailey (2003).

10832. *Donald E. Knuth, Stanford University, Stanford, CA*. Evaluate

$$\sum_{k=1}^{\infty}\left(\frac{k^k}{k!e^k}-\frac{1}{\sqrt{2\pi k}}\right).$$

1. A very rapid *Maple* computation yielded –0.08406950872765600 ... as the first 16 digits of the sum.
2. The Inverse Symbolic Calculator has a 'smart lookup' feature[23] that replied that this was probably $-(2/3)-\zeta(\frac{1}{2})/\sqrt{2\pi}$.
3. Ample experimental confirmation was provided by checking this to 50 digits. Thus within minutes we *knew* the answer.
4. As to why? A clue was provided by the surprising speed with which *Maple* computed the slowly convergent infinite sum. The package clearly knew something the user did not. Peering under

the covers revealed that it was using the *LambertW* function, W, which is the inverse of $w = z\exp(z)$.[24]

5. The presence of $\zeta(1/2)$ and standard Euler–MacLaurin techniques, using Stirling's formula (as might be anticipated from the question), led to

$$\sum_{k=1}^{\infty}\left(\frac{1}{\sqrt{2\pi k}} - \frac{1}{\sqrt{2}}\frac{(\frac{1}{2})_{k-1}}{(k-1)!}\right) = \frac{\zeta(\frac{1}{2})}{\sqrt{2\pi}}, \tag{7}$$

   where the binomial coefficients in (7) are those of $(1/\sqrt{2-2z})$. Now (7) is a formula *Maple* can 'prove'.

6. It remains to show

$$\sum_{k=1}^{\infty}\left(\frac{k^k}{k!e^k} - \frac{1}{\sqrt{2}}\frac{(\frac{1}{2})_{k-1}}{(k-1)!}\right) = -\frac{2}{3}. \tag{8}$$

7. Guided by the presence of W and its series $\sum_{k=1}^{\infty}\frac{(-k)^{k-1}z^k}{k!}$, an appeal to Abel's limit theorem lets one deduce the need to evaluate

$$\lim_{z\to 1}\left(\frac{d}{dz}\mathrm{W}(-\frac{z}{e}) + \frac{1}{\sqrt{2-2z}}\right) = \frac{2}{3} \tag{9}$$

   Again *Maple* happily does know (9).

Of course this all took a fair amount of human mediation and insight. It will be many years before such computational discovery can be fully automated.

8. PROOF VERSUS TRUTH

By some accounts Colin Percival's web-computation of π[25] is one of the largest computations ever done. It certainly shows the possibility to use inductive engineering-like methods in mathematics, if one keeps ones eye on the ball. As we saw, to assure accuracy the algorithm could be run twice starting at different points – say starting at 40 trillion minus 10. The overlapping digits will differ if any error has been made. If 20 hex-digits agree we can argue heuristically that the probability of error is roughly 1 part in 10^{25}.

While this is not a proof of correctness, it is certainly much less likely to be wrong than any really complicated piece of human mathematics. For example, perhaps 100 people alive can, given enough time, digest *all* of Andrew Wiles' extraordinarily sophisticated proof of *Fermat's Last Theorem* and it relies on a century long program. If there is even a 1% chance that each has overlooked the *same* subtle error[26] – probably in prior work not explicitly in Wiles' corrected version – then, clearly, many computational based ventures are much more secure.

This would seem to be a good place to address another common misconception. No amount of simple-minded case checking constitutes a proof (Figure 8). The 1976–1967 'proof' of the *Four Colour Theorem*[27] was a proof because prior mathematical analysis had reduced the problem to showing that a large but finite number of potentially bad configurations could be ruled out. The proof was viewed as somewhat flawed because the case analysis was inelegant, complicated and originally incomplete. In the last few years, the computation has been redone after a more satisfactory analysis.[28] Of course, Figure 7 is a proof for the USA.

Though many mathematicians still yearn for a simple proof in both cases, there is no particular reason to think that all elegant true conjectures have accessible proofs. Nor indeed given Goedel's or Turing's work need they have proofs at all.

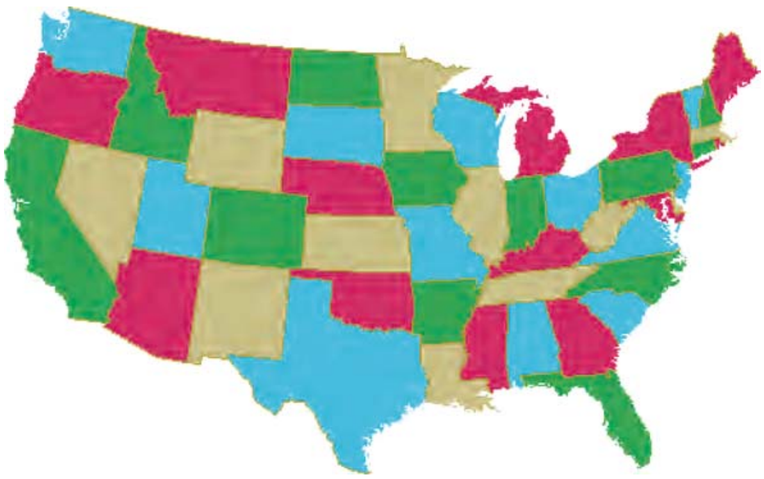

Figure 8. A four colouring of the continental USA.

8.1. *The Kepler Conjecture*

Kepler's conjecture that *The Densest Way to Stack Spheres is in a Pyramid* is the oldest problem in discrete geometry. It is also the most interesting recent example of computer-assisted proof. Published in the elite *Annals of Mathematics* with an "only 99% checked" disclaimer, this has triggered very varied reactions. While the several hundred pages of computer related work is clearly very hard to check, I do not find it credible that all other papers published in the Annals have exceeded such a level of verification.[29]

The proof of the *Kepler Conjecture*, that of the *Four Colour Theorem* and Clement Lams' computer-assisted proof of *The Non-existence of a Projective Plane of Order 10*, raise and answer quite distinct philosophical and mathematical questions – both real and specious. But one thing is certain such proofs will become more and more common.

8.2. *Kuhn and Planck on Paradigm Shifts*

Much of what I have described in detail or in passing involves changing set modes of thinking. Many profound thinkers view such changes as difficult:

> "The issue of paradigm choice can never be unequivocally settled by logic and experiment alone. ··· in these matters neither proof nor error is at issue. The transfer of allegiance from paradigm to paradigm is a conversion experience that cannot be forced." (Thomas Kuhn[30])

and

> "... a new scientific truth does not triumph by convincing its opponents and making them see the light, but rather because its opponents die and a new generation grows up that's familiar with it." (Albert Einstein quoting Max Planck[31])

8.3. *Hersh's Humanist Philosophy*

However hard such paradigm shifts and whatever the outcome of these discourses, mathematics is and will remain a uniquely human undertaking. Indeed Reuben Hersh's arguments for a humanist philosophy of mathematics, as paraphrased below, become more convincing in our setting:

1. *Mathematics is human*. It is part of and fits into human culture. It does not match Frege's concept of an abstract, timeless, tenseless, objective reality.
2. *Mathematical knowledge is fallible*. As in science, mathematics can advance by making mistakes and then correcting or even re-correcting them. The "fallibilism" of mathematics is brilliantly argued in Lakatos' *Proofs and Refutations*.
3. *There are different versions of proof or rigor*. Standards of rigor can vary depending on time, place, and other things. The use of computers in formal proofs, exemplified by the computer-assisted proof of the four color theorem in 1977, is just one example of an emerging nontraditional standard of rigor.
4. *Empirical evidence, numerical experimentation and probabilistic proof all can help us decide what to believe in mathematics*. Aristotelian logic isn't necessarily always the best way of deciding.
5. *Mathematical objects are a special variety of a social-cultural-historical object*. Contrary to the assertions of certain post-modern detractors, mathematics cannot be dismissed as merely a new form of literature or religion. Nevertheless, many mathematical objects can be seen as shared ideas, like Moby Dick in literature, or the Immaculate Conception in religion.[32]

To this I would add that for me mathematics is not ultimately about proof but about secure mathematical knowledge. Georg Friedrich Bernhard Riemann (1826–1866) was one of the most influential thinkers of the past 200 years. Yet he proved very few theorems, and many of the proofs were flawed. But his conceptual contributions, such as through Riemannian geometry and the Riemann zeta function, and to elliptic and Abelian function theory, were epochal. The experimental method is an addition not a substitute for proof, and its careful use is an example of Hersh's 'nontraditional standard of rigor'.

The recognition that 'quasi-intuitive' methods may be used to gain mathematical insight can dramatically assist in the learning and discovery of mathematics. Aesthetic and intuitive impulses are shot through our subject, and honest mathematicians will acknowledge their role. But a student who never masters proof will not be able to profitably take advantage of these tools.

8.4. *A Few Final Observations*

As we have already seen, the stark contrast between the deductive and the inductive has always been exaggerated. Herbert A. Simon, in the final edition of *The Sciences of the Artificial,*[33] wrote:

> "This skyhook-skyscraper construction of science from the roof down to the yet unconstructed foundations was possible because the behaviour of the system at each level depended only on a very approximate, simplified, abstracted characterization at the level beneath.[13] "[34]

> "This is lucky, else the safety of bridges and airplanes might depend on the correctness of the "Eightfold Way" of looking at elementary particles."

It is precisely this '*post hoc ergo propter hoc*' part of theory building that Russell so accurately typifies that makes him an articulate if surprising advocate of my own views.

And finally, I wish to emphasize that good software packages can make very difficult concepts accessible (e.g., *Mathematica, MatLab* and SketchPad) and radically assist mathematical discovery. Nonetheless, introspection is here to stay.

In Kieran Egan's words, "We are Pleistocene People." Our minds can subitize, but were not made for modern mathematics. We need all the help we can get. While proofs are often out of reach to students or indeed lie beyond present mathematics, understanding, even certainty, is not.

Perhaps indeed, "Progress is made 'one funeral at a time'."[35] In any event, as Thomas Wolfe put it "You can't go home again."

ACKNOWLEDGEMENTS

I wish to thank Nathalie Sinclair, Terry Stanway and Rina Zaskis for many pertinent discussions during the creation of this article.

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NOTES

[1] Henry Briggs is describing his first meeting in 1617 with Napier whom he had travelled from London to Edinburgh to meet. Quoted from H.W. Turnbull's *The Great Mathematicians,* Methuen, 1929.

[2] Chapter in *Making the Connection: Research and Practice in Undergraduate Mathematics*, MAA Notes, 2004 in Press.

[3] Ibid.

[4] CECM now averages well over a million accesses a month, many by humans.

[5] Quoted by Ram Murty in *Mathematical Conversations, Selections from The Mathematical Intelligencer*, compiled by Robin Wilson and Jeremy Gray, Springer-Verlag, New York, 2000.

[6] My own experience is principally at the tertiary level. An excellent middle school illustration is afforded by Nathalie Sinclair. (2001) "The aesthetics is relevant," *for the learning of mathematics*, 21: 25 – 32.

[7] E.g., `www.cecm.sfu.ca/personal/jborwein/talks.html`, `www.cs.dal.ca//, /jborwein, and personal/loki/Papers/Numbers/.`

[8] In *Mathematical Discovery: On Understanding, Learning and Teaching Problem Solving, 1968*.

[9] Illustration courtesy the Mathematical Association of America.

[10] Ibid.

[11] See Isaac Asimov (1988) book of science and nature quotations. In Isaac Asimov and J.A. Shulman (Eds), New York: Weidenfield and Nicolson, p. 115.

[12] In E. Borel, "Lecons sur la theorie des fonctions," 1928, quoted by George Polya (1981) in *Mathematical discovery: On understanding, learning, and teaching problem solving* (Combined Edition), New York: John Wiley, pp. 2–126.

[13] Others on a short list would include Poincaré and Weil.

[14] I originally typed this by mistake for Methodology.

[15] ISC space limits have changed from 10 Mb being a constraint in 1985 to 10 Gb being 'easily available' today. A version of 'Reveng' is available in current versions of *Maple*. Typing 'identify($\sqrt{2.0} + \sqrt{3.0}$)' will return the symbolic answer $\sqrt{2} + \sqrt{3}$ from the numerical input 3.146264370.

[16] Described as one of the top ten "Algorithm's for the Ages," Random Samples, Science, Feb. 4, 2000, and [10].

[17] From Peter Medawar's wonderful *Advice to a Young Scientist*, Harper (1979).

[18] *MAA Monthly*, November 2000, 241–242.

[19] In that the integrand and the summand agree.

[20] Many examples are given in P. Borwein and L. Jörgenson, "Visible Structures in Number Theory", `www.cecm.sfu.ca/preprints/1998pp.html.`

[21] A quote I agree with from his "A Tour of the Calculus," Pantheon Books, 1995.

[22] Fractal Cards: A Space for Exploration in Geometry and Discrete Mathematics, *Mathematics Teacher* 91 (198), 102–108.

[23] Alternatively, a sufficiently robust integer relation finder could be used.

[24] A search for 'Lambert W function' on MathSciNet provided 9 references – all since 1997 when the function appears named for the first time in *Maple* and *Mathematica*.

[25] Along with *Toy Story 2*.

[26] And they may be psychologically predisposed so to do!

27 Every planar map can be coloured with four colours so adjoining countries are never the same colour.

28 This is beautifully described at www.math.gatech.edu/personal/thomas/FC/fourcolor.html.

29 See "In Math, Computers Don't Lie. Or Do They?" *New York Times*, April 6, 2004.

30 In Ed Regis, *Who got Einstein's Office?* Addison-Wesley, 1986.

31 From F.G. Major, *The Quantum Beat,* Springer, 1998.

32 From "Fresh Breezes in the Philosophy of Mathematics," *American Mathematical Monthly*, August–September 1995, 589–594.

33 MIT Press, 1996, page 16.

34 Simon quotes Russell at length ...
[13] "... More than fifty years ago Bertrand Russell made the same point about the architecture of mathematics. See the "Preface" to *Principia Mathematica* "... the chief reason in favour of any theory on the principles of mathematics must always be inductive, i.e., it must lie in the fact that the theory in question allows us to deduce ordinary mathematics. In mathematics, the greatest degree of self-evidence is usually not to be found quite at the beginning, but at some later point; hence the early deductions, until they reach this point, give reason rather for believing the premises because true consequences follow from them, than for believing the consequences because they follow from the premises." Contemporary preferences for deductive formalisms frequently blind us to this important fact, which is no less true today than it was in 1910."

35 This harsher version of Planck's comment is sometimes attributed to Niels Bohr.

36 All journal references are available at www.cecm.sfu.ca/preprints/.

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Chapter 1
Aesthetics for the Working Mathematician

Jonathan M. Borwein

> If my teachers had begun by telling me that mathematics was pure play with presuppositions, and wholly in the air, I might have become a good mathematician, because I am happy enough in the realm of essence. But they were over-worked drudges, and I was largely inattentive, and inclined lazily to attribute to incapacity in myself or to a literary temperament that dullness which perhaps was due simply to lack of initiation. (Santayana, 1944, p. 238)

Most research mathematicians neither think deeply about nor are terribly concerned with either pedagogy or the philosophy of mathematics. Nonetheless, as I hope to indicate, aesthetic notions have always permeated (pure and applied) mathematics. And the top researchers have always been driven by an aesthetic imperative. Many mathematicians over time have talked about the 'elegance' of certain proofs or the 'beauty' of certain theorems, but my analysis goes deeper: I aim to show how the aesthetic imperative interacts with utility and intuition, as well as indicate how it serves to shape my own mathematical experiences. These analyses, rather than being retrospective and passive, will provide a living account of the aesthetic dimension of mathematical work.

> We all believe that mathematics is an art. The author of a book, the lecturer in a classroom tries to convey the structural beauty of mathematics to his readers, to his listeners. In this attempt, he must always fail. Mathematics is logical to be sure; each conclusion is drawn from previously derived statements. Yet the whole of it, the real piece of art, is not linear; worse than that, its perception should be instantaneous. We all have experienced on some rare occasions the feeling of elation in realizing that we have enabled our listeners to see at a moment's glance the whole architecture and all its ramifications. (Emil Artin, in Murty, 1988, p. 60)

I shall similarly argue for aesthetics before utility. Through a suite of examples drawn from my own research and interests, I aim to illustrate how and what this means on the front lines of research. I also will argue that the opportunities to evoke the mathematical aesthetic in research and teaching are almost boundless – at all levels of the curriculum. (An excellent middle-school illustration, for instance, is described in Sinclair, 2001.)

In part, this is due to the increasing power and sophistication of visualisation, geometry, algebra and other mathematical software. Indeed, by drawing on 'hot topics' as well as 'hot methods' (i.e. computer technology), I also provide a contemporary perspective which I hope will complement the more classical contributions to our understanding of the mathematical aesthetic offered by writers such as G. H. Hardy and Henri Poincaré (as discussed in Chapter α).

Webster's dictionary (1993, p. 19) first provides six different meanings of the word 'aesthetic', used as an adjective. However, I want to react to these two definitions of 'aesthetics', used as a noun:

1. The branch of philosophy dealing with such notions as the beautiful, the ugly, the sublime, the comic, etc., as applicable to the fine arts, with a view to establishing the meaning and validity of critical judgments concerning works of art, and the principles underlying or justifying such judgments.
2. The study of the mind and emotions in relation to the sense of beauty.

Personally, for my own definition of the aesthetic, I would require (unexpected) simplicity or organisation in apparent complexity or chaos – consistent with views of Dewey (1934), Santayana (1944) and others. I believe we need to integrate this aesthetic into mathematics education at every level, so as to capture minds for other than utilitarian reasons. I also believe detachment to be an important component of the aesthetic experience: thus, it is important to provide some curtains, stages, scaffolding and picture frames – or at least their mathematical equivalents. Fear of mathematics certainly does not hasten an aesthetic response.

Gauss, Hadamard and Hardy

Three of my personal mathematical heroes, very different individuals from different times, all testify interestingly on the aesthetic and the nature of mathematics.

Gauss

Carl Friedrich Gauss is claimed to have once confessed, "I have had my results for a long time, but I do not yet know how I am to arrive at them" (in Arber, 1954, p. 47). [1] One of Gauss's greatest discoveries, in 1799, was the relationship between the lemniscate sine function and the arithmetic–geometric mean iteration. This was based on a purely computational observation. The young Gauss wrote in his diary that "a whole new field of analysis will certainly be opened up" (*Werke,* X, p. 542; in Gray, 1984, p. 121).

He was right, as it pried open the whole vista of nineteenth-century elliptic and modular function theory. Gauss's specific discovery, based on tables of integrals provided by Scotsman James Stirling, was that the reciprocal of the integral

$$\frac{2}{\pi}\int_0^1 \frac{1}{\sqrt{1-t^4}}\,dt$$

agreed numerically with the limit of the rapidly convergent iteration given by setting $a_0 := 1$, $b_0 := \sqrt{2}$ and then computing:

$$a_{n+1} := \frac{a_n + b_n}{2}, \qquad b_{n+1} := \sqrt{a_n b_n}$$

It transpires that the two sequences $\{a_n\}$, $\{b_n\}$ have a common limit of 1.1981402347355922074…

Which object, the integral or the iteration, is the more familiar and which is the more elegant – then and now? Aesthetic criteria change with time (and within different cultures) and these changes manifest themselves in the concerns and discoveries of mathematicians. Gauss's discovery of the relationship between the lemniscate function and the arithmetic–geometric mean iteration illustrates how the traditionally preferred 'closed form' (here, the integral form) of equations have yielded centre stage, in terms both of elegance and utility, to recursion. This parallels the way in which biological and computational metaphors (even 'biology envy') have now replaced Newtonian mental images, as described and discussed by Richard Dawkins (1986) in his book *The Blind Watchmaker.*

In fact, I believe that mathematical thought patterns also change with time and that these in turn affect aesthetic criteria – not only in terms of what counts as an interesting problem, but also what methods the mathematician can use to approach these problems, as well as how a mathematician judges their solutions. As mathematics becomes more 'biological', and more computational, aesthetic criteria will continue to change.

Hadamard

A constructivist, experimental and aesthetically-driven rationale for mathematics could hardly do better than to start with Hadamard's claim that:

> The object of mathematical rigor is to sanction and legitimize the conquests of intuition, and there was never any other object for it. (in Pólya, 1981, p. 127)

Jacques Hadamard was perhaps the greatest mathematician other than Poincaré to think deeply and seriously about cognition in mathematics. He is quoted as saying, "in arithmetic, until the seventh grade, I was last or nearly last" (in MacHale, 1993, p. 142). Hadamard was co-prover (independently with Charles de la Vallée Poussin, in 1896) of the Prime Number theorem (the number of primes not exceeding n is asymptotic to $n/\log n$), one of the culminating results of nineteenth-century mathematics and one that relied on much preliminary computation and experimentation. He was also the author of *The Psychology of Invention in the Mathematical Field* (1945), a book that still rewards close inspection.

Hardy's *Apology*

Correspondingly, G. H. Hardy, the leading British analyst of the first half of the twentieth century, was also a stylish author who wrote compellingly in defence of pure mathematics. He observed that:

> All physicists and a good many quite respectable mathematicians are contemptuous about proof. (1945/1999, pp. 15-16)

His memoir, entitled *A Mathematician's Apology,* provided a spirited defence of beauty over utility:

> Beauty is the first test. There is no permanent place in the world for ugly mathematics. (1940, p. 84)

That said, although the sentiment behind it being perfectly understandable from an anti-war mathematician in war-threatened Britain, Hardy's claim that real mathematics is almost wholly useless has been over-played and, to my mind, is now very dated, given the importance of cryptography and other pieces of algebra and number theory devolving from very pure study.

In his tribute to Srinivasa Ramanujan entitled *Ramanujan: Twelve Lectures on Subjects Suggested by His Life and Work,* Hardy (1945/1999) offered the so-called 'Skewes number' as a "striking example of a false conjecture" (p. 15). The logarithmic integral function, written Li(x), is specified by:

$$Li(x) = \int_0^x \frac{1}{\log(t)}\, dt$$

Li(x) provides a very good approximation to the number of primes that do not exceed x. For example, Li(10^8) = 5,762,209.375…, while the number of primes not exceeding 10^8 is 5,761,455. It was conjectured that the inequality

> Li(x) > the number of primes not exceeding x

holds for all x and, indeed, it does so for many x. In 1933, Skewes showed the first explicit crossing occurs before $10^{10^{10^{34}}}$. This has been reduced to a relatively tiny number, a mere 10^{1167} (and, most recently, even lower), though one still vastly beyond direct computational reach.

Such examples show forcibly the limits on numerical experimentation, at least of a naïve variety. Many readers will be familiar with the 'law of large numbers' in statistics. Here, we see an instance of what some number theorists (e.g. Guy, 1988) call the 'strong law of small numbers': *all small numbers are special,* many are primes and direct experience is a poor guide. And sadly (or happily, depending on one's attitude), even 10^{1167} may be a small number.

Research Motivations and Goals

As a computational and experimental pure mathematician, my main goal is *insight.* Insight demands speed and, increasingly, parallelism (see Borwein and Borwein, 2001, on the challenges for mathematical computing). The mathematician's 'aesthetic buzz' comes not only from simply contemplating a beautiful piece of mathematics, but, additionally, from achieving insight. The computer, with its capacities for visualisation and computation, can encourage the aesthetic buzz of insight, by offering the mathematician the possibility of visual contact with mathematics and by allowing the mathematician to experiment with, and thus to become intimate with, mathematical ideas, equations and objects.

What is 'easy' is changing and I see an exciting merging of disciplines, levels and collaborators. Mathematicians are more and more able to:

- marry theory and practice, history and philosophy, proofs and experiments;
- match elegance and balance with utility and economy;
- inform all mathematical modalities computationally – analytic, algebraic, geometric and topological.

This is leading us towards what I term an *experimental m**a**thodology* as a philosophy and a practice (Borwein and Corless, 1999). This methodology is based on the following three approaches:

- meshing computation and mathematics, so that intuition is acquired;
- visualisation – three is a lot of dimensions and, nowadays, we can exploit pictures, sounds and haptic stimuli to get a 'feel' for relationships and structures (see also Chapter 7);
- 'exception barring' and 'monster barring' (using the terms of Lakatos, 1976).

Two particularly useful components of this third approach include graphical and randomised checks. For example, comparing $2\sqrt{y} - y$ and $-\sqrt{y}\ln(y)$ (for $0 < y < 1$) pictorially is a much more rapid way to divine which is larger than by using traditional analytic methods. Similarly, randomised checks of equations, inequalities, factorisations or primality can provide enormously secure knowledge or counter-examples when deterministic methods are doomed. As with traditional mathematical methodologies, insight and certainty are still highly valued, yet achieved in different ways.

Pictures and symbols

> If I can give an abstract proof of something, I'm reasonably happy. But if I can get a concrete, computational proof and actually produce numbers I'm much happier. I'm rather an addict of doing things on the computer, because that gives you an explicit criterion of what's going on. I have a visual way of thinking, and I'm happy if I can see a picture of what I'm working with. (John Milnor, in Regis, 1986, p. 78)

I have personally had this experience, in the context of studying the distribution of zeroes of the Riemann zeta function. Consider more explicitly the following image (see Figure 1), which shows the densities of zeroes for polynomials in powers of x with –1 and 1 as coefficients (they are manipulable at: www.cecm.sfu.ca/interfaces/). All roots of polynomials, up to a given degree, with coefficients of either –1 or 1 have been calculated by permuting through all possible combinations of polynomials, then solving for the roots of each. These roots are then plotted on the complex plane (around the origin).

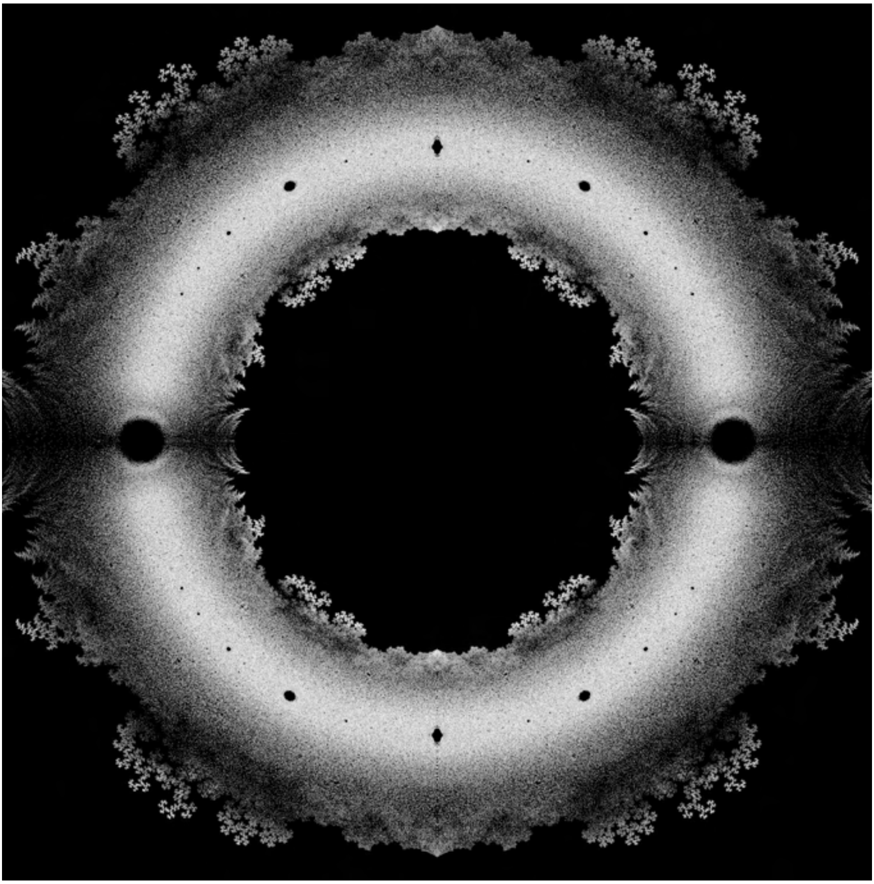

Figure 1: Density of zeroes for polynomials with coefficients of –1 and 1

In this case, graphical output from a computer allows a level of insight no amount of numbers could.

Some colleagues and I have been building educational software with these precepts embedded, such as *LetsDoMath* (see: www.mathresources.com). The intent is to challenge students honestly (e.g. through allowing subtle explorations within John Conway's 'Game of Life'), while making things tangible (e.g. 'Platonic solids' offers virtual manipulables that are more robust and expressive than the standard classroom solids).

Evidently, though, symbols are often more reliable than pictures. The picture opposite purports to give evidence that a solid can fail to be polyhedral at only one point. It shows the steps up to pixel level of inscribing a

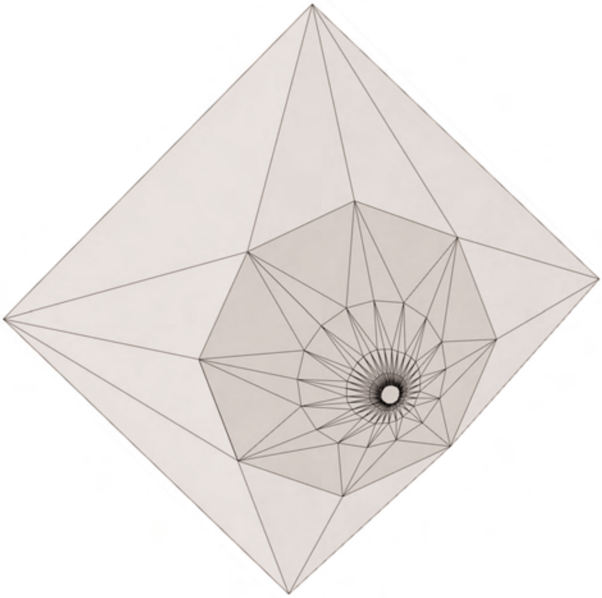

Figure 2: A misleading picture

regular 2^{n+1}-gon at height 2^{1-n}. However, ultimately, such a construction fails and produces a right circular cone. The false evidence in this picture held back a research project for several days – and might have derailed it.

Two Things about $\sqrt{2}$ and One Thing about π

Remarkably, one can still find new insights in the oldest areas. I discuss three examples of this. The first involves a new proof of the irrationality of $\sqrt{2}$ and the way in which it provides insight into a previously known result. The second invokes the strange interplay between rational and irrational numbers. Finally, the third instance reveals how the computer can make opaque some properties that were previously transparent, and *vice versa.*

Irrationality

Below is a graphical representation of Tom Apostol's (2000) lovely new geometric proof of the irrationality of $\sqrt{2}$. This example may seem routine at first, with respect to the literature on the mathematical aesthetic. Writers such as Hardy (1940), King (1992) and Wells (1990) have also talked about the beauty of quadratics such as $\sqrt{2}$. These writers have emphasised aesthetic criteria (such as economy and unexpectedness) that contribute to that judgement of beauty. On the other hand, Apostol's new proof, prefigured in others, shows how aesthetics can also serve to *motivate* mathematical inquiry.

PROOF Consider the *smallest* right-angled isosceles triangle with integer sides. Circumscribe a circle of length equal to the vertical side and construct the tangent to the circle where the hypotenuse cuts it (see Figure 3). The *smaller* isosceles triangle once again has integer sides.

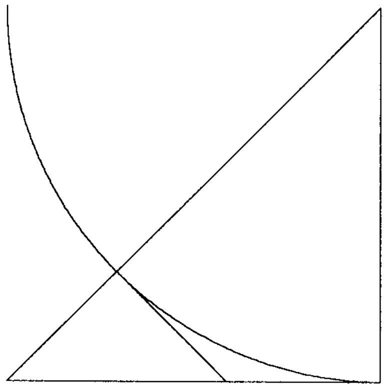

Figure 3: The square root of two is irrational

The proof is lovely because it offers new insight into a result that was first proven over two thousand years ago. It also verges on being a 'proof without words' (Nelsen, 1993), proofs which are much admired – yet infrequently encountered and not always trusted – by mathematicians (see Brown, 1999). Apostol's work demonstrates how mathematicians are not only motivated to find ground-breaking results, but that they also strive for better ways to say things or to show things, as Gauss was surely doing when he worked out his fourth, fifth and sixth proof of the law of quadratic reciprocity.

Rationality

By a variety of means, including the one above, we know that the square root of two is irrational. But mathematics is always full of surprises: $\sqrt{2}$ can also *make* things rational (a case of two wrongs making a right?).

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{(\sqrt{2}\sqrt{2})} = \sqrt{2}^{2} = 2.$$

Hence, by the principle of the excluded middle:

$$\text{Either} \quad \sqrt{2}^{\sqrt{2}} \in Q \quad \text{or} \quad \sqrt{2}^{\sqrt{2}} \notin Q.$$

In either case, we can deduce that there are irrational numbers a and b with a^b rational. But how do we know which ones? One may build a whole mathematical philosophy project around this. Yet, as *Maple* (the computer algebra system) confirms:

setting $\alpha := \sqrt{2}$ and $\beta := 2\ln_2 3$ yields $\alpha^\beta = 3$.

This illustrates nicely that verification is often easier than discovery. (Similarly, the fact that multiplication is easier than factorisation is at the base of secure encryption schemes for e-commerce.)

π and two integrals

Even *Maple* knows $\pi \neq 22/7$, since:

$$0 < \int_0^1 \frac{(1-x)^4\, x^4}{1+x^2}\, dx = \frac{22}{7} - \pi.$$

Nevertheless, it would be prudent to ask 'why' *Maple* is able to perform the evaluation and whether to trust it. In contrast, *Maple* struggles with the following *sophomore's dream:*

$$\int_0^1 \frac{1}{x^x}\, dx = \sum_{n=1}^{\infty} \frac{1}{n^n}.$$

Students asked to confirm this typically mistake numerical validation for symbolic proof.

Again, we see that computing adds reality, making the abstract concrete, and makes some hard things simple. This is strikingly the case with Pascal's Triangle. Figure 4 (from: www.cecm.sfu.ca/interfaces/) affords an emphatic example where deep fractal structure is exhibited in the elementary binomial coefficients.

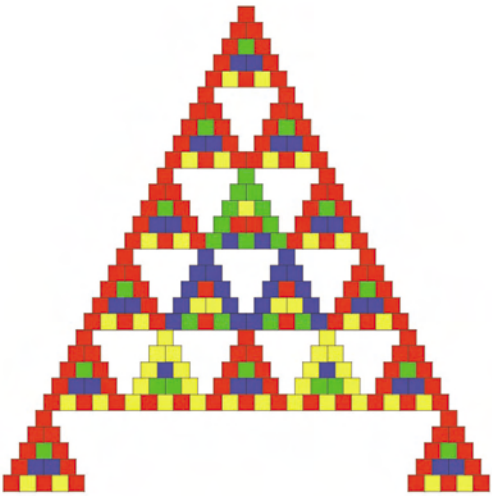

Figure 4: Thirty rows of Pascal's triangle (modulo five)

Berlinski (1997) comments on some of the effects of such visual–experimental possibilities in mathematics:

> The computer has in turn changed the very nature of mathematical experience, suggesting for the first time that mathematics, like physics, may yet become an empirical discipline, a place where things are discovered because they are seen. (p. 39)

Berlinski (1995) had earlier suggested, in his book *A Tour of the Calculus,* that there will be long-term effects:

> The body of mathematics to which the calculus gives rise embodies a certain swashbuckling style of thinking, at once bold and dramatic, given over to large intellectual gestures and indifferent, in large measure, to any very detailed description of the world. It is a style that has shaped the physical but not the biological sciences, and its success in Newtonian mechanics, general relativity, and quantum mechanics is among the miracles of mankind. But the era in thought that the calculus made possible is coming to an end. Everyone feels this is so, and everyone is right. (p. xiii)

π and Its Friends

My research on π with my brother, Peter Borwein, also offers aesthetic and empirical opportunities. In this example, my personal fascinations provide compelling illustrations of an aesthetic imperative in my own work. I first discuss the algorithms I have co-developed to compute the digits of π. These algorithms, which consist of simple algebraic equations, have made it possible for researchers to compute its first 2^{36} digits. I also discuss some of the methods and algorithms I have used to gain insight into relationships involving π.

A quartic algorithm (Borwein and Borwein, 1984)

The next algorithm I present grew out of work of Ramanujan. Set $a_0 = 6 - 4\sqrt{2}$ and $y_0 = \sqrt{2} - 1$. Iterate:

$$(1) \qquad y_{k+1} = \frac{1-(1-y_k^4)^{1/4}}{1+(1-y_k^4)^{1/4}}$$

$$(2) \qquad a_{k+1} = a_k(1+y_{k+1})^4\ b - 2^{2k+3}\ y_{k+1}(1+y_{k+1}+y_{k+1}^2)$$

Then the sequence $\{a_k\}$ converges *quartically* to 1/π.

There are nineteen pairs of simple algebraic equations (1, 2) as *k* ranges from 0 to 18. After seventeen years, this still gives me an aesthetic buzz. Why? With less than one page of equations, I have a tool for computing a number that differs from π (the most celebrated transcendental number) only after seven hundred billion digits. It is not only the economy of the tool that delights me, but also the stirring idea of 'almost-ness' – that even after seven hundred billion digits we still cannot nail π. The difference might seem trivial, but mathematicians know that it is not and they continue to improve their algorithms and computational tools.

This iteration has been used since 1986, with the Salamin–Brent scheme, by David Bailey (at the Lawrence Berkeley Labs) and by Yasumasa Kanada (in Tokyo). In 1997, Kanada computed over 51 billion digits on a Hitachi supercomputer (18 iterations, 25 hrs on 210 cpus). His penultimate world record was 2^{36} digits in April, 1999. A billion (2^{30}) digit computation has been performed on a single Pentium II PC in less than nine days. The present record is 1.24 trillion digits, computed by Kanada in December 2002 using quite different methods, and is described in my new book, co-authored with David Bailey (2003).

The fifty-billionth decimal digit of π *or* of 1/π is 04<u>2</u>! And after eighteen billion digits, the string 0123456789 has finally appeared and so Brouwer's famous intuitionist example *now* converges. [2] (Details such as this about π can be found at: www.cecm.sfu.ca/personal/jborwein/pi_cover.html.) From a probability perspective, such questions may seem uninteresting, but they continue to motivate and amaze mathematicians.

A further taste of Ramanujan

G. N. Watson, in discussing his response to similar formulae of the wonderful Indian mathematical genius Srinivasa Ramanujan, describes:

> a thrill which is indistinguishable from the thrill which I feel when I enter the Sagrestia Nuova of the Capelle Medicee and see before me the austere beauty of [the four statues representing] 'Day,' 'Night,' 'Evening,' and 'Dawn' which Michelangelo has set over the tombs of Giuliano de' Medici and Lorenzo de' Medici. (in Chandrasekhar, 1987, p. 61)

One of these is Ramanujan's remarkable formula, based on the elliptic and modular function theory initiated by Gauss.

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103 + 26390k)}{(k!)^4 \, 396^{4k}}.$$

Each term of this series produces an additional *eight* correct digits in the result – and only the ultimate multiplication by $\sqrt{2}$ is not a *rational* operation. Bill Gosper used this formula to compute seventeen million terms of the continued fraction for π in 1985. This is of interest, because we still cannot prove that the continued fraction for π is unbounded. Again, everyone *knows* that this is true.

That said, Ramanujan preferred related explicit forms for approximating π, such as the following:

$$\frac{\log(640320^3)}{\sqrt{163}} = 3.14159265358979\underline{30164} \approx \pi.$$

This equation is correct until the underlined places. *Inter alia,* the number e^{π} is the easiest transcendental to fast compute (by elliptic methods). One 'differentiates' $e^{-\pi t}$ to obtain algorithms such as the one above for π, via the arithmetic–geometric mean.

Integer relation detection

I make a brief digression to describe what integer relation detection methods do. (These may be tried at: www.cecm.sfu.ca/projects/IntegerRelations/.) I then apply them to π (see Borwein and Lisonek, 2000).

> DEFINITION A vector $(x_1, x_2, \ldots, x_n)$ of real numbers *possesses an integer relation,* if there exist integers a_i (not all zero) with:
>
> $$a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0$$
>
> PROBLEM Find a_i if such integers exist. If not, obtain lower 'exclusion' bounds on the size of possible a_i.
>
> SOLUTION For $n = 2$, *Euclid's algorithm* gives a solution. For $n \geq 3$, Euler, Jacobi, Poincaré, Minkowski, Perron and many others sought methods. The *first general algorithm* was found (in 1977) by Ferguson and Forcade. Since 1977, one has many variants: I will mainly be talking about two algorithms, LLL ('Lenstra, Lenstra and Lovász'; also available in *Maple* and *Mathematica*) and PSLQ ('Partial sums using matrix LQ decomposition', 1991; *parallelised,* 1999).

Integer relation detection was recently ranked among:

> the 10 algorithms with the greatest influence on the development and practice of science and engineering in the 20th century. (Dongarra and Sullivan, 2000, p. 22)

It could be interesting for the reader to compare these algorithms with the theorems on the list of the most 'beautiful' theorems picked out by Wells (1990) in his survey, in terms of criteria such as applicability, unexpectedness and fruitfulness.

Determining whether or not a number is algebraic is one problem that can be attacked using integer relation detection. Asking about algebraicity is handled by computing α to sufficiently high precision ($O(n = N^2)$) and applying LLL or PSLQ to the vector $(1, \alpha, \alpha^2, \ldots, \alpha^{N-1})$. Solution integers a_i are coefficients of a polynomial likely satisfied by α. If one has computed α to $n + m$ digits and run LLL using n of them, one has m digits to confirm the result heuristically. I have never seen this method return an honest 'false positive' for $m > 20$, say. If no relation is found, exclusion bounds are obtained, saying, for example, that any polynomial of degree less than N must have the Euclidean norm of its coefficients in excess of L (often astronomical). If we know or suspect an identity exists, then integer relations methods are very powerful. Let me illustrate this in the context of approximating π.

Machin's formula

We use *Maple* to look for the linear dependence of the following quantities:

[arctan(1), arctan(1/5), arctan(1/239)]

and 'recover' [1, –4, 1]. In other words, we can establish the following equation:

$$\pi/4 = 4\arctan(1/5) - \arctan(1/239).$$

Machin's formula was used on all serious computations of π from 1706 (a hundred digits) to 1973 (a million digits), as well as more abstruse but similar formulae used in creating Kanada's present record. After 1980, the methods described above started to be used instead.

Dase's formula

Again, we use *Maple* to look for the linear dependence of the following quantities:

[π/4, arctan(1/2), arctan(1/5), arctan(1/8)].

and recover [–1, 1, 1, 1]. In other words, we can establish the following equation:

$$\pi/4 = \arctan(1/2) + \arctan(1/5) + \arctan(1/8).$$

This equation was used by Dase to compute two hundred digits of π in his head in perhaps the greatest feat of mental arithmetic ever – 1/8 is apparently better than 1/239 (as in Machin's formula) for this purpose.

Who was Dase? Another burgeoning component of modern research and teaching life is having access to excellent data bases, such as the MacTutor History Archive maintained at: www-history.mcs.st-andrews.ac.uk (alas, not all sites are anywhere near so accurate and informative as this one). One may find details there on almost all of the mathematicians appearing in this chapter. I briefly illustrate its value by showing verbatim what it says about Dase.

> Zacharias Dase (1824–1861) had incredible calculating skills but little mathematical ability. He gave exhibitions of his calculating powers in Germany, Austria and England. While in Vienna in 1840 he was urged to use his powers for scientific purposes and he discussed projects with Gauss and others.
>
> Dase used his calculating ability to calculate π to 200 places in 1844. This was published in Crelle's Journal for 1844. Dase also constructed 7 figure log tables and produced a table of factors of all numbers between 7 000 000 and 10 000 000.

> Gauss requested that the Hamburg Academy of Sciences allow Dase to devote himself full-time to his mathematical work but, although they agreed to this, Dase died before he was able to do much more work.

Pentium farming

I finish this sub-section with another result obtained through integer relations methods or, as I like to call it, 'Pentium farming'. Bailey, Borwein and Plouffe (1997) discovered a series for π (and corresponding ones for some other *polylogarithmic* constants), which somewhat disconcertingly allows one to compute hexadecimal digits of π *without* computing prior digits. (This feels like magic, being able to tell the seventeen-millionth digit of π, say, without having to calculate the ones before it; it is like seeing God reach her hand deep into π.)

The algorithm needs very little memory and no multiple precision. The running time grows only slightly faster than linearly in the order of the digit being computed. The key, found by PSLQ as described above, is:

$$\pi = \sum_{k=0}^{\infty} \left(\frac{1}{16}\right)^k \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right).$$

Knowing an algorithm would follow, Bailey, Borwein and Plouffe spent several months hunting by computer for such a formula. Once found, it is easy to prove in *Mathematica,* in *Maple* or by hand – and provides a very nice calculus exercise.

This was a most successful case of *reverse mathematical engineering* and is entirely practicable. In September 1997, Fabrice Bellard (at INRIA) used a variant of this formula to compute one hundred and fifty-two binary digits of π, starting at the *trillionth* (10^{12}) place. This took twelve days on twenty work-stations working in parallel over the internet. In August 1998, Colin Percival (Simon Fraser University, age 17) finished a 'massively parallel' computation of the *five-trillionth bit* (using twenty-five machines at roughly ten times the speed of Bellard). In *hexadecimal notation,* he obtained:

$\underline{0}$7E45 733CC790B5B5979.

The corresponding binary digits of π starting at the forty-trillionth bit are:

$\underline{0}$ 0000 1111 1001 1111.

By September 2000, the quadrillionth bit had been found to be the digit 0 (using 250 cpu years on a total of one thousand, seven hundred and thirty-four machines from fifty-six countries). Starting at the 999,999,999,999,997th bit of π, we find:

111$\underline{0}$0 0110 0010 0001 0110 1011 0000 0110.

Solid and Discrete Geometry – and Number Theory

Although my own primary research interests are in numerical, classical and functional analysis, I find that the fields of solid and discrete geometry, as well as number theory, offer many examples of the kinds of concrete, insightful ideas I value. In the first example, I argue for the computational affordances available to study of solid geometry. I then discuss the genesis of an elegant proof in discrete geometry. Finally, I illustrate a couple of deep results in partition theory.

de Morgan

Augustus de Morgan, one of the most influential educators of his period and first president of the London Mathematical Society, wrote:

> Considerable obstacles generally present themselves to the beginner, in studying the elements of Solid Geometry, from the practice which has hitherto uniformly prevailed in this country, of never submitting to the eye of the student, the figures on whose properties he is reasoning, but of drawing perspective representations of them upon a plane. [...] I hope that I shall never be obliged to have recourse to a perspective drawing of any figure whose parts are not in the same plane. (in Rice, 1999, p. 540)

His comment illustrates the importance of concrete experiences with mathematical objects, even when the ultimate purpose is to abstract. There is a sense in which insight lies in physical manipulation. I imagine that de Morgan would have been happier using JavaViewLib (see: www.cecm.sfu.ca/interfaces/). This is Konrad Polthier's modern version of Felix Klein's famous set of geometric models. Correspondingly, a modern interactive version of Euclid is provided by *Cinderella* (a software tool which is largely comparable with *The Geometer's Sketchpad;* the latter is discussed in detail in Chapter 7 of this volume). Klein, like de Morgan, was equally influential as an educator and as a researcher.

Sylvester's theorem

Sylvester's theorem is worth mentioning because of its elegant visual proof, but also because of Sylvester's complex relationship to geometry: "The early study of Euclid made me a hater of geometry" (quoted in MacHale, 1993, p. 135). James Joseph Sylvester, who was the second president of the London Mathematical Society, may have hated Euclidean geometry, but discrete geometry (now much in fashion under the name 'computational geometry', offering another example of very useful pure mathematics) was different. His strong, emotional preference nicely illustrates how the aesthetic is involved in a mathematician's choice of fields.

Sylvester (1893) came up with the following conjecture, which he posed in *The Educational Times:*

> THEOREM Given n non-collinear points in the plane, then there is always at least one (*elementary* or *proper*) line going through exactly two points of the set.

Sylvester's conjecture was, so it seems, forgotten for fifty years. It was first established – 'badly', in the sense that the proof is much more complicated – by T. Grünwald (Gallai) in 1933 (see editorial comment in Steinberg, 1944) and also by Paul Erdös. Erdös, an atheist, named 'the Book' the place where God keeps aesthetically perfect proofs. L. Kelly's proof (given below), which Erdös accepted into 'the Book', was actually published by Donald Coxeter (1948) in the *American Mathematical Monthly*. This is a fine example of how the archival record may rapidly get obscured.

> PROOF Consider the point *closest* to a line it is not on and then suppose that line has three points on it (the horizontal line). The middle of those three points is clearly *closer* to the other line.

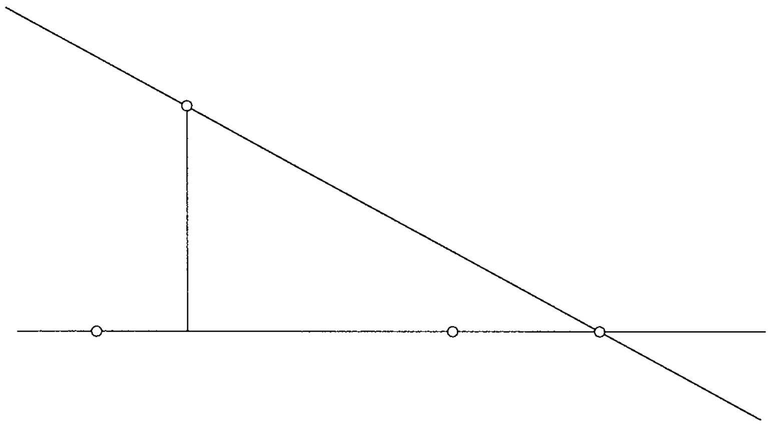

Figure 5. Kelly's proof from 'the Book'

As with Apostol's proof of the irrationality of $\sqrt{2}$, we can see the power of the right *minimal configuration.* Aesthetic appeal often comes from having this characteristic: that is, its appeal stems from being able to reason about an unknown number of objects by identifying a restricted view that captures all the possibilities. This is a process that is not so very different from that powerful method of proof known as mathematical induction.

Another example worth mentioning in this context (one that belongs in 'the Book') is Niven's (1947) marvellous (simple and short), half-page proof that π is *irrational* (see: www.cecm.sfu.ca/personal/jborwein/pi.pdf).

Partitions and patterns

Another subject that can be made highly accessible through experimental methods is additive number theory, especially *partition theory.* The number of *additive partitions* of q, $P(q)$, is generated by the following equation:

$$P(q) := \prod_{n\geq 1} (1-q^n)^{-1}$$

Thus, P(5) = 7, since:

5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1

= 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1

QUESTION How hard is P(q) to compute? Consider this question as it might apply in 1900 (for Major MacMahon, the father of our modern combinatorial analysis) and in 2000 (for *Maple*).

ANSWER Seconds for *Maple,* months for MacMahon. It is interesting to ask if development of the beautiful asymptotic analysis of partitions by Hardy, Ramanujan and others would have been helped or impeded by such facile computation.

Ex-post-facto algorithmic analysis can be used to facilitate independent student discovery of *Euler's pentagonal number theorem.*

$$\prod_{n\geq 1} (1-q^n) = \sum_{n=-\infty}^{\infty} (-1)^n \; q^{(3n+1)n/2}$$

Ramanujan used MacMahon's table of P(q) to intuit remarkable and deep congruences, such as:

P($5n$ + 4) ≡ 0 (mod 5)

P($7n$ + 5) ≡ 0 (mod 7)

and

P($11n$ + 6) ≡ 0 (mod 11)

from data such as:

$$\begin{aligned} P(q) &= 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 + 15q^7 + 22q^8 + 30q^9 \\ &+ 42q^{10} + 56q^{11} + 77q^{12} + 101q^{13} + 135q^{14} + 176q^{15} + 231q^{16} \\ &+ 297q^{17} + 385q^{18} + 490q^{19} + 627q^{20}b + 792q^{21}b + 792q^{21}b \\ &+ 1002q^{22} + 1255q^{23} + \ldots \end{aligned}$$

Nowadays, if introspection fails, we can recognise the *pentagonal numbers* occurring above in Sloane and Plouffe's on-line *Encyclopaedia of Integer Sequences* (see: www.research.att.com/personal/njas/sequences/eisonline.html). Here, we see a very fine example of *Mathematics: the Science of Patterns,* which is the title of Keith Devlin's (1994) book. And much more may similarly be done.

Some Concluding Discussion

In recent years, there have been revolutionary advances in cognitive science – advances that have a profound bearing on our understanding of mathematics. (More serious curricular insights should come from neuro-biology – see Dehaene *et al.,* 1999.) Perhaps the most profound of these new insights are the following, presented in Lakoff and Nuñez (2000).

1. *The embodiment of mind* The detailed nature of our bodies, our brains and our everyday functioning in the world structures human concepts and human reason. This includes mathematical concepts and mathematical reason. (See also Chapter 6.)
2. *The cognitive unconscious* Most thought is unconscious – not repressed in the Freudian sense, but simply inaccessible to direct conscious introspection. We cannot look directly at our conceptual systems and at our low-level thought processes. This includes most mathematical thought.
3. *Metaphorical thought* For the most part, human beings conceptualise abstract concepts in concrete terms, using ideas and modes of reasoning grounded in sensori-motor systems. The mechanism by which the abstract is comprehended in terms of the concept is called *conceptual metaphor.* Mathematical thought also makes use of conceptual metaphor: for instance, when we conceptualise numbers as points on a line.

Lakoff and Nuñez subsequently observe:

> What is particularly ironic about this is it follows from the empirical study of numbers as a product of mind that it is natural for people to believe that numbers are not a product of mind! (p. 81)

I find their general mathematical schema pretty persuasive but their specific accounting of mathematics forced and unconvincing (see also Schiralli and Sinclair, 2003). Compare this with a more traditional view, one that I most certainly espouse:

> The price of metaphor is eternal vigilance. (Arturo Rosenblueth and Norbert Wiener, in Lewontin, 2001, p. 1264)

Form follows function

> The waves of the sea, the little ripples on the shore, the sweeping curve of the sandy bay between the headlands, the outline of the hills, the shape of the clouds, all these are so many riddles of form, so many problems of morphology, and all of them the physicist can more or less easily read and adequately solve [...] (Thompson, 1917/1968, p. 10)

A century after biology started to think physically, how will mathematical thought patterns change?

> The idea that we could make biology mathematical, I think, perhaps is not working, but what is happening, strangely enough, is that maybe mathematics will become biological! (Chaitin, 2002)

To appreciate Greg Chaitin's comment, one has only to consider the metaphorical or actual origin of current 'hot topics' in mathematics research: simulated annealing ('protein folding'); genetic algorithms ('scheduling problems'); neural networks ('training computers'); DNA computation ('travelling salesman problems'); quantum computing ('sorting algorithms').

Humanistic philosophy of mathematics

However extreme the current paradigm shifts are and whatever the outcome of these discourses, mathematics is and will remain a uniquely human undertaking. Indeed, Reuben Hersh's (1995) full argument for a humanist philosophy of mathematics, as paraphrased below, becomes all the more convincing in this setting.

1. *Mathematics is human* It is part of and fits into human culture. It does not match Frege's concept of an abstract, timeless, tenseless and objective reality (see Resnik, 1980, and Chapter 8).
2. *Mathematical knowledge is fallible* As in science, mathematics can advance by making mistakes and then correcting or even re-correcting them. The 'fallibilism' of mathematics is brilliantly argued in Imre Lakatos's (1976) *Proofs and Refutations.*
3. *There are different versions of proof or rigour* Standards of rigour can vary depending on time, place and other things. Using computers in formal proofs, exemplified by the computer-assisted proof of the four-colour theorem in 1977, is just one example of an emerging, non-traditional standard of rigour.
4. *Aristotelian logic is not always necessarily the best way of deciding* Empirical evidence, numerical experimentation and probabilistic proof can all help us decide what to believe in mathematics.
5. *Mathematical objects are a special variety of a social–cultural–historical object* Contrary to the assertions of certain post-modern detractors, mathematics cannot be dismissed as merely a new form of literature or religion. Nevertheless, many mathematical objects can be seen as shared ideas, like *Moby Dick* in literature or the Immaculate Conception in religion.

The recognition that 'quasi-intuitive' methods may be used to gain good mathematical insight can dramatically assist in the learning and discovery of mathematics. Aesthetic and intuitive impulses are shot through our subject and honest mathematicians will acknowledge their role.

Some Final Observations

> When we have before us, for instance, a fine map, in which the line of coast, now rocky, now sandy, is clearly indicated, together with the windings of the rivers, the elevations of the land, and the distribution of the population, we have the simultaneous suggestion of so many facts, the sense of mastery over so much reality, that we gaze at it with delight, and need no practical motive to keep us studying it, perhaps for hours together. A map is not naturally thought of as an æsthetic object; it is too exclusively expressive. (Santayana, 1896/1910, p. 209)

This Santayana quotation was my earliest, and still favourite, encounter with aesthetic philosophy. It may be old fashioned and un-deconstructed in tone, but to me it rings true. He went on:

> And yet, let the tints of it be a little subtle, let the lines be a little delicate, and the masses of land and sea somewhat balanced, and we really have a beautiful thing; a thing the charm of which consists almost entirely in its meaning, but which nevertheless pleases us in the same way as a picture or a graphic symbol might please. Give the symbol a little intrinsic worth of form, line, and color, and it attracts like a magnet all the values of the things it is known to symbolize. It becomes beautiful in its expressiveness. (p. 210)

However, in conclusion, and to avoid possible accusations of mawkishness at the close, I also quote Jerry Fodor (1985):

> It is, no doubt, important to attend to the eternally beautiful and to believe the eternally true. But it is more important not to be eaten. (p. 4)

Notes

[1] This quotation is commonly attributed to Gauss, but it has proven remarkably resistant to being tracked down. Arber, the citation I give here, a philosopher of biology, acknowledges in a footnote (p. 47) that, “the present writer has been unable to trace this dictum to its original source”. Interestingly, even the St. Andrews history of mathematics site cites Arber. See also Dunnington (1955/2004).

[2] In *Brouwer’s Cambridge Lectures on Intuitionism,* the editor van Dalen (1981, p. 95) comments in a footnote:

> 3. The first use of undecidable properties of effectively presented objects (such as the decimal expansion of π) occurs in Brouwer (1908 [/1975]).

these cases. We must weigh the apparent security purchased by requiring predicative definitions against the burden of having to abandon in many cases what we, as mathematicians, consider natural definitions.

2. It is unclear exactly what objects we are committed to when we are committed to Peano Arithmetic. There are plenty of problems in number theory whose proofs use analytic means, for instance. Does commitment to Peano Arithmetic entail commitment to whatever objects are needed for these proofs? More generally, does commitment to a mathematical theory mean commitment to any objects needed for solving problems of that theory? If so, then Gödel's incompleteness theorems suggest that it is open what objects commitment to Peano Arithmetic entails.
3. As Feferman admits, it is unclear how to account predicatively for some mathematics used in currently accepted scientific practice, for instance, in quantum mechanics. In addition, I think that Feferman would not want to make the stronger claim that *all future* scientifically applicable mathematics will be accountable for by predicative means. However, the claim that *currently* scientifically applicable mathematics can be accounted for predicatively seems too time-bound to play an important role in a foundation of mathematics. Though it is impossible to predict all future scientific advances, it is reasonable to aim at a foundation of mathematics that has the potential to support these advances. Whether or not predicativity is such a foundation should be studied critically.
4. Whether the use of impredicative sets, and the uncountable more generally, is needed for ordinary finite mathematics, depends on whether by "ordinary" we mean "current." If so, then this is subject to the same worry I raised for (3). It also depends on where we draw the line on what counts as finite mathematics. If, for instance, Goldbach's conjecture counts as finite mathematics, then we have a statement of finite mathematics for which it is completely open whether it can be proved predicatively or not.

In emphasizing the degree to which concerns about predicativism shape this book, I should not overemphasize it. There is much besides predicativism in this book, as I have tried to indicate. In fact, Feferman advises that we not read his predicativism too strongly. In the preface, he describes his interest in predicativity as concerned with seeing how far in mathematics we can get without resorting to the higher infinite, whose justification he thinks can only be platonic. It may turn out that uncountable sets are needed for doing valuable mathematics, such as solving currently unsolved problems. In that case, Feferman writes, we "should look to see where it is necessary to use them and what we can say about what it is we know when we do use them" (p. ix).

Nevertheless, Feferman's committed anti-platonism is a crucial influence on the book. For mathematics right now, Feferman thinks, "a little bit goes a long way," as one of the essay titles puts it. The full universe of sets admitted by the platonist is unnecessary, he thinks, for doing the mathematics for which we must currently account. Time will tell if future developments will support that view, or whether, like Brouwer's view, it will require the alteration or outright rejection of too much mathematics to be viable. Feferman's book shows that, far from being over, work on the foundations of mathematics is vibrant and continuing, perched deliciously but precariously between mathematics and philosophy.

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The SIAM 100-Digit Challenge: A Study in High-Accuracy Numerical Computing

by Folkmar Bornemann, Dirk Laurie, Stan Wagon, and Jörg Waldvogel

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REVIEWED BY JONATHAN M. BORWEIN

Lists, challenges, and competitions have a long and primarily lustrous history in mathematics. This is the story of a recent highly successful challenge. The book under review makes it clear that with the continued advance of computing power and accessibility, the view that "real mathematicians don't compute" has little traction, especially for a newer generation of mathematicians who may readily take advantage of the maturation of computational packages such as *Maple*, *Mathematica*, and MATLAB.

Numerical Analysis Then and Now

George Phillips has accurately called Archimedes the first numerical analyst [2, pp. 165–169]. In the process of obtaining his famous estimate $3 + 10/71 < \pi < 3 + 1/7$, he had to master notions of recursion without computers, interval analysis without zero or positional arithmetic, and trigonometry without any of our modern analytic scaffolding. . . . Two millennia later, the same estimate can be obtained by a computer algebra system [3].

Example 1. A modern computer algebra system can tell one that

$$(1.1) \qquad 0 < \int_0^1 \frac{(1-x)^4 x^4}{1+x^2}\,dx = \frac{22}{7} - \pi,$$

since the integral may be interpreted as the area under a positive curve.

This leaves us no wiser as to why! If, however, we ask the same system to compute the indefinite integral, we are likely to be told that

$$\int_0^t \cdot = \frac{1}{7}t^7 - \frac{2}{3}t^6 + t^5 - \frac{4}{3}t^3 + 4t - 4\arctan(t).$$

Then (1.1) is now rigorously established by differentiation and an appeal to Newton's Fundamental theorem of calculus. □

While there were many fine arithmeticians over the next 1500 years, this anecdote from Georges Ifrah reminds us that mathematical culture in Europe had not sustained Archimedes's level up to the Renaissance.

> *A wealthy (15th-century) German merchant, seeking to provide his son with a good business education, consulted a learned man as to which European institution offered the best training. "If you only want him to be able to cope with addition and subtraction," the expert replied, "then any French or German university will do. But if you are intent on your son going on to multiplication and division—assuming that he has sufficient gifts—then you will have to send him to Italy.*[1]

By the 19th century, Archimedes had finally been outstripped both as a theorist and as an (applied) numerical analyst, see [7].

> *In 1831, Fourier's posthumous work on equations showed 33 figures of solution, got with enormous labour. Thinking this a good opportunity to illustrate the superiority of the method of W. G. Horner, not yet known in France, and not much known in England, I proposed to one of my classes, in 1841, to beat Fourier on this point, as a Christmas exercise. I received several answers, agreeing with each other, to 50 places of decimals. In 1848, I repeated the proposal, requesting that 50 places might be exceeded: I obtained answers of 75, 65, 63, 58, 57, and 52 places.* (Augustus De Morgan[2])

De Morgan seems to have been one of the first to mistrust William Shanks's epic computations of Pi—to 527, 607, and 727 places [2, pp. 147–161], noting there were too few sevens. But the error was only confirmed three quarters of a century later in 1944 by Ferguson with the help of a calculator in the last pre-computer calculations of π—though until around 1950 a "computer" was still a person and ENIAC was an "Electronic Numerical Integrator and Calculator" [2, pp. 277–281] on which Metropolis and Reitwiesner computed Pi to 2037 places in 1948 and confirmed that there were the expected number of sevens.

Reitwiesner, then working at the Ballistics Research Laboratory, Aberdeen Proving Ground in Maryland, starts his article [2, pp. 277–281] with

> *Early in June, 1949, Professor* JOHN VON NEUMANN *expressed an interest in the possibility that the ENIAC might sometime be employed to determine the value of π and e to many decimal places with a view toward obtaining a statistical measure of the randomness of distribution of the digits.*

The paper notes that *e* appears to be *too* random—this is now proven—and ends by respecting an oft-neglected "best-practice":

> *Values of the auxiliary numbers* arccot 5 *and* arccot 239 *to* 2035*D . . . have been deposited in the library of Brown University and the UMT file of MTAC.*

The 20th century's "Top Ten"

The digital computer, of course, greatly stimulated both the appreciation of and the need for algorithms and for algorithmic analysis. At the beginning of this century, Sullivan and Dongarra could write, "Great algorithms are the poetry of computation," when they compiled a list of the 10 algorithms having "the greatest influence on the development and practice of science and engineering in the 20th century".[3] Chronologically ordered, they are:

#1. 1946: **The Metropolis Algorithm for Monte Carlo**. Through the use of random processes, this algorithm offers an efficient way to stumble toward answers to problems that are too complicated to solve exactly.

#2. 1947: **Simplex Method for Linear Programming.** An elegant solution to a common problem in planning and decision making.

#3. 1950: **Krylov Subspace Iteration Method.** A technique for rapidly solving the linear equations that abound in scientific computation.

#4. 1951: **The Decompositional Approach to Matrix Computations.** A suite of techniques for numerical linear algebra.

#5. 1957: **The Fortran Optimizing Compiler.** Turns high-level code into efficient computer-readable code.

#6. 1959: **QR Algorithm for Computing Eigenvalues.** Another crucial matrix operation made swift and practical.

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[1]From page 577 of *The Universal History of Numbers: From Prehistory to the Invention of the Computer*, translated from French, John Wiley, 2000.

[2]Quoted by Adrian Rice in "What Makes a Great Mathematics Teacher?" on page 542 of The *American Mathematical Monthly*, June–July 1999.

[3]From "Random Samples," *Science* page 799, February 4, 2000. The full article appeared in the January/February 2000 issue of *Computing in Science & Engineering.*

#7. 1962: **Quicksort Algorithms for Sorting.** For the efficient handling of large databases.

#8. 1965: **Fast Fourier Transform.** Perhaps the most ubiquitous algorithm in use today, it breaks down waveforms (like sound) into periodic components.

#9. 1977: **Integer Relation Detection.** A fast method for spotting simple equations satisfied by collections of seemingly unrelated numbers.

#10. 1987: **Fast Multipole Method.** A breakthrough in dealing with the complexity of n-body calculations, applied in problems ranging from celestial mechanics to protein folding.

I observe that eight of these ten winners appeared in the first two decades of serious computing, and that Newton's method was apparently ruled ineligible for consideration.[4] Most of the ten are multiply embedded in every major mathematical computing package.

Just as layers of software, hardware, and middleware have stabilized, so have their roles in scientific, and especially mathematical, computing. When I first taught the simplex method thirty years ago, the texts concentrated on "Y2K"-like tricks for limiting storage demands. Now serious users and researchers will often happily run large-scale problems in MATLAB and other broad-spectrum packages, or rely on *NAG* library routines embedded in *Maple*.

While such out-sourcing or commoditization of scientific computation and numerical analysis is not without its drawbacks, I think the analogy with automobile driving in 1905 and 2005 is apt. We are now in possession of mature—not to be confused with "error-free"—technologies. We can be fairly comfortable that *Mathematica* is sensibly handling round-off or cancelation error, using reasonable termination criteria and the like. Below the hood, *Maple* is optimizing polynomial computations using tools like Horner's rule, running multiple algorithms when there is no clear best choice, and switching to reduced complexity (Karatsuba or FFT-based) multiplication when accuracy so demands. Wouldn't it be nice, though, if all vendors allowed as much peering under the bonnet as *Maple* does!

Example 2. The number of *additive partitions* of n, $p(n)$, is generated by

$$P(q) = 1 + \sum_{n\geq 1} p(n)q^n = \prod_{n\geq 1} (1 - q^n)^{-1}. \quad (1.2)$$

Thus $p(5) = 7$, because

$$5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1$$
$$= 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1,$$

as we ignore "0" and permutations. Additive partitions are less tractable than multiplicative ones, for there is no analogue of unique prime factorization nor the corresponding structure. Partitions provide a wonderful example of why Keith Devlin calls mathematics "the science of patterns."

Formula (1.2) is easily seen by expanding $(1 - q^n)^{-1}$ and comparing coefficients. A modern computational temperament leads to

Question: How hard is $p(n)$ to compute—in 1900 (for MacMahon the "father of combinatorial analysis") or in 2000 (for *Maple* or *Mathematica*)?

Answer: The computation of $p(200) = 3972999029388$ took MacMahon months and intelligence. Now, however, we can use the most naïve approach: Computing 200 terms of the series for the inverse product in (1.2) instantly produces the result, using either *Mathematica* or *Maple*. Obtaining the result $p(500) = 2300165032574323995027$ is not much more difficult, using the *Maple* code

```
N : =500; coeff(series(1/product
          (1-q^n,n=1..N+1),q,N+1),q,N);
```

Euler's Pentagonal number theorem

Fifteen years ago computing $P(q)$ in *Maple*, was very slow, while taking the series for the reciprocal $Q(q) = \Pi_{n\geq 1}(1 - q^n)$ was quite manageable! Why? Clearly the series for Q must have special properties. Indeed it is *lacunary*:

$$Q(q) = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + q^{26} - q^{35} - q^{40} + q^{51} + q^{57} - q^{70} - q^{77} + q^{92} + O(q^{100}). \quad (1.3)$$

This lacunarity is now recognized automatically by *Maple*, so the platform works much better, but we are much less likely to discover Euler's gem:

$$\prod_{n=1}^{\infty} (1 - q^n) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2}.$$

If we do not immediately recognize these *pentagonal numbers*, then Sloane's online *Encyclopedia of Integer Sequences*[5] immediately comes to the rescue, with abundant references to boot.

This sort of mathematical computation is still in its reasonably early days, but the impact is palpable—and no more so than in the contest and book under review.

About the Contest

For a generation Nick Trefethen has been at the vanguard of developments in scientific computation, both through his own research, on topics such as pseudo-spectra, and through much thoughtful and vigorous activity in the community. In a 1992 essay "The Definition of Numerical Analysis"[6] Trefethen engagingly demolishes the conventional definition of Numerical Analysis as "the science of rounding errors." He explores how this hyperbolic view emerged, and finishes by writing,

> *I believe that the existence of finite algorithms for certain problems, together with other historical forces, has*

[4]It would be interesting to construct a list of the ten most influential earlier algorithms.

[5]A fine model for of 21st-century databases, it is available at www.research.att.com/~njas/sequences

[6]*SIAM News*, November 1992.

distracted us for decades from a balanced view of numerical analysis. Rounding errors and instability are important, and numerical analysts will always be the experts in these subjects and at pains to ensure that the unwary are not tripped up by them. But our central mission is to compute quantities that are typically uncomputable, from an analytical point of view, and to do it with lightning speed. For guidance to the future we should study not Gaussian elimination and its beguiling stability properties, but the diabolically fast conjugate gradient iteration, or Greengard and Rokhlin's O(N) multipole algorithm for particle simulations, or the exponential convergence of spectral methods for solving certain PDEs, or the convergence in O(N) iterations achieved by multigrid methods for many kinds of problems, or even Borwein and Borwein's[7] magical AGM iteration for determining 1,000,000 digits of π in the blink of an eye. That is the heart *of numerical analysis.*

In the January 2002 issue of *SIAM News*, Nick Trefethen, by then of Oxford University, presented ten diverse problems used in teaching *modern* graduate numerical analysis students at Oxford University, the answer to each being a certain real number. Readers were challenged to compute ten digits of each answer, with a $100 prize to be awarded to the best entrant. Trefethen wrote, "If anyone gets 50 digits in total, I will be impressed."

And he was. A total of 94 teams, representing 25 different nations, submitted results. Twenty of these teams received a full 100 points (10 correct digits for each problem). They included the late John Boersma, working with Fred Simons and others; Gaston Gonnet (a Maple founder) and Robert Israel; a team containing Carl Devore; and the authors of the book under review variously working alone and with others. These results were much better than expected, but an originally anonymous donor, William J. Browning, provided funds for a $100 award to each of the twenty perfect teams. The present author, David Bailey,[8] and Greg Fee entered, but failed to qualify for an award.[9]

The ten challenge problems

The purpose of computing is insight, not numbers. (Richard Hamming[10])

The ten problems are:

#1. What is $\lim_{\epsilon\to 0}\int_\epsilon^1 x^{-1}\cos(x^{-1}\log x)dx$?

#2. A photon moving at speed 1 in the x-y plane starts at $t = 0$ at $(x,y) = (1/2, 1/10)$ heading due east. Around every integer lattice point (i, j) in the plane, a circular mirror of radius 1/3 has been erected. How far from the origin is the photon at $t = 10$?

#3. The infinite matrix A with entries $a_{11} = 1$, $a_{12} = 1/2$, $a_{21} = 1/3$, $a_{13} = 1/4$, $a_{22} = 1/5$, $a_{31} = 1/6$, etc., is a bounded operator on ℓ^2. What is $\|A\|$?

#4. What is the global minimum of the function $\exp(\sin(50x)) + \sin(60e^y) + \sin(70 \sin x) + \sin(\sin(80y)) - \sin(10(x + y)) + (x^2 + y^2)/4$?

#5. Let $f(z) = 1/\Gamma(z)$, where $\Gamma(z)$ is the gamma function, and let $p(z)$ be the cubic polynomial that best approximates $f(z)$ on the unit disk in the supremum norm $\|\cdot\|_\infty$. What is $\|f - p\|_\infty$?

#6. A flea starts at (0,0) on the infinite 2-D integer lattice and executes a biased random walk: At each step it hops north or south with probability 1/4, east with probability $1/4 + \epsilon$, and west with probability $1/4 - \epsilon$. The probability that the flea returns to (0,0) sometime during its wanderings is 1/2. What is ϵ?

#7. Let A be the 20000×20000 matrix whose entries are zero everywhere except for the primes $2, 3, 5, 7, \cdots, 224737$ along the main diagonal and the number 1 in all the positions a_{ij} with $|i - j| = 1, 2, 4, 8, \cdots, 16384$. What is the (1,1) entry of A^{-1}?

#8. A square plate $[-1,1] \times [-1,1]$ is at temperature $u = 0$. At time $t = 0$ the temperature is increased to $u = 5$ along one of the four sides while being held at $u = 0$ along the other three sides, and heat then flows into the plate according to $u_t = \Delta u$. When does the temperature reach $u = 1$ at the center of the plate?

#9. The integral $I(\alpha) = \int_0^2 [2 + \sin(10\alpha)]x^\alpha \sin(\alpha/(2 - x))\, dx$ depends on the parameter α. What is the value $\alpha \in [0,5]$ at which $I(\alpha)$ achieves its maximum?

#10. A particle at the center of a 10×1 rectangle undergoes Brownian motion (i.e., 2-D random walk with infinitesimal step lengths) till it hits the boundary. What is the probability that it hits at one of the ends rather than at one of the sides?

Answers correct to 40 digits to the problems are available at http://web.comlab.ox.ac.uk/oucl/work/nick.trefethen/hundred.html

Quite full details on the contest and the now substantial related literature are beautifully recorded on Bornemann's Web site

http://www-m8.ma.tum.de/m3/bornemann/challengebook/

which accompanies *The SIAM 100-digit Challenge: A Study In High-accuracy Numerical Computing*, which, for brevity, I shall call *The Challenge.*

About the Book and Its Authors

Success in solving these problems requires a broad knowledge of mathematics and numerical analysis, together with

[7] As in many cases, this eponym is inaccurate, if flattering: it really should be Gauss-Brent-Salamin.

[8] Bailey wrote the introduction to the book under review.

[9] We took Nick at his word and turned in 85 digits! We thought that would be a good enough entry and returned to other activities.

[10] In *Numerical Methods for Scientists and Engineers*, 1962.

significant computational effort, to obtain solutions and ensure correctness of the results. The strengths and limitations of *Maple, Mathematica*, MATLAB (The *3Ms*), and other software tools such as PARI or GAP, are strikingly revealed in these ventures. Almost all of the solvers relied in large part on one or more of these three packages, and while most solvers attempted to confirm their results, there was no explicit requirement for proofs to be provided. In December 2002, Keller wrote:

> *To the Editor:*
> *Recently, SIAM News published an interesting article by Nick Trefethen (July/August 2002, page 1) presenting the answers to a set of problems he had proposed previously (January/February 2002, page 1). The answers were computed digits, and the clever methods of computation were described.*
> *I found it surprising that no proof of the correctness of the answers was given. Omitting such proofs is the accepted procedure in scientific computing. However, in a contest for calculating precise digits, one might have hoped for more.*
> *Joseph B. Keller, Stanford University*

In my view Keller's request for proofs as opposed to compelling evidence of correctness is, in this context, somewhat unreasonable, and even in the long term counterproductive [3, 4]. Nonetheless, the authors of *The Challenge* have made a complete and cogent response to Keller and much much more. The interest generated by the contest has with merit extended to *The Challenge*, which has already received reviews in places such as *Science*, where mathematics is not often seen.

Different readers, depending on temperament, tools, and training, will find the same problem more or less interesting and more or less challenging. The book is arranged so the ten problems can be read independently. In all cases multiple solution techniques are given; background, mathematics, implementation details—variously in each of the *3Ms* or otherwise—and extensions are discussed, all in a highly readable and engaging way.

Each problem has its own chapter with its own lead author. The four authors, Folkmar Bornemann, Dirk Laurie, Stan Wagon, and Jörg Waldvogel, come from four countries on three continents and did not know each other as they worked on the book, though Dirk did visit Jörge and Stan visited Folkmar as they were finishing their manuscript. This illustrates the growing power of the collaboration, networking, and the grid—both human and computational.

Some high spots

As we saw, Joseph Keller raised the question of proof. On careful reading of the book, one may discover proofs of correctness for all problems except for #1, #3, and #5. For problem #5, one difficulty is to develop a robust interval implementation for both complex number computation and, more importantly, for the *Gamma function*. While error bounds for #1 may be out of reach, an analytic solution to #3 seems to this reviewer tantalizingly close.

The authors ultimately provided 10,000-digit solutions to nine of the problems. They say that this improved their knowledge on several fronts as well as being "cool." When using Integer Relation Methods, ultrahigh precision computations are often needed [3]. One (and only one) problem remains totally intractable[11]—at press time, getting more than 300 digits for #3 was impossible.

Some surprises

According to the authors,[12] they were surprised by the following, listed by problem:

#1. The best algorithm for 10,000 digits was the trusty *trapezoidal rule*—a not uncommon personal experience of mine.

#2. Using *interval arithmetic* with starting intervals of size smaller than 10^{-5000}, one can still find the position of the particle at time 2000 (not just time ten), which makes a fine exercise for very high-precision interval computation.

#4. Interval analysis algorithms can handle similar problems in higher dimensions. As a foretaste of future graphic tools, one can solve this problem using current *adaptive 3-D plotting* routines which can catch all the bumps. As an optimizer by background, this was the first problem my group solved using a damped Newton method.

#5. While almost all canned optimization algorithms failed, *differential evolution*, a relatively new type of evolutionary algorithm, worked quite well.

#6. This problem has an almost-closed form in terms of elliptic integrals and leads to a study of random walks on hypercubic lattices, and Watson integrals [3, 4, 5].

#9. The maximum parameter is expressible in terms of a *MeijerG function*. While this was not common knowledge among the contestants, *Mathematica* and *Maple* both will figure this out. This is another measure of the changing environment. It is usually a good idea—and not at all immoral—to data-mine[13] and find out what your favourite one of the *3Ms* knows about your current object of interest. For example, Maple tells one that:

---

[11] If only by the authors' new gold standard of 10,000 digits.

[12] Stan Wagon, private communication.

[13] By its own count, Wal-Mart has 460 terabytes of data stored on Teradata mainframes, made by NCR, at its Bentonville headquarters. To put that in perspective, the Internet has less than half as much data . . . ," Constance Hays, "What Wal-Mart Knows About Customers' Habits," *New York Times*, Nov. 14, 2004. Mathematicians also need databases.

```
  The Meijer G function is defined by the inverse
                           Laplace transform
  MeijerG([as,bs],[cs,ds],z)
                  /
        1         |  GAMMA(1-as+y) GAMMA(cs-y)   y
  = ---------     |  -------------------------- z  dy
    2 Pi  I       0  GAMMA(bs-y) GAMMA(1-ds+y)
                  |
                  /
                   L
where
  as = [a1, ..., am], GAMMA(1-as+y) = GAMMA(1-a1+y) ... GAMMA(1-am+y)
  bs = [b1, ..., bn], GAMMA(bs-y) = GAMMA(b1-y)     ... GAMMA(bn-y)
  cs = [c1, ..., cp], GAMMA(cs-y) = GAMMA(c1-y)     ... GAMMA(cp-y)
  ds = [d1, ... dq],  GAMMA(1-ds+y) = GAMMA(1-d1+y) ... GAMMA(1-dq+y)
```

Another excellent example of how packages are changing mathematics is the *Lambert W function* [4], whose properties and development are very nicely described in a recent article by Brian Hayes [8], *Why W?*

Two big surprises

I finish this section by discussing in more detail the two problems whose resolution most surprised the authors.

The essay on Problem #7, whose principal author was Bornemann, is titled: "Too Large to be Easy, Too Small to Be Hard." Not so long ago a $20{,}000 \times 20{,}000$ matrix was large enough to be hard. Using both *congruential* and *p-adic* methods, Dumas, Turner, and Wan obtained a fully *symbolic* answer, a rational with a 97,000-digit numerator and like denominator. Wan has reduced the time to obtain this to about 15 minutes on one machine, from using many days on many machines. While *p-adic* analysis is susceptible to parallelism, it is less easily attacked than are congruential methods; the need for better parallel algorithms lurks below the surface of much modern computational mathematics.

The surprise here, though, is not that the solution is rational, but that it can be explicitly constructed. The chapter, like the others, offers an interesting menu of numeric and exact solution strategies. Of course, in any numeric approach *ill-conditioning* rears its ugly head, while *sparsity* and other core topics come into play.

My personal favourite, for reasons that may be apparent, is:

Problem #10: "Hitting the Ends." Bornemann starts the chapter by exploring *Monte-Carlo methods*, which are shown to be impracticable. He then reformulates the problem *deterministically* as the value at the center of a 10×1 rectangle of an appropriate harmonic measure of the ends, arising from a 5-point discretization of Laplace's equation with Dirichlet boundary conditions. This is then solved by a well-chosen *sparse Cholesky* solver. At this point a reliable numerical value of $3.837587979 \cdot 10^{-7}$ is obtained. And the posed problem is solved numerically to the requisite 10 places.

But this is only the warm-up. We proceed to develop two analytic solutions, the first using *separation of variables* on the underlying PDE on a general $2a \times 2b$ rectangle. We learn that

$$p(a,b) = \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\,\mathrm{sech}\left(\frac{\pi(2n+1)}{2}\rho\right) \tag{3.4}$$

where $\rho := a/b$. A second method using *conformal mappings* yields

$$\operatorname{arccot}\rho = p(a,b)\,\frac{\pi}{2} + \arg K(e^{ip(a,b)\pi}), \tag{3.5}$$

where K is the *complete elliptic integral* of the first kind. It will not be apparent to a reader unfamiliar with inversion of elliptic integrals that (3.4) and (3.5) encode the same solution; but they must, as the solution is unique in (0,1); each can now be used to solve for $p = 10$ to arbitrary precision.

Bornemann finally shows that, for far from simple reasons, the answer is

$$p = \frac{2}{\pi}\arcsin(k_{100}), \tag{3.6}$$

where

$$k_{100} := \left((3-2\sqrt{2})\,(2+\sqrt{5})\,(-3+\sqrt{10})\,(-\sqrt{2}+\sqrt[4]{5})^2\right)^2$$

a simple composition of one arcsin and a few square roots. No one anticipated a closed form like this.

Let me show how to finish up. An apt equation is [5, (3.2.29)] showing that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\,\mathrm{sech}\left(\frac{\pi(2n+1)}{2}\rho\right) = \frac{1}{2}\arcsin k, \tag{3.7}$$

exactly when $k = k_{\rho^2}$ is parametrized by *theta functions* in terms of the so-called *nome*, $q = \exp(-\pi\rho)$, as Jacobi discovered. We have

$$k_{\rho^2} = \frac{\theta_2^2(q)}{\theta_3^2(q)} = \frac{\sum_{n=-\infty}^{\infty} q^{(n+1/2)^2}}{\sum_{n=-\infty}^{\infty} q^{n^2}} \tag{3.8}$$

Comparing (3.7) and (3.4), we see that the solution is

$$k_{100} = 6.02806910155971082882540712292\ldots \cdot 10^{-7},$$

as asserted in (3.6). The explicit form now follows from classical nineteenth-century theory as discussed in [1, 5]. In fact k_{210} is the singular value sent by Ramanujan to Hardy in his famous letter of introduction [2, 5]—if only Trefethen had asked for a $\sqrt{210} \times 1$ box, or even better a $\sqrt{15} \times \sqrt{14}$ one!

Alternatively, armed only with the knowledge that the singular values are always algebraic, we may finish with an *au courant* proof: numerically obtain the minimal polynomial from a high-precision computation with (3.8), and recover the surds [4].

Example 3. *Maple* allows the following

```
> Digits:=100:with(PolynomialTools):
> k:=s->evalf(EllipticModulus(exp(-Pi*sqrt(s)))):
> p:=latex(MinimalPolynomial(k(100),12)):
> 'Error',fsolve(p)[1]-evalf(k(100)); galois(p);
                 Error, 4 10^-106
"8T9", {"D(4)[x]2", "E(8):2"}, "+", 16, {"4 5)(6 7)", "(4 8)(1 5)(2 6)(3 7)",
      "(1 8)(2 3)(4 5)(6 7)", "(2 8)(1 3)(4 6)(5 7)"}
```

which finds the minimal polynomial for k_{100}, checks it to 100 places, tells us the *galois group*, and returns a latex expression 'p' which sets as:

$$p(\_X) = 1 - 1658904\ \_X - 3317540\ \_X^2 + 1657944\ \_X^3 + 6637254\ \_X^4 + 1657944\ \_X^5 - 3317540\ \_X^6 - 1658904\ \_X^7 + \_X^8,$$

and is *self-reciprocal*: it satisfies $p(x) = x^8 p(1/x)$. This suggests taking a square root, and we discover that $y = \sqrt{k_{100}}$ satisfies

$$1 - 1288y + 20y^2 - 1288y^3 - 26y^4 + 1288y^5 + 20y^6 + 1288y^7 + y^8.$$

Now life is good. The prime factors of 100 are 2 and 5, prompting

```
subs(_X=z,
    [op(((factor(p,{sqrt(2),sqrt(5)}))))]))
```

This yields four quadratic terms, the desired one being

$$q = z^2 + 322\,z - 228\,z\sqrt{2} + 144\,z\sqrt{5} - 102\,z\sqrt{2}\sqrt{5} + 323 - 228\,\sqrt{2} + 144\sqrt{5} - 102\sqrt{2}\sqrt{5}.$$

For security,

```
w:=solve(q)[2]: evalf[1000](k(100)-w^2);
```

gives a 1000-digit error check of $2.20226255 \cdot 10^{-998}$.

We leave it to the reader to find, using one of the 3Ms, the more beautiful form of k_{100} given above in (3.6). □

Considering also the many techniques and types of mathematics used, we have a wonderful advertisement for multi-field, multi-person, multi-computer, multi-package collaboration.

Concrete Constructive Mathematics

Elsewhere Kronecker said "In mathematics, I recognize true scientific value only in concrete mathematical truths, or to put it more pointedly, only in mathematical formulas." . . . I would rather say "computations" than "formulas," but my view is essentially the same. (Harold M. Edwards [6, p. 1])

Edwards comments elsewhere in his recent *Essays on Constructive Mathematics* that his own preference for constructivism was forged by experience of computing in the fifties, when computing power was, as he notes, "trivial by today's standards." My own similar attitudes were cemented primarily by the ability in the early days of personal computers to decode—with the help of *APL*—exactly the sort of work by Ramanujan which finished #10.

The SIAM 100-Digit Challenge: A Study In High-accuracy Numerical Computing is a wonderful and well-written book full of living mathematics by lively mathematicians. It shows how far we have come computationally and hints tantalizingly at what lies ahead. Anyone who has been interested enough to finish this review, and had not yet read the book, is strongly urged to buy and plunge in—computer in hand—to this fine advertisement for constructive mathematics 21st-century style. I would equally strongly suggest a cross-word solving style—pick a few problems from the list given, and try them before peeking at the answers and extensions given in *The Challenge*. Later, use it to illustrate a course or just for a refresher; and be pleasantly reminded that challenging problems rarely have only one path to solution and usually reward study.

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Conversations on Mind, Matter, and Mathematics

by Jean-Pierre Changeux & Alain Connes
edited and translated by M. B. DeBevoise

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REVIEWED BY JEAN PETITOT

What exactly is the type of reality of mathematical ideal entities? This problem remains largely an open question. Any ontology of abstract entities will encounter certain antinomies which have been well known for centuries if not millennia. These antinomies have led the various schools of contemporary epistemology increasingly to deny any reality to mathematical ideal objects, structures, constructions, proofs, and to justify this denial philosophically, thus rejecting the spontaneous naïve Platonism of most professional mathematicians. But they throw out the baby with the bath water. Contrary to such figures as Poincaré, Husserl, Weyl, Borel, Lebesgue, Veronese, Enriques, Cavaillès, Lautman, Gonseth, or the late Gödel, the dominant epistemology of mathematics is no longer an epistemology of mathematical content. For quite serious and precise philosophical reasons, it refuses to take into account what the great majority of creative brilliant mathematicians consider to be the true nature of mathematical knowledge. And yet, to quote the subtitle of Hao Wang's (1985) book *Beyond Analytic Philosophy*, one might well ask whether the imperative of any valid epistemology should not be "doing justice to what we know."

The remarkable debate *Conversations on Mind, Matter, and Mathematics* between Alain Connes and Jean-Pierre Changeux, both scientific minds of the very first rank and professors at the Collège de France in Paris, takes up the old question of the reality of mathematical idealities in a rather new and refreshing perspective. To be sure, since it is designed to be accessible to a wide audience, the debate is not framed in technical terms; the arguments often employ a broad brush and are not always sufficiently developed. Nevertheless, thanks to the exceptional standing of the protagonists, the debate manages to be compelling and relevant.

Jean-Pierre Changeux's Neural Materialism

Let me begin by summarizing some of Jean-Pierre Changeux's arguments.

Because mathematics is a human and cognitive activity, it is natural first to analyze it in psychological and neurocognitive terms. Psychologism, which formalists and logicists have decried since the time of Frege and Husserl, develops the reductionist thesis that mathematical objects and the logical idealities that formulate them can be reduced—as far as their reality is concerned—to mental states and processes. Depending on whether or not mental representations are themselves conceived as reducible to the underlying neural activity, this psychologism is either a materialist reductionism or a mentalist functionalism.

J-P. Changeux defends a variant of materialist reductionism. His aim is twofold: first, to inquire into the nature of mathematics, but also, at a more strategic level, to put mathematics in its place, so to speak. He has never concealed his opposition to Cartesian or Leibnizian rationalisms that have made mathematics the "queen" of the sciences. In his view, mathematics must abdicate its overly arrogant sovereignty, stop laying claim to universal validity and absolute truth, and accept the humbler role assigned to it by Bacon and Diderot—that of "servant" to the natural sciences (p. 7). And what better way to make mathematics surrender its prestigious seniority than to demonstrate scientifically that its claims to absolute truth have no more rational basis than do those of religious faith?

Pursuing his mission with great conviction, Changeux revisits all the traditional touchstones of the empiricist, materialist, and nominalist critiques of Platonist idealism in mathematics. He cites an impressive mass of scientific data along the way, including results from neurobiology and cognitive psychology in which he has played a leading role. It is this aspect of his approach which commands attention.

1. The empiricist and constructivist theses hold that mathematical objects are "creatures of reason" whose reality is purely cerebral (p. 11). They are representations, that is, mental objects that exist materially in the brain, and "corresponding to physical [i.e., neural] states" (p. 14).

> Mental representations—memory objects—are coded in the brain as forms in the Gestalt sense, and stored in the neurons and synapses, despite significant variability in synaptic efficacy (p. 128).

Their object-contents are reflexively analyzable and their properties can be clarified axiomatically. But that is possible only because, as mental representations, they are endowed with a material reality (pp. 11–15). What's more, the axiomatic method of analysis is itself a "cerebral process" (p. 30).

2. One might try to salvage an autonomy for the formal logical and mathematical levels by admitting, in line with

Ten Problems in Experimental Mathematics

David H. Bailey, Jonathan M. Borwein, Vishaal Kapoor, and Eric W. Weisstein

1. INTRODUCTION. This article was stimulated by the recent SIAM "100 Digit Challenge" of Nick Trefethen, beautifully described in [**12**] (see also [**13**]). Indeed, these ten numeric challenge problems are also listed in [**15**, pp. 22–26], where they are followed by the ten symbolic/numeric challenge problems that are discussed in this article. Our intent in [**15**] was to present ten problems that are characteristic of the sorts of problems that commonly arise in "experimental mathematics" [**15**], [**16**]. The challenge in each case is to obtain a high precision numeric evaluation of the quantity and then, if possible, to obtain a symbolic answer, ideally one with proof. Our goal in this article is to provide solutions to these ten problems and, at the same time, to present a concise account of how one combines symbolic and numeric computation, which may be termed "hybrid computation," in the process of mathematical discovery.

The passage from object α to answer ω often relies on being able to compute the object to sufficiently high precision, for example, to determine numerically whether α is algebraic or is a rational combination of known constants. While some of this is now automated in mathematical computing software such as *Maple* and *Mathematica*, in most cases intelligence is needed, say in choosing the search space and in deciding the degree of polynomial to hunt for. In a similar sense, using symbolic computing tools such as those incorporated into *Maple* and *Mathematica* often requires significant human interaction to produce material results. Such matters are discussed in greater detail in [**15**] and [**16**].

Integer relation detection. Several of these solutions involve the usage of integer relation detection schemes to find experimentally a likely relationship. For a given real vector $(x_1, x_2, \ldots, x_n)$ an integer relation algorithm is a computational scheme that either finds the n-tuple of integers $(a_1, a_2, \ldots, a_n)$, not all zero, such that $a_1x_1 + a_2x_2 + \cdots a_nx_n = 0$ or else establishes that there is no such integer vector within a ball of some radius about the origin, where the metric is the Euclidean norm $(a_1^2 + a_2^2 + \cdots + a_n^2)^{1/2}$.

At the present time, the best known integer relation algorithm is the PSLQ algorithm [**25**] of Helaman Ferguson, who is well known in the community for his mathematical sculptures. Simple formulations of the PSLQ algorithm and several variants are given in [**7**]. Another widely used integer relation detection scheme involves the Lenstra-Lenstra-Lovasz (LLL) algorithm. The PSLQ algorithm, together with related lattice reduction schemes such as LLL, was recently named one of ten "algorithms of the century" by the publication *Computing in Science and Engineering* [**3**].

Perhaps the best-known application of PSLQ is the 1995 discovery, by means of a PSLQ computation, of the "BBP" formula for π:

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k}\left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right).$$

This formula permits one to calculate directly binary or hexadecimal digits beginning at the nth digit, without the need to calculate any of the first $n-1$ digits [**6**]. This result has, in turn, led to more recent results that suggest a possible route to a proof that π and some other mathematical constants are 2-normal (i.e., that every m-long binary string occurs in the binary expansion with limiting frequency b^{-m} [**8**], [**9**]). The BBP formula even has some practical applications: it is used, for example, in the g95 compiler for transcendental function evaluations [**34**].

All integer relation schemes require very high precision arithmetic, both in the input data and in the operation of the algorithms. Simple reckoning shows that if an integer relation solution vector $(a_i, a_2, \ldots, a_n)$ has Euclidean norm 10^d, then the input data must be specified to at least dn digits, lest the true solution be lost in a sea of numerical artifacts. In some cases, including one mentioned at the end of the next section, thousands of digits are required before a solution can be found with these methods. This is the principal reason for the great interest in high-precision numerical evaluations in experimental mathematics research. It is the also the motivation behind this set of ten challenge problems.

2. THE BIFURCATION POINT B_3.

Problem 1. *Compute the value of r for which the chaotic iteration*

$$x_{n+1} = r x_n (1 - x_n),$$

starting with some x_0 in $(0, 1)$, exhibits a bifurcation between four-way periodicity and eight-way periodicity. Extra credit: This constant is an algebraic number of degree not exceeding twenty. Find the minimal polynomial with integer coefficients that it satisfies.

History and context. The chaotic iteration $x_{n+1} = r x_n (1 - x_n)$ has been studied since the early days of chaos theory in the 1950s. It is often called the "logistic iteration," since it mimics the behavior of an ecological population that, if its growth one year outstrips its food supply, often falls back in numbers for the following year, thus continuing to vary in a highly irregular fashion. When r is less than one iterates of the logistic iteration converge to zero. For r in the range $1 < r < B_1 = 3$ iterates converge to some nonzero limit. If $B_1 < r < B_2 = 1 + \sqrt{6} = 3.449489\ldots$, the limiting behavior bifurcates—every other iterate converges to a distinct limit point. For r with $B_2 < r < B_3$ iterates hop between a set of four distinct limit points; when $B_3 < r < B_4$, they select between a set of eight distinct limit points; this pattern repeats until $r > B_\infty = 3.569945672\ldots$, when the iteration is completely chaotic (see Figure 1). The limiting ratio $\lim_n (B_n - B_{n-1})/(B_{n+1} - B_n) = 4.669201\ldots$ is known as *Feigenbaum's delta constant*.

A very readable description of the logistic iteration and its role in modern chaos theory are given in Gleick's book [**26**]. Indeed, John von Neumann had suggested using the logistic map as a random number generator in the late 1940s. Work by W. Ricker in 1954 and detailed analytic studies of logistic maps beginning in the 1950s with Paul Stein and Stanislaw Ulam showed the existence of complicated properties of this type of map beyond simple oscillatory behavior [**35**, pp. 918–919].

Solution. We first describe how to obtain a highly accurate numerical value of B_3 using a relatively straightforward search scheme. Other schemes could be used to find B_3; we present this one to underscore the fact that computational results sufficient for the purposes of experimental mathematics can often be obtained without resorting to highly sophisticated techniques.

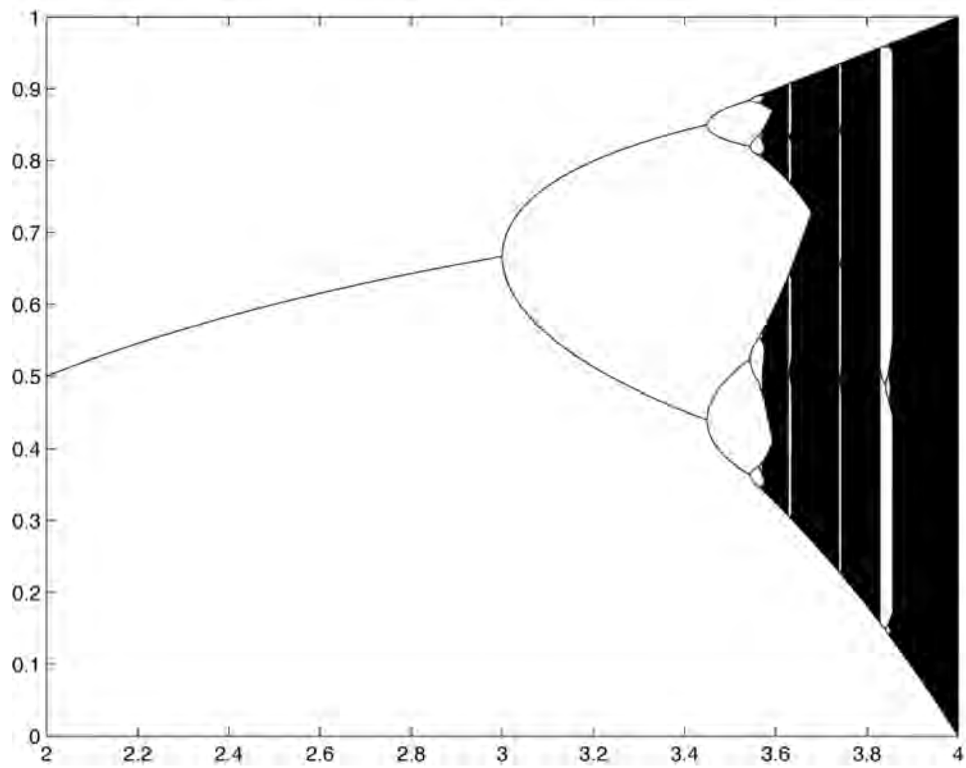

Figure 1. Bifurcation in the logistic iteration.

Let $f_8(r, x)$ be the eight-times iterated evaluation of $rx(1 - x)$, and let $g_8(r, x) = f_8(r, x) - x$. Imagine a three-dimensional graph, where r ranges from left to right and x ranges from bottom to top (as in Figure 1), and where $g_8(r, x)$ is plotted in the vertical (out-of-plane) dimension. Given some initial r slightly less than B_3, we compute a "comb" of function values at n evenly spaced x values (with spacing h_x) near the limit of the iteration $x_{n+1} = f_8(r, x_n)$. In our implementation, we use $n = 12$, and we start with $r = 3.544$, $x = 0.364$, $h_r = 10^{-4}$, and $h_x = 5 \times 10^{-4}$. With this construction, the comb has $n/2$ negative function values, followed by $n/2$ positive function values. We then increment r by h_r and reevaluate the "comb," continuing in this fashion until two sign changes are observed among the n function values of the "comb." This means that a bifurcation occurred just prior to the current value of r, so we restore r to its previous value (by subtracting h_r), reduce h_r, say by a factor of four, and also reduce the h_x roughly by a factor of 2.5. We continue in this fashion, moving the value of r and its associated "comb" back and forth near the bifurcation point with progressively smaller intervals h_r. The center of the comb in the x-direction must be adjusted periodically to ensure that $n/2$ negative function values are followed by $n/2$ positive function values, and the spacing parameter h_x must be adjusted as well to ensure that two sign changes are disclosed when this occurs. We quit when the smallest of the n function values is within two or three orders of magnitude of the "epsilon" of the arithmetic (e.g., for 2000-digit working precision, "epsilon" is 10^{-2000}). The final value of r is then the desired value B_3, accurate to within a tolerance given by the final value of r_h. With 2000-digit working precision, our implementation of this scheme finds B_3 to 1330-digit accuracy in about five minutes on a 2004-era computer. The first hundred digits are as follows:

$$B_3 = 3.544090359551922853615965986604804540583099845444573675457812530305842942858863012256258566424891799962 6\ldots.$$

With even a moderately accurate value of r in hand (at least two hundred digits or so), one can use a PSLQ program (such as the PSLQ programs available at the URL `http://crd.lbl.gov/~dhbailey/mpdist`) to check whether r is an algebraic constant. This is done by computing the vector $(1, r, r^2, \ldots, r^n)$ for various n, beginning with a small value such as two or three, and then searching for integer relations among these $n+1$ real numbers. When $n \geq 12$, the relation

$$0 = r^{12} - 12r^{11} + 48r^{10} - 40r^9 - 193r^8 + 392r^7 + 44r^6 + 8r^5 - 977r^4 - 604r^3 + 2108r^2 + 4913 \tag{1}$$

can be recovered.

A symbolic solution that explicitly produces the polynomial (1) can be obtained as follows. We seek a sequence $x_1, x_2, \ldots, x_4$ that satisfies the equations

$$x_2 = rx_1(1-x_1), \quad x_3 = rx_2(1-x_2), \quad x_4 = rx_3(1-x_3), \quad x_1 = rx_4(1-x_4),$$

and

$$1 = \left|\prod_{i=1}^{4} r(1-2x_i)\right|.$$

The first four conditions represent a period-4 sequence in the logistic equation $x_{n+1} = rx_n(1-x_n)$, and the last condition represents the stability of the cycle, which must be 1 or -1 for a bifurcation point (see [**33**] for details).

First, we deal with the system corresponding to $1 + \prod_{i=1}^{4} r(1-2x_i) = 0$. We compute the lexicographic Groebner basis in *Maple*:

```
with(Groebner):
L := [x2 - r*x1*(1-x1),x3 - r*x2*(1-x2),x4 - r*x3*(1-x3),
      x1 - r*x4*(1-x4),r^4*(1-2*x1)*(1-2*x2)*(1-2*x3)*(1-2*x4) + 1];
      gbasis(L,plex(x1,x2,x3,x4,r));
```

After a cup of coffee, we discover the univariate element

$$(r^4+1)(r^4-8r^3+24r^2-32r+17) \times (r^4-4r^3-4r^2+16r+17) \times (r^{12} - 12r^{11} + 48r^{10} - 40r^9 - 193r^8 + 392r^7 + 44r^6 + 8r^5 - 977r^4 - 604r^3 + 2108r^2 + 4913)$$

in the Groebner basis, in which the monomial ordering is lexicographical with r last.

The first three of these polynomials have no real roots, and the fourth has four real roots. Using trial and error, it is easy to determine that B_3 is the root of the minimal polynomial

$$r^{12} - 12r^{11} + 48r^{10} - 40r^9 - 193r^8 + 392r^7 + 44r^6 + 8r^5 - 977r^4 - 604r^3 + 2108r^2 + 4913,$$

which has the numerical value stated earlier. The corresponding *Mathematica* code reads:

```
GroebnerBasis[{x2 - r x1(1 - x1), x3 - r x2(1 - x2),
 x4 - r x3(1 - x3), x1 - r x4(1 - x4),
 r^4(1 - 2x1)(1 - 2x2)(1 - 2x3)(1 - 2x4) + 1},
 r,
{x1, x2, x3, x4}, MonomialOrder -> EliminationOrder]//Timing
```

This requires only 1.2 seconds on a 3 GHz computer. These computations can also be recreated very quickly in *Magma*, an algebraic package available at `http://magma.maths.usyd.edu.au/magma`:

```
Q := RationalField(); P<x,y,z,w,r> := PolynomialRing(Q,5);
 I:= ideal< P| y - r*x*(1-x), z - r*y*(1-y), w - r*z*(1-z),
     x - r*w*(1-w), r^4*(1-2*x)*(1-2*y)*(1-2*z)*(1-2*w)+1>;
time B := GroebnerBasis(I);
```

This took 0.050 seconds on a 2.4Ghz Pentium 4.

The significantly more challenging problem of computing and analyzing the constant $B_4 = 3.564407266095\ldots$ is discussed in [**7**]. In this study, conjectural reasoning suggested that B_4 might satisfy a 240-degree polynomial, and, in addition, that $\alpha = -B_4(B_4 - 2)$ might satisfy a 120-degree polynomial. The constant α was then computed to over 10,000-digit accuracy, and an advanced three-level multi-pair PSLQ program was employed, running on a parallel computer system, to find an integer relation for the vector $(1, \alpha, \alpha^2, \ldots, \alpha^{120})$. A numerically significant solution was obtained, with integer coefficients descending monotonically from 257^{30}, which is a 73-digit integer, to the final value, which is one (a striking result that is exceedingly unlikely to be a numerical artifact). This experimentally discovered polynomial was recently confirmed in a large symbolic computation [**30**].

Additional information on the Logistic Map is available at `http://mathworld.wolfram.com/LogisticMap.html`.

3. MADELUNG'S CONSTANT.

Problem 2. *Evaluate*

$$\sum_{(m,n,p)\neq 0} \frac{(-1)^{m+n+p}}{\sqrt{m^2+n^2+p^2}}, \tag{2}$$

where convergence means the limit of sums over the integer lattice points enclosed in increasingly large cubes surrounding the origin. Extra credit: Usefully identify this constant.

History and context. Highly conditionally convergent sums like this are very common in physical chemistry, where they are usually written down with no thought of convergence. The sum in question arises as an idealization of the electrochemical stability of NaCl. One computes the total potential at the origin when placing a positive or negative charge at each nonzero point of the cubic lattice [**16**, chap. 4].

Solution. It is important to realize that this sum must be viewed as the limit of the sum in successively larger cubes. The sum diverges when spheres are used instead. To clarify this consider, for complex s, the series

$$b_2(s) = \sum_{(m,n)\neq 0} \frac{(-1)^{m+n}}{(m^2+n^2)^{s/2}}, \qquad b_3(s) = \sum_{(m,n,p)\neq 0} \frac{(-1)^{m+n+p}}{(m^2+n^2+p^2)^{s/2}}. \tag{3}$$

These converge in two and three dimensions, respectively, over increasing "cubes," provided that $\mathrm{Re}\, s > 0$. When $s = 1$, one *may* sum over circles in the plane but not spheres in three-space, and one may *not* sum over diamonds in dimension two. Many chemists do not know that $b_3(1) \neq \sum_n (-1)^n r_3(n)/\sqrt{n}$, a series that arises by summing over increasing spheres but that diverges. Indeed, the number $r_3(n)$ of representations of n as a sum of three squares is quite irregular—no number of the form $8n+7$ has such a representation—and is not $O(n^{1/2})$. This matter is somewhat neglected in the discussion of Madelung's constant in Julian Havil's deservedly popular recent book *Gamma: Exploring Euler's Constant* [**27**], which contains a wealth of information related to each of our problems in which Euler had a hand.

Straightforward methods to compute (3) are extremely unproductive. Such techniques produce at most three digits—indeed, the physical model should have a solar-system sized salt crystal to justify ignoring the boundary. Thus, we are led to using more sophisticated methods. We note that

$$b_3(s) = \sum{}' \frac{(-1)^{i+j+k}}{(i^2+j^2+k^2)^{s/2}},$$

where $\sum'$ signifies a sum over $\mathbb{Z}^3\backslash\{(0,0,0)\}$, and let $M_s(f)$ denote the Mellin transform

$$M_s(f) = \int_0^\infty f(x) x^{s-1}\, dx.$$

The quantity that we wish to compute is $b_3(1)$. It follows by symmetry that

$$\begin{aligned} b_3(1) &= \sum{}' \frac{(-1)^{i+j+k}(i^2+j^2+k^2)}{(i^2+j^2+k^2)^{3/2}} \\ &= 3\sum{}' \frac{(-1)^i(i^2)(-1)^{j+k}}{(i^2+j^2+k^2)^{3/2}}. \end{aligned} \tag{4}$$

We observe that $M_s(e^{-t}) = \Gamma(s)$, so

$$M_{3/2}\big(q^{n^2+j^2+k^2}\big) = \Gamma\left(\frac{3}{2}\right)(n^2+j^2+k^2)^{-3/2},$$

where n, j, and k are arbitrary integers and $q = e^{-t}$. Continuing, we rewrite equation (4) as

$$\Gamma\left(\frac{3}{2}\right) b_3(1) = 3M_{3/2}\left(\sum_{n=-\infty}^{\infty} (-1)^n n^2 q^{n^2} \theta_4^2(x)\right),$$

where $\theta_4(x) = \sum_{-\infty}^{\infty} (-1)^n x^{n^2}$ is the usual Jacobi theta-function. Since the *theta transform*—a form of Poisson summation—yields $\theta_4(e^{-\pi/s}) = \sqrt{s}\,\theta_2(e^{-s\pi})$, it follows that

$$\Gamma\left(\frac{3}{2}\right) b_3(1) = 3 \sum_{n=-\infty}^{\infty} n^2 M_{3/2}\left(\sum (-1)^n n^2 q^{n^2} \frac{\pi}{x}\theta_2^2\left(\frac{\pi^2}{x}\right)\right).$$

Also, $\Gamma(3/2) = \sqrt{\pi}/2$, so

$$b_3(1) = 12\sqrt{\pi}\sum_{n=1}^{\infty}(-1)^n n^2 \sum_{(j,k)\text{ odd}} \int_0^{\infty} \left[e^{-n^2x-(\pi^2/4x)(j^2+k^2)}\right] x^{-1/2}\, dx.$$

The integral is evaluated in [**19**, Exercise 4, sec. 2.2] and is $(\pi/n^2)^{1/2}e^{-\pi n\sqrt{j^2+k^2}}$, whence

$$b_3(1) = 48\pi \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{n=1}^{\infty}(-1)^n n e^{-\pi n\sqrt{(2j+1)^2+(2k+1)^2}}.$$

Finally, when $a > 0$,

$$4\sum_{n=1}^{\infty}(-1)^{n+1} n e^{-an} = \frac{4e^{-a}}{(1+e^{-a})^2} = \operatorname{sech}^2\left(\frac{a}{2}\right),$$

from which we obtain

$$b_3(1) = 12\pi \sum_{\substack{m,n\geq 1\\ m,n\text{ odd}}} \operatorname{sech}^2\left(\frac{\pi}{2}(m^2+n^2)^{1/2}\right). \tag{5}$$

Summing over m and n from 1 up to 81 in (5) gives

$$b_3(1) = 1.74756459463318219063621203554439740348516143662474175\\ 815282535076504062353276117989075836269460789 1\ldots.$$

It is possible to accelerate the convergence further still. Details can be found in [**19**], [**16**].

There are closed forms for sums with an even number of variables, up to 24 and beyond. For example, $b_2(2s) = -4\,\alpha(s)\,\beta(s)$, where

$$\alpha(s) = \sum_{n\geq 0}(-1)^n/(n+1)^s$$

and

$$\beta(s) = \sum_{n\geq 0}(-1)^n/(2n+1)^s.$$

In particular, $b_2(2) = -\pi\log 2$. No such closed form for b_3 is known, while much work has been expended looking for one. The formula for b_2 is due to Lorenz (1879). It was rediscovered by G. H. Hardy and is equivalent to Jacobi's Lambert series formula for $\theta_3^2(q)$:

$$\theta_3^2(q) - 1 = 4\sum_{n\geq 0}(-1)^n \frac{q^{2n+1}}{1-q^{2n+1}}.$$

This, in turn, is equivalent to the formula for the number $r_2(n)$ of representations of n as a sum of two squares, counting order and sign,

$$r_2(n) = 4(d_1(n) - d_3(n)),$$

where d_k is the number of divisors of n congruent to k modulo four. The analysis of three squares is notoriously harder.

Additional information on Madelung's constant and lattice sums is available at `http://mathworld.wolfram.com/MadelungConstants.html` and `http://mathworld.wolfram.com/LatticeSum.html`.

4. DOUBLE EULER SUMS.

Problem 3. *Evaluate the sum*

$$C = \sum_{k=1}^{\infty} \left(1 - \frac{1}{2} + \cdots + (-1)^{k+1}\frac{1}{k}\right)^2 \frac{1}{(k+1)^3}. \tag{6}$$

Extra credit: Evaluate this constant as a multiterm expression involving well-known mathematical constants. This expression has seven terms and involves π, $\log 2$, $\zeta(3)$, *and* $\mathrm{Li}_5(1/2)$, *where* $\mathrm{Li}_n(x) = \sum_{k>0} x^n/n^k$ *is the nth polylogarithm. (Hint: The expression is "homogenous," in the sense that each term has the same total "degree." The degrees of* π *and* $\log 2$ *are each 1, the degree of* $\zeta(3)$ *is* 3, *the degree of* $\mathrm{Li}_5(1/2)$ *is 5, and the degree of* α^n *is n times the degree of* α*.)*

History and context. In April 1993, Enrico Au-Yeung, an undergraduate at the University of Waterloo, brought to the attention of one of us (Borwein) the curious result

$$\sum_{k=1}^{\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right)^2 \frac{1}{k^2} = 4.59987\ldots \approx \frac{17}{4}\zeta(4) = \frac{17\pi^4}{360}. \tag{7}$$

The function $\zeta(s)$ in (7) is the classical *Riemann zeta-function*:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Euler had solved Bernoulli's *Basel problem* when he showed that, for each positive integer n, $\zeta(2n)$ is an explicit rational multiple of π^{2n} [**16**, sec. 3.2].

Au-Yeung had computed the sum in (7) to 500,000 terms, giving an accuracy of five or six decimal digits. Suspecting that his discovery was merely a modest numerical coincidence, Borwein sought to compute the sum to a higher level of precision. Using Fourier analysis and Parseval's equation, he obtained

$$\frac{1}{2\pi}\int_0^{\pi} (\pi - t)^2 \log^2\left(2\sin\frac{t}{2}\right) dt = \sum_{n=1}^{\infty} \frac{\left(\sum_{k=1}^{n} \frac{1}{k}\right)^2}{(n+1)^2}. \tag{8}$$

The idea here is that the series on the right of (8) permits one to evaluate (7), while the integral on the left can be computed using the numerical quadrature facility of *Mathematica* or *Maple*. When he did this, Borwein was surprised to find that the conjectured identity holds to more than thirty digits. We should add here that, by good fortune, $17/360 = 0.047222\ldots$ has period one and thus can plausibly be recognized from its first six digits, so that Au-Yeung's numerical discovery was not entirely far-fetched.

Solution. We define the multivariate zeta-function by

$$\zeta(s_1, s_2, \ldots, s_k) = \sum_{n_1 > n_2 > \cdots > n_k > 0} \prod_{j=1}^{k} n_j^{-|s_j|} \sigma_j^{-n_j},$$

where the $s_1, s_2, \ldots, s_k$ are nonzero integers and $\sigma_j = \text{signum}(s_j)$. A fast method for computing such sums based on Hölder convolution is discussed in [**20**] and implemented in the EZFace+ interface, which is available as an online tool at the URL `http://www.cecm.sfu.ca/projects/ezface+`. Expanding the squared term in (6), we have

$$C = \sum_{0 < i, j < k} \frac{(-1)^{i+j}}{ijk^3} = 2\,\zeta(3, -1, -1) + \zeta(3, 2). \tag{9}$$

Evaluating this in EZFace+ we quickly obtain

$$C = 0.15616693338117691588103590968798819368577670984030387295752935449707503744029579145520565370935814757 8\ldots.$$

Given this numerical value, PSLQ or some other integer-relation-finding tool can be used to see if this constant satisfies a rational linear relation with the following constants (as suggested in the hint): π^5, $\pi^4 \log(2)$, $\pi^3 \log^2(2)$, $\pi^2 \log^3(2)$, $\pi \log^4(2)$, $\log^5(2)$, $\pi^2\zeta(3)$, $\pi \log(2)\zeta(3)$, $\log^2(2)\zeta(3)$, $\zeta(5)$, $\text{Li}_5(1/2)$. The result is quickly found to be

$$C = 4\text{Li}_5\left(\frac{1}{2}\right) - \frac{1}{30}\log^5(2) - \frac{17}{32}\zeta(5) - \frac{11}{720}\pi^4\log(2) + \frac{7}{4}\zeta(3)\log^2(2) + \frac{1}{18}\pi^2\log^3(2) - \frac{1}{8}\pi^2\zeta(3).$$

This result has been proved in various ways, both analytic and algebraic. Indeed, all evaluations of sums of the form $\zeta(\pm a_1, \pm a_2, \ldots, \pm a_m)$ with *weight* $w = \sum_k a_k$ $(w < 8)$, as in (9), have been established.

Further history and context. What Borwein did not know at the time was that Au-Yeung's suspected identity follows directly from a related result proved by De Doelder in 1991. In fact, it had cropped up even earlier as a problem in this MONTHLY, but the story goes back further still. Some historical research showed that Euler considered these summations. In response to a letter from Goldbach, he examined sums that are equivalent to

$$\sum_{k=1}^{\infty}\left(1 + \frac{1}{2^m} + \cdots + \frac{1}{k^m}\right)\frac{1}{(k+1)^n}. \tag{10}$$

The great Swiss mathematician was able to give explicit values for certain of these sums in terms of the Riemann zeta-function.

Starting from where we left off in the previous section provides some insight into evaluating related sums. Recall that the Taylor expansion of

$$f(x) = -\frac{1}{2}\log(1-x)\log(1+x)$$

takes the form

$$f(x) = \sum_{k=1}^{\infty}\left(1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2k-1}\right)\frac{x^{2k}}{2k}.$$

Applying Parseval's identity to $f(e^{it})$, we have an effective way of computing

$$\sum_{k=1}^{\infty}\frac{\left(1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2k-1}\right)^2}{(2k)^2}$$

in terms of an integral that can be rapidly evaluated in *Maple* or *Mathematica*.

Alternatively, we may compute

$$\sum_{k=1}^{\infty}\frac{\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}\right)^2}{k^2}.$$

The Fourier expansions of $(\pi - t)/2$ and $-\log|2\sin(t/2)|$ are

$$\sum_{n=1}^{\infty}\frac{\sin(nt)}{n} = \frac{\pi - t}{2} \qquad (0 < t < 2\pi)$$

and

$$\sum_{n=1}^{\infty}\frac{\cos(nt)}{n} = -\log|2\sin(t/2)| \qquad (0 < t < 2\pi), \tag{11}$$

respectively. Multiplying these together, simplifying, and doing a partial fraction decomposition gives

$$-\log|2\sin(t/2)| \cdot \frac{\pi - t}{2} = \sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=1}^{n-1}\frac{1}{k}\sin(nt)$$

on $(0, 2\pi)$. Applying Parseval's identity results in

$$\frac{1}{4\pi}\int_0^{2\pi}(\pi - t)^2\log^2(2\sin(t/2))dt = \sum_{n=1}^{\infty}\frac{\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)^2}{(n+1)^2}.$$

The integral may be computed numerically in *Maple* or *Mathematica*, delivering an approximation to the sum.

The *Clausen functions* defined by

$$\mathrm{Cl}_2(\theta) = \sum_{n=1}^{\infty}\frac{\sin(n\theta)}{n^2}, \qquad \mathrm{Cl}_3(\theta) = \sum_{n=1}^{\infty}\frac{\cos(n\theta)}{n^3}, \qquad \mathrm{Cl}_4(\theta) = \sum_{n=1}^{\infty}\frac{\sin(n\theta)}{n^4}, \cdots$$

arise as repeated antiderivatives of (11). They are useful throughout harmonic analysis and elsewhere. For example, with $\alpha = 2\arctan\sqrt{7}$, one discovers with the aid of

PSLQ that

$$6\mathrm{Cl}_2(\alpha) - 6\mathrm{Cl}_2(2\alpha) + 2\mathrm{Cl}_2(3\alpha) \stackrel{?}{=} 7\mathrm{Cl}_2\left(\frac{2\pi}{7}\right) + 7\mathrm{Cl}_2\left(\frac{4\pi}{7}\right) - 7\mathrm{Cl}_2\left(\frac{6\pi}{7}\right) \quad (12)$$

(here the question mark is used because no proof is yet known) or, in what can be shown to be equivalent, that

$$\frac{24}{7\sqrt{7}} \int_{\pi/3}^{\pi/2} \log\left(\left|\frac{\tan(t)+\sqrt{7}}{\tan(t)-\sqrt{7}}\right|\right) dt \stackrel{?}{=} L_{-7}(2) = 1.151925470\ldots. \quad (13)$$

This arises from the volume of an ideal tetrahedron in hyperbolic space [**15**, pp. 90–91]. (Here $L_{-7}(s) = \sum_{n>0} \chi_{-7}(n) n^{-s}$ is the primitive L-series modulo seven, whose character pattern is 1, 1, −1, 1, −1, −1, 0, which is given by

$$\chi_{-7}(k) = 2(\sin(k\tau) + \sin(2k\tau) - \sin(3k\tau))/\sqrt{7}$$

with $\tau = 2\pi/7$.)

Although (13) has been checked to twenty thousand decimal digits, by using a numerical integration scheme we shall describe in section 8, and although it is known for K-theoretic reasons that the ratio of the left- and right-hand sides of (12) is rational [**14**], to the best of our knowledge there is no proof of either (12) or (13). We might add that recently two additional conjectured identities related to (13) have been discovered by PSLQ computations. Let I_n be the definite integral of (13), except with limits $n\pi/24$ and $(n+1)\pi/24$. Then

$$-2I_2 - 2I_3 - 2I_4 - 2I_5 + I_8 + I_9 - I_{10} - I_{11} \stackrel{?}{=} 0,$$
$$I_2 + 3I_3 + 3I_4 + 3I_5 + 2I_6 + 2I_7 - 3I_8 - I_9 \stackrel{?}{=} 0. \quad (14)$$

Readers who attempt to calculate numerical values for either the integral in (13) or the integral I_9 in (14) should note that the integrand has a nasty singularity at $t = \arctan\sqrt{7}$.

In retrospect, perhaps it was for the better that Borwein had not known of De Doelder's and Euler's results, because Au-Yeung's intriguing numerical discovery launched a fruitful line of research by a number of researchers that has continued until the present day. Sums of this general form are known nowadays as "Euler sums" or "Euler-Zagier sums." Euler sums can be studied through a profusion of methods: combinatorial, analytic, and algebraic. The reader is referred to [**16**, chap. 3] for an overview of Euler sums and their applications. We take up the story again in Problem 9.

Additional information on Euler sums is available at `http://mathworld.wolfram.com/EulerSum.html`.

5. KHINTCHINE'S CONSTANT.

Problem 4. *Evaluate*

$$K_0 = \prod_{k=1}^{\infty}\left[1 + \frac{1}{k(k+2)}\right]^{\log_2 k} = \prod_{k=1}^{\infty} k^{[\log_2(1+1/k(k+2))]}. \quad (15)$$

Extra credit: Evaluate this constant in terms of a less-well-known mathematical constant.

History and context. Given some particular continued fraction expansion $\alpha = [a_0, a_1, \ldots]$, consider forming the limit

$$K_0(\alpha) = \lim_{n\to\infty} (a_0\, a_1 \cdots a_n)^{1/n}.$$

Based on the *Gauss-Kuzmin distribution*, which establishes that the digit distribution of a random continued fraction satisfies $\text{Prob}\{a_k = n\} = \log_2(1 + 1/k(k+2))$, Khintchine showed that the limit exists for almost all continued fractions and is a certain constant, which we now denote K_0. This circle of ideas is accessibly developed in [**27**]. As such a constant has an interesting interpretation, computation seems like the next step.

Taking logarithms of both sides of (15) and simplifying, we have

$$\log 2 \cdot \log K_0 = \sum_{n=1}^{\infty} \log n \cdot \log\left(1 + \frac{1}{n(n+2)}\right).$$

Such a series converges extremely slowly. Computing the sum of the first 10000 terms gives only two digits of $\log 2 \cdot \log K_0$. Thus, direct computation again proves to be quite difficult.

Solution. Rewriting $\log n$ as the telescoping sum

$$\log n = (\log n - \log(n-1)) + \cdots + (\log 2 - \log 1) = \sum_{k=2}^{n} \log \frac{k}{k-1},$$

we see that

$$\log 2 \cdot \log K_0 = \sum_{n=2}^{\infty} \sum_{k=2}^{n} \log \frac{k}{k-1} \cdot \log \frac{(n+1)^2}{n(n+2)}.$$

We interchange the order of summation to obtain

$$\log 2 \cdot \log K_0 = \sum_{k=2}^{\infty} \sum_{n=k}^{\infty} \log \frac{(n+1)^2}{n(n+2)} \log \frac{k}{k-1}. \tag{16}$$

But

$$\sum_{n=k}^{\infty} \log \frac{(n+1)^2}{n(n+2)} = \log \frac{k+1}{k} = \log\left(1 + \frac{1}{k}\right),$$

so (16) transforms into

$$\log 2 \cdot \log K_0 = -\sum_{k=2}^{\infty} \log\left(1 - \frac{1}{k}\right) \log\left(1 + \frac{1}{k}\right). \tag{17}$$

The Maclaurin series for $-\log(1-x)\log(1+x)$ is

$$\sum_{k=1}^{\infty} \left(1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2k-1}\right) \frac{x^{2k}}{k}.$$

This allows us to rewrite $\log 2 \cdot \log K_0$ as

$$\begin{aligned}\log 2 \cdot \log K_0 &= \sum_{k=1}^{\infty}\left(1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2k-1}\right)\frac{1}{k}\sum_{n=2}^{\infty} n^{-2k} \\ &= \sum_{k=1}^{\infty}\left(1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2k-1}\right)\frac{1}{k}(\zeta(2k) - 1).\end{aligned}$$

Appealing to either *Maple* or *Mathematica*, we can easily compute this sum. Taking the first 161 terms, we obtain one hundred digits of K_0:

$$\begin{aligned}K_0 = 2.&685452001065306445309714835481795693820382293994462953 \\ &051152345557218859537152002801141174931847709\ldots.\end{aligned}$$

However, faster convergence is possible, and the constant has now been computed to more than seven thousand places. Moreover, the harmonic and other averages are similarly treated. It appears to satisfy its own predicted behavior (for details, see [**5**], [**32**]). Correspondingly, using 10^8 terms one can obtain the approximation $K_0(\pi) \approx 2.675\ldots$. Note however that $K_0(e) = \infty = \lim_{n\to\infty} \sqrt[3n]{(2n)!}$, since e is a member of the measure zero set of exceptions not having $K_0(\alpha) = K_0$, as a result of the non-Gauss-Kuzmin distribution of terms in the continued fraction $e = [2, 1, 2, 1, 1, 4, 1, 1, 6, \ldots]$.

We emphasize that while it is known that almost all numbers α have limits $K_0(\alpha)$ that equal K_0, this has not been exhibited for any explicit number α, excluding artificial examples constructed using their continued fractions [**5**].

6. RAMANUJAN'S AGM CONTINUED FRACTION.

Problem 5. *For positive real numbers a, b, and η define $R_\eta(a, b)$ by*

$$R_\eta(a, b) = \cfrac{a}{\eta + \cfrac{b^2}{\eta + \cfrac{4a^2}{\eta + \cfrac{9b^2}{\eta + \ddots}}}}.$$

Calculate $R_1(2, 2)$. *Extra credit: Evaluate this constant as a two-term expression involving a well-known mathematical constant.*

History and context. This continued fraction arises in Ramanujan's *Notebooks*. He discovered the beautiful fact that

$$\frac{R_\eta(a, b) + R_\eta(b, a)}{2} = R_\eta\left(\frac{a+b}{2}, \sqrt{ab}\right).$$

The authors wished to record this in [**15**] and to check the identity computationally. A first attempt to find $R_1(1, 1)$ by direct numerical computation failed miserably, and with some effort only three reliable digits were obtained: $0.693\ldots$. With hindsight, it was realized that the slowest convergence of the fraction occurs in the mathematically simplest case, namely, when $a = b$. Indeed, $R_1(1, 1) = \log 2$, as the first primitive numerics had tantalizingly suggested.

Solution. Attempting a direct computation of $R_1(2,2)$ using a depth of twenty thousand gives only two digits. Thus we must seek more sophisticated methods. From [**16**, (1.11.70)] we learn that when $0 < b < a$,

$$\mathcal{R}_1(a,b) = \frac{\pi}{2}\sum_{n\in Z}\frac{aK(k)}{K^2(k)+a^2n^2\pi^2}\operatorname{sech}\left(n\pi\frac{K(k')}{K(k)}\right), \tag{18}$$

where $k = b/a = \theta_2^2/\theta_3^2$ and $k' = \sqrt{1-k^2}$. Here θ_2 and θ_3 are Jacobian theta-functions, and K is a complete elliptic integral of the first kind.

Writing (18) as a Riemann sum, we find that

$$\begin{aligned}\mathcal{R}(a) = \mathcal{R}_1(a,a) &= \int_0^\infty \frac{\operatorname{sech}(\pi x/(2a))}{1+x^2}\,dx \\ &= 2a\sum_{k=1}^\infty \frac{(-1)^{k+1}}{1+(2k-1)a},\end{aligned} \tag{19}$$

where the final equality follows from the Cauchy-Lindelöf theorem. This sum can also be written as

$$\mathcal{R}(a) = \frac{2a}{1+a}\,{}_2F_1\left(\frac{1}{2a}+\frac{1}{2},1;\frac{1}{2a}+\frac{3}{2};-1\right),$$

where ${}_2F_1(\cdot)$ denotes the hypergeometric function [**1**, p. 556]. The latter form is what we use in *Maple* or *Mathematica* to determine

$$\begin{aligned}\mathcal{R}(2) = {} & 0.97499098879872209671990033452921084400592021999471060574526\,8 \\ & 25128587738745570859435232532091112936\,2\ldots.\end{aligned}$$

This constant, as written, is a bit difficult to recognize, but if one first divides by $\sqrt{2}$ and exploits the *Inverse Symbolic Calculator*, an online tool available at the URL `http://www.cecm.sfu.ca/projects/ISC/ISCmain.html`, it becomes apparent that the quotient is $\pi/2 - \log(1+\sqrt{2})$. Thus we conclude, experimentally, that

$$\mathcal{R}(2) = \sqrt{2}[\pi/2 - \log(1+\sqrt{2})].$$

Indeed, it follows (see [**18**]) that

$$\mathcal{R}(a) = 2\int_0^1 \frac{t^{1/a}}{1+t^2}\,dt.$$

Note that $\mathcal{R}(1) = \log 2$. No nontrivial closed-form expression is known for $\mathcal{R}(a,b)$ when $a \neq b$, although

$$\mathcal{R}_1\left(\frac{1}{4\pi}\beta\left(\frac{1}{4},\frac{1}{4}\right),\frac{\sqrt{2}}{8\pi}\beta\left(\frac{1}{4},\frac{1}{4}\right)\right) = \frac{1}{2}\sum_{n\in\mathbf{Z}}\frac{\operatorname{sech}(n\pi)}{1+n^2}$$

is almost closed. It would be pleasant to find a direct proof of (19). Further details are to be found in [**18**], [**17**], and [**16**].

7. EXPECTED DISTANCE ON A UNIT SQUARE.

Problem 6. *Calculate the expected distance E_2 between two random points on different sides of the unit square:*

$$E_2 = \frac{2}{3}\int_0^1\int_0^1 \sqrt{x^2+y^2}\,dx\,dy + \frac{1}{3}\int_0^1\int_0^1 \sqrt{1+(u-v)^2}\,du\,dv. \tag{20}$$

Extra credit: Express this constant as a three-term expression involving algebraic constants and an evaluation of the natural logarithm with an algebraic argument.

History and context. This evaluation and the next were discovered, in slightly more complicated form, by James D. Klein [**16**, p. 66]. He computed the numerical integral and compared it with the result of a Monte Carlo simulation. Indeed, a straightforward approach to a quick numerical value for an arbitrary iterated integral is to use a Monte-Carlo simulation, which entails approximating the integral by a sum of function values taken at pseudo-randomly generated points within the region. It is important to use a good pseudo-random number generator for this purpose. We tried doing a Monte Carlo evaluation for this problem, using a pseudo-random number generator based on the recently discovered class of provably normal numbers [**9**], [**15**, pp. 169–70]. The results we obtained for the two integrals in question, with 10^8 pseudo-random pairs, are $0.765203\ldots$ and $1.076643\ldots$, respectively, yielding an expected distance of $0.869017\ldots$. Unfortunately, none of these three values immediately suggests a closed form, and they are not sufficiently accurate (because of statistical limitations) to be suitable for PSLQ or other constant recognition tools. More digits are needed.

Solution. It is possible to calculate high-precision numerical values for these two integrals using a two-dimensional quadrature (numerical integration) program. In our program, we employed a two-dimensional version of the "tanh-sinh" quadrature algorithm, which we will discuss in more detail in Problem 8. Two-dimensional quadrature is usually much more expensive than one-dimensional quadrature, at a given precision level, because many more function evaluations must be performed. Often a highly parallel computer system must be used to obtain a high-precision result in reasonable run time [**11**]. Nonetheless, in this case we were able to evaluate the first of the two integrals to 108-digit accuracy in twenty-one minutes run time on a 2004-era computer, and the second to 118-digit accuracy in just twenty seconds. The first is more difficult due to nondifferentiability of the integrand at the origin.

Indeed, in this case both *Maple* and *Mathematica* are able to evaluate each of these integrals numerically, as is, to over one hundred decimal digit accuracy in just a few minutes of run time, either by evaluating the inner integral symbolically and the outer integral numerically or else by performing full two-dimensional numerical quadrature. *Maple*, *Mathematica*, and the two-dimensional quadrature program all agreed on the following numerical value for the expected distance:

$$\alpha = 0.86900905527453446388497059434540662485671927963168056$$
$$96603508645841798221746930531132135548754357 54\ldots.$$

Using PSLQ, with the basis elements α, $\sqrt{2}$, $\log(\sqrt{2}+1)$, and 1, we obtain

$$\alpha = \frac{1}{9}\sqrt{2} + \frac{5}{9}\log(\sqrt{2}+1) + \frac{2}{9}. \tag{21}$$

An alternate solution is to attempt to evaluate the integrals symbolically! In fact, in this case Version 5.1 of *Mathematica* can do both the integrals "out of the box," whereas in the first case *Maple* appears to need coaxing, for instance, by converting to polar coordinates:

$$2\int_0^{\pi/4}\int_0^{\sec\theta} r^2\,dr\,d\theta = \frac{2}{3}\int_0^{\pi/4}\sec^3\theta\,d\theta = \frac{1}{3}\sqrt{2} - \frac{1}{6}\log(2) + \frac{1}{3}\log(2+\sqrt{2}),$$

since the radius for a given θ is $1/\cos\theta$. As for the second integral, *Maple* and *Mathematica* both give

$$-\frac{1}{3}\sqrt{2} - \frac{1}{2}\log(\sqrt{2}-1) + \frac{1}{2}\log(1+\sqrt{2}) + \frac{2}{3}.$$

To obtain the second integral analytically, write it as $2\int_0^1\int_0^u\sqrt{1+(u-v)^2}\,dv\,du$. Now change variables (set $t = u - v$) to obtain $1/2\int_0^1\{u\sqrt{1+u^2} + \operatorname{arcsinh} u\}\,du$. Thus, the expected distance is

$$\frac{1}{9}\sqrt{2} - \frac{1}{9}\log(2) + \frac{2}{9}\log(2+\sqrt{2}) - \frac{1}{6}\log(\sqrt{2}-1) + \frac{1}{6}\log(1+\sqrt{2}) + \frac{2}{9},$$

which can be simplified to the formula (21).

Additional information on the problem is available at `http://mathworld.wolfram.com/SquareLinePicking.html`.

8. EXPECTED DISTANCE ON A UNIT CUBE.

Problem 7. *Calculate the expected distance between two random points on different faces of the unit cube. (Hint: This can be expressed in terms of integrals as*

$$\begin{aligned} E_3 := {}& \frac{4}{5}\int_0^1\int_0^1\int_0^1\int_0^1\sqrt{x^2+y^2+(z-w)^2}\,dw\,dx\,dy\,dz \\ &+ \frac{1}{5}\int_0^1\int_0^1\int_0^1\int_0^1\sqrt{1+(y-u)^2+(z-w)^2}\,du\,dw\,dy\,dz.) \end{aligned}$$

Extra credit: Express this constant as a six-term expression involving algebraic constants and two evaluations of the natural logarithm with algebraic arguments.

History and context. As we noted earlier, this evaluation was discovered, in essentially the same form, by Klein [**16**, p. 66]. As with Problem 6, a Monte Carlo integration scheme can be used to obtain quick approximations to the integrals. The values we obtained were 0.870792... and 1.148859..., respectively, yielding an expected distance of 0.926406.... Once again, however, these numerical values do not immediately suggest a closed-form evaluation, yet the accuracy is too low to apply PSLQ or other constant recognition schemes. What's more, in this case, unlike Problem 6, neither *Maple* nor *Mathematica* are able to evaluate these four-fold integrals directly—though *Mathematica* comes close. As in most cases "help" is needed, in the form of mathematical manipulation to render these integrals in a form where mathematical computing software can evaluate them—numerically or symbolically.

Solution. Let ${}_2F_1(\cdot)$ again denote the hypergeometric function [**1**, p. 556]. One can show that the first integral evaluates to

$$\frac{\sqrt{2\pi}}{5}\sum_{n=2}^{\infty}\frac{{}_2F_1(1/2,-n+2;3/2;1/2)}{(2n+1)\Gamma(n+2)\Gamma(5/2-n)}+\frac{4}{15}\sqrt{2}+\frac{2}{5}\log\left(\sqrt{2}+1\right)-\frac{1}{75}\pi$$

and the second generalized hypergeometric function formally evaluates to

$$\frac{\sqrt{\pi}}{10}\sum_{n=0}^{\infty}\frac{{}_4F_3(1,1/2,-1/2-n,-n-1;2,1/2-n,3/2;-1)}{(2n+1)\Gamma(n+2)\Gamma(3/2-n)}$$
$$-\frac{2}{25}+\frac{\sqrt{2}}{50}+\frac{1}{10}\log\left(\sqrt{2}+1\right).$$

(Although the second diverges as a Riemann sum, both *Maple* and *Mathematica* can handle it, with some human help, producing numerical values of the corresponding Borel sum.) Both expressions are consequences of the binomial theorem, modulo an initial integration with respect to z in the first case. These expansions allow one to compute the expectation to high precision numerically and to express both of the individual integrals in terms of the same set of constants. The numerical value of the desired expectation is

0.92639005517404672921816358654777901444496019010733504673252192127
1418504594036683829313473075349968212 . . . .

An integer relation search in the span of $\left\{1,\pi,\sqrt{2},\sqrt{3},\log(1+\sqrt{2}),\log(2+\sqrt{3})\right\}$ produces

$$\frac{4}{75}+\frac{17}{75}\sqrt{2}-\frac{2}{25}\sqrt{3}-\frac{7}{75}\pi+\frac{7}{25}\log\left(1+\sqrt{2}\right)+\frac{7}{25}\log\left(7+4\sqrt{3}\right).$$

With substantial effort we were able to nurse the symbolic integral out of *Maple*. We started, as in the previous problem, by integrating with respect to w over $[0,z]$, doubling, and continuing in this fashion until we reduced the problem to showing that

$$\begin{aligned}E_3 &= -\int_0^1\left(2x^3+6x^2+3\right)\ln\left(\sqrt{2+x^2}-1\right)dx\\ &\quad+\int_0^1 3\frac{-(x^2+1)\ln\left(\sqrt{2+x^2-1}\right)+\ln\left(\sqrt{2}-1\right)}{x^2(x^2+1)}\,dx\\ &= -\frac{5}{3}\pi+\frac{7}{6}\sqrt{2}+\frac{7}{2}\ln\left(1+\sqrt{2}\right)-\frac{3}{2}\ln(2)+\ln\left(1+\sqrt{3}\right)+\frac{37}{24}\\ &\quad+\frac{3}{4}\ln\left(1+\sqrt{2}\right)\pi,\end{aligned}$$

which we leave to the reader to establish.

Mathematica was more helpful: consider

```
4/5 Integrate[Sqrt[x^2 + y^2 + (z - w)^2], {x, 0, 1}, {y, 0, 1},
  {w, 0, 1}, {z, 0, 1}]// Timing
  {52.483021*Second, (168*Sqrt[2] - 24*Sqrt[3] - 44*Pi + 72*ArcSinh[1] +
  162*ArcSinh[1/Sqrt[2]] + 24*Log[2] - 240*Log[-1 + Sqrt[3]] +
  192*Log[1 + Sqrt[3]] + 20*Log[26 + 15*Sqrt[3]] + 3*Log[70226 +
  40545*Sqrt[3]])/900}
```

This form is what the shipping version of *Mathematica* 5.1 returns on a 3.0 GHz Pentium 4. It evaluates the first integral directly, while the second one can be done with a little help. The combined outcomes can then be simplified symbolically to the result shown.

There is also an ingenious method due to Michael Trott using a Laplace transform to reduce the four-dimensional integrals to integrals over one-dimensional integrands. It proceeds by eliminating the square roots (which cause most of the difficulty in symbolic evaluation of the multiple integrals) at the expense of introducing one additional (but "easy") integral. The original problem can then be written in terms of the *single* integral

$$\int_0^\infty \left[-\frac{14}{25} e^{-z^2} \sqrt{\pi}\, \mathrm{erf}^2(z) + \frac{28 e^{-2z^2} \mathrm{erf}(z)}{25 z} + \frac{7 e^{-z^2} \mathrm{erf}(z)}{25 z} - \frac{12 e^{-3z^2}}{25\sqrt{\pi}} + \frac{68 e^{-2z^2}}{75\sqrt{\pi}} + \frac{8 e^{-z^2}}{75\sqrt{\pi}} \right] dz,$$

which can be evaluated directly in *Mathematica* to produce the symbolic expression for E_3.

Nonetheless, we must emphasize (i) that one needs to proceed with confidence, since such symbolic computations can take several minutes, and (ii) that phrases like "*Maple* can not" or "*Mathematica* can" are release-specific and may also depend on the skill of the human user to make use of expert knowledge in mathematics, symbolic computation, or both, in order to produce a form of the problem that is most amenable to computation in a given software system. This explains our desire to illustrate various solution paths here and elsewhere.

Additional information on this problem is available at `http://mathworld.wolfram.com/CubeLinePicking.html`. For more information about the Laplace transform trick applied to the related problem of expected distance in a unit hypercube, see `http://mathworld.wolfram.com/HypercubeLinePicking.html`.

9. AN INFINITE COSINE PRODUCT.

Problem 8. *Calculate*

$$\pi_2 = \int_0^\infty \cos(2x) \prod_{n=1}^{\infty} \cos\left(\frac{x}{n}\right) dx.$$

History and context. The challenge of showing that $\pi_2 < \pi/8$ was posed by Bernard Mares, Jr., along with the problem of demonstrating that

$$\pi_1 = \int_0^\infty \prod_{n=1}^{\infty} \cos\left(\frac{x}{n}\right) dx < \frac{\pi}{4}.$$

This is indeed true, although the error is remarkably small, as we shall see.

Solution. The computation of a high-precision numerical value for this integral is rather challenging, owing in part to the oscillatory behavior of $\prod_{n \geq 1} \cos(x/n)$ (see Figure 2) but mostly because of the difficulty of computing high-precision evaluations of the integrand. Note that evaluating thousands of terms of the infinite product would

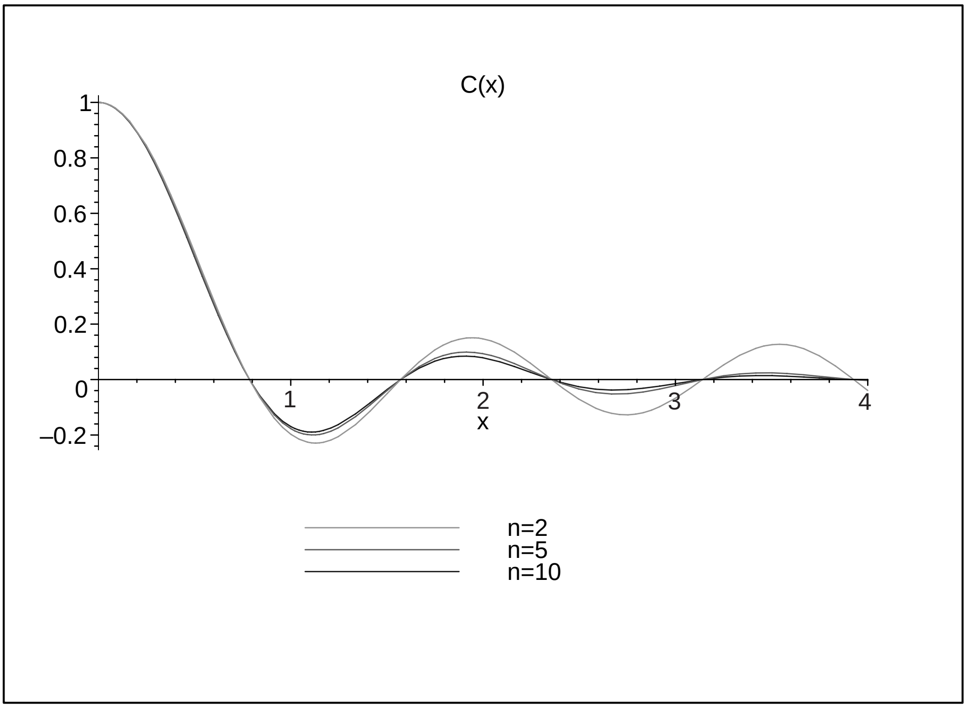

Figure 2. Approximations to $\prod_{n\geq 1} \cos(x/n)$.

produce only a few correct digits. Thus it is necessary to rewrite the integrand in a form more suitable for computation.

Let $f(x)$ signify the integrand. We can express $f(x)$ as

$$f(x) = \cos(2x) \left[\prod_{k=1}^{m} \cos\left(\frac{x}{k}\right) \right] \exp(f_m(x)), \tag{22}$$

where we choose m greater than x and where

$$f_m(x) = \sum_{k=m+1}^{\infty} \log\cos\left(\frac{x}{k}\right). \tag{23}$$

The kth summand can be expanded in a Taylor series [**1**, p. 75], as follows:

$$\log\cos\left(\frac{x}{k}\right) = \sum_{j=1}^{\infty} \frac{(-1)^j 2^{2j-1}(2^{2j}-1)B_{2j}}{j(2j)!} \left(\frac{x}{k}\right)^{2j},$$

in which B_{2j} are Bernoulli numbers. Observe that since $k > m > x$ in (23), this series converges. We can then write

$$f_m(x) = \sum_{k=m+1}^{\infty} \sum_{j=1}^{\infty} \frac{(-1)^j 2^{2j-1}(2^{2j}-1)B_{2j}}{j(2j)!} \left(\frac{x}{k}\right)^{2j}. \tag{24}$$

After applying the identity [**1**, p. 807]

$$B_{2j} = \frac{(-1)^{j+1} 2(2j)!\zeta(2j)}{(2\pi)^{2j}}$$

and interchanging the sums, we obtain

$$f_m(x) = -\sum_{j=1}^{\infty} \frac{(2^{2j}-1)\zeta(2j)}{j\pi^{2j}} \left[\sum_{k=m+1}^{\infty} \frac{1}{k^{2j}}\right] x^{2j}.$$

Note that the inner sum can also be written in terms of the zeta-function, as follows:

$$f_m(x) = -\sum_{j=1}^{\infty} \frac{(2^{2j}-1)\zeta(2j)}{j\pi^{2j}} \left[\zeta(2j) - \sum_{k=1}^{m} \frac{1}{k^{2j}}\right] x^{2j}.$$

This can now be reduced to a compact form for purposes of computation as

$$f_m(x) = -\sum_{j=1}^{\infty} a_j b_{j,m} x^{2j}, \tag{25}$$

where

$$a_j = \frac{(2^{2j}-1)\zeta(2j)}{j\pi^{2j}}, \tag{26}$$

$$b_{j,m} = \zeta(2j) - \sum_{k=1}^{m} 1/k^{2j}. \tag{27}$$

We remark that $\zeta(2j)$, a_j, and $b_{j,m}$ can all be precomputed, say for j up to some specified limit and for a variety of m. In our program, which computes this integral to 120-digit accuracy, we precompute $b_{j,m}$ for $m = 1, 2, 4, 8, 16, \ldots, 256$ and for j up to 300. During the quadrature computation, the function evaluation program picks m to be the first power of two greater than the argument x, and then applies formulas (22) and (25). It is not necessary to compute $f(x)$ for x larger than 200, since for these large arguments $|f(x)| < 10^{-120}$ and thus may be presumed to be zero.

The computation of values of the Riemann zeta-function can be done using a simple algorithm due to Peter Borwein [**21**] or, since what we really require is the entire set of values $\{\zeta(2j) : 1 \le j \le n\}$ for some n, by a convolution scheme described in [**5**]. It is important to note that the computation of both the zeta values and the $b_{j,m}$ must be done with a much higher working precision (in our program, we use 1600-digit precision) than the 120-digit precision required for the quadrature results, since the two terms being subtracted in formula (27) are very nearly equal. These values need to be calculated to a *relative* precision of 120 digits.

With this evaluation scheme for $f(x)$ in hand, the integral (8) can be computed using, for instance, the tanh-sinh quadrature algorithm, which can be implemented fairly easily on a personal computer or workstation and is also well suited to highly parallel processing [**10**], [**11**], [**16**, p. 312]. This algorithm approximates an integral $f(x)$ on $[-1, 1]$ by transforming it to an integral on $(-\infty, \infty)$ via the change of variable $x = g(t)$, where $g(t) = \tanh(\pi/2 \cdot \sinh t)$:

$$\int_{-1}^{1} f(x)\,dx = \int_{-\infty}^{\infty} f(g(t))g'(t)\,dt = h \sum_{j=-\infty}^{\infty} w_j f(x_j) + E(h). \tag{28}$$

Here $x_j = g(hj)$ and $w_j = g'(hj)$ are abscissas and weights for the tanh-sinh quadrature scheme (which can be precomputed), and $E(h)$ is the error in this approximation.

The function $g'(t) = \pi/2 \cdot \cosh t \cdot \operatorname{sech}^2(\pi/2 \cdot \sinh t)$ and its derivatives tend to zero very rapidly for large $|t|$. Thus, even if the function $f(t)$ has an infinite derivative, a blow-up discontinuity, or oscillatory behavior at an endpoint, the product function $f(g(t))g'(t)$ is in many cases quite well behaved, going rapidly to zero (together with all of its derivatives) for large $|t|$. In such cases, the Euler-Maclaurin summation formula [**2**, p. 180] can be invoked to conclude that the error $E(h)$ in the approximation (28) decreases very rapidly—faster than any power of h. In many applications, the tanh-sinh algorithm achieves quadratic convergence (i.e., reducing the size h of the interval in half produces twice as many correct digits in the result).

The tanh-sinh quadrature algorithm is designed for a finite integration interval. In this problem, where the interval of integration is $[0, \infty)$, it is necessary to convert the integral to a problem on a finite interval. This can be done with the simple substitution $s = 1/(x+1)$, which yields an integral from 0 to 1.

In spite of the substantial computation required to construct the zeta- and b-arrays, as well as the abscissas x_j and weights w_j needed for tanh-sinh quadrature, the entire calculation requires only about one minute on a 2004-era computer, using the ARPREC arbitrary precision software package available at `http://crd.lbl.gov/~dhbailey/mpdist`. The first hundred digits of the result are the following:

$$0.3926990816987241548078304229099378605246454341872315959268122851$$
$$620932471399385461790165127474553667777\ldots.$$

A *Mathematica* program capable of producing 100 digits of this constant is available on Michael Trott's website: `http://www.mathematicaguidebooks.org/downloads/N_2_01_Evaluated.nb`.

Using the Inverse Symbolic Calculator, for instance, one finds that this constant is likely to be $\pi/8$. But a careful comparison with a high-precision value of $\pi/8$, namely,

$$0.3926990816987241548078304229099378605246461749218882276218680740$$
$$384770507857761248285043531677646334 97\ldots,$$

reveals that they are *not* equal—the two values differ by approximately 7.407×10^{-43}. Indeed, these two values are provably distinct. This follows from the fact that

$$\sum_{n=1}^{55} 1/(2n+1) > 2 > \sum_{n=1}^{54} 1/(2n+1).$$

See [**16**, chap. 2] for additional details. We do not know a concise closed-form expression for this constant.

Further history and context. Recall the *sinc* function

$$\operatorname{sinc} x = \frac{\sin x}{x},$$

and consider, the seven highly oscillatory integrals:

$$I_1 = \int_0^\infty \operatorname{sinc} x \, dx = \frac{\pi}{2},$$
$$I_2 = \int_0^\infty \operatorname{sinc} x \operatorname{sinc}\left(\frac{x}{3}\right) \, dx = \frac{\pi}{2},$$
$$I_3 = \int_0^\infty \operatorname{sinc} x \operatorname{sinc}\left(\frac{x}{3}\right) \operatorname{sinc}\left(\frac{x}{5}\right) \, dx = \frac{\pi}{2},$$
$$\vdots$$
$$I_6 = \int_0^\infty \operatorname{sinc} x \operatorname{sinc}\left(\frac{x}{3}\right) \cdots \operatorname{sinc}\left(\frac{x}{11}\right) \, dx = \frac{\pi}{2},$$
$$I_7 = \int_0^\infty \operatorname{sinc} x \operatorname{sinc}\left(\frac{x}{3}\right) \cdots \operatorname{sinc}\left(\frac{x}{13}\right) \, dx = \frac{\pi}{2}.$$

It comes as something of a surprise, therefore, that

$$\begin{aligned} I_8 &= \int_0^\infty \operatorname{sinc} x \operatorname{sinc}\left(\frac{x}{3}\right) \cdots \operatorname{sinc}\left(\frac{x}{15}\right) \, dx \\ &= \frac{467807924713440738696537864469}{935615849440640907310521750000}\pi \approx 0.499999999992646\pi. \end{aligned}$$

When this was first discovered by a researcher, using a well-known computer algebra package, both he and the software vendor concluded there was a “bug” in the software. Not so! It is fairly easy to see that the limit of the sequence of such integrals is $2\pi_1$. Our analysis, via Parseval’s theorem, links the integral

$$I_N = \int_0^\infty \operatorname{sinc}(a_1 x) \operatorname{sinc}(a_2 x) \cdots \operatorname{sinc}(a_N x) \, dx$$

with the volume of the polyhedron P_N described by

$$P_N = \left\{ x : \left| \sum_{k=2}^N a_k x_k \right| \le a_1, |x_k| \le 1, 2 \le k \le N \right\}$$

for $x = (x_2, x_3, \ldots, x_N)$. If we let

$$C_N = \{(x_2, x_3, \ldots, x_N) : -1 \le x_k \le 1, 2 \le k \le N\},$$

then

$$I_N = \frac{\pi}{2a_1} \frac{\operatorname{Vol}(P_N)}{\operatorname{Vol}(C_N)}.$$

Thus, the value drops precisely when the constraint $\sum_{k=2}^N a_k x_k \le a_1$ becomes *active* and bites the hypercube C_N. That occurs when $\sum_{k=2}^N a_k > a_1$. In the foregoing,

$$\frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{13} < 1,$$

but on addition of the term $1/15$, the sum exceeds 1, the volume drops, and $I_N = \pi/2$ no longer holds. A similar analysis applies to π_2. Moreover, it is fortunate that we began with π_1 or the falsehood of $\pi_2 = 1/8$ would have been much harder to see.

Additional information on this problem is available at `http://mathworld.wolfram.com/InfiniteCosineProductIntegral.html` and `http://mathworld.wolfram.com/BorweinIntegrals.html`.

10. A MULTIVARIATE ZETA-FUNCTION.

Problem 9. *Calculate*

$$\sum_{i>j>k>l>0} \frac{1}{i^3 j k^3 l}.$$

Extra credit: Express this constant as a single-term expression involving a well-known mathematical constant.

History and context. We resume the discussion from Problem 3. In the notation introduced there, we ask for the value of $\zeta(3, 1, 3, 1)$. The study of such sums in two variables, as we noted, originated with Euler. These investigations were apparently due to a serendipitous mistake. Goldbach wrote to Euler [**15**, pp. 99–100]:

> When I recently considered further the indicated sums of the last two series in my previous letter, I realized immediately that the same series arose due to a mere writing error, from which indeed the saying goes, "Had one not erred, one would have achieved less [*Si non errasset, fecerat ille minus*]."

Euler's *reduction formula* is

$$\zeta(s, 1) = \frac{s}{2}\zeta(s+1) - \frac{1}{2}\sum_{k=1}^{s-2} \zeta(k+1)\zeta(s+1-k),$$

which *reduces* the given double Euler sums to a sum of products of classical ζ-values. Euler also noted the first *reflection formulas*

$$\zeta(a, b) + \zeta(b, a) = \zeta(a)\zeta(b) - \zeta(a+b),$$

certainly valid when $a > 1$ and $b > 1$. This is an easy algebraic consequence of adding the double sums. Another marvelous fact is the *sum formula*

$$\sum_{\Sigma a_i = n, a_i \geq 0} \zeta(a_1 + 2, a_2 + 1, \ldots, a_r + 1) = \zeta(n + r + 1) \tag{29}$$

for nonnegative integers n and r. This, as David Bradley observes, is equivalent to the generating function identity

$$\sum_{n>0} \frac{1}{n^r(n-x)} = \sum_{k_1>k_2>\cdots k_r>0} \prod_{j=1}^{r} \frac{1}{k_j - x}.$$

The first three nontrivial cases of (29) are $\zeta(3) = \zeta(2, 1)$, $\zeta(4) = \zeta(3, 1) + \zeta(2, 2)$, and $\zeta(2, 1, 1) = \zeta(4)$.

Solution. We notice that such a function is a generalization of the zeta-function. Similar to the definition in section 4, we define

$$\zeta(s_1, s_2, \ldots, s_k; x) = \sum_{n_1 > n_2 > \cdots > n_k > 0} \frac{x_1^n}{n_1^{s_1} n_2^{s_2} \cdots n_r^{s_r}}, \tag{30}$$

for $s_1, s_2, \ldots, s_k$ nonnegative integers. We see that we are asked to compute the value $\zeta(3, 1, 3, 1; 1)$. Such a sum can be evaluated directly using the EZFace+ interface at `http://www.cecm.sfu.ca/projects/ezface+`, which employs the Hölder convolution, giving us the numerical value

$$\begin{aligned}&0.005229569563530960100930652283899231589890420784634635522547448\\ &9721488695446601500749754543248561040162 7\ldots.\end{aligned} \tag{31}$$

Alternatively, we may proceed using differential equations. It is fairly easy to see [**16**, sec. 3.7] that

$$\frac{d}{dx}\zeta(n_1, n_2, \ldots, n_r; x) = \frac{1}{x}\zeta(n_1 - 1, n_2, \ldots, n_r; x) \quad (n_1 > 1), \tag{32}$$

$$\frac{d}{dx}\zeta(n_1, n_2, \ldots, n_r; x) = \frac{1}{1-x}\zeta(n_2, \ldots, n_r; x) \quad (n_1 = 1), \tag{33}$$

with initial conditions $\zeta(n_1; 0) = \zeta(n_1, n_2; 0) = \cdots = \zeta(n_1, \ldots, n_r; 0) = 0$ and $\zeta(\cdot; x) \equiv 1$. Solving

```
> dsys1 =
> diff(y3131(x),x) = y2131(x)/x,
> diff(y2131(x),x) = y1131(x)/x,
> diff(y1131(x),x) = 1/(1-x)*y131(x),
> diff(y131(x),x) = 1/(1-x)*y31(x),
> diff(y31(x),x) = y21(x)/x,
> diff(y21(x),x) = y11(x)/x,
> diff(y11(x),x) = y1(x)/(1-x),
> diff(y1(x),x) = 1/(1-x);
> init1 = y3131(0) = 0,y2131(0) = 0, y1131(0) = 0,
>                    y131(0) = 0,y31(0) = 0,y21(0) = 0,y11(0) = 0,y1(0) = 0;
```

in *Maple*, we obtain 0.005229569563518039612830536519667669502942 (this is valid to thirteen decimal places). Maple's `identify` command is unable to identify portions of *this* number, and the inverse symbolic calculator does not return a result. It should be mentioned that both *Maple* and the ISC identified the constant $\zeta(3, 1)$ (see the remark under the "history and context" heading). From the hint for this question, we know this is a single-term expression. Suspecting a form similar to $\zeta(3, 1)$, we search for a constants c and d such that $\zeta(3, 1, 3, 1) = c\pi^d$. This leads to $c = 1/81440 = 2/10!$ and $d = 8$.

Further history and context. We start with the simpler value, $\zeta(3, 1)$. Notice that

$$-\log(1 - x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots,$$

so

$$f(x) = -\log(1-x)/(1-x) = x + \left(1+\frac{1}{2}\right)x^2 + \left(1+\frac{1}{2}+\frac{1}{3}\right)x^3 + \cdots$$
$$= \sum_{n \geq m > 0} \frac{x^n}{m}.$$

As noted in the section on double Euler sums,

$$\frac{(-1)^{m+1}}{\Gamma(m)} \int_0^1 x^n \log^{m-1} x \, dx = \frac{1}{(n+1)^m},$$

so integrating f using this transform for $m = 3$, we obtain

$$\zeta(3,1) = \frac{1}{2} \int_0^1 f(x) \log^2 x \, dx$$
$$= 0.270580808427784547879000924 \ldots.$$

The corresponding generating function is

$$\sum_{n \geq 0} \zeta(\{3,1\}_n) x^{4n} = \frac{\cosh(\pi x) - \cos(\pi x)}{\pi^2 x^2},$$

equivalent to Zagier's conjectured identity

$$\zeta(\{3,1\}_n) = \frac{2\pi^{4n}}{(4n+2)}.$$

Here $\{3,1\}_n$ denotes n-fold concatenation of $\{3,1\}$.

The proof of this identity (see [**16**, p. 160]) derives from a remarkable factorization of the generating function in terms of hypergeometric functions:

$$\sum_{n \geq 0} \zeta(\{3,1\}_n) x^{4n} = {}_2F_1\left(x\frac{(1+i)}{2}, -x\frac{(1+i)}{2}; 1; 1\right)$$
$$\times \, {}_2F_1\left(x\frac{(1-i)}{2}, -x\frac{(1-i)}{2}; 1; 1\right).$$

Finally, it can be shown in various ways that

$$\zeta(\{3\}_n) = \zeta(\{2,1\}_n)$$

for all n, while a proof of the numerically-confirmed conjecture

$$\zeta(\{2,1\}_n) \stackrel{?}{=} 2^{3n} \zeta(\{-2,1\}_n) \tag{34}$$

remains elusive. Only the first case of (34), namely,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{m=1}^{n-1} \frac{1}{m} = 8 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \sum_{m=1}^{n-1} \frac{1}{m} \quad (= \zeta(3))$$

has a self-contained proof [**16**]. Indeed, the only other established case is

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{m=1}^{n-1} \frac{1}{m} \sum_{p=1}^{m-1} \frac{1}{p^2} \sum_{q=1}^{p-1} \frac{1}{q} = 64 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \sum_{m=1}^{n-1} \frac{1}{m} \sum_{p=1}^{m-1} \frac{(-1)^p}{p^2} \sum_{q=1}^{p-1} \frac{1}{q} \quad (= \zeta(3,3)).$$

This is an outcome of a complete set of equations for multivariate zeta-functions of depth four.

There has been abundant evidence amassed to support identity (34) since it was found in 1996. For example, very recently Petr Lisonek checked the first eighty-five cases to one thousand places in about forty-one hours with only the *expected roundoff error*. And he checked $n = 163$ in ten hours. This is the *only* identification of its type of an Euler sum with a distinct multivariate zeta-function.

11. A WATSON INTEGRAL.

Problem 10. *Evaluate*

$$W = \frac{1}{\pi^3} \int_0^{\pi} \int_0^{\pi} \int_0^{\pi} \frac{1}{3 - \cos x - \cos y - \cos z} \, dx\, dy\, dz. \tag{35}$$

History and context. The integral arises in Gaussian and spherical models of ferromagnetism and in the theory of random walks. It leads to one of the most impressive closed-form evaluations of an equivalent multiple integral due to G. N. Watson:

$$\begin{aligned} \widehat{W} &= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{1}{3 - \cos x - \cos y - \cos z} \, dx\, dy\, dz \\ &= \frac{1}{96} (\sqrt{3} - 1)\, \Gamma^2\left(\frac{1}{24}\right) \Gamma^2\left(\frac{11}{24}\right) \\ &= 4\pi \left(18 + 12\sqrt{2} - 10\sqrt{3} - 7\sqrt{6}\right) \mathrm{K}^2(k_6), \end{aligned} \tag{36}$$

where $k_6 = (2 - \sqrt{3})(\sqrt{3} - \sqrt{2})$ is the sixth singular value. The most self-contained derivation of this very subtle result is due to Joyce and Zucker in [**28**] and [**29**], where more background can also be found.

Solution. In [**31**], it is shown that a simplification can be obtained by applying the formula

$$\frac{1}{\lambda} = \int_0^{\infty} e^{-\lambda t} \, dt \qquad (\operatorname{Re} \lambda > 0) \tag{37}$$

to W_3. The three-dimensional integral is then reducible to a single integral by using the identity

$$\frac{1}{\pi} \int_0^{\infty} \exp(t \cos\theta) d\theta = I_0(t), \tag{38}$$

in which $I_0(t)$ is the modified Bessel function of the first kind. It follows from this that $W = \int_0^{\infty} \exp(-3t) I_0^3(t) dt$. This integral can be evaluated to one hundred digits in *Maple*, giving

$$W_3 = 0.505462019717326006052004053227140259985129014817420892188993487886028773451173816800537247069896038 0\ldots. \tag{39}$$

Finally, an integer relation hunt to express $\log W$ in terms of $\log \pi$, $\log 2$, $\log \Gamma(k/24)$, and $\log(\sqrt{3}-1)$ will produce (36).

We may also write W_3 as a product solely of values of the gamma function. This is what our *Mathematician's ToolKit* returned:

```
0 = -1.* log[w3] + -1.* log[gamma[1/24]] + 4.*log[gamma[3/24]] +
-8.*log[gamma[5/24]] + 1.* log[gamma[7/24]] + 14.*log[gamma[9/24]] +
-6.*log[gamma[11/24]] + -9.*log[gamma[13/24]] + 18.*log[gamma[15/24]] +
-2.*log[gamma[17/24]] + -7.*log[gamma[19/24]]
```

Proving this is achieved by comparing the result with (36) and establishing the implicit gamma representation of $(\sqrt{3}-1)^2/96$.

Similar searches suggest there is no similar four-dimensional closed form—the relevant Bessel integral is $W_4 = \int_0^\infty \exp(-4t) I_0^4(t)\,dt$. (N.B. $\int_0^\infty \exp(-2t) I_0^2(t)\,dt = \infty$.) In this case it is necessary to compute $\exp(-t)I_0(t)$ carefully, using a combination of the formula

$$\exp(-t)I_0(t) = \exp(-t)\sum_{n=0}^{\infty} \frac{t^{2n}}{2^{2n}(n!)^2}$$

for t up to roughly $1.2 \cdot d$, where d is the number of significant digits desired for the result, and

$$\exp(-t)I_0(t) \approx \frac{1}{\sqrt{2\pi t}} \sum_{n=0}^{N} \frac{\prod_{k=1}^{n}(2k-1)^2}{(8t)^n n!}$$

for large t, where the upper limit N of the summation is chosen to be the first index n such that the summand is less than 10^{-d} (since this is an asymptotic expansion, taking more terms than N may increase, not decrease the error). We have implemented this as 'besselexp' in our *Mathematician's ToolKit*, available at `http://crd.lbl.gov/~dhbailey/mpdist`. Using this software, which includes a PSLQ facility, we found that W_4 is not expressible as a product of powers of $\Gamma(k/120)$ $(0 < k < 120)$ with coefficients having fewer than 80 digits. This result does not, of course, rule out the possibility of a larger relation, but it does cast some doubt, in an experimental sense, that such a relation exists—enough to stop looking.

Additional information on this problem is available at `http://mathworld.wolfram.com/WatsonsTripleIntegrals.html`.

12. CONCLUSION. While all the problems described herein were studied with a great deal of experimental computation, clean proofs are known for the final results given (except for Problem 7), and in most cases a lot more has by now been proved. Nonetheless, in each case the underlying object suggests plausible generalizations that are still open.

The "hybrid computations" involved in these solutions are quite typical of modern experimental mathematics. Numerical computations by themselves produce no insight, and symbolic computations frequently fail to produce full-fledged, closed-form solutions. But when used together, with significant human interaction, they are often successful in discovering new facts of mathematics and in suggesting routes to formal proof.

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Book Reviews

Edited by Robert E. O'Malley, Jr.

Featured Review: Oxford Users' Guide to Mathematics. *Edited by Eberhard Zeidler.* Oxford University Press, Oxford, 2004. $59.50. xxii+1285 pp., softcover. ISBN 0-19-850763-1.

1. Dictionaries, Handbooks, Encyclopedias, and Tables.

For thousands of years, dictionaries, encyclopedias, handbooks, manuals, outlines, primers, problem books, review journals, synopses, tables, and users' guides have played a central role in mathematics—from the Rhind or Ahmes papyruses (see [3]) of the Egyptians (c. 3400 BCE), through Leonardo (Fibonacci) Pisano's *Liber Abaci* (1202–1228) to the Jahrbuch and later Zentralblatt, Math Reviews/MathSciNet, and even Wikipedia and MathWorld.

In his preface to Fibonacci's work, L. E. Sigler, the translator, writes, "*Liber Abaci* is an encyclopedic work treating most of the known mathematics of the thirteenth century on arithmetic, algebra, and problem solving. . . . *Liber Abaci* was good mathematics when it was written and it is good mathematics today." What a model to emulate!

Let me continue with a discursive discussion of reference material, and of the issues any author of such has to deal with, before turning to the (*Oxford*) guide in question. Given my own twenty years as book editor, book author, and "accidental lexicographer"—as described below—I feel reasonably comfortable doing this. There is a community of compilers and collators, and I write as one of this generally supportive fraternity.

This is in part because as a compiler one largely asserts facts without substantiation. Thus, there is much need for judgment and great room for error—as there is indeed in a textbook, particularly in exercises (quoted in a book review in *Science*, 1994):

> "[T]he proof is left as an exercise" occurred in *De Triangulis Omnimodis* by Regiomontanus, written 1464 and published 1533. He is quoted as saying "This is seen to be the converse of the preceding. Moreover, it has a straightforward proof, as did the preceding. Whereupon I leave it to you for homework."

Unlike a paper in one's specialty or even a book in one's favorite topic, where one can aspire to mastery of the subject, a compiler is constantly skating on thin ice. Replacement of an "a" by a "the" can show how thin the veneer of knowledge is. A reversed inequality might well not be invidious, but mistranscription of a superscript may well leave an expression mangled as, for example, in equation (1) below. Above all, placing material in one's own deft words is a recipe for original sin.

Publishers are invited to send books for review to Book Reviews Editor, SIAM, 3600 *University City Science Center, Philadelphia, PA* 19104-2688.

1.1. The Collins Dictionary: A Disclaimer and My Own Experiences. I feel somewhat in an at-least-apparent conflict of interest. I am coauthor with Ephraim Borowski[1] of the *Collins Dictionary of Mathematics* [1]. It is now in its 15th printing and was relaunched in an updated Smithsonian edition late in 2005. Since I write a less than glowing review it may well seem self-interested and disingenuous. But a conflict of interest announced is one at least half-resolved.

We started writing the Collins Dictionary in 1985 after a reader of the general Collins dictionary complained justifiably about certain of the mathematical and logical entries therein. Borowski[2] and I were asked to revise the thousand or so mathematical terms, which we did. At the end we had a stack of handwritten file cards and a mild addiction which grew into the dictionary. This was typed on four Macintoshes (one a repentant Lisa), using the chalkboard as a database manager, with frequent airmailing of floppy disks across the Atlantic. We ended up having written a 9,000-or-so-term[3] book which became the first text set from disk in Europe—an interesting if not a pretty process. Through ignorance on Collins' part, we had been left the "electronic and musical rights." By the mid-nineties this had resulted in an interactive CD version, the *MathResource*, which embeds student *Maple* (see www.mathresouces.com). Ten years later the dictionary is sitting symmetrically inside *Maple*.

After "finishing" the first edition of our dictionary in 1988, I found I could not enjoy a single colloquium or seminar for more than three years. I would constantly ask myself, "Did I define that term correctly, should I have included their result?" I felt like a giant hamster on a never-ending lexical treadwheel. Such is the life of a lexicographer or a compiler.

1.2. Dictionaries in General. Neglecting entirely Denis Diderot and his *Encyclopédie*[4] (1745–1772), let us revisit some of the central events in English. *Roget's Thesaurus*, published in 1852 as "Treasury of Words" by the remarkable Peter Mark Roget (1779–1869), had rapid and enormous success, even as a fashion accessory for the cultured; for a period it was good social style to consult it openly in drawing room conversation. Roget never expected it to be used except by the well educated! It is now online free at http://thesaurus.reference.com.

Samuel Johnson's Dictionary (written between 1747 and 1755) is generally viewed as the first English dictionary. Since only Scotland educated the middle-class in those days, five out of six paid assistants were Scots, and definitions like *oats*, as a food that sustains horses in England but people in Scotland, must be read with this knowledge. As with *Fowler's Modern English Usage*, it had some effect in standardizing usage and spelling. While most authors aim to be descriptive, not prescriptive, readers often take prescriptions. Over the last two centuries, Canada has veered between "math" and "maths," "analyse" and "analyze," "cancelled" and "canceled," "-metre" and "-meter," never quite finding the "centre/center."[5]

Johnson (1709–1784), immortalized by Boswell's marvelous 1791 *Life of Samuel Johnson*, had all our modern troubles with funding his projects,[6] and these are re-

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[1]With the assistance of many others. We met for lexicographic reasons—our names were listed next to each other on Oxford class lists.

[2]Who was already engaged in revising philosophy, religion, and other entries.

[3]Counting dictionary entries is not an exact science.

[4]Diderot's original co-editor was the mathematician d'Alembert.

[5]All *MathResources* software has to have a bilingual "units toggle."

[6]Paid by the chapter for a book on *The Snakes of Europe*, one chapter *in extenso* reads "There are no snakes in Ireland."

flected in the dictionary. The entry for *patron* was aimed at the Earl of Chesterton, who offered patronage only at the end of the day—when success was assured:

> **patron**, n., one who countenances, supports or protects. Commonly a wretch who supports with insolence, and is paid with flattery.

The dictionary was far from error-free, about which Johnson was refreshingly honest. When challenged as to why he had defined a *pastern* to be a horse's kneecap he replied, "Ignorance, madam, pure ignorance."

Johnson's American competitor Noah Webster (1758–1843) had a dramatic impact on English through his 1840 dictionary.[7] The Webster dictionary standardized spellings such as "colour" and "favour" and led to their acceptance in the UK and largely in Canada, but by early in the last century had lost out in the U.S. save for the occasional faux-Victorian "icecream parlour." Together Webster and Johnson had spawned the modern dictionary, while the *Oxford English Dictionary* (OED) of 1928 inarguably nursed it to term.

As charmingly described in Simon Winchester's best-seller *The Professor and the Madman*, the OED was and remains a monumental project that took the better part of forty years to see the light of day. The OED was perhaps the first clearly open-source project. Readers everywhere sent paper slips recording what became the *earliest usages* one finds today in the OED. The slips arrived from such as W.C. Minor (the Madman), who contributed thousands of entries from Broadmoor prison, at the Scriptorium in Oxford, where they were inserted in pigeonholes before being compared, contrasted, and digested under the direction after 1879 of (the Professor) James Augustus Henry Murray (1837–1915). What a worthy ancestor to the open-source Wikipedia[8] and shared computations like those at www.mersenne.org.[9]

1.3. Some "Recent" Mathematical Dictionaries. Mathematics is an ancient subject and so for me "recent" means roughly since World War Two. This is consonant with my student days in Oxford, when "modern literature" ended with *Ulysses*.

When Borowski and I began our work there had been no new one-volume college-mathematics dictionary for a generation, since the Van Nostrand *Mathematics Dictionary* by Glen and Robert C. James (1942–1959). In that case a distinguished mathematician son assisted an older lexicographer father.[10] Unlike our predecessors we opted for a full lexical structure rather than Britannica-like topic entries. I think that in this we were farsighted, certainly in light of Internet reading habits.

News of our impending Collins volume immediately triggered a similar slimmer dictionary from Penguin (1989) and Chris Clapham's *Concise Oxford Dictionary of Mathematics* (1990). Volumes followed from Barrons (1995) and McGraw-Hill (1997) among others. A more modern entrant was Eric Weisstein's *Concise Encyclopedia of Mathematics* (CRC, 1998), which has a CD version and has developed—after an intellectual property tussle between Wolfram and CRC—into a lovely and comprehensive set of well-maintained[11] and much-visited resources on the *Mathematica* website http://mathworld.wolfram.com. It now has over 12,000 entries. A more specialized

[7] After his death and having eschewed copyright protection, it was acquired by the Merriams in 1847, whence the *Merriam-Webster Dictionary*.

[8] Derived from the Hawaiian *wiki wiki* meaning "quick" or "informal," a *wiki* is "the simplest online database that works" (see http://en.wikipedia.org).

[9] The most recent Mersenne prime was found by an ophthalmologist.

[10] So also did George assist Tobias Danzig.

[11] Maintaining a website is in some ways easier (ease of correction and user input) and in others much harder (pressure to correct) than with print.

but highly rewarding volume is Stephen Finch's *Mathematical Constants* (Cambridge University Press, 2003).

At the other end of the spectrum is the more advanced two-volume topic-based *Encyclopedic Dictionary of Mathematics* (1993) from the Mathematical Society of Japan, which I find unwieldy: too big to use easily and with less information about more topics than a row of subject books. By contrast, I found the *VNR Concise Encyclopedia of Mathematics* (1977), which is aimed at a high-school/early college market, very nicely illustrated and it proved very useful in my own lexical work. Unlike all the others mentioned, lamentably it does not seem to have a recent edition.

1.4. The Issues for Authors. These are enormous in ambit. They include the desired *depth* and *breadth* of coverage. Is it fair to suppose that a user consulting "affine variety" has no need to be told much about "affine"? What *x-refs* are needed? Achieving *balanced coverage* is also a huge headache. When I would show our manuscript to an analyst she would tell me the algebra coverage was excellent but the analysis was wanting.... Especially with multiple authors one should add *uniformity of style* and convention.

Originality (authorship) and *accuracy* (authority) are often in conflict. Collins used "eight words in sequela" as a definition of plagiarism in trade books. In the interest of correctness, precise science and engineering are typically excluded from this impossible constraint—try defining an abelian group; you are appropriately apt to give the same definition as I did.

Determining which of competing definitions and theorems to trust is problematic: Is a *topology* implicitly assumed *Hausdorff*? Does a *field* in the given context always have *characteristic zero*? Is a *partial order* taken to be *antisymmetric*? May a *Banach space* be complex? And so on. Book authors notoriously make running assumptions that frequent readers become aware of, but not so the innocent compiler or assistant. Even the best older sources such as Whittacker and Watson's *Modern Analysis* are terribly prone to this. Of course the ideal future includes complete semantics and wonderful metadata.

In our 2002 edition we added an appendix on the millennium problems to accompany the one on the Hilbert problems. My coauthor wanted to write his own descriptions, but I wished to copy those on the Clay Institute website. We compromised. The definition of plagiarism was plagiarized (from Tom Lehrer and the *New York Times*) and I hope it is the only case of plagiarism in the volume. Incidentally, Noah Webster is accused of great gobs of plagiarism, but he also gets deserved credit for uniformizing spelling and much else in American English.

Plagiarism is only one of many *copyright issues*.[12] Wikipedia writes,

> Copyrights currently last for seventy years after the death of an author, or seventy-five to ninety-five years in the case of works of corporate authorship and works first published before January 1, 1978. All works in the United States before 1923 are in the public domain.... Some material from as recently as 1963 has entered the public domain but some as old as 1923 remains copyrighted if renewals were filed.... No additional material will enter the public domain until 2019 due to changes in the applicable laws.

Such is the "Mickey Mouse" Act introduced in Congress by the late Sonny Bono. By contrast:

[12]See http://en.wikipedia.org/wiki/United_States_copyright_law and www.ceic.math.ca.

> The U.S. Congress first exercised its power to enact copyright legislation with the Copyright Act of 1790. The Act secured an author the exclusive right to publish and vend "maps, charts and books" for a term of 14 years, with the right of renewal for one additional 14 year term if the author was still alive. The act did not regulate other kinds of writings, such as musical compositions or newspapers and specifically noted that it did not prohibit copying the works of foreign authors. The vast majority of writings were never copyrighted—between 1790 and 1799, of 13,000 titles published in the United States, only 556 were copyrighted.

The law's "14 + 14" formula was very much akin the 1710 British *Act of Anne*. Many of us would like to see a return to the spirit of Anne.

Both clearing and asserting copyright[13] itself can be excruciating. It took three years to get all permissions needed[14] for *Pi: A Sourcebook* [3]. The laws differ over many jurisdictions. While the U.S. has First Amendment Rights and notions of fair use, the EU has Moral Rights, I live in the British Commonwealth (which has neither), China has not signed the Berne Convention, and nothing is entirely clear on the World Wide Web. It is not always certain who owns the rights or even sometimes who the author is. Illustrations are worse; we had to get permission from the British Museum to place a picture of the Rhind papyrus in [3].[15] Our publisher asked us not even to try to put a picture of Winnie the Pooh doing math in [6]—it meant asking Disney. We were refused permission by Fox "for reasons we are not at liberty to share with you" to use a fax (which depicted Bart Simpson) sent to my coauthor requesting the 40,000th digit of pi—this despite the fact that the answer was used in an episode of *The Simpsons*.

Maintenance and *enhancement* are terrible problems. Errors arise in may ways—from Johnson's "pasterns" to discontinuity of authorship and changing formats over the years (in our case from MacWord to HyperCards, PageMaker, TeX, VisualBasic, MathML, and beyond). Especially without the use of relational databases and other IT tools (we now benefit from having the dictionary fully hyperlinked so missing or stray x-refs are much easier to find) it is a nightmare to update computations of pi or Mersenne primes, solutions to once open problems (true and false) such as Fermat's last theorem or recent work on the Poincaré conjecture, deaths of living mathematicians such as Paul Erdős or Claude Shannon, and the like. For each major revision this process has entailed hiring assistants, often at our own expense. Keeping prices down for authors is frequently used by book publishers as a reason for resisting enhancements such as color or paying for more copyediting and fact checking.

A more vexing problem is to capture past lacunae (or is it lacunas?) and to chart the changing boundary of the relevant collection. For example, between 1985 and 2000 the following entries (which were arguably not needed in our dictionary in 1985) were among those that had migrated into many undergraduate curricula and were added or dramatically revamped in the 2002 edition.

MATHEMATICAL NEOLOGIA: Erdős graph, fractal dimension, genetic algorithm, interior point methods, monster group, q-bit, quantum computer, RSA code, and Andrew Wiles.

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[13] Charles Dickens was among the foreign (and U.S.) authors who railed at the exclusion of foreign authors, but it was only in 1891 that this law was changed.

[14] Even though Springer-Verlag would settle for three active attempts.

[15] Thereby setting a mathematical record perhaps, since we needed permissions over a five millennium span.

Terms such as Groebner basis, integer relation, internet graph, neural network, and the new polynomial primality algorithm of Aggarwal, Kayal, and Saxenna (AKS) [7, pp. 300–303] are on the list be added in the next edition.

A compendium is also the easiest of books for a reviewer to tear apart—you just look for a few maladroit terms in your specialty and build your review around them. When we were fixing the Collins dictionary, I would read other, say, medical, entries, as I waited for my colleague to move to the next math term. I always trusted the medical terms and rarely the math ones. I did have the pleasure of replacing an erroneous Anglocentric definition of a *home run* with the compact "*A four base hit.*"

2. The Pleasures and Perils of Compendia.

Samuel Johnson observed that dictionaries are like watches in that "the best do not run true, and the worst are better than none." The same is true of handbooks, tables, and databases. That is in part why we all need many!

> Several years ago I was invited to contemplate being marooned on the proverbial desert island. What book would I most wish to have there, in addition to the Bible and the complete works of Shakespeare? My immediate answer was: Abramowitz and Stegun's *Handbook of Mathematical Functions*. If I could substitute for the Bible, I would choose Gradsteyn and Ryzhik's *Table of Integrals, Series and Products*. Compounding the impiety, I would give up Shakespeare in favor of Prudnikov, Brychkov, and Marichev's *Tables of Integrals and Series.*
>
> ...
>
> On the island, there would be much time to think about waves on the water that carve ridges on the sand beneath and focus sunlight there; shapes of clouds; subtle tints in the sky.... With the arrogance that keeps us theorists going, I harbor the delusion that it would be not too difficult to guess the underlying physics and formulate the governing equations. It is when contemplating how to solve these equations—to convert formulations into explanations—that humility sets in. Then, compendia of formulas become indispensable.[16]

Prudnikov, Brychkov, and Marichev's excellent three-volume compendium is printed in a mediocre Soviet format. It contains as Entry 9 on page 750 of Volume 1,

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{1}{k^2\left(k^2-kl+l^2\right)}=\frac{\pi^{?}\sqrt{3}}{30}, \tag{1}$$

where the "?" is *probably* "4". Integer relation methods (see [6, sect. 6.3]) strongly suggest that no *reasonable* value of "?" works. I still do not know what is intended in equation (1).[17] There are many such examples in the literature from Lewin's attempt to understand an enticing polylogarithmic assertion of Landen (see [7, p. 210]) to Ramanujan and of course Fermat's last theorem. We would benefit from a well-developed set of *Forensic Mathematics* tools—such as would certainly exist for CSI-Oberwolfach?

[16] Michael Berry, "Why Are Special Functions Special?" *Physics Today,* April 2001; available online from http://www.physicstoday.org/pt/vol-54/iss-4/p11.html.

[17] I have intentionally not asked the authors directly, but return to the challenge from time to time.

Three quarters of a century ago G. H. Hardy, in his retirement lecture as *London Mathematical Society* Secretary, commented (see [8, p. 474]) that

> Harald Bohr is reported to have remarked, "Most analysts spend half their time hunting through the literature for inequalities they want to use, but cannot prove."

They still do and so have to choose among consulting good, bad, and indifferent compendia on inequalities.

2.1. When Good Things Come from Bad Sources. Ramanujan's form of inspiration may be a rather more common story given the presence of the World Wide Web and the cost of commercial material, especially in the developing world.

The Indian genius Srinivasa Ramanujan (1887–1920), growing up in Kumbakonam, about 250 km from Madras,[18] read what was available in his local library. He learned what he learned largely from two books: S. L. Loney's *Plane Trigonometry*, standard trigonometry of the time, and *A Synopsis of Elementary Results in Pure Mathematics* written by Carr, who was a "crammer" in Cambridge. This is a compilation of many thousands of results that "might be on exams." This source apparently contained no complex variables and so Ramanujan famously knew none when he arrived in Cambridge in 1914. He had, however, worked out marvelous new mathematics based on what he had gleaned from these unexceptional sources.

Today these sources might be replaced by Schaum's *Outlines*[19] and Sloane's wonderful online *Encyclopedia of Sequences*[20] (www.research.att.com/∼njas/sequences) or the soon-to-be-released *Digital Library of Mathematical Functions* (DLMF) being completed at NIST, originally the National Bureau of Standards; see http://dlmf.nist.gov. The DLMF is a massive print-CD-and-Web revision of Abramowitz and Stegun's *Handbook of Mathematical Functions*, partially funded by the NSF. The Web version will be freely available and will have quite sophisticated "math-aware" search capabilities.[21]

The original book has sold perhaps 750,000 copies between its NIST and Dover editions —making it the best-selling mathematics reference book ever. The new book is still over 1,000 pages long but the 500 pages of numerical tables in the original have almost disappeared (*Maple*, *Mathematica*, and MATLAB being broadly accessible) and been replaced by more and newer mathematics—with formula-level metadata and with the old grayscale illustrations replaced by fine colored graphics which have some dynamic functionality in the digital edition.

One hopes any new Ramanujan would also be able to call upon *JSTOR* (www.jstor.org) and *MathSciNet*, (e-math.ams.org/mathscinet), but this will depend on whether he has directly or indirectly paid for access. He would certainly have access to many of the resources in the emerging *World Digital Mathematics Library* (www.wdml.org).

Very recently David Bailey and I have been working on parallel quadrature implementations of Euler–Maclaurin summation [2]. We found that http://planetmath.org/encyclopedia/ProofOfEulerMaclaurinSummationFormula.html had correct and useful but nonstandard information, while other sites were less satisfactory. This was equally true, though, of books as it was of websites.

[18] He moved to Madras in 1910.

[19] Or by more dubious variants.

[20] Based on a 1985 *Academic Press* book with 5,000 entries, this immaculate database now has over 110,000 entries.

[21] And so if Michael Berry's island has wifi, he could keep Shakespeare in book form. . . .

3. The Oxford User's Guide.

The *Editorial Review From Book News, Inc.* of the *Guide* says

> Recognizing the importance of mathematics in research and commerce, the many instances in which different aspects of mathematics are coming to inform each other, and the prevalence of the personal computer, Zeidler (Max Planck Institute for Mathematics in the Sciences) and contributors offer a basic overview of mathematics for students, practitioners, and teachers.

This is fairly accurate as to the actual scope of the *Guide*, save for the odd reference to the prevalence of the personal computer. The Oxford University Press (OUP) description is less on target:

> The *Oxford User's Guide to Mathematics* in Science and Engineering represents a comprehensive handbook on mathematics. It covers a broad spectrum of mathematics.... The book offers a broad modern picture of mathematics starting from basic material up to more advanced topics.... The book addresses students in engineering, mathematics, computer science, natural sciences, high-school teachers, as well as a broad spectrum of practitioners in industry and professional researchers.... The bibliography represents a comprehensive collection of the contemporary standard literature in the main fields of mathematics.

Having made these expansive claims, a publisher has some obligation to ensure they have been met. Expectation management is an issue in all walks of life from academic publications to national elections.

3.1. Something of the Oxford Users' Guide. The claims made by OUP may have been close to true in 1958 but are not today. My quarrel is more with what is left out than with what is said. Many students view their texts as exoskeletons—what is not there does not exist. As the case of Ramanujan shows, even mediocre coverage is often better than complete omission. Zeidler is an excellent researcher, a fine scholar, and broadly knowledgable; but as I have already indicated, even modest dictionaries need sizeable and continuing teams.

Let me divide the *Guide* in four. My notional *Part* I contains roughly 225 pages of elementary mathematics and tabular information alluded to earlier. My *Part* II follows with 375 pages on analysis (of which less than 10 cover harmonic analysis), 125 pages on algebra and number theory, and 150 pages on geometry (elementary, algebraic, and differential). This core material is followed by *Part* III with 30 pages on foundations, 60 pages on calculus of variations and optimization (linear and nonlinear), and 70 pages on probability and statistics. *Part* IV comprises 125 pages on scientific computation: numerical methods for linear algebra, interpolation, nonlinear equations, and ordinary and partial differential equations. The book is then completed by a 25-page history of mathematics, a 27-page bibliography, and various indices.

The topics covered are thus somewhat staid. They are, I imagine, quite faithful to a thirty-year-old undergraduate German curriculum, but even undergraduate mathematics has moved on. Moreover, one uses a compendium especially to look up material with which one is not familiar—often in subjects not taken in college.

For example, point-set topology (other than metric), algebraic topology, combinatorics, dynamical systems and chaos, financial mathematics, game theory, and graph theory, are among the missing or get only the most cursory mention. Thus, on page 833 a footnote refers to another book by Zeidler[22] for the definition of topology which

[22] Published in 1995 in German, with a still only promised English version.

is needed to make sense of the Zariski topology! Likewise, complexity theory rates a paragraph on page 1050. So it is puzzling that Oxford recommends the book to groups such as "students... in computer science" or "practitioners in industry."

Moreover, even the entries on topics like scientific computation and optimization, whose coverage is touted, are somewhat limited and do not include interior point methods or much discovered since the simplex method or the singular value decomposition. A few references to computational science have been sprinkled in rather at random. For instance, the totality of practical numeric guidance appears to be on the bottom of page 1049:

> **Numerical mathematics with Mathematica**: *With this software package you are able to perform many of the numerical standard procedures on your home PC.*
>
> ......
>
> *For every imaginable numerical procedure, no matter how elegant it appears, there are* counterexamples *for which the method does not work at all.*

The Chinese remainder theorem? Newton's method for the square root? The AKS primality algorithm? This is false or at best true but somewhat fatuous. I suppose this is the sort of thing that justifies OUP's saying that the *Guide* "offers a broad modern picture of mathematics."

To be fair my serious criticisms are directed largely at OUP—and Teubner before it—and the process by which both had the *Guide* refereed and produced. Additionally, the translation, while largely very good, is a trifle Teutonic and seemingly done without adequate *mathematical* copyediting. On page 239 one reads about the irrationality of $\sqrt{2}$ that

> This discovery destroyed the harmonic picture of the universe by the Pythagoreans and triggered a deep shock.

On page 878 we learn that the method of indirect proof "then leads this assumption to a contradiction." Such stiltedness is sometimes to the point of obscuring the meaning: on page 823, I have no idea what—in the context of Pythagorean triples—an accord is, despite it being in the index.

Typographically, the *Guide* has masses of white space and gratuitous boxes of a kind that probably looked fine at one time. They now only add needless heft to an already weighty book with too small margins. Unlike the DLMF's decision that tables were obsolete, the *Guide* still has roughly 150 pages of material much better found online or on a personal computer. Even the binding is dubious: my cover tore in the first week of very mild use!

A more thorough review and production process would surely have adequately addressed this last set of issues. I can no better make this point than to quote Simkin and Fiske quoting the late Stephen J. Gould in a review of Simon Winchester's *Krakatoa*:[23]

> In his review of Winchester's previous book, *The Map That Changed the World,* Stephen Jay Gould wrote: "I don't mean to sound like an academic sourpuss, but I just don't understand the priorities of publishers who spare no expense to produce an elegantly illustrated and beautifully designed book and then permit the text to wallow in simple, straight-out factual

[23]Tom Simkin and Richard S. Fiske, "Clouded Picture of a Big Bang," *Science*, July 4 (2003), pp. 50–51. These reviews do make me question the reliability of *The Professor and the Madman.*

> errors, all easily corrected for the minimal cost of one scrutiny of the galleys by a reader with professional expertise. . . ."
>
> With *Krakatoa*, the publisher clearly spared considerable expense, and this new book also wallows in errors. Perhaps, given our popular culture's appetite for sensationalized disasters, a modern publisher would rather not see all those pesky details corrected.

It seems this is a somewhat under-considered "economy" English adaptation of a ten-year-old Teubner book which was itself already somewhat dated, having had its first of eighteen German editions in 1958. As I have said, I have great respect for Zeidler and his colleagues. But like a university department's set of teaching notes, this *Guide* has decayed over time. Will current and future generations have a taste for information served up as it is in the *Guide*? Would contestants in the recent *SIAM* 100-*Digit Challenge* [4, 5] have found the *Guide* helpful? I suspect not. I decided to sample *Google*, *MathWorld*, and the *Guide* on the terms in the *Neologia* above. I did better on the Web.

4. Conclusion.

There are many positive things to be said about the book under review. The price is good. What it covers it usually covers well and it seems largely error-free. It contains several attractive extra features such as a useful biography[24] of books on the subjects it does cover and an amusing brief history of mathematics.[25] [26]

On balance, I am happy to add the *Users' Guide* to my reference shelf—right next to the computer and its online resources. I'll look in it for topics where it is strong, such as analysis and classical applied physical mathematics, and avoid its advice on topics like numerics.

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[7] J. M. Borwein, D. H. Bailey, and R. Girgensohn, *Experimentation in Mathematics: Computational Paths to Discovery*, A. K. Peters, Natick, MA, 2004.

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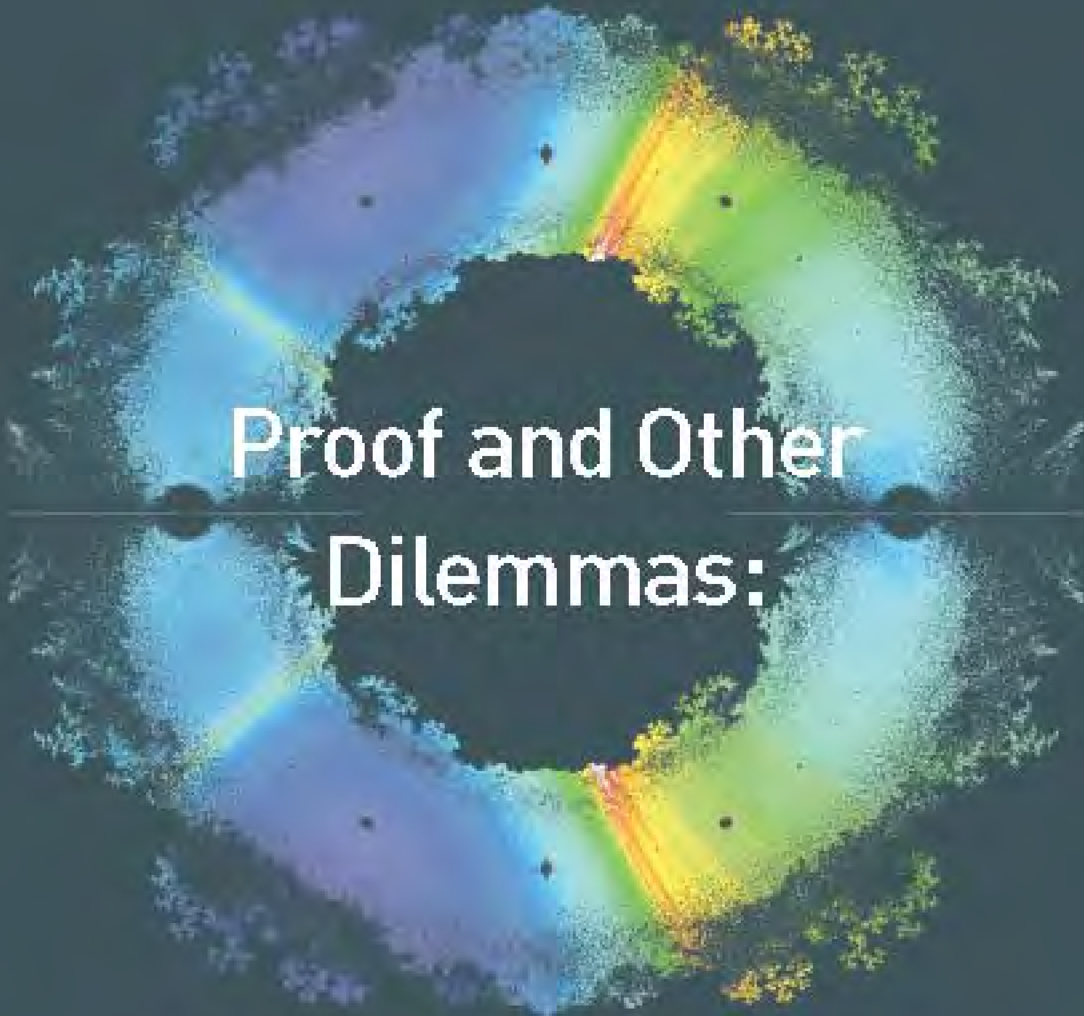
[24] I had trouble discerning the rationale for a book to be included in this large, but in no sense comprehensive, unannotated list.

[25] Which for some reason includes Mary Queen of Scot's execution in 1587 and Shakespeare's dates, and a lot more Nobel Prize winners than Fields medalists.

[26] Again, here one sees the difficulty of the task attempted: Claude Shannon (1916–2001) is listed among the living.

SPECTRUM



Proof and Other Dilemmas:

Mathematics and Philosophy

Bonnie Gold & Roger Simons, Editors

Implications of Experimental Mathematics for the Philosophy of Mathematics[1]

Jonathan Borwein, FRSC[2]

Christopher Koch [35] accurately captures a great scientific distaste for philosophizing:

> *"Whether we scientists are inspired, bored, or infuriated by philosophy, all our theorizing and experimentation depends on particular philosophical background assumptions. This hidden influence is an acute embarrassment to many researchers, and it is therefore not often acknowledged."* (Christopher Koch, 2004)

That acknowledged, I am of the opinion that mathematical philosophy matters more now than it has in nearly a century. The power of modern computers matched with that of modern mathematical software and the sophistication of current mathematics is changing the way we do mathematics.

In my view it is now both necessary and possible to admit quasi-empirical inductive methods fully into mathematical argument. In doing so carefully we will enrich mathematics and yet preserve the mathematical literature's deserved reputation for reliability—even as the methods and criteria change. What do I mean by reliability? Well, research mathematicians still consult Euler or Riemann to be informed, anatomists only consult Harvey[3] for historical reasons. Mathematicians happily quote old papers as core steps of arguments, physical scientists expect to have to confirm results with another experiment.

1 Mathematical Knowledge as I View It

Somewhat unusually, I can exactly place the day at registration that I became a mathematician and I recall the reason why. I was about to deposit my punch cards in the 'honours history bin'. I remember thinking

> *"If I do study history, in ten years I shall have forgotten how to use the calculus properly. If I take mathematics, I shall still be able to read competently about the War of 1812 or the Papal schism."* (Jonathan Borwein, 1968)

The inescapable reality of objective mathematical knowledge is still with me. Nonetheless, my view then of the edifice I was entering is not that close to my view of the one I inhabit forty years later.

[1]The companion web site is at **www.experimentalmath.info**

[2]Canada Research Chair, Faculty of Computer Science, 6050 University Ave, Dalhousie University, Nova Scotia, B3H 1W5 Canada. E-mail: `jborwein@cs.dal.ca`

[3]William Harvey published the first accurate description of circulation, "An Anatomical Study of the Motion of the Heart and of the Blood in Animals," in 1628.

I also know when I became a computer-assisted fallibilist. Reading Imre Lakatos' *Proofs and Refutations*, [38], a few years later while a very new faculty member, I was suddenly absolved from the grave sin of error, as I began to understand that missteps, mistakes and errors are the grist of all creative work.[4] The book, his doctorate posthumously published in 1976, is a student conversation about the Euler characteristic. The students are of various philosophical stripes and the discourse benefits from his early work on Hegel with the Stalinist Lukács in Hungary and from later study with Karl Popper at the London School of Economics. I had been prepared for this dispensation by the opportunity to learn a variety of subjects from Michael Dummett. Dummett was at that time completing his study rehabilitating Frege's status, [23].

A decade later the appearance of the first 'portable' computers happily coincided with my desire to decode Srinivasa Ramanujan's (1887–1920) cryptic assertions about theta functions and elliptic integrals, [13]. I realized that by coding his formulae and my own in the *APL* programming language[5], I was able to rapidly confirm and refute identities and conjectures and to travel much more rapidly and fearlessly down potential blind alleys. I had become a computer-assisted fallibilist; at first somewhat falteringly but twenty years have certainly honed my abilities.

Today, while I appreciate fine proofs and aim to produce them when possible, I no longer view proof as the royal road to secure mathematical knowledge.

2 Introduction

I first discuss my views, and those of others, on the nature of mathematics, and then illustrate these views in a variety of mathematical contexts. A considerably more detailed treatment of many of these topics is to be found in my book with Dave Bailey entitled *Mathematics by Experiment: Plausible Reasoning in the 21st Century*—especially in Chapters One, Two and Seven, [9]. Additionally, [2] contains several pertinent case studies as well as a version of this current chapter.

Kurt Gödel may well have overturned the mathematical apple cart entirely deductively, but nonetheless he could hold quite different ideas about legitimate forms of mathematical reasoning, [28]:

> *"If mathematics describes an objective world just like physics, there is no reason why inductive methods should not be applied in mathematics just the same as in physics."* (Kurt Gödel[6], 1951)

[4]Gila Hanna [30] takes a more critical view placing more emphasis on the role of proof and certainty in mathematics; I do not disagree, so much as I place more value on the role of computer-assisted refutation. Also 'certainty' usually arrives late in the development of a proof.

[5]Known as a 'write only' very high level language, APL was a fine tool; albeit with a steep learning curve whose code is almost impossible to read later.

[6]Taken from a previously unpublished work, [28].

While we mathematicians have often separated ourselves from the sciences, they have tended to be more ecumenical. For example, a recent review of *Models. The Third Dimension of Science*, [17], chose a mathematical plaster model of a Clebsch diagonal surface as its only illustration. Similarly, authors seeking examples of the aesthetic in science often choose iconic mathematics formulae such as $\mathrm{E} = \mathrm{MC}^2$.

Let me begin by fixing a few concepts before starting work in earnest. Above all, I hope to persuade you of the power of mathematical experimentation—it is also fun—and that the traditional accounting of mathematical learning and research is largely an ahistorical caricature. I recall three terms.

mathematics, n. *a group of related subjects, including algebra, geometry, trigonometry and calculus, concerned with the study of number, quantity, shape, and space, and their inter-relationships, applications, generalizations and abstractions.*

This definition–taken from my Collins Dictionary [6]—makes no immediate mention of proof, nor of the means of reasoning to be allowed. The Webster's Dictionary [54] contrasts:

induction, n. *any form of reasoning in which the conclusion, though supported by the premises, does not follow from them necessarily.*; and

deduction, n. *a process of reasoning in which a conclusion follows necessarily from the premises presented, so that the conclusion cannot be false if the premises are true.*
b. a conclusion reached by this process.

Like Gödel, I suggest that both should be entertained in mathematics. This is certainly compatible with the general view of mathematicians that in some sense "mathematical stuff is out there" to be discovered. In this paper, I shall talk broadly about experimental and heuristic mathematics, giving accessible, primarily visual and symbolic, examples.

3 Philosophy of Experimental Mathematics

> *"The computer has in turn changed the very nature of mathematical experience, suggesting for the first time that mathematics, like physics, may yet become an empirical discipline, a place where things are discovered because they are seen."*
> (David Berlinski, [4])

The shift from *typographic* to *digital culture* is vexing for mathematicians. For example, there is still no truly satisfactory way of displaying mathematics on the web–and certainly not of asking mathematical questions. Also, we respect *authority*, [29], but value *authorship* deeply—however much the two values are in conflict, [16]. For example, the more I recast someone else's ideas in my own words, the more I enhance my authorship while undermining the original authority of the notions. Medieval scribes had the opposite concern and so took care to attribute their ideas to such as Aristotle or Plato.

And we care more about the *reliability* of our literature than does any other science, Indeed I would argue that we have over-subscribed to this notion and often pay lip-service not real attention to our older literature. How often does one see original sources sprinkled like holy water in papers that make no real use of them–the references offering a false sense of scholarship?

The traditional central role of proof in mathematics is arguably and perhaps appropriately under siege. Via examples, I intend to pose and answer various questions. I shall conclude with a variety of quotations from our progenitors and even contemporaries:

My Questions. What constitutes secure mathematical knowledge? When is computation convincing? Are humans less fallible? What tools are available? What methodologies? What of the 'law of the small numbers'? Who cares for certainty? What is the role of proof? How is mathematics actually done? How should it be? I mean these questions both about the apprehension (discovery) and the establishment (proving) of mathematics. This is presumably more controversial in the formal proof phase.

My Answers. To misquote D'Arcy Thompson (1860–1948) 'form follows function', [52]: rigour (proof) follows reason (discovery); indeed, excessive focus on rigour has driven us away from our wellsprings. Many good ideas are wrong. Not all truths are provable, and not all provable truths are worth proving. Gödel's incompleteness results certainly showed us the first two of these assertions while the third is the bane of editors who are frequently presented with correct but unexceptional and unmotivated generalizations of results in the literature. Moreover, near certainty is often as good as it gets—intellectual context (community) matters. Recent complex human proofs are often very long, extraordinarily subtle and fraught with error—consider, Fermat's last theorem, the Poincaré conjecture, the classification of finite simple groups, presumably any proof of the Riemann hypothesis, [25]. So while we mathematicians publicly talk of certainty we really settle for security.

In all these settings, modern computational tools dramatically change the nature and scale of available evidence. Given an interesting identity buried in a long and complicated paper on an unfamiliar subject, which would give you more confidence in its correctness: staring at the proof, or confirming computationally that it is correct to 10,000 decimal places?

Here is such a formula, [3, p. 20]:

$$\frac{24}{7\sqrt{7}}\int_{\pi/3}^{\pi/2}\log\left|\frac{\tan t+\sqrt{7}}{\tan t-\sqrt{7}}\right|dt \stackrel{?}{=} L_{-7}(2) = \tag{1}$$
$$\sum_{n=0}^{\infty}\left[\frac{1}{(7n+1)^2}+\frac{1}{(7n+2)^2}-\frac{1}{(7n+3)^2}+\frac{1}{(7n+4)^2}-\frac{1}{(7n+5)^2}-\frac{1}{(7n+6)^2}\right].$$

This identity links a volume (the integral) to an arithmetic quantity (the sum). It arose out of some studies in quantum field theory, in analysis of the volumes of ideal tetrahedra in hyperbolic space. The question mark is used because, while no hint of a path to a formal proof is yet known, it has been verified numerically to 20,000 digit precision–using 45 minutes on 1024 processors at Virginia Tech.

A more inductive approach can have significant benefits. For example, as there is still some doubt about the proof of the classification of finite simple groups it is important to ask whether the result is true but the proof flawed, or rather if there is still perhaps an 'ogre' sporadic group even larger than the 'monster'? What heuristic, probabilistic or computational tools can increase our confidence that the ogre does or does not exist? Likewise, there are experts who still believe the *Riemann hypothesis*[7] (RH) may be false and that the billions of zeroes found so far are much too small to be representative.[8] In any event, our understanding of the complexity of various crypto-systems relies on (RH) and we should like secure knowledge that any counter-example is enormous.

Peter Medawar (1915–87)—a Nobel prize winning oncologist and a great expositor of science—writing in *Advice to a Young Scientist*, [44], identifies four forms of scientific experiment:

1. The Kantian experiment: generating "the classical non-Euclidean geometries (hyperbolic, elliptic) by replacing Euclid's axiom of parallels (or something equivalent to it) with alternative forms." All mathematicians perform such experiments while the majority of computer explorations are of the following Baconian form.

2. The Baconian experiment is a contrived as opposed to a natural happening, it "is the consequence of 'trying things out' or even of merely messing about." Baconian experiments are the explorations of a happy if disorganized beachcomber and carry little predictive power.

3. Aristotelian demonstrations: "apply electrodes to a frog's sciatic nerve, and lo, the leg kicks; always precede the presentation of the dog's dinner with the ringing of a bell, and lo, the bell alone will soon make the dog dribble." Arguably our 'Corollaries' and 'Examples' are Aristotelian, they reinforce but do not predict. Medawar then says the most important form of experiment is:

4. The Galilean experiment is "a critical experiment – one that discriminates between possibilities and, in doing so, either gives us confidence in the view we are taking or makes us think it in need of correction." The Galilean the only form of experiment which stands to make Experimental Mathematics a serious enterprise. Performing careful, replicable Galilean experiments requires work and care.

Reuben Hersh's arguments for a humanist philosophy of mathematics, especially [31, pp. 590–591] and [32, p. 22], as paraphrased below, become even more convincing in our highly computational setting.

1. Mathematics is human. *It is part of and fits into human culture. It does not match Frege's concept of an abstract, timeless, tenseless, objective reality.*[9]

2. Mathematical knowledge is fallible. *As in science, mathematics can advance by making mistakes and then correcting or even re-correcting them. The "fallibilism" of mathematics is brilliantly argued in Lakatos'* Proofs and Refutations.

3. There are different versions of proof or rigor. *Standards of rigor can vary depending on time, place, and other things. The use of computers in formal proofs, exemplified by*

[7] All non-trivial zeroes—not negative even integers—of the zeta function lie on the line with real part 1/2.

[8] See [45] and various of Andrew Odlyzko's unpublished but widely circulated works.

[9] That Frege's view of mathematics is wrong, for Hersh as for me, does not diminish its historical importance.

the computer-assisted proof of the four color theorem in 1977,[10] *is just one example of an emerging nontraditional standard of rigor.*

4. Empirical evidence, numerical experimentation and probabilistic proof all can help us decide what to believe in mathematics. *Aristotelian logic isn't necessarily always the best way of deciding.*

5. Mathematical objects are a special variety of a social-cultural-historical object. *Contrary to the assertions of certain post-modern detractors, mathematics cannot be dismissed as merely a new form of literature or religion. Nevertheless, many mathematical objects can be seen as shared ideas, like Moby Dick in literature, or the Immaculate Conception in religion.*

I entirely subscribe to points 2., 3., 4., and with certain caveats about objective knowledge[11] to points 1. and 5. In any event mathematics is and will remain a uniquely human undertaking.

This version of humanism sits *fairly* comfortably along-side current versions of **social-constructivism** as described next.

> *"The social constructivist thesis is that mathematics is a social construction, a cultural product, fallible like any other branch of knowledge."* (Paul Ernest, [26, §3])

But only if I qualify this with "*Yes, but much-much less fallible than most branches of knowledge.*" Associated most notably with the writings of Paul Ernest—an English Mathematician and Professor in the Philosophy of Mathematics Education who in [27] traces the intellectual pedigree for his thesis, a pedigree that encompasses the writings of Wittgenstein, Lakatos, Davis, and Hersh among others—social constructivism seeks to define mathematical knowledge and epistemology through the social structure and interactions of the mathematical community and society as a whole.

This interaction often takes place over very long periods. Many of the ideas our students—and some colleagues—take for granted took a great deal of time to gel. The Greeks suspected the impossibility of the three *classical construction problems* [12] and the irrationality of the golden mean was well known to the Pythagoreans.

While concerns about potential and completed infinities are very old, until the advent of the calculus with Newton and Leibnitz and the need to handle fluxions or infinitesimals, the level of need for rigour remained modest. Certainly Euclid is in its geometric domain generally a model of rigour, while also Archimedes' numerical analysis was not equalled until the 19th century.

[10]Especially, since a new implementation by Seymour, Robertson and Thomas in 1997 which has produced a simpler, clearer and less troubling implementation.

[11]While it is not Hersh's intention, a superficial reading of point 5. hints at a cultural relativism to which I certainly do not subscribe.

[12]Trisection, circle squaring and cube doubling were taken by the educated to be impossible in antiquity. Already in 414 BCE, in his play *The Birds*, Aristophanes uses 'circle-squarers' as a term for those who attempt the impossible. Similarly, the French Academy stopped accepting claimed proofs a full two centuries before the 19th century achieved proofs of their impossibility.

The need for rigour arrived in full force in the time of Cauchy and Fourier. The treacherous countably infinite processes of analysis and the limitations of formal manipulation came to the fore. It is difficult with a modern sensibility to understand how Cauchy's proof of the continuity of pointwise-limits could coexist in texts for a generation with clear counter-examples originating in Fourier's theory of heat.[13]

By the end of the 19th century Frege's (1848-1925) attempt to base mathematics in a linguistically based *logicism* had foundered on Russell and other's discoveries of the paradoxes of naive set theory. Within thirty five years Gödel—and then Turing's more algorithmic treatment[14]—had similarly damaged both Russell and Whitehead's and Hilbert's programs.

Throughout the twentieth century, bolstered by the armor of abstraction, the great ship Mathematics has sailed on largely unperturbed. During the last decade of the 19th and first few decades of the 20th century the following main streams of philosophy emerged explicitly within mathematics to replace logicism, but primarily as the domain of philosophers and logicians.

- *Platonism.* Everyman's idealist philosophy—stuff exists and we must find it. Despite being the oldest mathematical philosophy, Platonism—still predominant among working mathematicians—was only christened in 1934 by Paul Bernays.[15]

- *Formalism.* Associated mostly with Hilbert—it asserts that mathematics is invented and is best viewed as formal symbolic games without intrinsic meaning.

- *Intuitionism.* Invented by Brouwer and championed by Heyting, intuitionism asks for inarguable monadic components that can be fully analyzed and has many variants; this has interesting overlaps with recent work in cognitive psychology such as Lakoff and Nunez' work, [39], on 'embodied cognition'.[16]

- *Constructivism.* Originating with Markoff and especially Kronecker (1823–1891), and refined by Bishop it finds fault with significant parts of classical mathematics. Its 'I'm from Missouri, tell me how big it is' sensibility is not to be confused with Paul Ernest's 'social constructivism', [27].

The last two philosophies deny the principle of the *excluded middle*, "A or not A", and resonate with computer science—as does some of formalism. It is hard after all to run a deterministic program which does not know which disjunctive logic-gate to follow.

[13]Cauchy's proof appeared in his 1821 text on analysis. While counterexamples were pointed out almost immediately, Stokes and Seidel were still refining the missing uniformity conditions in the late 1840s.

[14]The modern treatment of incompleteness leans heavily on Turing's analysis of the *Halting problem* for so-called Turing machines.

[15]See Karlis Podnieks, "Platonism, Intuition and the Nature on Mathematics", available at `http://www.ltn.lv/ podnieks/gt1.html`

[16]The cognate views of Henri Poincaré (1854–1912), [47, p. 23] on the role of the *subliminal* are reflected in "*The mathematical facts that are worthy of study are those that, by their analogy with other facts are susceptible of leading us to knowledge of a mathematical law, in the same way that physical facts lead us to a physical law.*" He also wrote *"It is by logic we prove, it is by intuition that we invent,"* [48].

By contrast the battle between a Platonic idealism (a 'deductive absolutism') and various forms of 'fallibilism'(a quasi-empirical 'relativism') plays out across all four, but fallibilism perhaps lives most easily within a restrained version of intuitionism which looks for 'intuitive arguments' and is willing to accept that 'a proof is what convinces'. As Lakatos shows, an argument that was convincing a hundred years ago may well now be viewed as inadequate. And one today trusted may be challenged in the next century.

As we illustrate in the next section or two, it is only perhaps in the last twenty five years, with the emergence of powerful mathematical platforms, that any approach other than a largely undigested Platonism and a reliance on proof and abstraction has had the tools[17] to give it traction with working mathematicians.

In this light, Hales' proof of Kepler's conjecture that *the densest way to stack spheres is in a pyramid* resolves the oldest problem in discrete geometry. It also supplies the most interesting recent example of intensively computer-assisted proof, and after five years with the review process was published in the *Annals of Mathematics*—with an "only 99% checked" disclaimer.

This process has triggered very varied reactions [34] and has provoked Thomas Hales to attempt a formal computational proof which he expects to complete by 2011, [25]. Famous earlier examples of fundamentally computer-assisted proof include the *Four color theorem* and proof of the *Non-existence of a projective plane of order 10*. The three raise and answer quite distinct questions about computer-assisted proof—both real and specious. For example, there were real concerns about the completeness of the search in the 1976 proof of the Four color theorem but there should be none about the 1997 reworking by Seymour, Robertson and Thomas.[18] Correspondingly, Lam deservedly won the 1992 *Lester R. Ford award* for his compelling explanation of why to trust his computer when it announced there was no plane of order ten, [40]. Finally, while it is reasonable to be concerned about the certainty of Hales' conclusion, was it really the *Annal's* purpose to suggest all other articles have been more than 99% certified?

To make the case as to how far mathematical computation has come we trace the changes over the past half century. The 1949 computation of π to 2,037 places suggested by von Neumann, took 70 hours. A billion digits may now be computed in much less time on a laptop. Strikingly, it would have taken roughly 100,000 ENIAC's to store the Smithsonian's picture—as is possible thanks to *40 years of Moore's law* in action . . ..[19]
This is an astounding record of sustained exponential progress without peer in the history of technology. Additionally, mathematical tools are now being implemented on parallel platforms, providing *much* greater power to the research mathematician. Amassing huge amounts of processing power will not alone solve many mathematical problems. There are very few mathematical 'Grand-challenge problems', [12] where, as in the physical sciences, a few more orders of computational power will resolve a problem.

[17] That is, to broadly implement Hersh's central points (2.-4.).

[18] See `http://www.math.gatech.edu/ thomas/FC/fourcolor.html`.

[19] **Moore's Law** is now taken to be the assertion that *semiconductor technology approximately doubles in capacity and performance roughly every 18 to 24 months*.

For example, an order of magnitude improvement in computational power currently translates into one more day of accurate weather forecasting, while it is now common for biomedical researchers to design experiments today whose outcome is predicated on 'peta-scale' computation being available by say 2010, [51]. There is, however, much more value in *very rapid 'Aha's'* as can be obtained through "micro-parallelism"; that is, where we benefit by being able to compute many simultaneous answers on a neurologically-rapid scale and so can hold many parts of a problem in our mind at one time.

To sum up, in light of the discussion and terms above, I now describe myself a a social-constructivist, and as a computer-assisted fallibilist with constructivist leanings. I believe that more-and-more the interesting parts of mathematics will be less-and-less susceptible to classical deductive analysis and that Hersh's 'non-traditional standard of rigor' must come to the fore.

4 Our Experimental Mathodology

Despite Picasso's complaint that "computers are useless, they only give answers," the main goal of computation in pure mathematics is arguably to yield *insight*. This demands speed or, equivalently, substantial *micro-parallelism* to provide answers on a cognitively relevant scale; so that we may ask and answer more questions while they remain in our consciousness. This is relevant for rapid verification; for validation; for *proofs* and *especially for refutations* which includes what Lakatos calls "monster barring", [38]. Most of this goes on in the daily small-scale accretive level of mathematical discovery but insight is gained even in cases like the proof of the Four color theorem or the Non-existence of a plane of order ten. Such insight is not found in the case-enumeration of the proof, but rather in the algorithmic reasons for believing that one has at hand a tractable unavoidable set of configurations or another effective algorithmic strategy. For instance, Lam [40] ran his algorithms on known cases in various subtle ways, and also explained why built-in redundancy made the probability of machine-generated error negligible. More generally, the act of programming—if well performed—always leads to more insight about the structure of the problem.

In this setting it is enough to equate *parallelism* with access to requisite *more* space and speed of computation. Also, we should be willing to consider all computations as 'exact' which provide truly reliable answers.[20] This now usually requires a careful *hybrid* of symbolic and numeric methods, such as achieved by *Maple*'s liaison with the *Numerical Algorithms Group* (NAG) Library[21], see [5, 8]. There are now excellent tools for such purposes throughout analysis, algebra, geometry and topology, see [9, 10, 5, 12, 15].

Along the way questions required by—or just made natural by—computing start to force out older questions and possibilities in the way beautifully described a century ago by Dewey regarding evolution.

[20] If careful interval analysis can certify that a number known to be integer is larger that 2.5 and less than 3.5, this constitutes an exact computational proof that it is 3.

[21] See http://www.nag.co.uk/.

"Old ideas give way slowly; for they are more than abstract logical forms and categories. They are habits, predispositions, deeply engrained attitudes of aversion and preference. Moreover, the conviction persists—though history shows it to be a hallucination—that all the questions that the human mind has asked are questions that can be answered in terms of the alternatives that the questions themselves present. But in fact intellectual progress usually occurs through sheer abandonment of questions together with both of the alternatives they assume; an abandonment that results from their decreasing vitality and a change of urgent interest. We do not solve them: we get over them. Old questions are solved by disappearing, evaporating, while new questions corresponding to the changed attitude of endeavor and preference take their place. Doubtless the greatest dissolvent in contemporary thought of old questions, the greatest precipitant of new methods, new intentions, new problems, is the one effected by the scientific revolution that found its climax in the 'Origin of Species.' " (John Dewey, [20])

Lest one think this a feature of the humanities and the human sciences, consider the artisanal chemical processes that have been lost as they were replaced by cheaper industrial versions. And mathematics is far from immune. Felix Klein, quoted at length in the introduction to [11], laments that "now the younger generation hardly knows abelian functions." He goes on to explain that:

"In mathematics as in the other sciences, the same processes can be observed again and again. First, new questions arise, for internal or external reasons, and draw researchers away from the old questions. And the old questions, just because they have been worked on so much, need ever more comprehensive study for their mastery. This is unpleasant , and so one is glad to turn to problems that have been less developed and therefore require less foreknowledge—even if it is only a matter of axiomatics, or set theory, or some such thing." (Felix Klein, [33, p. 294])

Freeman Dyson has likewise gracefully described how taste changes:

"I see some parallels between the shifts of fashion in mathematics and in music. In music, the popular new styles of jazz and rock became fashionable a little earlier than the new mathematical styles of chaos and complexity theory. Jazz and rock were long despised by classical musicians, but have emerged as art-forms more accessible than classical music to a wide section of the public. Jazz and rock are no longer to be despised as passing fads. Neither are chaos and complexity theory. But still, classical music and classical mathematics are not dead. Mozart lives, and so does Euler. When the wheel of fashion turns once more, quantum mechanics and hard analysis will once again be in style." (Freeman Dyson, [24])

For example recursively defined objects were once anathema—Ramanujan worked very hard to replace lovely iterations by sometimes-obscure closed-form approximations. Additionally, what is "easy" changes: high performance computing and networking are blurring, merging disciplines and collaborators. This is democratizing mathematics but further challenging authentication—consider how easy it is to find information on *Wikipedia*[22] and how hard it is to validate it.

Moving towards a well articulated Experimental *Mathodology*—both in theory and practice—will take much effort. The need is premised on the assertions that intuition is acquired—we can and must better mesh computation and mathematics, and that visualization is of growing importance—in many settings even three is a lot of dimensions.

"Monster-barring" (Lakatos's term, [38], for refining hypotheses to rule out nasty counterexamples[23]) and "caging" (Nathalie Sinclair tells me this is my own term for imposing needed restrictions in a conjecture) are often easy to enhance computationally, as for example with randomized checks of equations, linear algebra, and primality or graphic checks of equalities, inequalities, areas, etc. Moreover, our mathodology fits well with the kind of pedagogy espoused at a more elementary level (and without the computer) by John Mason in [43].

4.1 Eight Roles for Computation

I next recapitulate eight roles for computation that Bailey and I discuss in our two recent books [9, 10]:

#1. Gaining insight and intuition or just knowledge. Working algorithmically with mathematical objects almost inevitably adds insight to the processes one is studying. At some point even just the careful aggregation of data leads to better understanding.

#2. Discovering new facts, patterns and relationships. The number of *additive partitions* of a positive integer n, $p(n)$, is *generated* by

$$P(q) := 1 + \sum_{n\geq 1} p(n)q^n = \frac{1}{\prod_{n=1}^{\infty}(1-q^n)}. \tag{2}$$

Thus, $p(5) = 7$ since

$$5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = \ 2+1+1+1 = 1+1+1+1+1.$$

Developing (2) is a fine introduction to enumeration via *generating functions*. Additive partitions are harder to handle than multiplicative factorizations, but they are very interesting, [10, Chapter 4]. Ramanujan used Major MacMahon's table of $p(n)$ to intuit remarkable deep congruences such as

$$p(\mathbf{5n{+}4}) \equiv 0 \mod \mathbf{5}, \quad p(\mathbf{7n{+}5}) \equiv 0 \mod \mathbf{7}, \quad p(\mathbf{11n{+}6}) \equiv 0 \mod \mathbf{11},$$

[22] *Wikipedia* is an open source project at `http://en.wikipedia.org/wiki/Main_Page`; "wiki-wiki" is Hawaiian for "quickly".

[23] Is, for example, a polyhedron always convex? Is a curve intended to be simple? Is a topology assumed Hausdorff, a group commutative?

from relatively limited data like

$$
\begin{aligned}
P(q) &= 1 + q + 2\,q^2 + 3\,q^3 + \underline{5}\,q^{\mathbf{4}} + \overline{7}\,q^{\mathbf{5}} + 11\,q^6 + 15\,q^7 \\
&+ 22\,q^8 + \underline{30}\,q^{\mathbf{9}} + 42\,q^{10} + 56\,q^{11} + \overline{77}\,q^{\mathbf{12}} + 101\,q^{13} + \underline{135}\,q^{\mathbf{14}} \\
&+ 176\,q^{15} + 231\,q^{16} + 297\,q^{17} + 385\,q^{18} + \overline{\underline{490}}\,q^{\mathbf{19}} \\
&+ 627\,q^{20}b + 792\,q^{21} + 1002\,q^{22} + \cdots + p(200)q^{200} + \cdots \qquad (3)
\end{aligned}
$$

Cases $5n+4$ and $7n+5$ are flagged in (3). Of course, it is markedly easier to (heuristically) confirm than find these fine examples of *Mathematics: the science of patterns.*[24] The study of such congruences—much assisted by symbolic computation—is very active today.

#3. Graphing to expose mathematical facts, structures or principles. Consider Nick Trefethen's fourth challenge problem as described in [5, 8]. It requires one to find ten good digits of:

4. What is the global minimum of the function

$$\exp(\sin(50x)) + \sin(60e^y) + \sin(70\sin x) + \sin(\sin(80y)) - \sin(10(x+y)) + (x^2+y^2)/4?$$

As a foretaste of future graphic tools, one can solve this problem graphically and interactively using current *adaptive 3-D plotting* routines which can catch all the bumps. This does admittedly rely on trusting a good deal of software.

#4. Rigourously testing and especially falsifying conjectures. I hew to the Popperian scientific view that we primarily falsify; but that as we perform more and more testing experiments without such falsification we draw closer to firm belief in the truth of a conjecture such as: *the polynomial* $P(n) = n^2 - n + p$ *has prime values for all* $n = 0, 1, \ldots, p-2$, *exactly for* Euler's lucky prime numbers, *that is, p= 2, 3, 5, 11, 17, and 41.*[25]

#5. Exploring a possible result to see if it *merits* formal proof. A conventional deductive approach to a hard multi-step problem really requires establishing all the subordinate lemmas and propositions needed along the way—especially if they are highly technical and un-intuitive. Now some may be independently interesting or useful, but many are only worth proving if the entire expedition pans out. Computational experimental mathematics provides tools to survey the landscape with little risk of error: only if the view from the summit is worthwhile, does one lay out the route carefully. I discuss this further at the end of the next Section.

#6. Suggesting approaches for formal proof. The proof of the *cubic theta function identity* discussed on [10, pp. 210] shows how a fully intelligible human proof can be obtained entirely by careful symbolic computation.

[24] The title of Keith Devlin's 1996 book, [21].

[25] See [55] for the answer.

#7. Computing replacing lengthy hand derivations. Who would wish to verify the following prime factorization by hand?

$$\begin{aligned} & 6422607578676942838792549775208734746307 \\ = & (2140992015395526641)(1963506722254397)(1527791). \end{aligned}$$

Surely, what we value is understanding the underlying algorithm, not the human work?

#8. Confirming analytically derived results. This is a wonderful and frequently accessible way of confirming results. Even if the result itself is not computationally checkable, there is often an accessible corollary. An assertion about bounded operators on Hilbert space may have a useful consequence for three-by-three matrices. It is also an excellent way to error correct, or to check calculus examples before giving a class.

5 Finding Things versus Proving Things

I now illuminate these eight roles with eight mathematical examples. At the end of each I note some of the roles illustrated.

1. Pictorial comparison of $y - y^2$ and $y^2 - y^4$ to $-y^2 \ln(y)$, when y lies in the unit interval, is a much more rapid way to divine which function is larger than by using traditional analytic methods.

Figure 1 below shows that it is clear in the latter case the functions cross, and so it is futile to try to prove one majorizes the other. In the first case, evidence is provided to motivate attempting a proof and often the picture serves to guide such a proof—by showing monotonicity or convexity or some other salient property. ■

This certainly illustrates roles #3 and #4, and perhaps role #5.

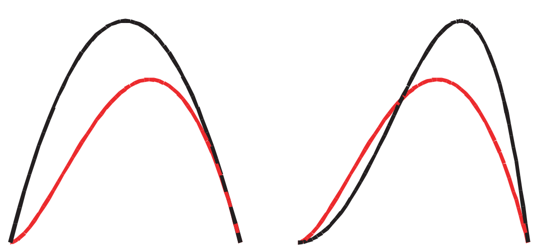


Figure 1. (Ex. 1.): Graphical comparison of $-x^2 \ln(x)$ (lower local maximum in both graphs) with $x - x^2$ (left graph) and $x^2 - x^4$ (right graph)

2. A proof and a disproof. Any modern computer algebra can tell one that

$$0 < \int_0^1 \frac{(1-x)^4 x^4}{1+x^2}\, dx = \frac{22}{7} - \pi, \tag{4}$$

since the integral may be interpreted as the area under a positive curve. We are however no wiser as to why! If however we ask the same system to compute the indefinite integral, we are likely to be told that

$$\int_0^t \cdot = \frac{1}{7}\,t^7 - \frac{2}{3}\,t^6 + t^5 - \frac{4}{3}\,t^3 + 4\,t - 4\,\arctan(t)\,.$$

Then (4) is now rigourously established by differentiation and an appeal to the Fundamental theorem of calculus. ■

This illustrates roles #1 and #6. It also falsifies the bad conjecture that $\pi = 22/7$ and so illustrates #4 again. Finally, the computer's proof is easier (#7) and very nice, though probably it is not the one we would have developed by ourselves. The fact that 22/7 is a continued fraction approximation to π has led to many hunts for generalizations of (4), see [10, Chapter 1]. None so far are entirely successful.

3. **A computer discovery and a 'proof' of the series for** $\arcsin^2(x)$. We compute a few coefficients and observe that there is a regular power of 4 in the numerator, and integers in the denominator; or equivalently we look at $\arcsin(x/2)^2$. The generating function package 'gfun' in *Maple*, then predicts a recursion, r, for the denominators and solves it, as R.

```
>with(gfun):
>s:=[seq(1/coeff(series(arcsin(x/2)^2,x,25),x,2*n),n=1..6)]:
>R:=unapply(rsolve(op(1, listtorec(s,r(m))),r(m)),m);[seq(R(m),m=0..8)];
```

yields, $s := [4, 48, 360, 2240, 12600, 66528]$,

$$R := m \mapsto 8\,\frac{4^m\,\Gamma(3/2+m)(m+1)}{\pi^{1/2}\Gamma(1+m)},$$

where Γ is the Gamma function, and then returns the sequence of values

$$[4, 48, 360, 2240, 12600, 66528, 336336, 1647360, 7876440].$$

We may now use Sloane's *Online Encyclopedia of Integer Sequences*[26] to reveal that the coefficients are $R(n) = 2n^2\binom{2n}{n}$. More precisely, sequence A002544 identifies $R(n+1)/4 = \binom{2n+1}{n}(n+1)^2$.

```
> [seq(2*n^2*binomial(2*n,n),n=1..8)];
```

confirms this with

$$[4, 48, 360, 2240, 12600, 66528, 336336, 1647360].$$

Next we write

[26] At `www.research.att.com/~njas/sequences/index.html`

```
> S:=Sum((2*x)^(2*n)/(2*n^2*binomial(2*n,n)),n=1..infinity):S=values(S);
```

which returns

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{(2\,x)^{2n}}{n^2\binom{2n}{n}} = \arcsin^2(x).$$

That is, we have discovered—and proven if we trust or verify *Maple*'s summation algorithm—the desired Maclaurin series.

As prefigured by Ramanujan, it transpires that there is a beautiful closed form for $\arcsin^{2m}(x)$ for all $m = 1, 2, \ldots$. In [14] there is a discussion of the use of *integer relation methods*, [9, Chapter 6], to find this closed form and associated proofs are presented. ■

Here we see an admixture of all of the roles save #3, but above all #2 and #5.

4. Discovery without proof. Donald Knuth[27] asked for a closed form evaluation of:

$$\sum_{k=1}^{\infty}\left\{\frac{k^k}{k!\,e^k} - \frac{1}{\sqrt{2\,\pi\,k}}\right\} = -0.084069508727655\ldots. \tag{5}$$

Since about 2000 CE it has been easy to compute 20—or 200—digits of this sum in *Maple* or *Mathematica*; and then to use the 'smart lookup' facility in the *Inverse Symbolic Calculator*(ISC). The ISC at `http://oldweb.cecm.sfu.ca/projects/ISC` uses a variety of search algorithms and heuristics to predict what a number might actually be. Similar ideas are now implemented as 'identify' in *Maple* and (for algebraic numbers only) as 'Recognize' in *Mathematica*, and are described in [8, 9, 15, 1]. In this case it *rapidly* returns

$$0.084069508727655 \approx \frac{2}{3} + \frac{\zeta\,(1/2)}{\sqrt{2\,\pi}}.$$

We thus have a prediction which *Maple* 9.5 on a 2004 laptop *confirms* to 100 places in under 6 seconds and to 500 in 40 seconds. Arguably we are done. After all we were asked to *evaluate* the series and we now know a closed-form answer.

Notice also that the 'divergent' $\zeta(1/2)$ term is formally to be expected in that while $\sum_{n=1}^{\infty} 1/n^{1/2} = \infty$, the *analytic continuation* of $\zeta(s) := \sum_{n=1}^{\infty} 1/n^s$ for $s > 1$ evaluated at $1/2$ does occur! ■

We have discovered and tested the result and in so doing gained insight and knowledge while illustrating roles #1, #2 and #4. Moreover, as described in [10, pp. 15], one can also be led by the computer to a very satisfactory computer-assisted but also very human proof,

[27] Posed as an MAA Problem [36].

thus illustrating role #6. Indeed, the first hint is that the computer algebra system returned the value in (5) very quickly even though the series is very slowly convergent. This suggests the program is doing something intelligent—and it is! Such a use of computing is termed "instrumental" in that the computer is fundamental to the process, see [41].

5. **A striking conjecture with no known proof strategy** (as of spring 2007) given in [10, p. 162] is: for $n = 1, 2, 3 \cdots$

$$8^n \, \zeta\left(\{\overline{2}, 1\}_n\right) \stackrel{?}{=} \zeta\left(\{2, 1\}_n\right). \tag{6}$$

Explicitly, the first two cases are

$$8 \sum_{n>m>0} \frac{(-1)^n}{n^2 m} = \sum_{n>0} \frac{1}{n^3} \qquad \text{and} \qquad 64 \sum_{n>m>o>p>0} \frac{(-1)^{n+o}}{n^2 m \, o^2 p} = \sum_{n>m>0} \frac{1}{n^3 m^3}.$$

The notation should now be clear—we use the 'overbar' to denote an alternation. Such alternating sums are called *multi-zeta values* (MZV) and positive ones are called *Euler sums* after Euler who first studied them seriously. They arise naturally in a variety of modern fields from combinatorics to mathematical physics and knot theory.

There is abundant evidence amassed since 'identity' (6) was found in 1996. For example, very recently Petr Lisonek checked the first 85 cases to 1000 places in about 41 HP hours with only the *predicted round-off error*. And the case $n = 163$ was checked in about ten hours. These objects are very hard to compute naively and require substantial computation as a precursor to their analysis.

Formula (6) is the *only* identification of its type of an Euler sum with a distinct MZV and we have no idea why it is true. Any similar MZV proof has been both highly non-trivial and illuminating. To illustrate how far we are from proof: can just the case $n = 2$ be proven *symbolically* as has been the case for $n = 1$? ■

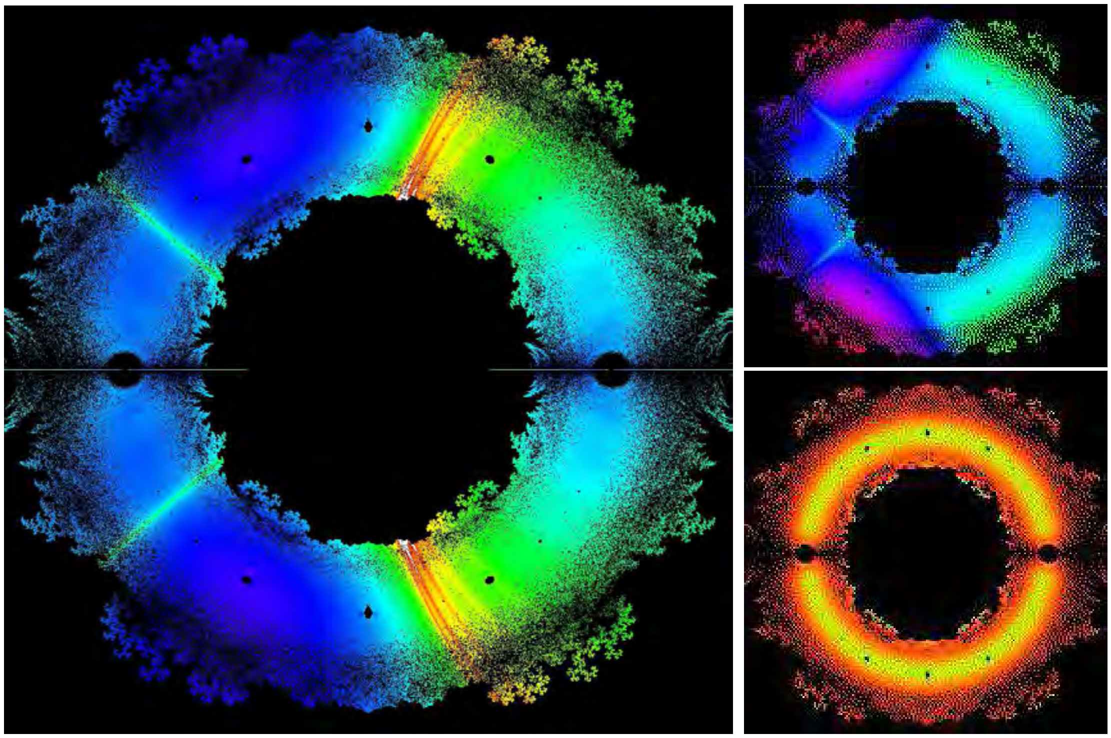


Figure 2. (Ex. 6.): "The price of metaphor is eternal vigilance."
(Arturo Rosenblueth & Norbert Wiener, [42])

This identity was discovered by the British quantum field theorist David Broadhurst and me during a large hunt for such objects in the mid-nineties. In this process we discovered and proved many lovely results (see [9, Chapter 2] and [10, Chapter 4]), thereby illustrating #1,#2, #4, #5 and #7. In the case of 'identity' (6) we have failed with #6, but we have ruled out many sterile approaches. It is one of many examples where we can now have (near) certainty without proof. Another was shown in equation (1) above.

6. **What you draw *is* what you see.** *Roots of polynomials with coefficients 1 or -1 up to degree 18.*

   As the quote suggests, pictures are highly metaphorical. The shading in Figure 2 is determined by a normalized sensitivity of the coefficients of the polynomials to slight variations around the values of the zeros with red indicating low sensitivity and violet indicating high sensitivity.[28] It is hard to see how the structure revealed in the pictures above[29] would be seen other than through graphically data-mining. Note the different shapes—now proven—of the holes around the various roots of unity.

   The striations are unexplained but all re-computations expose them! And the fractal structure is provably there. Nonetheless different ways of measuring the stability of

[28] Colour versions may be seen at `http://oldweb.cecm.sfu.ca/personal/loki/Projects/Roots/Book/`.

[29] We plot all complex zeroes of polynomials with only -1 and 1 as coefficients up to a given degree. As the degree increases some of the holes fill in—at different rates.

the calculations reveal somewhat different features. This is very much analogous to a chemist discovering an unexplained but robust spectral line. ■

This certainly illustrates #2 and #7, but also #1 and #3.

7. **Visual Dynamics.** In recent continued fraction work, Crandall and I needed to study the *dynamical system* $t_0 := t_1 := 1$:

$$t_n := \frac{1}{n}\, t_{n-1} + \omega_{n-1}\left(1 - \frac{1}{n}\right) t_{n-2},$$

where $\omega_n = a^2, b^2$ for n even, odd respectively, are two unit vectors. Think of this as a **black box** which we wish to examine scientifically. Numerically, all one *sees* is $t_n \to 0$ slowly. Pictorially, in Figure 3, we *learn* significantly more.[30] If the iterates are plotted with colour changing after every few hundred iterates,[31] it is clear that they spiral roman-candle like in to the origin:

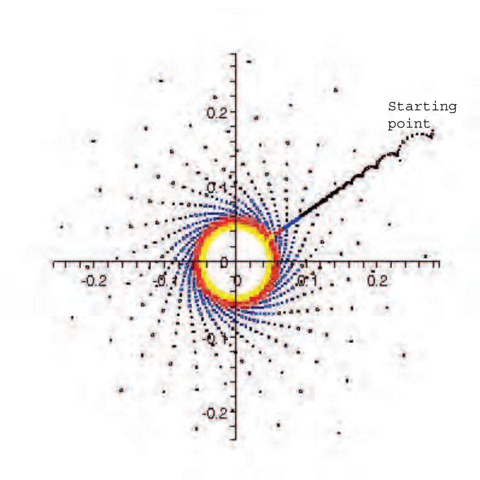


Figure 3. (Ex. 7.): "Visual convergence in the complex plane"

Scaling by $\sqrt{n}$, and distinguishing even and odd iterates, *fine structure* appear in Figure 4. We now observe, predict and validate that the outcomes depend on whether or not one or both of a and b are roots of unity (that is, rational multiples of π). Input a p-th root of unity and out come p spirals, input a non-root of unity and we see a circle. ■

This forceably illustrates role #2 but also roles #1, #3, #4. It took my coauthors and me, over a year and 100 pages to convert this intuition into a rigorous formal proof, [3]. Indeed, the results are technical and delicate enough that I have more faith in the facts than in the finished argument. In this sentiment, I am not entirely alone.

[30]... "Then felt I like a watcher of the skies, when a new planet swims into his ken." From John Keats (1795-1821) poem *On first looking into Chapman's Homer*.

[31]A colour version may be seen on the cover of [2].

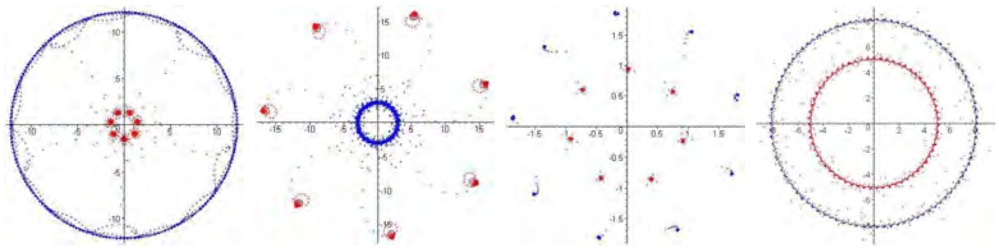


Figure 4. (Ex. 7.): The attractors for various $|a| = |b| = 1$

Carl Friedrich Gauss, who drew (carefully) and computed a great deal, is said to have noted, *I **have** the result, but I do not yet know how to get it.*[32] An excited young Gauss writes: "A new field of analysis has appeared to us, self-evidently, in the study of functions etc." (October 1798, reproduced in [9, Fig. 1.2, p.15]). It had and the consequent proofs pried open the doors of much modern elliptic function and number theory.

My penultimate and more comprehensive example is more sophisticated and I beg the less-expert analyst's indulgence. Please consider its structure and not the details.

8. **A full run.** Consider the *unsolved* **Problem 10738** from the 1999 *American Mathematical Monthly*, [10]:

   Problem: For $t > 0$ let

$$m_n(t) = \sum_{k=0}^{\infty} k^n \exp(-t)\, \frac{t^k}{k!}$$

   be the *nth* moment of a *Poisson distribution* with parameter t. Let $c_n(t) = m_n(t)/n!$. Show

   a) $\{m_n(t)\}_{n=0}^{\infty}$ is log-convex[33] for all $t > 0$.
   b) $\{c_n(t)\}_{n=0}^{\infty}$ is not log-concave for $t < 1$.
   c*) $\{c_n(t)\}_{n=0}^{\infty}$ is log-concave for $t \geq 1$.

   Solution. (a) Neglecting the factor of $\exp(-t)$ as we may, this reduces to

$$\sum_{k,j\geq 0} \frac{(jk)^{n+1} t^{k+j}}{k!j!} \leq \sum_{k,j\geq 0} \frac{(jk)^n t^{k+j}}{k!\, j!}\, k^2 = \sum_{k,j\geq 0} \frac{(jk)^n t^{k+j}}{k!j!}\, \frac{k^2 + j^2}{2},$$

   and this now follows from $2jk \leq k^2 + j^2$.

   (b) As

$$m_{n+1}(t) = t \sum_{k=0}^{\infty} (k+1)^n \exp(-t)\, \frac{t^k}{k!},$$

[32]Like so many attributions, the quote has so far escaped exact isolation!

[33]A sequence $\{a_n\}$ is *log-convex* if $a_{n+1}a_{n-1} \geq a_n^2$, for $n \geq 1$ and log-concave when the inequality is reversed.

on applying the binomial theorem to $(k+1)^n$, we see that $m_n(t)$ satisfies the recurrence

$$m_{n+1}(t) = t\sum_{k=0}^{n}\binom{n}{k}m_k(t), \qquad m_0(t) = 1.$$

In particular for $t = 1$, we computationally obtain as many terms of the sequence

$$1, 1, 2, 5, 15, 52, 203, 877, 4140\ldots$$

as we wish. These are the *Bell numbers* as was discovered again by consulting *Sloane's Encyclopedia* which can also tell us that, for $t = 2$, we have the *generalized Bell numbers*, and gives the exponential generating functions.[34] Inter alia, an explicit computation shows that

$$t\,\frac{1+t}{2} = c_0(t)\,c_2(t) \le c_1(t)^2 = t^2$$

exactly if $t \ge 1$, which completes (b).

Also, preparatory to the next part, a simple calculation shows that

$$\sum_{n\ge 0} c_n u^n = \exp\left(t(e^u - 1)\right). \tag{7}$$

(c*)[35] We appeal to a recent theorem, [10, p. 42], due to E. Rodney Canfield which proves the lovely and quite difficult result below. A self-contained proof would be very fine.

Theorem 1 *If a sequence* $1, b_1, b_2, \cdots$ *is non-negative and log-concave then so is the sequence* $1, c_1, c_2, \cdots$ *determined by the generating function equation*

$$\sum_{n\ge 0} c_n u^n = \exp\left(\sum_{j\ge 1} b_j \frac{u^j}{j}\right).$$

Using equation (7) above, we apply this to the sequence $\mathbf{b_j = t/(j-1)!}$ which is log-concave exactly for $t \ge 1$. ■

A search in 2001 on *MathSciNet* for "Bell numbers" since 1995 turned up 18 items. Canfield's paper showed up as number 10. Later, *Google* found it immediately!

Quite unusually, the given solution to (c) was the only one received by the *Monthly*. The reason might well be that it relied on the following sequence of steps:

[34]Bell numbers were known earlier to Ramanujan—an example of *Stigler's Law of Eponymy*, [10, p. 60]. Combinatorially they count the number of nonempty subsets of a finite set.

[35]The '*' indicates this was the unsolved component.

A **(Question Posed)** $\Rightarrow$ Computer Algebra System $\Rightarrow$ Interface $\Rightarrow$
Search Engine $\Rightarrow$ Digital Library $\Rightarrow$ Hard New Paper $\Rightarrow$ **(Answer)**

Without going into detail, we have visited most of the points elaborated in Section 4.1. Now if only we could already automate this process!

Jacques Hadamard, describes the role of proof as well as anyone—and most persuasively given that his 1896 proof of the Prime number theorem is an inarguable apex of rigorous analysis.

> *"The object of mathematical rigor is to sanction and legitimize the conquests of intuition, and there was never any other object for it."* (Jacques Hadamard[36])

Of the eight uses of computers instanced above, let me reiterate the central importance of heuristic methods for determining what is true and whether it merits proof. I tentatively offer the following surprising example which is very very likely to be true, offers no suggestion of a proof and indeed may have no reasonable proof.

9. **Conjecture.** *Consider*

$$x_n = \left\{ 16x_{n-1} + \frac{120n^2 - 89n + 16}{512n^4 - 1024n^3 + 712n^2 - 206n + 21} \right\} \tag{8}$$

The sequence $\beta_n = (\lfloor 16x_n \rfloor)$, where (x_n) is the sequence of iterates defined in equation (8), precisely generates the hexadecimal expansion of $\pi - 3$.

(Here $\{\cdot\}$ denotes the fractional part and $(\lfloor\cdot\rfloor)$ denotes the integer part.) In fact, we know from [9, Chapter 4] that the first million iterates are correct and in consequence:

$$\sum_{n=1}^{\infty} \|x_n - \{16^n \pi\}\| \leq 1.46 \times 10^{-8} \ldots . \tag{9}$$

where $\|a\| = \min(a, 1-a)$. By the first Borel-Cantelli lemma this shows that the hexadecimal expansion of π only finitely differs from (β_n). Heuristically, the probability of any error is very low. ■

6 Conclusions

To summarize, I do argue that reimposing the primacy of mathematical knowledge over proof is appropriate. So I return to the matter of what it takes to persuade an individual to adopt new methods and drop time honoured ones. Aptly, we may start by consulting Kuhn on the matter of paradigm shift:

[36] J. Hadamard, in E. Borel, Lecons sur la theorie des fonctions, 3rd ed. 1928, quoted in [49, (2), p. 127] . See also [47].

> *"The issue of paradigm choice can never be unequivocally settled by logic and experiment alone. ⋯ in these matters neither proof nor error is at issue. The transfer of allegiance from paradigm to paradigm is a conversion experience that cannot be forced."* (Thomas Kuhn[37])

As we have seen, the pragmatist philosopher John Dewey eloquently agrees, while Max Planck, [46], has also famously remarked on the difficulty of such paradigm shifts. This is Kuhn's version[38]:

> *"And Max Planck, surveying his own career in his Scientific Autobiography, sadly remarked that "a new scientific truth does not triumph by convincing its opponents and making them see the light, but rather because its opponents eventually die, and a new generation grows up that is familiar with it."* (Albert Einstein, [37, 46])

This transition is certainly already apparent. It is certainly rarer to find a mathematician under thirty who is unfamiliar with at least one of *Maple*, *Mathematica* or *MatLab*, than it is to one over sixty five who is really fluent. As such fluency becomes ubiquitous, I expect a re-balancing of our community's valuing of deductive proof over inductive knowledge.

In his famous lecture to the Paris International Congress in 1900, Hilbert writes[39]

> *"Moreover a mathematical problem should be difficult in order to entice us, yet not completely inaccessible, lest it mock our efforts. It should be to us a guidepost on the mazy path to hidden truths, and ultimately a reminder of our pleasure in the successful solution."* (David Hilbert, [56])

Note the primacy given by a most exacting researcher to discovery and to truth over proof and rigor. More controversially and most of a century later, Greg Chaitin invites us to be bolder and act more like physicists.

> *"I believe that elementary number theory and the rest of mathematics should be pursued more in the spirit of experimental science, and that you should be willing to adopt new principles...* And the Riemann Hypothesis isn't self-evident either, but it's very useful. *A physicist would say that there is ample experimental evidence for the Riemann Hypothesis and would go ahead and take it as a working assumption. ⋯ We may want to introduce it formally into our mathematical system."* (Greg Chaitin, [9, p. 254])

Ten years later:

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[37] In [50], *Who Got Einstein's Office?* The answer is Arne Beurling.

[38] Kuhn is quoting Einstein quoting Planck. There are various renderings of this second-hand German quotation.

[39] See the late Ben Yandell's fine account of the twenty-three "*Mathematische Probleme*" lecture, Hilbert Problems and their solvers, [56]. The written lecture (given in [56]) is considerably longer and further ranging that the one delivered in person.

> *"[Chaitin's] "Opinion" article proposes that the Riemann hypothesis (RH) be adopted as a new axiom for mathematics. Normally one could only countenance such a suggestion if one were assured that the RH was undecidable. However, a proof of undecidability is a logical impossibility in this case, since if RH is false it is provably false. Thus, the author contends, one may either wait for a proof, or disproof, of RH—both of which could be impossible—or one may take the bull by the horns and accept the RH as an axiom. He prefers this latter course as the more positive one."* (Roger Heath Brown[40])

Much as I admire the challenge of Greg Chaitin's statements, I am not yet convinced that it is helpful to add axioms as opposed to proving conditional results that start "Assuming the continuum hypothesis" or emphasize that "without assuming the Riemann hypothesis we are able to show ...". Most important is that we lay our cards on the table. We should explicitly and honestly indicate when we believe our tools to be heuristic, we should carefully indicate why we have confidence in our computations—and where our uncertainty lies— and the like.

On that note, Hardy is supposed to have commented—somewhat dismissively—that Landau, a great German number theorist, would never be the first to prove the Riemann Hypothesis, but that if someone else did so then Landau would have the best possible proof shortly after. I certainly hope that a more experimental methodology will better value independent replication and honour the first transparent proof[41] of Fermat's last theorem as much as Andrew Wiles' monumental proof. Hardy also commented that he did his best work past forty. Inductive, accretive, tool-assisted mathematics certainly allows brilliance to be supplemented by experience and—as in my case—stands to further undermine the notion that one necessarily does one's best mathematics young.

6.1 As for Education

The main consequence for me is that a *constructivist educational curriculum*—supported by both good technology and reliable content—is both possible and highly desirable. In a traditional instructivist mathematics classroom there are few opportunities for realistic discovery. The current sophistication of dynamic geometry software such as *Geometer's Sketchpad*, *Cabri* or *Cinderella*, of many fine web-interfaces, and of broad mathematical computation platforms like *Maple* and *Mathematica* has changed this greatly—though in my opinion both *Maple* and *Mathematica* are unsuitable until late in high-school, as they presume too much of both the student and the teacher. A thoughtful and detailed discussion of many of the central issues can be found in J.P. Lagrange's article [41] on teaching functions in such a milieu.

Another important lesson is that we need to teach procedural or *algorithmic thinking*. Although some vague notion of a computer program as a repeated procedure is probably

[40] Roger Heath-Brown's *Mathematical Review* of [18], 2004.

[41] Should such exist and as you prefer be discovered or invented.

ubiquitous today, this does not carry much water in practice. For example, five years or so ago, while teaching future elementary school teachers (in their final year), I introduced only one topic not in the text: extraction of roots by Newton's method. I taught this in class, tested it on an assignment and repeated it during the review period. About half of the students participated in both sessions. On the final exam, I asked the students to compute $\sqrt{3}$ using Newton's method starting at $x_0 = 3$ to estimate $\sqrt{3} = \underline{1.732}050808\ldots$ so that the first three digits after the decimal point were correct. I hoped to see $x_1 = 2$, $x_2 = 7/4$ and $x_3 = 97/56 = \underline{1.732}142857\ldots$. I gave the students the exact iteration in the form

$$x_{\text{NEW}} = \frac{x + 3/x_{\text{OLD}}}{2}, \tag{10}$$

and some other details. The half of the class that had been taught the method had no trouble with the question. The rest almost without exception "guessed and checked". They tried $x_{\text{OLD}} = 3$ and then rather randomly substituted many other values in (10). If they were lucky they found some x_{OLD} such that x_{NEW} did the job.

My own recent experiences with technology-mediated curriculum are described in Jen Chang's 2006 MPub, [19]. There is a concurrent commercial implementation of such a middle-school *Interactive School Mathematics* currently being completed by *MathResources*.[42] Many of the examples I have given, or similar ones more tailored to school [7], are easily introduced into the curriculum, but only if the teacher is not left alone to do so. Technology also allows the same teacher to provide enriched material (say, on fractions, binomials, irrationality, fractals or chaos) to the brightest in the class while allowing more practice for those still struggling with the basics. That said, successful mathematical education relies on active participation of the learner and the teacher and my own goal has been to produce technological resources to support not supplant this process; and I hope to make learning or teaching mathematics more rewarding and often more fun.

6.2 Last Words

To reprise, I hope to have made convincing arguments that the traditional deductive accounting of Mathematics is a largely ahistorical caricature—Euclid's millennial sway not withstanding.[43] Above all, mathematics is primarily about *secure knowledge* not proof, and that while the aesthetic is central, we must put much more emphasis on notions of supporting evidence and attend more closely to the reliability of witnesses.

Proofs are often out of reach—but understanding, even certainty, is not. Clearly, computer packages can make concepts more accessible. A short list includes linear relation algorithms, Galois theory, Groebner bases, etc. While progress is made "*one funeral at a time,*"[44] in Thomas Wolfe's words "*you can't go home again*" and as the co-inventor of the

[42] See `http://www.mathresources.com/products/ism/index.html`. I am a co-founder of this ten-year old company. Such a venture is very expensive and thus relies on commercial underpinning.

[43] Most of the cited quotations are stored at `jborwein/quotations.html`

[44] This grim version of Planck's comment is sometimes attributed to Niels Bohr but this seems specious. It is also spuriously attributed on the web to Michael Milken, and I imagine many others

Fast Fourier transform properly observed, in [53][45]

> *"Far better an approximate answer to the right question, which is often vague, than the exact answer to the wrong question, which can always be made precise."*
> (J. W. Tukey, 1962)

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bolically. I've been trading similar worksheets for this course with Jason Grout at Iowa State University and Robert Mařík at Mendel University in the Czech Republic, often by publishing worksheets off the Sage public server. Sage will see significant action as I finish an integral calculus course this term with infinite series and Taylor polynomials. Preparing an upcoming presentation will give me an excuse to learn more about Sage's graph theory routines.

Sage is big, and there is much to explore and to use in your professional activities as a mathematician. It is an impressive concentration and unification of mathematical knowledge. The reliance on mature open-source packages and open standards provides a measure of confidence and future-proofing. There are a few rough edges as the project matures, but this also provides the opportunity to get involved and influence development. But see for yourself by experimenting at the public server (sagenb.org) along with the over 5,000 others who have accounts there, or simply install your own copy on your favorite hardware. Either way, it's free.

Acknowledgments. This review has benefited greatly from the help of the Sage community, specifically Michael Abshoff, Robert Bradshaw, Craig Citro, Ahmed Fasih, Jason Grout, Mike Hansen, David Joyner, Josh Kantor, Nancy Neudauer, Harald Schilly, and William Stein. Their assistance is greatly appreciated.

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The Princeton Companion to Mathematics. *Edited by Timothy Gowers, with June Barrow-Green and Imre Leader.* Princeton University Press, Princeton, NJ, 2008. $99.00. xxii+1034 pp., hardcover. ISBN 978-0-691-11880-2.

Tim Gowers was a 1998 Fields medalist for his marvelous resolution of long-standing problems in Banach space theory—such as whether it is possible for a Banach space to have no isomorphic hyperplane (it is)—and in combinatorics; and while he continues such work, in exemplary fashion he has also found time for various more didactic and expository projects such as *Mathematics: A Very Short Introduction* (2002) and the book under review, activity with various media, and much else. Both associate editors, June Barrow-Green (Deputy Director for the Centre and Research Fellow in History of Mathematics at the Open University) and the combinatorist Imre Leader (Professor at Trinity College, Cambridge), have distinguished records.

This work, which I shall refer to below, as Gowers does, as "*The Companion*," is a fine validation of the well-known proposition that if you want a job done right you should ask a busy person to do it. In this case many very busy people have performed an invaluable job very, very well. This handsome, hefty, and attractively priced volume received Honorable Mention for the 2008 PROSE Award for Professional and Scholarly Excellence for Single Volume Reference/Science, Association of American Publishers. In his excellent preface Gowers describes the painstaking six-year process which led to this work and writes that "the central focus of this book is *modern*, *pure* mathematics," both highlighted terms being lucidly discussed. Since this review is appearing in *SIAM Review* I should emphasize that a great deal of less pure mathematics is captured. He also points out that a "companion is not an encyclopaedia" and that

> "[*t*]*he Princeton Companion to Mathematics* could be said to be about everything that Russell's definition [of pure Mathematics] leaves out."

Let me complete my review prematurely. Every research mathematician, every university student of mathematics, and every serious amateur of mathematical science should own a least one copy of *The Companion*. Indeed, the sheer weight of the volume suggests that it is advisable to own two: one for work and one at home. You may want to get a copy of *The Companion* for a friend. I bought a copy as an 85th birthday present for my mathematician father.

Reviews of *The Companion*, both professional and on Amazon.com (which also has a good selection of superlative comments extracted from professional reviews), have been generally laudatory, as indeed they should be. Princeton University Press also maintains a web site at http://press.princeton.edu/TOCs/c8350.html (whose scope and intention Gowers describes in his preface) from which the careful potential buyer can make a fully informed decision to purchase. Additionally, Princeton University Press provides the full table of contents, the preface, the list of contributors (a most impressive collection which includes mathematical household names such as Atiyah, Connes, Daubechies, Lax, and Tao, as well as many very distinguished authors most of whose names are probably not familiar to any given reader), and very representative sample articles consisting of I.2 "The Language and Grammar of Mathematics," II.2 "Geometry," IV.5 "Arithmetic Geometry," IV.21 "Numerical Analysis," V.10 "Fermat's Last Theorem," VI.61 "Jules Henri Poincaré (1854–1912)," VII.2 "Mathematical Biology," and VIII.6 "Advice to a Young Mathematician" (by Sir Michael Atiyah, Bela Bollobas, and others). From this list the reader of this review can already probably glean the structure of the book, which consists of eight parts.

Part I, "Introduction," contains "What Is Mathematics About?," "The Language and Grammar of Mathematics," "Some Fundamental Mathematical Definitions" (32 pages), and "The General Goals of Mathematical Research."

Part II, "The Origins of Modern Mathematics," has seven entries commencing with "From Numbers to Number Systems and Geometry" and culminating with "The Crisis in the Foundations of Mathematics."

Part III, "Mathematical Concepts," consists of 99 brief entries arranged alphabetically. These entries are typically between one and three pages. They start with "The Axiom of Choice" and visit topics such as "Calabi–Yau Manifolds," "Countable and Uncountable Sets," "Dynamical Systems and Chaos," "The Fast Fourier Transform," "Homology and Cohomology," "The Ising Model," "The Leech Lattice," "Matroids," "Number Fields," "Probability Distributions," "Quantum Computation," "Ricci Flow," and "Special Functions," before finishing up with "Von Neumann Algebras," "Wavelets," and a Joyce-like revisiting of the axioms of set theory with "The Zermelo–Fraenkel Axioms" (on page 314, which should please the pi lover).

Part IV, "Branches of Mathematics," occupies pages 315 through 680 and covers 26 topics including "Algebraic Numbers," 'Representation Theory," "Harmonic Analysis," "General Relativity and the Einstein Equations," "Enumerative and Algebraic Combinatorics," "Numerical Analysis," and "High-Dimensional Geometry and Its Probabilistic Analogues."

Part V, "Theorems and Problems," has 36 alphabetic entries of between one and three pages. It starts as it must with "The ABC Conjecture," and touches upon "The

Banach–Tarski Paradox," "Carleson's Theorem," "The Classification of Finite Simple Fermat's Last Theorem," "The Four-Color Theorem," "Gödel's Theorem," "The Insolubility of the Halting Problem," "Mostow's Strong Rigidity Theorem," "The P versus NP Problem," "The Poincaré Conjecture," "The Prime Number Theorem and the Riemann Hypothesis," "The Resolution of Singularities," "The Robertson–Seymour Theorem," and "The Weil Conjectures."

In Part VI, "Mathematicians" are arranged chronologically from Pythagoras (ca. 569 B.C.E.–ca. 494 B.C.E.) and Euclid (ca. 325 B.C.E.–ca. 265 B.C.E.) through Abu Ja'far Muhammad ibn Musa al-Khwarizmi (800–847), Leonardo of Pisa (known as Fibonacci) (ca. 1170–ca. 1250), François Viète (1540–1603), Pierre Fermat (160?–1665), the Bernoullis (fl. 18th century), Leonhard Euler (1707–1783), Jean-Baptiste Joseph Fourier (1768–1830), Carl Friedrich Gauss (1777–1855), Nicolai Ivanovich Lobachevskii (1792–1856), William Rowan Hamilton (1805–1865), Eduard Kummer (1810–1893), James Joseph Sylvester (1814–1897), William Burnside (1852–1927), Jacques Hadamard (1865–1963), Emmy Noether (1882–1935), Norbert Wiener (1894–1964), William Vallance Douglas Hodge (1903–1975), Abraham Robinson (1918–1974), and, finally, as the 96th entry and the only living member of the list, Nicolas Bourbaki (1935–). If your mathematical hero is missing above, that is likely to be because of my selection, not the editors' oversight. Indeed, when I proofread this review I wondered why my own favorite G.H. Hardy was not listed above, but in fact he was indeed included.

Part VII, "The Influence of Mathematics," has fourteen entries, each of approximately ten pages, including "Mathematics and Chemistry," "The Mathematics of Traffic in Networks," "Mathematics and Economic Reasoning," "Mathematics and Medical Statistics," and "Mathematics and Art."

Part VIII, "Final Perspectives," comprises seven essays, each between about five and ten pages in length, entitled "The Art of Problem Solving," "Why Mathematics? You Might Ask," "The Ubiquity of Mathematics," "Numeracy," "Mathematics: An Experimental Science," "Advice to a Young Mathematician" (perhaps in homage to Peter Medawar's wonderful 1979 *Advice to a Young Scientist*), and "A Chronology of Mathematical Events."

The sheer scale and scope of the book, which finishes with a very good index, should now be fully apparent. In my 2006 featured review in *SIAM Review* of *The Oxford Users' Guide to Mathematics* (*SIAM Rev.*, 48 (2006), pp. 585–594) I wrote generally of the issues involved with such projects and, despite great sympathy, I found much to be critical of with regards to the roles of both its editors and its publisher. Indeed, I wrote:

> A more thorough review and production process would surely have adequately addressed this last set of issues. I can no better make this point than to quote Simkin and Fiske quoting [in *Science*] the late Stephen J. Gould in a review of Simon Winchester's Krakatoa. (... These reviews do make me question the reliability of *The Professor and the Madman*.)
>
>> In his review of Winchester's previous book, *The Map That Changed the World*, Stephen Jay Gould wrote: "I don't mean to sound like an academic sourpuss, but I just don't understand the priorities of publishers who spare no expense to produce an elegantly illustrated and beautifully designed book and then permit the text to wallow in simple, straight-out factual errors, all easily corrected for the minimal cost of one scrutiny of the galleys by a reader with professional expertise...."
>>
>> With *Krakatoa*, the publisher clearly spared considerable expense, and this new book also wallows in errors. Perhaps, given our popular culture's appetite for sensationalized disasters, a modern publisher would rather not see all those pesky details corrected.

Even an academic sourpuss should be pleased with the attention to detail of *The Companion*'s publishers, editors, and authors and with many judicious decisions—about the level of exposition, level of detail, what to include and what to omit, and much more—which have led to a well-integrated and highly readable volume. Gowers writes:

> [T]he editorial process has been a very active one: we have not just commissioned the articles and accepted whatever we have been sent. Some drafts have had to be completely discarded and new articles written in the light of editorial comments. Others have needed substantial changes, which have sometimes been made by the authors and sometimes by the editors. A few articles were accepted with only trivial changes, but these were a very small minority.

I described in my 2006 review how hard it is to produce such a volume—let alone to do it so splendidly—and how easy it is to find fault in any project with such audacious goals. This I know full well from my own more prosaic efforts as a co-author of *The Collins–Smithsonian Dictionary of Mathematics*. Thus, in an attempt to limit bias, I left a copy in my office for several months and sampled it with students or colleagues who dropped in for a chat or with a query. I found little missing. Indeed, the only item I did not find during this process but thought I should have found was "Turing test," and that term is perhaps not fairly within the compass of modern pure mathematics. I finish by quoting again from Gowers' own preface.

> **6 Who Is The Companion Aimed At?**
>
> The original plan for *The Companion* was that all of it should be accessible to anybody with a good background in high school mathematics (including calculus). However, it soon became apparent that this was an unrealistic aim: there are branches of mathematics that are so much easier to understand when one knows at least some university-level mathematics that it does not make good sense to attempt to explain them at a lower level. On the other hand, there are other parts of the subject that decidedly can be explained to readers without this extra experience. So in the end we abandoned the idea that the book should have a uniform level of difficulty.
>
> Accessibility has, however, remained one of our highest priorities, and throughout the book we have tried to discuss mathematical ideas at the lowest level that is practical. *In particular, the editors have tried very hard not to allow any material into the book that they do not themselves understand, which has turned out to be a very serious constraint* [my emphasis]. Some readers will find some articles too hard and other readers will find other articles too easy, but we hope that all readers from advanced high school level onwards will find that they enjoy a substantial proportion of the book.
>
> What can readers of different levels hope to get out of *The Companion*? If you have embarked on a university level mathematics course, you may find that you are presented with a great deal of difficult and unfamiliar material without having much idea why it is important and where it is all going. Then you can use *The Companion* to provide yourself with some perspective on the subject. (For example, many more people know what a ring is than can give a good reason for caring about rings. But there are very good reasons, which you can read about in RINGS, IDEALS, AND MODULES [III.81] and ALGEBRAIC NUMBERS [IV.1].)
>
> If you are coming to the end of the course, you may be interested in doing research in mathematics. But undergraduate courses typically give you very little idea of what research is actually like. So how do you decide which areas of mathematics truly interest you at the research level? It is not easy, but the decision can make the

> difference between becoming disillusioned and ultimately not getting a Ph.D., and going on to a successful career in mathematics. This book, especially Part IV, tells you what mathematicians of many different kinds are thinking about at the research level, and may help you to make a more informed decision.
>
> If you are already an established research mathematician, then your main use for this book will probably be to understand better what your colleagues are up to. Most non-mathematicians are very surprised to learn how extraordinarily specialized mathematics has become. Nowadays it is not uncommon for a very good mathematician to be completely unable to understand the papers of another mathematician, even from an area that appears to be quite close. This is not a healthy state of affairs: anything that can be done to improve the level of communication among mathematicians is a good idea. The editors of this book have learned a huge amount from reading the articles carefully, and we hope that many others will avail themselves of the same opportunity.

Judging by the sales numbers shown on Amazon.com, however they are actually computed, a great many copies are already with readers, if perhaps not all yet read. Everyone involved with this project deserves our deep gratitude.

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Simulation and Inference for Stochastic Differential Equations. *By Stefano M. Iacus*. Springer, New York, 2008. $79.95. xviii+286 pp., hardcover. ISBN 978-0-387-75838-1.

I jumped at the chance to review this book. It deals with two themes that deserve a much higher profile in applied and computational mathematics: *uncertainty* and *inference*. You don't need to delve into stochastic models in order to appreciate their importance. A typical deterministic model will involve initial data and physical coefficients that are either

(a) completely unknown, or

(b) known only up to some level of error.

Case (a) would arise, for example, where one or more rate constants in a chemical reaction system could not be measured. In this context, the unknown parameters could be fitted to time series data relating to the observed concentrations of various species. A standard approach in computational and applied mathematics is to treat this as an optimization problem and seek the parameter values that best fit the data, for example, in a least squares sense subject to some realistic constraints. However, the resulting "point estimate" would be frowned upon by many experts in statistical inference [2], who would argue, quite reasonably, that returning just a single number (or even a single number plus some sort of local sensitivity estimate) is an inadequate summary, with the language and tools of probability theory providing a more appropriate setting.

Case (b) could of course arise after observed data has been used to deal with case (a). It may also arise when coefficients can be observed directly, but the measurements are subject to experimental errors. In either circumstance, it seems illogical to focus all our energies on theoretical or numerical analysis of a single "best guess" of the underlying problem specification. Instead, we should deal with questions such as: Given a quantitative representation of the uncertainty in the model, can we find a quantitative representation of the uncertainty in the output? Of course, analyzing or computing the solution for a fixed instance of the model will be an important subproblem. But the bigger picture, which sits at the intersection between statistics, probability, computer science, and applied

Exploratory Experimentation: Digitally-Assisted Discovery and Proof

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June 7, 2009

Abstract

I believe that the mathematical community (appropriately defined) is facing a great challenge to re-evaluate the role of proof in light of the power of current computer systems, of modern mathematical computing packages and of the growing capacity to data-mine on the internet. Add to that the enormous complexity of many modern mathematical results such as the Poincaré conjecture, Fermat's last theorem, and the classification of finite simple groups. With great challenges come great opportunities. I intend to touch upon the current challenges and opportunities for the learning and doing of mathematics. As the prospects for inductive mathematics blossom, the need to ensure that the role of proof is properly founded remains undiminished.

1 Digitally-assisted Discovery and Proof

Exploratory Experimentation

> *"[I]ntuition comes to us much earlier and with much less outside influence than formal arguments which we cannot really understand unless we have reached a relatively high level of logical experience and sophistication.*
>
> *Therefore, I think that in teaching high school age youngsters we should emphasize intuitive insight more than, and long before, deductive reasoning."* — George Polya (1887-1985) [22, 2 p. 128]

I share this view with Polya who goes on to say that, nonetheless, proof should certainly be taught in school. I begin with some observations many of which have been fleshed out in *The Computer as Crucible* [9], *Mathematics by Experiment* [7], and *Experimental Mathematics in Action* [4]. My musings focus on the changing nature of mathematical knowledge and in consequence asks the questions such as "How do we come to believe and trust pieces of mathematics?", "Why do we wish to prove things?" and "How do we teach what and why to students?"

While I have described myself in [4] and elsewhere as a "computationally assisted fallibilist", I am far from a social-constructivist. Like Richard Brown, I believe that Science "*at least* attempts *to faithfully represent reality*" [10, p. 7]. I am, though, persuaded by various notions of embodied cognition. Smail [26, p. 113] writes:

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> *"[T]he large human brain evolved over the past 1.7 million years to allow individuals to negotiate the growing complexities posed by human social living."*

In consequence we find various modes of argument more palatable than others, and are more prone to make certain kinds of errors than others. Likewise, Steve Pinker's observation about language [21, p. 83] as founded on

> *"...the ethereal notions of space, time, causation, possession, and goals that appear to make up a language of thought."*

remain equally potent within mathematics. The computer offers to provide scaffolding both to enhance mathematical reasoning and to restrain mathematical error.

To begin with let me briefly reprise what I mean by discovery, and by proof. The following attractive definition of *discovery* has the satisfactory consequence that a student can certainly discovery results whether known to the teacher or not. Nor is it necessary to demand that each dissertation be original (only independently discovered):

> *"In short, discovering a truth is coming to believe it in an independent, reliable, and rational way"*—Marcus Giaquinto [13, p. 50]

A standard definition[1] of *proof* follows.

> **PROOF**, n. a sequence of statements, each of which is either validly derived from those preceding it or is an axiom or assumption, and the final member of which, the conclusion, is the statement of which the truth is thereby established.

As a working definition of mathematics itself, I offer the following in which the word *proof* does not enter. Nor should it; mathematics is much more than proof alone:

> **MATHEMATICS**, n. a group of subjects, including algebra, geometry, trigonometry and calculus, concerned with number, quantity, shape, and space, and their inter-relationships, applications, generalizations and abstractions.
>
> **DEDUCTION**, n. 1. the process of reasoning typical of mathematics and logic, in which a conclusion follows necessarily from given premises so that it cannot be false when the premises are true.
>
> **INDUCTION**, n. 3. (Logic) a process of reasoning in which a general conclusion is drawn from a set of particular premises, often drawn from experience or from experimental evidence. The conclusion goes beyond the information contained in the premises and does not follow necessarily from them. Thus an inductive argument may be highly probable yet lead to a false conclusion; for example, large numbers of sightings at widely varying times and places provide very strong grounds for the falsehood that all swans are white.

It awaited the discovery of Australia to confound the seemingly compelling inductive conclusion that all swans are white. I observe that we typically take for granted the distinction between *induction* and *deduction* and rarely discuss their roles with either our colleagues or our students.

Despite the conventional identification of Mathematics with deductive reasoning, in his 1951 Gibbs Lecture Kurt Gödel (1906-1978) said:

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[1] All definitions below are taken from the *Collin's Dictionary of Mathematics* which I co-authored. It is available as software with student *Maple* inside at:
`http://www.mathresources.com/products/mathresource/index.html`.

> *"If mathematics describes an objective world just like physics, there is no reason why inductive methods should not be applied in mathematics just the same as in physics."*

He held this view until the end of his life despite the epochal deductive achievement of his incompleteness results. And this opinion has been echoed or amplified by logicians as different as Willard Quine and Greg Chaitin. More generally, one discovers a substantial number of great mathematicians from Archimedes and Galileo—who apparently said *"All truths are easy to understand once they are discovered; the point is to discover them."*—to Poincaré and Carleson have emphasized how much it helps to "know" the answer. Over two millennia ago Archimedes wrote to Eratosthenes in the introduction to his long-lost and recently re-constituted *Method of Mechanical Theorems* [19]

> *"I will send you the proofs of the theorems in this book. Since, as I said, I know that you are diligent, an excellent teacher of philosophy, and greatly interested in any mathematical investigations that may come your way, I thought it might be appropriate to write down and set forth for you in this same book a certain special method, by means of which you will be enabled to recognize certain mathematical questions with the aid of mechanics. I am convinced that this is no less useful for finding proofs of these same theorems.*
> *For some things, which first became clear to me by the mechanical method, were afterwards proved geometrically, because their investigation by the said method does not furnish an actual demonstration.* For it is easier to supply the proof when we have previously acquired, by the method, some knowledge of the questions than it is to find it without any previous knowledge.*"*

Think of the *Method* as an ur-precursor to today's interactive geometry software—with the caveat that, for example, *Cinderella* actually does provide certificates for much Euclidean geometry. As 2006 Abel Prize winner Leonard Carleson describes in his 1966 ICM speech on his positive resolution of Luzin's 1913 conjecture, about the pointwise convergence of Fourier series for square-summable functions, after many years of seeking a counter-example he decided none could exist. The importance of this confidence is expressed as follows:

> *"The most important aspect in solving a mathematical problem is the conviction of what is the true result. Then it took 2 or 3 years using the techniques that had been developed during the past 20 years or so.*

Digitally Mediated Mathematics I shall now assume that all proofs discussed are "non-trivial" in some fashion appropriate to the level of the material—since the issue of using inductive methods is really only of interest with this caveat. Armed with these terms, it remains to say that by *digital assistance* I intend the use of such *artefacts* as

- *Modern Mathematical Computer Packages*—be they Symbolic, Numeric, Geometric, or Graphical. I would capture all as "modern hybrid workspaces". One should also envisage much more use of stereo visualization, *haptics*[2], and auditory devices.

- *More Specialist Packages* or *General Purpose Languages* such as Fortran, C++, CPLEX, GAP, PARI, SnapPea, Graffiti, and MAGMA. The story of the *SIAM 100-Digits Challenge* [6] illustrates the degree to which mathematicians now start

[2]With the growing realization of the importance of gesture in mathematics *"as the very texture of thinking,"* [25, p. 92] it is time to seriously explore tactile devices.

computational work within a hybrid platform such as *Maple*, *Mathematica* or MATLAB and make only sparing recourse to more specialist packages when the hybrid work spaces prove too limited.

- *Web Applications* such as: Sloane's Online Encyclopedia of Integer Sequences, the Inverse Symbolic Calculator, Fractal Explorer, Jeff Weeks' Topological Games, or Euclid in Java.[3]

- *Web Databases* including Google, MathSciNet, ArXiv, JSTOR, Wikipedia, MathWorld, Planet Math, Digital Library of Mathematical Functions (DLMF), MacTutor, Amazon, and many more sources that are not always viewed as part of the palette. Nor is necessary that one approve unreservedly, say of the historical reliability of MacTutor, to acknowledge that with appropriate discrimination in its use it is a very fine resource.

All entail *data-mining* in various forms. Franklin [12] argues that what Steinle has termed *"exploratory experimentation"* facilitated by *"widening technology"* as in pharmacology, astrophysics, biotechnology is leading to a reassessment of what is viewed as a legitimate experiment; in that a "*local model*" is not a prerequisite for a legitimate experiment. Hendrik Sørenson [27] cogently makes the case that *experimental mathematics*—as 'defined' below—is following similar tracks.

> *"These aspects of exploratory experimentation and wide instrumentation originate from the philosophy of (natural) science and have not been much developed in the context of experimental mathematics. However, I claim that e.g. the importance of wide instrumentation for an exploratory approach to experiments that includes concept formation also pertain to mathematics."*

Danny Hillis is quoted as saying recently that:

> *"Knowing things is very 20th century. You just need to be able to find things."*

on how *Google* has already changed how we think.[4] This is clearly not yet true and will never be, yet it catches something of the changing nature of cognitive style in the 21st century.

In consequence, the boundaries between mathematics and the natural sciences and between inductive and deductive reasoning are blurred and getting blurrier. This is discussed at some length by Jeremy Avigad [1]. A very useful discussion of similar issues from a more explicitly pedagogical perspective is given by de Villiers [11] who also provides a quite extensive bibliography.

Experimental Mathodology We started *The Computer as Crucible* [9] with then United States Supreme court Justice Potter Stewart's famous if somewhat dangerous 1964 comment on pornography:

> *"I know it when I see it."*

[3] A cross-section of such resources is available through `http://ddrive.cs.dal.ca/~isc/portal/`.

[4] In Achenblog `http://blog.washingtonpost.com/achenblog/` of July 1 2008. Likewise, Chris Anderson, the Editor-in-Chief of *Wired*, recently wrote "There's no reason to cling to our old ways. It's time to ask: What can science learn from Google?" in a provocative article *The end of Theory*, see `http://www.wired.com/science/discoveries/magazine/16-07/pb_theory`.

I complete this section by reprising from [7] what somewhat less informally we mean by *experimental mathematics*. I say 'somewhat' since I do not take up the perhaps vexing philosophical question of whether a true *experiment* in mathematics is even possible—without adopting a fully realist philosophy of mathematics—or if we should be better to refer to 'quasi-experiments'? Some of this is discussed in [4, Chapter 1] and [7, Chapters 1,2, and 8], wherein further limn the various ways in which the term 'experiment' is used and underline the need for mathematical experiments with predictive power.

What is experimental mathematics?

1. Gaining insight and *intuition*.[5]
2. *Discovering* new relationships.[6]
3. *Visualizing* math principles.[7]
4. *Testing* and especially *falsifying* conjectures.[8]
5. *Exploring* a possible result to see if it *merits* formal proof.[9]
6. *Suggesting* approaches for *formal proof*.[10]
7. *Computing* replacing lengthy hand derivations.[11]
8. *Confirming* analytically derived results.[12]

Of these the first five play a central role in the current context, and the sixth plays a significant one.

Cognitive Challenges Finally let me touch upon the *Stroop effect*[13] illustrating *directed attention* or *interference*. This classic cognitive psychology test, discovered by John Ridley Stroop in 1935, is as follows. Consider the picture in Figure 1 in which various coloured words are colored in one of the colours mentioned but not necessarily in the same one.

[5] I firmly believe that—in most important senses—intuition, far from being "*knowledge or belief obtained neither by reason nor by perception,*" as the Collin's English Dictionary and Kant would have it, is acquired not innate as is captured by Lewis Wolpert's 2000 title *The Unnatural Nature of Science*, see also [14].

[6] I use discovery in Giaquinto's terms as quoted above.

[7] I intend the fourth Random House sense of "*to make perceptible to the mind or imagination*" not just Giaquinto's more direct meaning.

[8] Karl Popper's "critical rationalism" asserts that induction can never lead to truth and hence that one can only falsify theories [10]. Whether one believes this is the slippery slope to *Post modernist interpretations of science* (Brown's term abbreviated *PIS*) or not is open to debate, but Mathematics, being based largely on deductive science, has little to fear and much to gain from more aggressive use of falsification.

[9] 'Merit' is context dependent. It may mean one thing in a classroom and quite another for a research mathematician.

[10] I refer to computer-assisted or computer-directed proof which is quite far from *Formal Proof*—the topic of a special issue of the *Notices of the AMS* in December 2008.

[11] Hales' recent solution of the *Kepler problem*, described in the 2008 *Notices* article pushes the boundary on when 'replacement' becomes qualitatively different from say factoring a very large prime. Independently,

[12] The a posteriori value of confirmation is huge, whether this be in checking answers while preparing a calculus class, or in confirming one's apprehension of a newly acquired fact.

[13] `http://www.snre.umich.edu/eplab/demos/st0/stroopdesc.html` has a fine overview.

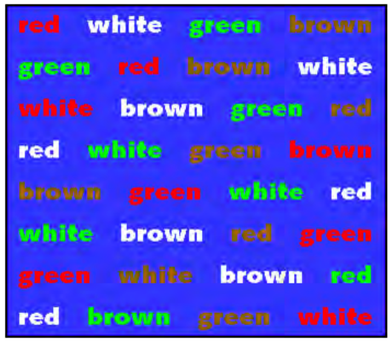

Figure 1: An Illustration of the Stroop test.

First, say the **colour** which the given **word mentions**.

Second, say the **colour** in which the **word is written**.

Most people find the second the harder. You may find yourself taking more time for each word, and may frequently say the word, rather than the color in which the word appears. Proficient (young) multitaskers find it easy to suppress information and so perform the second faster than traditionally. Indeed, Cliff Nass' work in the CHIME lab at Stanford suggests that neurological changes are taking place amongst the 'born-digital.'[14] If such cognitive changes are taking place there is even more reason to ensure that epistemology, pedagogy, and cognitive science are in concert.

Paradigm Shifts

> *"Old ideas give way slowly; for they are more than abstract logical forms and categories. They are habits, predispositions, deeply engrained attitudes of aversion and preference. Moreover, the conviction persists-though history shows it to be a hallucination that all the questions that the human mind has asked are questions that can be answered in terms of the alternatives that the questions themselves present. But in fact intellectual progress usually occurs through sheer abandonment of questions together with both of the alternatives they assume an abandonment that results from their decreasing vitality and a change of urgent interest. We do not solve them: we get over them.*
> *Old questions are solved by disappearing, evaporating, while new questions corresponding to the changed attitude of endeavor and preference take their place. Doubtless the greatest dissolvent in contemporary thought of old questions, the greatest precipitant of new methods, new intentions, new problems, is the one effected by the scientific revolution that found its climax in the "Origin of Species." "*—John Dewey (1859-1952)[15]

Thomas Kuhn (1922-1996) has noted that a true *paradigm shift*—as opposed to the cliché—is "a conversion experience."[16] You (and enough others) either have one or you

[14]See `http://www.snre.umich.edu/eplab/demos/st0/stroop_program/stroopgraphicnonshockwave.gif`.

[15]In Dewey's introduction to his 1910 *The Influence of Darwin on Philosophy and Other Essays.* Dewey, a leading pragmatist (or instrumentalist) philosopher and educational thinker of his period is also largely responsible for the Trotsky archives being at Harvard through his activities on the *Dewey Commission.*

[16]This was said in an interview in [23], not only in Kuhn's 1962 *The Structure of Scientific Revolutions* which Brown notes is *"the single most influential work in the history of science in the twentieth century."* In Brown's accounting [10] he bears more responsibility for the slide into PIS than either Dewey or Popper. An unpremeditated example of digitally assisted research is that—as I type—I am listening to *The Structure of Scientific Revolutions*, having last read it 35 years ago.

don't. Oliver Heaviside (1850-1925) said in defending his operator calculus before it could be properly justified: *"Why should I refuse a good dinner simply because I don't understand the digestive processes involved?* But please always remember as Arturo Rosenblueth and Norbert Wiener wrote: *"The price of metaphor is eternal vigilance."* [17] I may not convince you to reevaluate your view of Mathematics as an entirely deductive science—if so indeed you view it—but in the next section I will give it my best shot.

2 Mathematical Examples

I continue with various explicit examples. I leave it to the reader to decide how much or how frequently he or she wishes to exploit the processes I advertise. Nonetheless they all controvert Picasso's *"Computers are useless they can only give answers."* and confirm Hamming's *"The purpose of computing is insight not numbers."* As a warm-up illustration, consider Figure 2. The lower function in both graphs is $x \mapsto -x^2 \log x$ The left-hand graph compares $x \mapsto x - x^2$ while the right-hand graph compares $x \mapsto x^2 - x^4$ each on $0 \le x \le 1$.

Before the advent of plotting calculators if asked a question like *"Is $-x^2 \log x$ less than $x - x^2$ on the open interval between zero and one?"* one immediately had recourse to the calculus. Now that would be silly, clearly they cross. In the other case, if there is a problem it is at the right-hand end point. 'Zooming' will probably persuade you that $-x^2 \log x \le x^2 - x^4$ on $0 \le x \le 1$ and may even guide a calculus proof if a proof is needed.

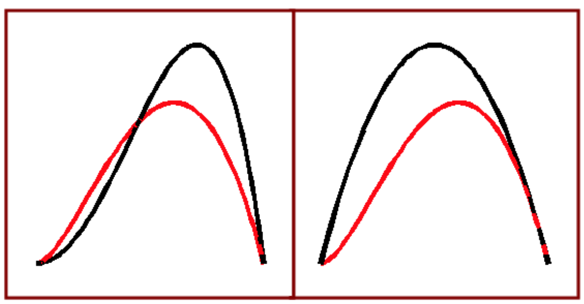

Figure 2: Try Visualization or Calculus First?

The examples below contain material on sequences, generating functions, special functions, continued fractions, partial fractions, definite and indefinite integrals, finite and infinite sums, combinatorics and algebra, matrix theory, dynamic geometry and recursions, differential equations, mathematical physics, among other things. So they capture the three main divisions of pure mathematical thinking: algebraic-symbolic, analytic, and topologic-geometric, while making contact with more applied issues in computation, numerical analysis and the like.

Example I: What Did the Computer Do?

> *"This computer, although assigned to me, was being used on board the International Space Station. I was informed that it was tossed overboard to be burned up in the atmosphere when it failed."*—anonymous NASA employee[18]

[17] Quoted by R. C. Leowontin, in *Science* p. 1264, Feb 16, 2001 (the *Human Genome Issue*).

[18] *Science*, August 3, 2007, p. 579: *"documenting equipment losses of more than $94 million over the past 10 years by the agency."*

In my own work computer experimentation and digitally-mediated research now invariably play a crucial part. Even in many seemingly non-computational areas of functional analysis and the like there is frequently a computable consequence whose verification provides confidence in the result under development. Moreover, the process of specifying my questions enough to program with them invariably enhances my understanding and sometimes renders the actual computer nearly superfluous. For example, in a recent study of expectation or "box integrals" [5] we were able to evaluate a quantity, which had defeated us for years, namely

$$\mathcal{K}_1 := \int_3^4 \frac{\operatorname{arcsec}(x)}{\sqrt{x^2-4\,x+3}}dx$$

in *closed-form* as

$$\begin{aligned}\mathcal{K}_1 &= \operatorname{Cl}_2(\theta) - \operatorname{Cl}_2\left(\theta+\frac{\pi}{3}\right) - \operatorname{Cl}_2\left(\theta-\frac{\pi}{2}\right) + \operatorname{Cl}_2\left(\theta-\frac{\pi}{6}\right) - \operatorname{Cl}_2\left(3\,\theta+\frac{\pi}{3}\right) \\ &+ \operatorname{Cl}_2\left(3\,\theta+\frac{2\pi}{3}\right) - \operatorname{Cl}_2\left(3\,\theta-\frac{5\pi}{6}\right) + \operatorname{Cl}_2\left(3\,\theta+\frac{5\pi}{6}\right) + \left(6\,\theta-\frac{5\,\pi}{2}\right)\log\left(2-\sqrt{3}\right).\end{aligned} \tag{1}$$

where $\operatorname{Cl}_2(\theta) := \sum_{n=1}^{\infty} \sin(n\theta)/n^2$ is the *Clausen function*, and $3\,\theta := \arctan\left(\frac{16-3\sqrt{15}}{11}\right)+\pi$.

Along the way to the evaluation above, after exploiting some insightful work by George Lamb, there were several stages of symbolic computation, at times involving an expression for $\mathcal{K}_1$ with over 28,000 characters (perhaps 25 standard novel pages). It may well be that the closed form in (1) can be further simplified. In any event, the very satisfying process of distilling the computer's 28,000 character discovery, required a mixture of art and technology and I would be hard pressed to assert categorically whether it constituted a conventional proof. Nonetheless, it is correct and has been checked numerically to over a thousand-digit decimal precision.

I turn next to a mathematical example which I hope will reinforce my assertion that there is already an enormous amount to be mined mathematically on the internet. And this is before any mathematical character recognition tools have been made generally available and when it is still very hard to search mathematics on the web.

Example II: What is That Number?

> *"The dictum that everything that people do is 'cultural' ... licenses the idea that every cultural critic can meaningfully analyze even the most intricate accomplishments of art and science. ... It is distinctly weird to listen to pronouncements on the nature of mathematics from the lips of someone who cannot tell you what a complex number is!"*—Norman Levitt[19]

In 1995 or so Andrew Granville emailed me the number

$$\alpha := 1.4331274267223\ldots \tag{2}$$

and challenged me to identify it; I think this was a test I could have failed. I asked *Maple* for its continued fraction. In the conventional concise notation I was rewarded with

$$\alpha = [1,2,3,4,5,6,7,8,9,10,11,...]. \tag{3}$$

Even if you are unfamiliar with continued fractions, you will agree that the changed representation in (3) has exposed structure not apparent from (2)! I reached for a good book on continued fractions and found the answer

$$\alpha = \frac{I_1(2)}{I_0(2)} \tag{4}$$

[19] In *The flight From Science and Reason.* See *Science*, Oct. 11, 1996, p. 183.

where I_0 and I_1 are *Bessel functions* of the first kind. Actually I remembered that all arithmetic continued fractions arise in such fashion, but as we shall see one now does not need to.

In 2009 there are at least three “zero-knowledge” strategies:

1. Given (3), type “arithmetic progression”, “continued fraction” into *Google*.

2. Type “$1, 4, 3, 3, 1, 2, 7, 4, 2$” into *Sloane’s Encyclopedia of Integer Sequences*.[20]

3. Type the decimal digits of α into the *Inverse Symbolic Calculator*.[21]

I illustrate the results of each strategy.

1. On October 15, 2008, on typing “arithmetic progression”, “continued fraction” into *Google*, the first three hits were those shown in Figure 3. Moreover, the MathWorld entry tells us that any arithmetic continued fraction is of a ratio of Bessel functions, as shown in the inset to Figure 3 which also refers to the second hit in Figure 3. The reader may wish to see what other natural search terms uncover (4)—perhaps in the newly unveiled *Wolfram Alpha*.

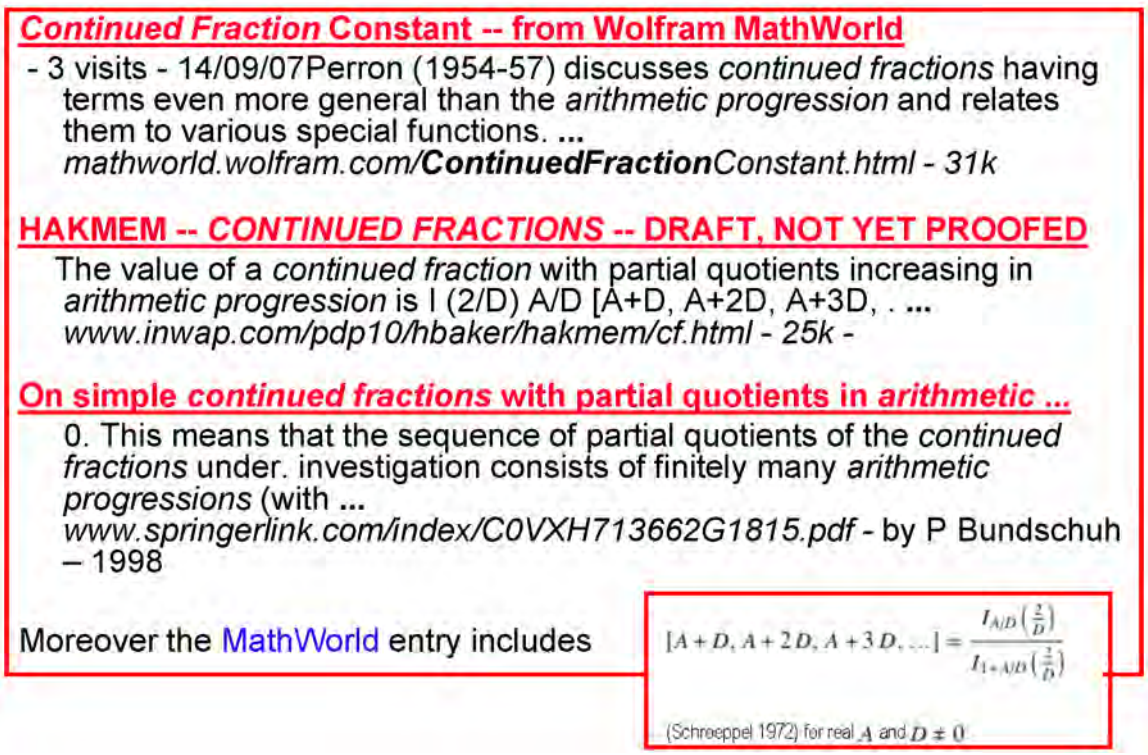

Figure 3: What Google and MathWorld offer.

2. Typing the first few digits into Sloane’s interface results in the response shown in Figure 4. In this case we are even told what the series representations of the requisite Bessel functions are, we are given sample code (in this case in *Mathematica*), and we are lead to many links and references. Moreover, the site is carefully moderated and continues to grow. Note also that this strategy only became viable after May 14th 2008 when the sequence was added to the database which now contains in excess of $158,000$ entries.

3. If one types the decimal representation of α into the Inverse Symbolic Calculator (ISC) it returns

[20] See `http://www.research.att.com/~njas/sequences/`.

[21] The online *Inverse Symbolic Calculator* `http://ddrive.cs.dal.ca/~isc` was newly web-accessible in 1995.

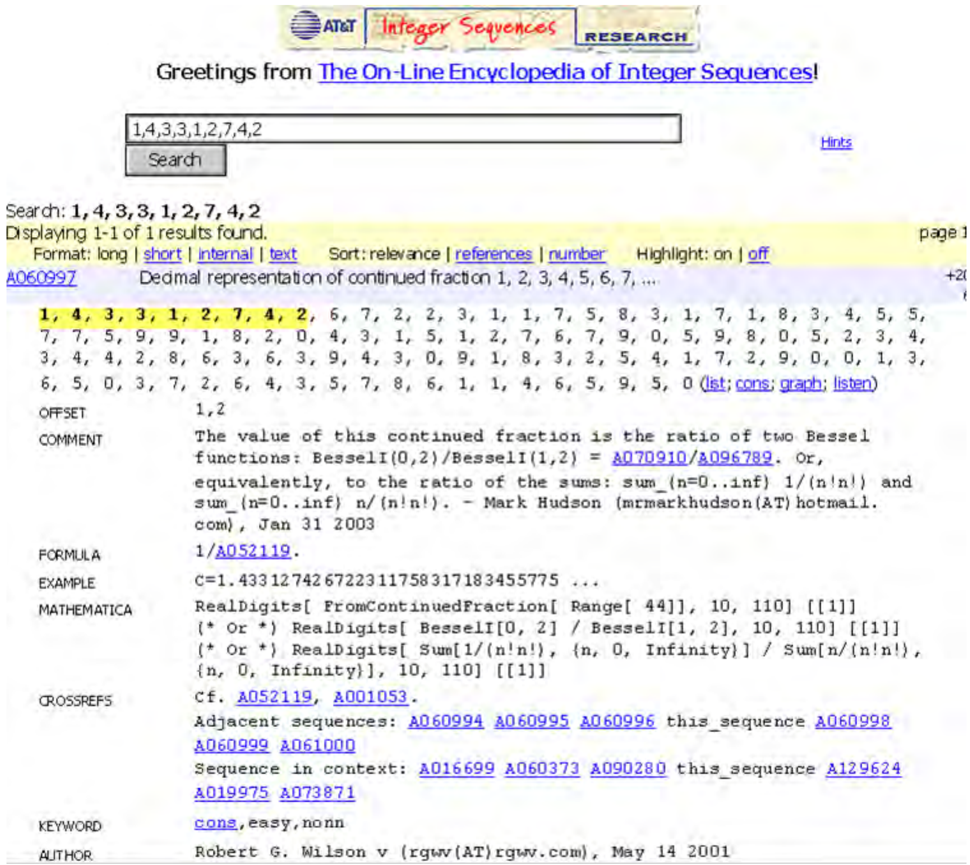

Figure 4: What *Sloane's Encyclopedia* offers.

```
Best guess:  BesI(0,2)/BesI(1,2)
```

Most of the functionality of the ISC is built into the "identify" function in versions of *Maple* starting with version 9.5. For example, `identify(4.45033263602792)` returns $\sqrt{3}+e$. As always, the experienced user will be able to extract more from this tool than the novice for whom the ISC will often produce more.

Example III: From Discovery to Proof

> *"Besides it is an error to believe that rigor in the proof is the enemy of simplicity."*—David Hilbert[22]

The following integral was made popular in a 1971 *Eureka*[23] article

$$0 < \int_0^1 \frac{(1-x)^4 x^4}{1+x^2}\,\mathrm{d}x \quad = \quad \frac{22}{7} - \pi \tag{5}$$

as described in [7]. As the integrand is positive on $(0,1)$ the integral yields an area and hence $\pi < 22/7$. Set on a 1960 Sydney honours mathematics final exam (5) perhaps originated in 1941 with the author of the 1971 article—Dalzeil who chose not reference his earlier self! Why should we trust this discovery? Well *Maple* and *Mathematica* both 'do it'. But this is *proof by appeal to authority* less imposing than, say, von Neumann [18] and a better answer is to ask *Maple* for the indefinite integral

$$\int_0^{\mathbf{t}} \frac{(1-x)^4 x^4}{1+x^2}\,\mathrm{d}x \quad = \quad ?$$

[22]In his *23 Mathematische Probleme* lecture to the Paris International Congress, 1900 [28].

[23]*Eureka* was an undergraduate Cambridge University journal.

The computer algebra system (CAS) will return

$$\int_0^t \frac{x^4 (1-x)^4}{1+x^2} \mathrm{d}x \quad = \quad \frac{1}{7} t^7 - \frac{2}{3} t^6 + t^5 - \frac{4}{3} t^3 + 4\,t - 4 \arctan(t) \tag{6}$$

and now differentiation and the *Fundamental theorem of calculus* proves the result.

This is probably not the proof one would find by hand, but it is a totally rigorous one, and represents an "instrumental use" of the computer. The fact that a CAS will quite possibly be able to evaluate an indefinite integral or a finite sum whenever it can evaluate the corresponding definite integral or infinite sum frequently allows one to provide a certificate for such a discovery. In the case of a sum the certificate often takes the form of a mathematical induction (deductive version). Another interesting feature of this example is that it appears to be quite irrelevant that 22/7 is an early and the most famous continued-fraction approximation to π [20]. Not every discovery is part of a hoped-for pattern.

Example IV: From Concrete to Abstract

"The plural of 'anecdote' is not 'evidence'."—Alan L. Leshner[24]

1. In April 1993, Enrico Au-Yeung, then an undergraduate at the University of Waterloo, brought to my attention the result

$$\sum_{k=1}^{\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right)^2 k^{-2} \quad = \quad 4.59987\ldots \quad \approx \quad \frac{17}{4}\zeta(4) \quad = \quad \frac{17\pi^4}{360}$$

He had spotted from six place accuracy that $0.047222\ldots = 17/360$. I was very skeptical, but Parsevals identity computations affirmed this to high precision. This is effectively a special case of the following class

$$\zeta(s_1, s_2, \cdots, s_k) \quad = \quad \sum_{n_1 > n_2 > \cdots > n_k > 0} \prod_{j=1}^{k} n_j^{-|s_j|} \sigma_j^{-n_j},$$

where s_j are integers and $\sigma_j = \mathrm{signum} s_j$. These can be rapidly computed as implemented at `www.cecm.sfu.ca/projects/ezface+`. [8]. In the past 20 years they have become of more and more interest in number theory, combinatorics, knot theory and mathematical physics. A marvellous example is Zagier's conjecture, found experimentally and now proven in [8], viz;

$$\zeta\left(\overbrace{3, 1, 3, 1, \cdots, 3, 1}^{n}\right) = \frac{2\pi^{4n}}{(4n+2)!} \tag{7}$$

Along the way to finding the proof we convinced ourselves that (7) held for many values including $n = 163$ which required summing a slowly convergent 326-dimensional sum to $1,000$ places with our fast summation method. Equation (7) is a remarkable non-commutative counterpart of the classical formula for $\zeta(2n)$ [8, Ch. 3].

2. In the course of proving empirically-discovered conjectures about such multiple zeta values [7] we needed to obtain the coefficients in the *partial fraction* expansion for

$$\frac{1}{x^s(1-x)^t} = \sum_{j \geq 0} \frac{a_j^{s,t}}{x^j} + \sum_{j \geq 0} \frac{b_j^{s,t}}{(1-x)^j} \tag{8}$$

[24] *Science*'s publisher speaking at the Canadian Federal Science & Technology Forum, Oct 2, 2002.

It transpires that

$$a_j^{s,t} = \binom{s+t-j-1}{s-j}$$

with a symmetric expression for $b_j^{s,t}$. This was know to Euler and once known is fairly easily proved by induction. But it can certainly be discovered in a CAS by considering various rows or diagonals in the matrix of coefficients—and either spotting the pattern or failing that by asking Sloane's Encyclopedia. Partial fractions like continued fractions and Gaussian elimination are the sort of task that *once mastered* are much better performed by computer while one focusses on more conceptual issues they expose.

3. We also needed to show that $M := A + B - C$ was invertible where the $n \times n$ matrices A, B, C respectively had entries

$$(-1)^{k+1}\binom{2n-j}{2n-k}, \quad (-1)^{k+1}\binom{2n-j}{k-1}, \quad (-1)^{k+1}\binom{j-1}{k-1}. \tag{9}$$

Thus, A and C are triangular while B is full. For example, in nine dimensions M is displayed below

$$\begin{bmatrix} 1 & -34 & 272 & -1360 & 4760 & -12376 & 24752 & -38896 & 48620 \\ 0 & -16 & 136 & -680 & 2380 & -6188 & 12376 & -19448 & 24310 \\ 0 & -13 & 105 & -470 & 1470 & -3458 & 6370 & -9438 & 11440 \\ 0 & -11 & 88 & -364 & 1015 & -2093 & 3367 & -4433 & 5005 \\ 0 & -9 & 72 & -282 & 715 & -1300 & 1794 & -2002 & 2002 \\ 0 & -7 & 56 & -210 & 490 & -792 & 936 & -858 & 715 \\ 0 & -5 & 40 & -145 & 315 & -456 & 462 & -341 & 220 \\ 0 & -3 & 24 & -85 & 175 & -231 & 203 & -120 & 55 \\ 0 & -1 & 8 & -28 & 56 & -70 & 56 & -28 & 9 \end{bmatrix}$$

After messing around futilely with lots of cases in an attempt to spot a pattern, it occurred to me to ask *Maple* for the *minimal polynomial* of M.

```
> linalg[minpoly](M(12),t);
```

returns $-2 + t + t^2$. Emboldened I tried

```
> linalg[minpoly](B(20),t); linalg[minpoly](A(20),t); linalg[minpoly](C(20),t);
```

and was rewarded with $-1 + t^3, -1 + t^2, -1 + t^2$. Since a typical matrix has a full degree minimal polynomial, we are quite assured that A, B, C really are roots of unity. Armed with this discovery we are lead to try to prove

$$A^2 = I, \quad BC = A, \quad C^2 = I, \quad CA = B^2 \tag{10}$$

which is a nice combinatorial exercise (by hand or computer). Clearly then we obtain also

$$B^3 = B \cdot B^2 = B(CA) = (BC)A = A^2 = I \tag{11}$$

and the requisite formula

$$M^{-1} = \frac{M+I}{2}$$

is again a fun exercise in formal algebra; in fact, we have

$$\begin{aligned} M^2 &= AA + AB - AC + BA + BB - BC - CA - CB + CC \\ &= I + C - B - A + I \\ &= 2I - M. \end{aligned}$$

It is also worth confirming that we have discovered an amusing presentation of the symmetric group S_3. Characteristic or minimal polynomials, entirely abstract for me as a student, now become members of a rapidly growing box of concrete symbolic tools, as do many matrix decomposition results, the use of Groebner bases, Robert Risch's 1968 decision algorithm for when an elementary function has an elementary indefinite integral, and so on.

Many algorithmic components of CAS are today extraordinarily effective when two decades ago they were more like 'toys'. This is equally true of extreme-precision calculation—a prerequisite for much of my own work [2, 5] and others [6]—or in combinatorics. Consider the *generating function* of the number of *additive partitions*, $p(n)$ of a natural number where we ignore order and zeroes. Thus,

$$5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1 = 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1$$

and so $p(5) = 7$. The *ordinary generating function* (12) discovered by Euler is

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{\prod_{k=1}^{\infty}(1 - q^k)}. \tag{12}$$

This is easily obtained by using the geometric formula for each $1/(1 - q^k)$ and observing how many powers of q^n are obtained. The famous computation by MacMahon of $p(200) = 3972999029388$ if done *symbolically and entirely naively* from (12) on a reasonable laptop took 20 minutes in 1991, and about 0.17 seconds today while

$$p(2000) = 4720819175619413888601432406799959512200344166$$

took about two minutes in 2009. Moreover, Crandall was able, in December 2008, to calculate $p(10^9)$ in 3 seconds on his laptop, using Rademacher's 'finite' series along with FFT methods. Likewise, the record for computation of π has gone from under 30 million decimal digits in 1986 to over 1.6 trillion places this year.

Example V: A Dynamic Discovery and Partial Proof

> *"Considerable obstacles generally present themselves to the beginner, in studying the elements of Solid Geometry, from the practice which has hitherto uniformly prevailed in this country, of never submitting to the eye of the student, the figures on whose properties he is reasoning, but of drawing perspective representations of them upon a plane. . . . I hope that I shall never be obliged to have recourse to a perspective drawing of any figure whose parts are not in the same plane."*—Augustus De Morgan (1806–1871) [24, p. 540]

In a wide variety of problems (protein folding, 3SAT, spin glasses, giant Sudoku, etc.) we wish to find a point in the intersection of two sets A and B where B is non-convex but "divide and concur" works better than theory can explain. Let $P_A(x)$ and $R_A(x) := 2\,P_A(x) - x$ denote respectively the *projector* and *reflector* on a set A as shown

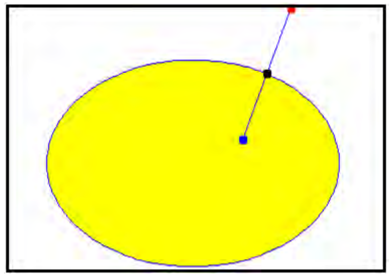

Figure 5: Reflector (interior) and Projector (boundary) of a point external to an ellipse.

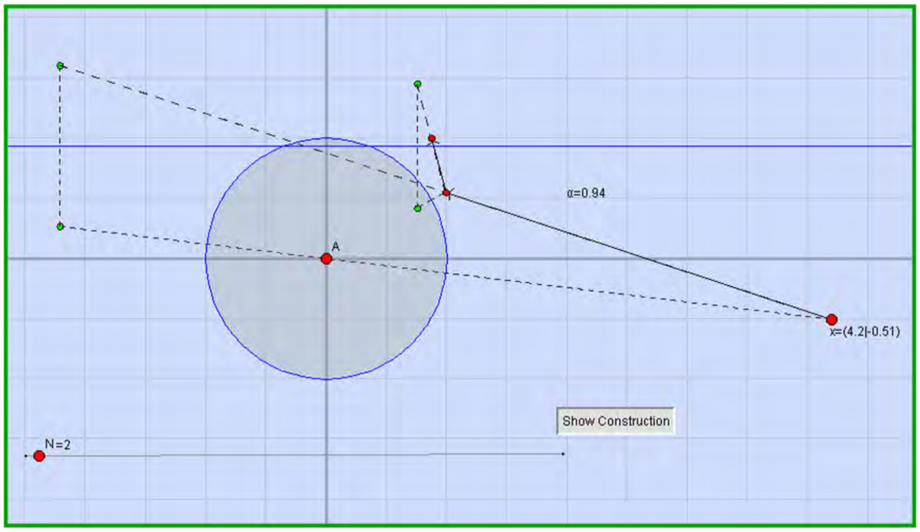

Figure 6: The first three iterates of (14) in *Cinderella*.

in Figure 5 where A is the boundary of the shaded ellipse. Then "divide and concur" is the natural geometric iteration "reflect-reflect-average":

$$x_{n+1} \Longrightarrow \frac{x_n + R_A\left(R_B(x_n)\right)}{2}. \tag{13}$$

Consider the simplest case of a line A of height α (all lines may be assumed horizontal) and the unit circle B. With $z_n := (x_n, y_n)$ we obtain the explicit iteration

$$x_{n+1} := \cos\theta_n, y_{n+1} := y_n + \alpha - \sin\theta_n, \quad (\theta_n := \arg z_n). \tag{14}$$

For the infeasible case with $\alpha > 1$ it is easy to see the iterates go to infinity vertically. For the tangent $\alpha = 1$ we provably converge to an infeasible point. For $0 < \alpha < 1$ the pictures are lovely but proofs escape me and my collaborators. Spiraling is ubiquitous in this case. Two representative *Maple* pictures follow:

For $\alpha = 0$ we can prove convergence to one of the two points in $A \cap B$ if and only if we do not start on the vertical axis, where we provably have *chaos*. The iteration is illustrated in Figure 6 starting at $(4.2, -0.51)$ with $\alpha = 0.94$. Let me sketch how the interactive geometry *Cinderella*[25] leads one both to discovery and a proof in this equatorial case. Interactive applets are easily made and the next two figures come from ones that are stored on line at

A1. `http://users.cs.dal.ca/~jborwein/reflection.html`; and

A2. `http://users.cs.dal.ca/~jborwein/expansion.html` respectively.

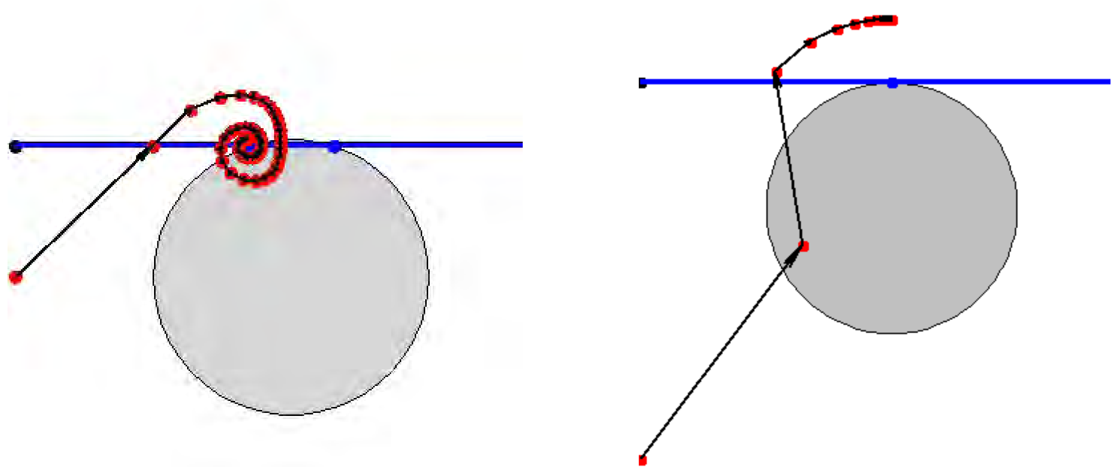

Figure 7: The behaviour of (14) for $\alpha = 0.95$ (L) and $\alpha = 1$ (R).

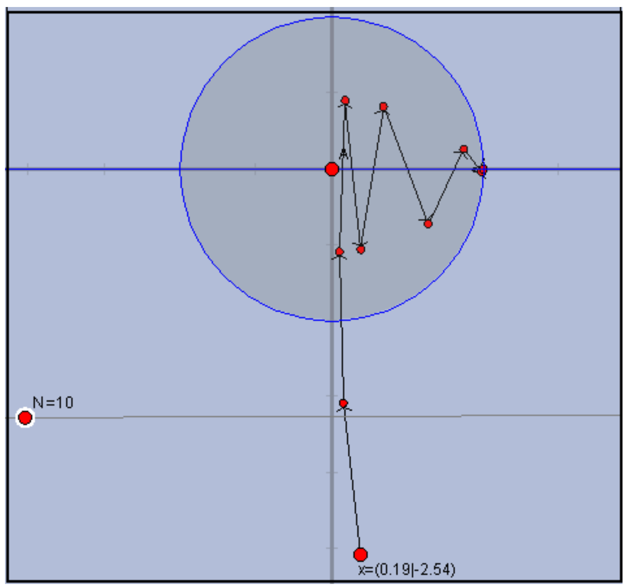

Figure 8: Discovery of the proof with $\alpha = 0$.

Figure 8 illustrates the applet **A1.** at work: by dragging the trajectory (with $N = 28$) one quickly discovers that

(i) as long as the iterate is outside the unit circle the next point is *always* closer to the origin;

(ii) once inside the circle the iterate *never* leaves;

(iii) the angle now *oscillates* to zero and the trajectory hence converges to $(1, 0)$.

All of this is quite easily made algebraic in the language of (14).

Figure 9 illustrates the applet **A2.** which takes up to 10,000 starting points in the rectangle $\{(x, y)\colon 0 \le x \le 1, |y - \alpha\| \le 1\}$ coloured by distance from the vertical axis with red on the axis and violet at $x = 1$, and produces the first hundred iterations in gestalt. Thus, we see clearly but I cannot yet prove, that all points not on the y-axis are swept into the feasible point $(\sqrt{1 - \alpha^2}, \alpha)$. It also shows that to accurately record the behaviour *Cinderella*'s double precision is inadequate and hence provides a fine if unexpected starting point for a discussion of numerical analysis and instability.

[25] Available at `http://www.cinderella.de`.

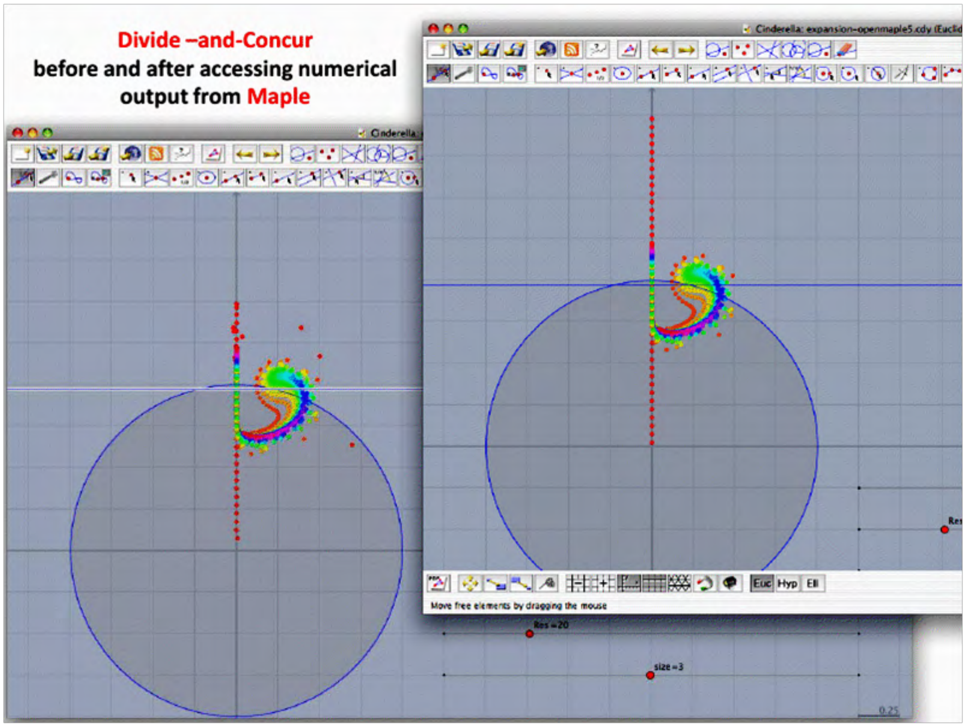

Figure 9: Gestalt of 400 third steps in *Cinderella* without (L) and with *Maple* data (R).

> *"A heavy warning used to be given [by lecturers] that pictures are not rigorous; this has never had its bluff called and has permanently frightened its victims into playing for safety. Some pictures, of course, are not rigorous, but I should say most are (and I use them whenever possible myself)."*—J. E. Littlewood, (1885-1977)[26]

Á la Littlewood, I find it hard to persuade myself that the applet **A2.** does not constitute a *generic proof* of what it displays in Figure 10.

We have also considered the analogous differential equation since asymptotic techniques for such differential equations are better developed. We decided

$$x'(t) = \frac{x(t)}{r(t)} - x(t) r(t) := \sqrt{x(t)^2 + y(t)^2}\,] \tag{15}$$
$$y'(t) = \alpha - \frac{y(t)}{r(t)}$$

was a reasonable counterpart to the Cartesian formulation of (14)—we have replaced the difference $x_{n+1} - x_n$ by $x'(t)$, etc.—as shown in Figure 11.

Now we have a whole other class of discoveries without explanations.

This is also an ideal problem to introduce early under-graduates to research as it involves only school geometry notions and has many accessible extensions in two or three dimensions. Much can be discovered and most of it will be both original and unproven. Consider what happens when B is a line segment or a finite set rather than a line or when A is a more general conic section. Corresponding algorithms, like "project-project-average", are representative of what was used to correct the Hubble telescope's early optical abberation problems.

[26]From p. 53 of the 1953 edition of Littlewood's *Miscellany* and so said long before the current fine graphic, geometric, and other visualization tools were available; also quoted in [18].

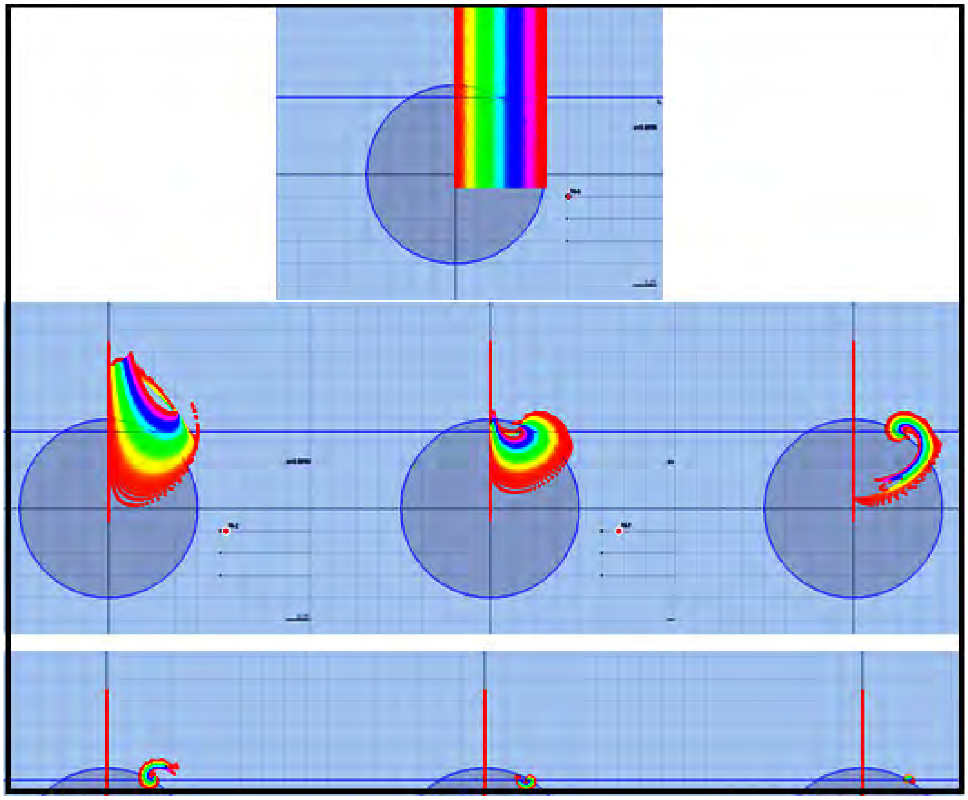

Figure 10: Snapshots of 10,000 points after 0, 2, 7, 13, 16, 21, and 27 steps in *Cinderella*.

Example VI: Knowledge without Proof

> *"All physicists and a good many quite respectable mathematicians are contemptuous about proof."* —G. H. Hardy (1877-1947)[27]

A few years ago Guillera found various Ramanujan-like identities for π, including three most basic ones:

$$\frac{128}{\pi^2} = \sum_{n=0}^{\infty}(-1)^n r(n)^5(13+180n+820n^2)\left(\frac{1}{32}\right)^{2n} \tag{16}$$

$$\frac{8}{\pi^2} = \sum_{n=0}^{\infty}(-1)^n r(n)^5(1+8n+20n^2)\left(\frac{1}{2}\right)^{2n} \tag{17}$$

$$\frac{32}{\pi^3} \stackrel{?}{=} \sum_{n=0}^{\infty} r(n)^7(1+14n+76n^2+168n^3)\left(\frac{1}{8}\right)^{2n}. \tag{18}$$

where

$$r(n) = \frac{(1/2)_n}{n!} = \frac{1/2\cdot 3/2\cdot \cdots \cdot (2n-1)/2}{n!} = \frac{\Gamma(n+1/2)}{\sqrt{\pi}\,\Gamma(n+1)}.$$

As far as we can tell there are no analogous formulae for $1/\pi^N$ with $N \geq 4$. Guillera proved (16) and (17) in tandem, by using very ingeniously the *Wilf–Zeilberger algorithm* for formally proving hypergeometric-like identities [7, 4]. He ascribed the third to Gourevich, who found it using*integer relation methods* [7, 4]. Formula (18) has been checked to

[27] In his famous *Mathematician's Apology* of 1940. I can not resist noting that modern digital assistance often makes more careful referencing unnecessary and sometimes even unhelpful!

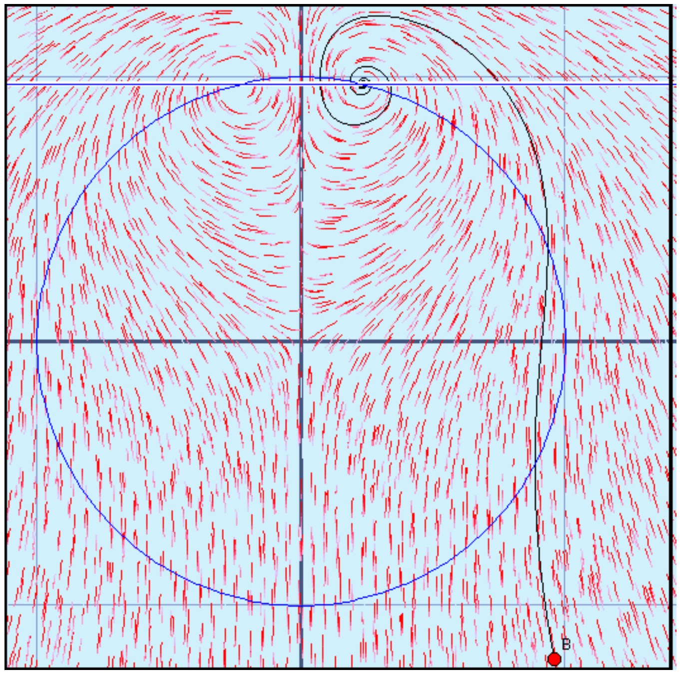

Figure 11: ODE solution and vector field for (15) with $\alpha = 0.97$ in *Cinderella*.

extreme precision. It is certainly true but has no proof, nor does anyone have an inkling of how to prove it especially as experiment suggests that it has no mate unlike (16) and (17). My intuition tells me that if a proof exists it is most probably more a verification than an explication and so I for one have stopped looking. I am happy just to know the beautiful identity is true. It may be so for no good reason. It might conceivably have no proof and be a very concrete Gödel statement.

Example VII. A Mathematical Physics Limit

> "*Anyone who is not shocked by quantum theory has not understood a single word.*"—Niels Bohr (1885–1962)

The following N-dimensional integrals arise independently in mathematical physics in statistical mechanics of the *Ising Model* and as we discovered later in *Quantum Field Theory*:

$$C_N = \frac{4}{N!} \int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^N (u_j + 1/u_j)\right)^2} \frac{du_1}{u_1} \cdots \frac{du_N}{u_N}. \tag{19}$$

We first showed that C_N can be transformed to a 1-D integral:

$$C_N = \frac{2^N}{N!} \int_0^\infty t K_0^N(t)\, dt \tag{20}$$

where K_0 is a *modified Bessel function*—Bessel functions which we met in Example I are quite ubiquitous. We then computed 400-digit numerical values. This is impossible for $n \geq 4$ from (19) but accessible from (20) and a good algorithm for K_0. Thence, we found the following, now proven, results [3]:

$$C_3 = \mathrm{L}_{-3}(2) := \sum_{n \geq 0} \left\{ \frac{1}{(3n+1)^2} - \frac{1}{(3n+2)^2} \right\}$$

$$C_4 = 14\,\zeta(3).$$

We also observed that

$$C_{1024} \quad = \quad 0.63047350337438679612204019271087890435458 7\ldots$$

and that the limit as $N \to \infty$ was the same to many digits. We then used the Inverse Symbolic Calculator, the aforementioned online numerical constant recognition facility, at

`http://ddrive.cs.dal.ca/~isc/portal`

which returned

```
Output: Mixed constants, 2 with elementary transforms.
        .6304735033743867 = sr(2)^2/exp(gamma)^2
```

from which we discovered that

$$C_{1024} \quad \approx \quad \lim_{n\to\infty} C_n = 2e^{-2\gamma}.$$

Here $\gamma = 0.57721566490153\ldots$ is *Euler's constant* and is perhaps the most basic constant which is not yet proven irrational [15]. The limit discovery showed the Bessel function representation to be fundamental. Likewise $\zeta(3) = \sum_{n=1}^{\infty} 1/n^3$ the value of the Riemann zeta-function at 3, also called Apéry's constant, was only proven irrational in 1978 and the irrationality of $\zeta(5)$ remains unproven. The limit discovery, and its appearance in the literature of Bessel functions, persuaded us the Bessel function representation (20) was fundamental—not just technically useful—and indeed this is the form in which C_N, for odd N appears in quantum field theory [3].

Example VIII: Apéry's formula

> *"Another thing I must point out is that you cannot prove a vague theory wrong. ... Also, if the process of computing the consequences is indefinite, then with a little skill any experimental result can be made to look like the expected consequences."*—Richard Feynman (1918–1988)

Margo Kondratieva found the following identity in 1890 papers of Markov [4]:

$$\sum_{n=0}^{\infty} \frac{1}{(n+a)^3} \quad = \quad \frac{1}{4}\sum_{n=0}^{\infty} \frac{(-1)^n\,(n!)^6}{(\,2n+1)!}\,\frac{\left(5\,(n+1)^2+6\,(a-1)\,(n+1)+2\,(a-1)^2\right)}{\prod_{k=0}^{n}(a+k)^4}. \tag{21}$$

Apéry's 1978 formula

$$\zeta(3) \quad = \quad \frac{5}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3\binom{2k}{k}}, \tag{22}$$

which played a key role in his proof of its irrationality, is the case with $a = 0$.

Luckily, by adopting Giaquinto's accounting of discovery we are still entitled to say that Apéry discovered the formula (22) which now bears his name.

We observe that *Maple* 'establishes' identity (21) in the hypergeometric formula

$$-\frac{1}{2}\,\Psi\left(2,a\right) = -\frac{1}{2}\,\Psi\left(2,a\right) - \zeta\left(3\right) + \frac{5}{4}\,{}_4\mathrm{F}_3\left(\begin{array}{c}1,1,1,1\\ 2,2,\frac{3}{2}\end{array}\middle|\,-\frac{1}{4}\right),$$

that is, it has reduced it to a form of (22). Like much of mathematics this example leads to something whose computational consequences are very far from indefinite. Indeed, it is the rigidity of much algorithmic mathematics that makes it so frequently the way hardware or software errors such as the 'Pentium Bug' are first uncovered.

3 Concluding Remarks

> *"We [Kaplansky and Halmos] share a philosophy about linear algebra: we think basis-free, we write basis-free, but when the chips are down we close the office door and compute with matrices like fury."*[28]

Theory and practice should be better comported!

The students of today live, as we do, in an information-rich, judgement-poor world in which the explosion of information, and of tools, is not going to diminish. So we have to teach judgement (not just concern with plagiarism) when it comes to using what is already possible digitally. This means mastering the sorts of tools I have illustrated. Additionally, it seems to me critical that we mesh our software design—and our teaching style more generally—with our growing understanding of our cognitive strengths and limitations as a species (as touched upon in the introduction). Judith Grabner has noted that a large impetus for the development of modern rigor in mathematics came with the Napoleonic introduction of regular courses: lectures and text books force a precision and a codification that apprenticeship obviates.

As Dave Bailey noted to me recently:

> *"Moreover, there is a growing consensus that human minds are fundamentally not very good at mathematics, and must be trained [17]. Given this fact, the computer can be seen as a perfect complement to humans—we can intuit but not reliably calculate or manipulate; computers are not yet very good at intuition, but are great at calculations and manipulations."*

We also have to acknowledge that most of our classes will contain a very broad variety of skills and interests (and relatively few future mathematicians). Properly balanced, discovery and proof, assisted by good software, can live side-by-side and allow for the ordinary and the talented to flourish in their own fashion. Impediments to the assimilation of the tools I have illustrated are myriad as I am only too aware from recent my own teaching experiences. These impediments include our own inertia and organizational and technical bottlenecks (this is often from poor IT design - not so much from too few dollars). The impediments certainly include under-prepared or misprepared colleagues and the dearth of good material from which to teach a modern syllabus.

Finally, it will never be the case that quasi-inductive mathematics supplants proof. We need to find a new equilibrium. Consider the following empirically-discovered identity

$$\sum_{n=-\infty}^{\infty} \operatorname{sinc}(n)\operatorname{sinc}(n/3)\operatorname{sinc}(n/5)\cdots\operatorname{sinc}(n/23)\operatorname{sinc}(n/29) \tag{23}$$
$$= \int_{-\infty}^{\infty} \operatorname{sinc}(x)\operatorname{sinc}(x/3)\operatorname{sinc}(x/5)\cdots\operatorname{sinc}(x/23)\operatorname{sinc}(x/29)\,dx$$

where the denumerators range over the primes.

Provably, the following is true: The analogous "sum equals integral" identity remains valid for more than the first $10,176$ primes but stops holding after some larger prime, and thereafter the 'sum minus integral" is positive but *much less than one part in a googolplex* [2]. It is hard to imagine that inductive mathematics alone will ever be able to handle such behaviour . Nor, for that matter, is it clear to me what it means psychologically to digest equations which are false by a near infinitesimal amount.

[28] In Paul Halmos' *Celebrating 50 Years of Mathematics.*

That said, we are only beginning to scratch the surface of a very exciting set of tools for the enrichment of mathematics, not to mention the growing power of formal proof engines. I conclude with one of my favourite quotes from George Polya and Jacques Hadamard [22]:

> *"This "quasi-experimental" approach to proof can help to de-emphasise a focus on rigor and formality for its own sake, and to instead support the view expressed by Hadamard when he stated* "The object of mathematical rigor is to sanction and legitimize the conquests of intuition, and there was never any other object for it." "

Unlike Frank Quinn perhaps, I believe that in the most complex modern cases certainty, in any reasonable sense, is unattainable through proof. I do believe that even then quasi-inductive methods and experimentation can help us improve our level of certainty. Like Reuben Hersh [16], I am happy to at least entertain some "*non-traditional forms of proof.*" Never before have we had such a cornucopia of fine tools to help us develop and improve our intuition. The challenge is to learn how to harness them, how to develop and how to transmit the necessary theory and practice.

Acknowledgements I owe many people thanks for helping refine my thoughts on this subject over many years. Four I must mention by name: my long-standing collaborators Brailey Sims, Richard Crandall and David Bailey, and my business partner Ron Fitzgerald from *MathResources* who has taught me a lot about balancing pragmatism and idealism in educational technology—among other things. I also thank Henrik Sørenson whose thought-provoking analysis gave birth to the title and the thrust of the paper, and my student Chris Maitland who built most of the *Cinderella* applets.

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EXPLORATORY EXPERIMENTATION AND COMPUTATION

DAVID H. BAILEY AND JONATHAN M. BORWEIN

ABSTRACT. We believe the mathematical research community is facing a great challenge to re-evaluate the role of proof in light of recent developments. On one hand, the growing power of current computer systems, of modern mathematical computing packages, and of the growing capacity to data-mine on the Internet, has provided marvelous resources to the research mathematician. On the other hand, the enormous complexity of many modern capstone results such as the Poincaré conjecture, Fermat's last theorem, and the classification of finite simple groups has raised questions as to how we can better ensure the integrity of modern mathematics. Yet as the need and prospects for inductive mathematics blossom, the requirement to ensure the role of proof is properly founded remains undiminished.

1. Exploratory Experimentation

> The authors' thesis—once controversial, but now a commonplace—is that computers can be a useful, even essential, aid to mathematical research.—Jeff Shallit

Jeff Shallit wrote this in his recent review `MR2427663` of [9]. As we hope to make clear, Shallit was entirely right in that many, if not most, research mathematicians now use the computer in a variety of ways to draw pictures, inspect numerical data, manipulate expressions symbolically, and run simulations. However, it seems to us that there has not yet been substantial and intellectually rigorous progress in the way mathematics is presented in research papers, textbooks and

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classroom instruction, or in how the mathematical discovery process is organized.

1.1. **Mathematicians are humans.** We share with George Pólya (1887-1985) the view [24, 2 p. 128] that while learned,

> intuition comes to us much earlier and with much less outside influence than formal arguments.

Pólya went on to reaffirm, nonetheless, that proof should certainly be taught in school.

We turn to observations, many of which have been fleshed out in coauthored books such *Mathematics by Experiment* [9], and *Experimental Mathematics in Action* [3], where we have noted the changing nature of mathematical knowledge and in consequence ask questions such as "How do we teach what and why to students?", "How do we come to believe and trust pieces of mathematics?", and "Why do we wish to prove things?" An answer to the last question is "That depends." Sometimes we wish insight and sometimes, especially with subsidiary results, we are more than happy with a certificate. The computer has significant capacities to assist with both.

Smail [26, p. 113] writes:

> the large human brain evolved over the past 1.7 million years to allow individuals to negotiate the growing complexities posed by human social living.

As a result, humans find various modes of argument more palatable than others, and are more prone to make certain kinds of errors than others. Likewise, the well-known evolutionary psychologist Steve Pinker observes that language [23, p. 83] is founded on

> ...the ethereal notions of space, time, causation, possession, and goals that appear to make up a language of thought.

This remains so within mathematics. The computer offers scaffolding both to enhance mathematical reasoning, as with the recent computation of the Lie group E_8, see `http://www.aimath.org/E8/computerdetails.html`, and to restrain mathematical error.

1.2. **Experimental mathodology.** Justice Potter Stewart's famous 1964 comment, *"I know it when I see it"* is the quote with which *The Computer as Crucible* [12] starts. A bit less informally, by *experimental mathematics* we intend [9]:

(a) Gaining insight and *intuition*;
(b) *Visualizing* math principles;
(c) *Discovering* new relationships;
(d) *Testing* and especially *falsifying* conjectures;

(e) *Exploring* a possible result to see if it *merits* formal proof;
(f) *Suggesting* approaches for formal proof;
(g) *Computing* replacing lengthy hand derivations;
(h) *Confirming* analytically derived results.

Of these items (a) through (e) play a central role, and (f) also plays a significant role for us, but connotes computer-assisted or computer-directed proof and thus is quite distinct from *formal proof* as the topic of a special issue of these *Notices* in December 2008; see, e.g., [19].

1.2.1. *Digital integrity, I.* For us (g) has become ubiquitous, and we have found (h) to be particularly effective in ensuring the integrity of published mathematics. For example, we frequently check and correct identities in mathematical manuscripts by computing particular values on the LHS and RHS to high precision and comparing results—and then if necessary use software to repair defects.

As a first example, in a current study of "character sums" we wished to use the following result derived in [13]:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{m+n-1}}{(2m-1)(m+n-1)^3} \stackrel{?}{=} 4\,\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{51}{2880}\pi^4-\frac{1}{6}\pi^2\log^2(2)+\frac{1}{6}\log^4(2)+\frac{7}{2}\log(2)\zeta(3). \tag{1.1}$$

Here $\mathrm{Li}_4(1/2)$ is a polylogarithmic value. However, a subsequent computation to check results disclosed that whereas the LHS evaluates to $-0.872929289\ldots$, the RHS evaluates to $2.509330815\ldots$. Puzzled, we computed the sum, as well as each of the terms on the RHS (sans their coefficients), to 500-digit precision, then applied the "PSLQ" algorithm, which searches for integer relations among a set of constants [15]. PSLQ quickly found the following:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{m+n-1}}{(2m-1)(m+n-1)^3} = 4\,\mathrm{Li}_4\left(\frac{1}{2}\right)-\frac{151}{2880}\pi^4-\frac{1}{6}\pi^2\log^2(2)+\frac{1}{6}\log^4(2)+\frac{7}{2}\log(2)\zeta(3). \tag{1.2}$$

In other words, in the process of transcribing (1.1) into the original manuscript, "151" had become "51." It is quite possible that this error would have gone undetected and uncorrected had we not been able to computationally check and correct such results. This may not always matter, but it can be crucial.

With a current Research Assistant, Alex Kaiser at Berkeley, we have started to design software to refine and automate this process and to run it before submission of any equation-rich paper. This semi-automated integrity checking becomes pressing when verifiable output from a symbolic manipulation might be the length of a Salinger novel. For instance, recently while studying expected radii of points in a hypercube [11], it was necessary to show the existence of a "closed form" for

$$J(t) \quad := \quad \int_{[0,1]^2} \frac{\log(t + x^2 + y^2)}{(1 + x^2)(1 + y^2)} \, \mathrm{d}x \, \mathrm{d}y. \tag{1.3}$$

The computer verification of [11, Thm. 5.1] quickly returned a 100000-character "answer" that could be numerically validated very rapidly to hundreds of places. A highly interactive process stunningly reduced a basic instance of this expression to the concise formula

$$J(2) = \frac{\pi^2}{8} \log 2 - \frac{7}{48} \zeta(3) + \frac{11}{24} \pi \, \mathrm{Cl}_2 \left(\frac{\pi}{6} \right) - \frac{29}{24} \pi \, \mathrm{Cl}_2 \left(\frac{5\pi}{6} \right), \tag{1.4}$$

where Cl_2 is the *Clausen function* $\mathrm{Cl}_2(\theta) := \sum_{n \geq 1} \sin(n\theta)/n^2$ (Cl_2 is the simplest non-elementary Fourier series). Automating such reductions will require a sophisticated simplification scheme with a very large and extensible knowledge base.

1.3. **Discovering a truth.** Giaquinto's [17, p. 50] attractive encapsulation

> In short, discovering a truth is coming to believe it in an independent, reliable, and rational way.

has the satisfactory consequence that a student can legitimately discover things already "known" to the teacher. Nor is it necessary to demand that each dissertation be absolutely original—only that it be independently discovered. For instance, a differential equation thesis is no less meritorious if the main results are subsequently found to have been accepted, unbeknown to the student, in a control theory journal a month earlier—provided they were independently discovered. Near-simultaneous independent discovery has occurred frequently in science, and such instances are likely to occur more and more frequently as the earth's "new nervous system" (Hillary Clinton's term in a recent policy address) continues to pervade research.

Despite the conventional identification of mathematics with deductive reasoning, Kurt Gödel (1906-1978) in his 1951 Gibbs Lecture said:

> If mathematics describes an objective world just like physics, there is no reason why inductive methods should not be applied in mathematics just the same as in physics.

He held this view until the end of his life despite—or perhaps because of—the epochal deductive achievement of his incompleteness results.

Also, we emphasize that many great mathematicians from Archimedes and Galileo—who reputedly said *"All truths are easy to understand once they are discovered; the point is to discover them."*—to Gauss, Poincaré, and Carleson have emphasized how much it helps to "know" the answer beforehand. Two millennia ago, Archimedes wrote, in the Introduction to his long-lost and recently reconstituted *Method* manuscript,

> For it is easier to supply the proof when we have previously acquired, by the method, some knowledge of the questions than it is to find it without any previous knowledge.

Archimedes' *Method* can be thought of as an uber-precursor to today's interactive geometry software, with the caveat that, for example, *Cinderella* actually does provide proof certificates for much of Euclidean geometry.

As 2006 Abel Prize winner Lennart Carleson describes in his 1966 ICM speech on his positive resolution of Luzin's 1913 conjecture (that the Fourier series of square-summable functions converge pointwise a.e. to the function), after many years of seeking a counterexample, he finally decided none could exist. He expressed the importance of this confidence as follows:

> The most important aspect in solving a mathematical problem is the conviction of what is the true result. Then it took 2 or 3 years using the techniques that had been developed during the past 20 years or so.

1.4. **Digital Assistance.** By *digital assistance*, we mean the use of:

(a) *Integrated mathematical software* such as *Maple* and *Mathematica*, or indeed MATLAB and their open source variants.
(b) *Specialized packages* such as CPLEX, PARI, SnapPea, Cinderella and MAGMA.
(c) *General-purpose programming languages* such as C, C++, and Fortran-2000.

(d) *Internet-based applications* such as: Sloane's Encyclopedia of Integer Sequences, the Inverse Symbolic Calculator,[1] Fractal Explorer, Jeff Weeks' Topological Games, or Euclid in Java.[2]

(e) *Internet databases and facilities* including Google, MathSciNet, arXiv, Wikipedia, MathWorld, MacTutor, Amazon, Amazon Kindle, and many more that are not always so viewed.

All entail data-mining in various forms. The capacity to consult the Oxford dictionary and Wikipedia instantly within Kindle dramatically changes the nature of the reading process. Franklin [16] argues that Steinle's "exploratory experimentation" facilitated by "widening technology" and "wide instrumentation," as routinely done in fields such as pharmacology, astrophysics, medicine, and biotechnology, is leading to a reassessment of what legitimates experiment; in that a "local model" is not now a prerequisite. Thus, a pharmaceutical company can rapidly examine and discard tens of thousands of potentially active agents, and then focus resources on the ones that survive, rather than needing to determine in advance which are likely to work well. Similarly, aeronautical engineers can, by means of computer simulations, discard thousands of potential designs, and submit only the best prospects to full-fledged development and testing.

Hendrik Sørenson [27] concisely asserts that experimental mathematics —as defined above—is following similar tracks with software such as *Mathematica*, *Maple* and MATLAB playing the role of wide instrumentation.

> These aspects of exploratory experimentation and wide instrumentation originate from the philosophy of (natural) science and have not been much developed in the context of experimental mathematics. However, I claim that e.g. the importance of wide instrumentation for an exploratory approach to experiments that includes concept formation also pertain to mathematics.

In consequence, boundaries between mathematics and the natural sciences and between inductive and deductive reasoning are blurred and becoming more so. (See also [2].) This convergence also promises some

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[1] Most of the functionality of the ISC, which is now housed at `http://carma-lx1.newcastle.edu.au:8087`, is now built into the "identify" function of *Maple* starting with version 9.5. For example, the *Maple* command `identify(4.45033263602792)` returns $\sqrt{3}+e$, meaning that the decimal value given is simply approximated by $\sqrt{3}+e$.

[2] A cross-section of Internet-based mathematical resources is available at `http://ddrive.cs.dal.ca/~isc/portal/` and `http://www.experimentalmath.info`.

relief from the frustration many mathematicians experience when attempting to describe their proposed methodology on grant applications to the satisfaction of traditional hard scientists. We leave unanswered the philosophically-vexing if mathematically-minor question as to whether genuine mathematical experiments (as discussed in [9]) truly exist, even if one embraces a fully idealist notion of mathematical existence. It surely seems to us that they do.

2. Pi, Partitions and Primes

The present authors cannot now imagine doing mathematics without a computer nearby. For example, characteristic and minimal polynomials, which were entirely abstract for us as students, now are members of a rapidly growing box of concrete symbolic tools. One's eyes may glaze over trying to determine structure in an infinite family of matrices including

$$M_4 = \begin{bmatrix} 2 & -21 & 63 & -105 \\ 1 & -12 & 36 & -55 \\ 1 & -8 & 20 & -25 \\ 1 & -5 & 9 & -8 \end{bmatrix} \quad M_6 = \begin{bmatrix} 2 & -33 & 165 & -495 & 990 & -1386 \\ 1 & -20 & 100 & -285 & 540 & -714 \\ 1 & -16 & 72 & -177 & 288 & -336 \\ 1 & -13 & 53 & -112 & 148 & -140 \\ 1 & -10 & 36 & -66 & 70 & -49 \\ 1 & -7 & 20 & -30 & 25 & -12 \end{bmatrix}$$

but a command-line instruction in a computer algebra system will reveal that both $M_4^3 - 3M_4 - 2I = 0$ and $M_6^3 - 3M_6 - 2I = 0$. Likewise, more and more matrix manipulations are profitably, even necessarily, viewed graphically. As is now well known in numerical linear algebra, graphical tools are essential when trying to discern qualitative information such as the block structure of very large matrices. See, for instance, Figure 1.

Equally accessible are many matrix decompositions, the use of Groebner bases, Risch's decision algorithm (to decide when an elementary function has an elementary indefinite integral), graph and group catalogues, and others. Many algorithmic components of a *computer algebra system* are today extraordinarily effective compared with two decades ago, when they were more like toys. This is equally true of extreme-precision calculation—a prerequisite for much of our own work [7, 10, 8]. As we will illustrate, during the three decades that we have seriously tried to integrate computational experiments into research, we have experienced at least 12 Moore's law doublings of computer power and memory capacity [9, 12], which when combined with the utilization of highly parallel clusters (with thousands of processing cores) and

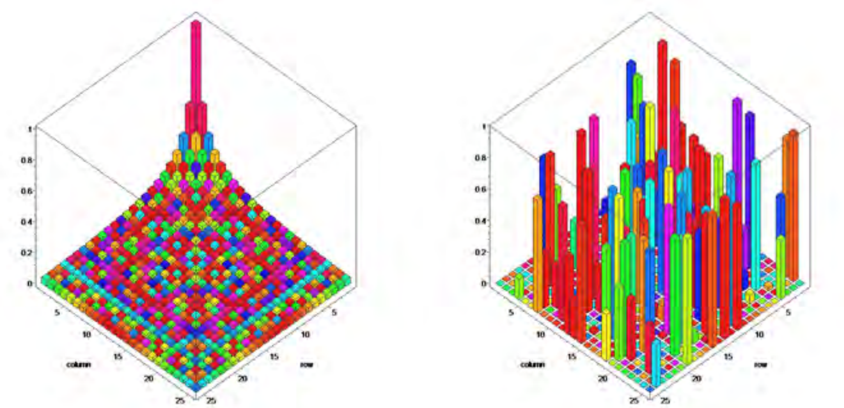

FIGURE 1. Plots of a 25×25 Hilbert matrix (L) and a matrix with 50% sparsity and random $[0, 1]$ entries (R).

fiber-optic networking, has resulted in six to seven orders of magnitude speedup for many operations.

2.1. **The partition function.** Consider the number of additive partitions, $p(n)$, of a natural number, where we ignore order and zeroes. For instance, $5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1$, so $p(5) = 7$. The ordinary generating function (2.1) discovered by Euler is

$$\sum_{n=0}^{\infty} p(n) q^n = \prod_{k=1}^{\infty} \left(1 - q^k\right)^{-1}. \tag{2.1}$$

(This can be proven by using the geometric formula for $1/(1-q^k)$ to expand each term and observing how powers of q^n occur.)

The famous computation by MacMahon of $p(200) = 3972999029388$ at the beginning of the 20th century, done symbolically and entirely naively from (2.1) on a reasonable laptop, took 20 minutes in 1991 but only 0.17 seconds today, while the many times more demanding computation

$$p(2000) = 4720819175619413888601432406799959512200344166$$

took just two minutes in 2009. Moreover, in December 2008, Crandall was able to calculate $p(10^9)$ in three seconds on his laptop, using the Hardy-Ramanujan-Rademacher 'finite' series for $p(n)$ along with FFT methods. Using these techniques, Crandall was also able to calculate the probable primes $p(1000046356)$ and $p(1000007396)$, each of which has roughly 35000 decimal digits.

Such results make one wonder when easy access to computation discourages innovation: Would Hardy and Ramanujan have still discovered their marvelous formula for $p(n)$ if they had powerful computers at hand?

2.2. **Quartic algorithm for π.** Likewise, the record for computation of π has gone from 29.37 *million* decimal digits in 1986, to over 2.7 *trillion* digits in 2010. Since the algorithm below was used as part of each computation, it is interesting to compare the performance in each case: Set $a_0 := 6 - 4\sqrt{2}$ and $y_0 := \sqrt{2} - 1$, then iterate

$$\begin{aligned} y_{k+1} &= \frac{1-(1-y_k^4)^{1/4}}{1+(1-y_k^4)^{1/4}}, \\ a_{k+1} &= a_k(1+y_{k+1})^4 - 2^{2k+3}y_{k+1}(1+y_{k+1}+y_{k+1}^2). \end{aligned} \tag{2.2}$$

Then a_k converges *quartically* to $1/\pi$—each iteration approximately quadruples the number of correct digits. Twenty-one full-precision iterations of (2.2), which was discovered on a 16K Radio Shack portable in 1983, produce an algebraic number that coincides with π to well more than six trillion places. This scheme and the 1976 Salamin–Brent scheme [9, Ch. 3] have been employed frequently over the past quarter century. Here is a highly abbreviated chronology:

- 1986: Computing 29.4 million digits required 28 hours on one CPU of the new Cray-2 at NASA Ames Research Center, using (2.2). Confirmation using another algorithm took 40 hours. This computation uncovered hardware and software errors on the Cray-2. Success required developing faster FFTs [9, Ch. 3].
- Jan. 2009: Computing 1.649 trillion digits using (2.2) required 73.5 hours on 1024 cores (and 6.348 Tbyte memory) of a Appro Xtreme-X3 system. This was checked with a computation via the Salamin-Brent scheme that took 64.2 hours and 6.732 Tbyte of main memory. The two computations differed only in the last 139 places.
- Apr. 2009: Takahashi increased his record to an amazing 2.576 trillion digits.
- Dec. 2009: Bellard computed nearly 2.7 trillion decimal digits of π (first in binary), using the Chudnovsky series given below in (2.9). This took 131 days, but he only used a single four-core workstation with lots of disk storage and even more human intelligence! Full details of these feats are available at `http://en.wikipedia.org/wiki/Chronology_of_computation_of_pi`.

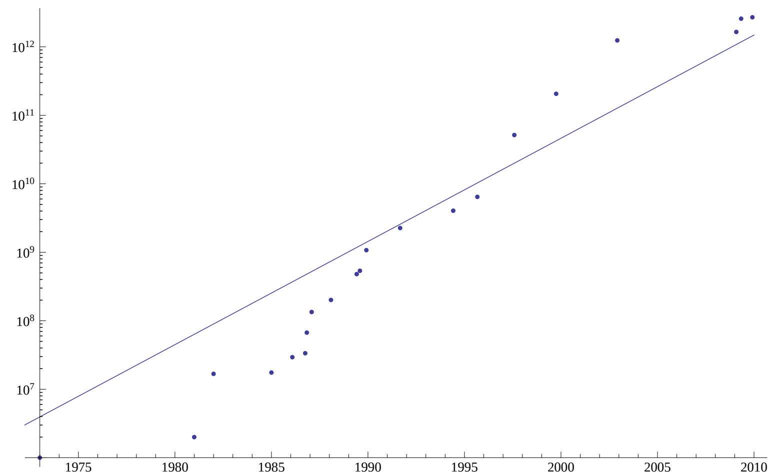

FIGURE 2. Plot of π calculations, in digits (dots), compared with the long-term slope of Moore's Law (line).

Daniel Shanks, who in 1961 computed π to over 100,000 digits, once told Phil Davis that a billion-digit computation would be "forever impossible." But both Kanada and the Chudnovskys achieved that in 1989. Similarly, the intuitionists Brouwer and Heyting asserted the "impossibility" of ever knowing whether the sequence 0123456789 appears in the decimal expansion of π, yet it was found in 1997 by Kanada, beginning at position 17387594880. As late as 1989, Roger Penrose ventured in the first edition of his book *The Emperor's New Mind* that we likely will never know if a string of ten consecutive sevens occurs in the decimal expansion of π. This string was found in 1997 by Kanada, beginning at position 22869046249.

Figure 2— shows the progress of π calculations since 1970, superimposed with a line that charts the long-term trend of Moore's Law. It is worth noting that whereas progress in computing π exceeded Moore's Law in the 1990s, it has lagged Moore's Law in the past decade. This may be due in part to the fact that π programs can no longer employ system-wide fast Fourier transforms for multiplication (since most state-of-the-art supercomputers have insufficient network bandwidth), and so less-efficient hybrid schemes must be used instead.

2.2.1. *Digital integrity, II.* There are many possible sources of errors in these and other large-scale computations:

- The underlying formulas used might conceivably be in error.
- Computer programs implementing these algorithms, which employ sophisticated algorithms such as fast Fourier transforms

to accelerate multiplication, are prone to human programming errors.

- These computations usually are performed on highly parallel computer systems, which require error-prone programming constructs to control parallel processing.
- Hardware errors may occur—this was a factor in the 1986 computation of π, as noted above.

So why would anyone believe the results of such calculations? The answer is that such calculations are always double-checked with an independent calculation done using some other algorithm, sometimes in more than one way. For instance, Kanada's 2002 computation of π to 1.3 trillion decimal digits involved first computing slightly over one trillion hexadecimal (base-16) digits. He found that the 20 hex digits of π beginning at position $10^{12} + 1$ are `B4466E8D21 5388C4E014`.

Kanada then calculated these hex digits using the "BBP" algorithm [6]. The BBP algorithm for π is based on the formula

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i} \left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6} \right), \qquad (2.3)$$

which was discovered using the "PSLQ" integer relation algorithm [15]. Integer relation methods find or exclude potential rational relations between vectors of real numbers. At the start of this millennium, they were named one of the top ten algorithms of the twentieth century by *Computing in Science and Engineering*. The most effective is Helaman Ferguson's PSLQ algorithm [9, 3].

Eventually PSLQ produced the formula

$$\pi = 4\,{}_2F_1\left(\left. \begin{matrix} 1, \frac{1}{4} \\ \frac{5}{4} \end{matrix} \right| -\frac{1}{4} \right) + 2 \tan^{-1}\left(\frac{1}{2}\right) - \log 5, \qquad (2.4)$$

where ${}_2F_1\left(\left. \begin{matrix} 1, \frac{1}{4} \\ \frac{5}{4} \end{matrix} \right| -\frac{1}{4} \right) = 0.955933837\ldots$ is a Gaussian hypergeometric function.

From (2.4), the series (2.3) almost immediately follows. The BBP algorithm, which is based on (2.3), permits one to calculate binary or hexadecimal digits of π beginning at an arbitrary starting point, without needing to calculate any of the preceding digits, by means of a simple scheme that does not require very high precision arithmetic.

The result of the BBP calculation was `B4466E8D21 5388C4E014`. Needless to say, in spite of the many potential sources of error in both computations, the final results dramatically agree, thus confirming (in a convincing but heuristic sense) that both results are almost certainly correct. Although one cannot rigorously assign a "probability" to this event, note that the chances that two random strings of 20 hex digits perfectly agree is one in $16^{20} \approx 1.2089 \times 10^{24}$.

This raises the following question: What is more securely established, the assertion that the hex digits of π in positions $10^{12}+1$ through $10^{12}+20$ are `B4466E8D21 5388C4E014`, or the final result of some very difficult work of mathematics that required hundreds or thousands of pages, that relied on many results quoted from other sources, and that (as is frequently the case) only a relative handful of mathematicians besides the author can or have carefully read in detail?

2.3. **Euler's totient function ϕ.** As another measure of what changes over time and what doesn't, consider two conjectures regarding $\phi(n)$, which counts the number of positive numbers less than and relatively prime to n:

2.3.1. **Giuga's conjecture (1950).** *An integer $n > 1$, is a prime if and only if $\mathcal{G}_n := \sum_{k=1}^{n-1} k^{n-1} \equiv n-1 \mod n$.*

Counterexamples are necessarily *Carmichael numbers*—rare birds only proven infinite in 1994—and much more. In [10, pp. 227] we exploited the fact that if a number $n = p_1 \cdots p_m$ with $m > 1$ prime factors p_i is a counterexample to Giuga's conjecture (that is, satisfies $s_n \equiv n-1 \mod n$), then for $i \neq j$ we have $p_i \neq p_j$,

$$\sum_{i=1}^{m} \frac{1}{p_i} \;>\; 1,$$

and the p_i form a *normal sequence*: $p_i \not\equiv 1 \mod p_j$ for $i \neq j$. Thus, the presence of '3' excludes $7, 13, 19, 31, 37, \ldots$, and of '5' excludes $11, 31, 41, \ldots$.

This theorem yielded enough structure, using some predictive experimentally discovered heuristics, to build an efficient algorithm to show—over several months in 1995—that any counterexample had at least 3459 prime factors and so exceeded 10^{13886}, extended a few years later to 10^{14164} in a five-day desktop computation. The heuristic is self-validating every time that the programme runs successfully. But this method necessarily fails after 8135 primes; someday we hope to exhaust its use.

While writing this piece, one of us was able to obtain almost as good a bound of 3050 primes in under 110 minutes on a laptop computer, and a bound of 3486 primes and 14,000 digits in less than 14 hours; this was extended to 3,678 primes and 17,168 digits in 93 CPU-hours on a Macintosh Pro, using *Maple* rather than C++, which is often orders-of-magnitude faster but requires much more arduous coding.

An equally hard related conjecture for which much less progress can be recorded is:

2.3.2. **Lehmer's conjecture (1932).** $\phi(n)\big|(n-1)$ *if and only if* n *is prime.* He called this *"as hard as the existence of odd perfect numbers."*

Again, prime factors of counterexamples form a normal sequence, but now there is little extra structure. In a 1997 Simon Fraser M.Sc. thesis, Erick Wong verified the conjecture for 14 primes, using normality and a mix of PARI, C++ and *Maple* to press the bounds of the 'curse of exponentiality.' This very clever computation subsumed the entire scattered literature in one computation but could only extend the prior bound from 13 primes to 14.

For Lehmer's related 1932 question: *when does* $\phi(n) \mid (n+1)$*?*, Wong showed there are eight solutions with no more than seven factors (six-factor solutions are due to Lehmer). Let

$$\mathcal{L}_m := \prod_{k=0}^{m-1} F_k$$

with $F_n := 2^{2^n} + 1$ denoting the *Fermat primes.* The solutions are

$$2, \mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_5,$$

and the rogue pair 4919055 and 6992962672132095, but analyzing just eight factors seems out of sight. Thus, in 70 years the computer only allowed the exclusion bound to grow by one prime.

In 1932 Lehmer couldn't factor 6992962672132097. If it had been prime, a ninth solution would exist: since $\phi(n)|(n+1)$ with $n+2$ prime implies that $N := n(n+2)$ satisfies $\phi(N)|(N+1)$. We say *couldn't* because the number is divisible by 73; which Lehmer—a father of much factorization literature–could certainly have discovered had he anticipated a small factor. Today discovering that

$$6992962672132097 = 73 \cdot 95794009207289$$

is nearly instantaneous, while fully resolving Lehmer's original question remains as hard as ever.

2.4. **Inverse computation and Apéry-like series.** Three intriguing formulae for the Riemann zeta function are

$$(a)\ \zeta(2) = 3\sum_{k=1}^{\infty}\frac{1}{k^2\binom{2k}{k}}, \quad (b)\ \zeta(3) = \frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3\binom{2k}{k}}, \tag{2.5}$$

$$(c)\ \zeta(4) = \frac{36}{17}\sum_{k=1}^{\infty}\frac{1}{k^4\binom{2k}{k}}.$$

Binomial identity (2.5)(a) has been known for two centuries, while (b)—exploited by Apéry in his 1978 proof of the irrationality of $\zeta(3)$—was discovered as early as 1890 by Markov, and (c) was noted by Comtet [3].

Using integer relation algorithms, bootstrapping, and the "Pade" function (*Mathematica* and *Maple* both produce rational approximations well), in 1996 David Bradley and one of us [3, 10] found the following unanticipated generating function for $\zeta(4n+3)$:

$$\sum_{k=0}^{\infty}\zeta(4k+3)\,x^{4k} = \frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3\binom{2k}{k}(1-x^4/k^4)}\prod_{m=1}^{k-1}\left(\frac{1+4x^4/m^4}{1-x^4/m^4}\right). \tag{2.6}$$

Note that this formula permits one to read off an infinity of formulas for $\zeta(4n+3)$, $n>0$, beginning with (2.5)(b), by comparing coefficients of x^{4k} on the LHS and the RHS.

A decade later, following a quite analogous but much more deliberate experimental procedure, as detailed in [3], we were able to discover a similar general formula for $\zeta(2n+2)$ that is pleasingly parallel to (2.6):

$$\sum_{k=0}^{\infty}\zeta(2k+2)\,x^{2k} = 3\sum_{k=1}^{\infty}\frac{1}{k^2\binom{2k}{k}(1-x^2/k^2)}\prod_{m=1}^{k-1}\left(\frac{1-4\,x^2/m^2}{1-x^2/m^2}\right). \tag{2.7}$$

As with (2.6), one can now read off an infinity of formulas, beginning with (2.5)(a). In 1996, the authors could reduce (2.6) to a finite form that they could not prove, but Almquist and Granville did a year later. A decade later, the Wilf-Zeilberger algorithm [28, 22]—for which the inventors were awarded the Steele Prize—directly (as implemented in *Maple*) certified (2.7) [9, 3]. In other words, (2.7) was both discovered and proven by computer.

We found a comparable generating function for $\zeta(2n+4)$, giving (2.5) (c) when $x=0$, but one for $\zeta(4n+1)$ still eludes us.

2.5. **Reciprocal series for π.** Truly novel series for $1/\pi$, based on elliptic integrals, were discovered by Ramanujan around 1910 [3, 9, 29].

One is:

$$\frac{1}{\pi} \;=\; \frac{2\sqrt{2}}{9801}\sum_{k=0}^{\infty}\frac{(4k)!\,(1103+26390k)}{(k!)^4 396^{4k}}. \tag{2.8}$$

Each term of (2.8) adds eight correct digits. Gosper used (2.8) for the computation of a then-record 17 million digits of π in 1985—thereby completing the first proof of (2.8) [9, Ch. 3]. Shortly thereafter, David and Gregory Chudnovsky found the following variant, which lies in the quadratic number field $Q(\sqrt{-163})$ rather than $Q(\sqrt{58})$:

$$\frac{1}{\pi} \;=\; 12\sum_{k=0}^{\infty}\frac{(-1)^k\,(6k)!\,(13591409+545140134k)}{(3k)!\,(k!)^3\,640320^{3k+3/2}}. \tag{2.9}$$

Each term of (2.9) adds 14 correct digits. The brothers used this formula several times, culminating in a 1994 calculation of π to over four billion decimal digits. Their remarkable story was told in a prizewinning *New Yorker* article [25]. Remarkably, as we already noted earlier, (2.9) was used again in late 2009 for the current record computation of π. In consequence, Fabrice Bellard has provided access to two trillion-digit integers whose ratio is bizarrely close to π.

2.5.1. **Wilf-Zeilberger at work.** A few years ago Jésus Guillera found various Ramanujan-like identities for π, using integer relation methods. The three most basic—and entirely rational—identities are:

$$\frac{4}{\pi^2} \;=\; \sum_{n=0}^{\infty}(-1)^n r(n)^5(13+180n+820n^2)\left(\frac{1}{32}\right)^{2n+1} \tag{2.10}$$

$$\frac{2}{\pi^2} \;=\; \sum_{n=0}^{\infty}(-1)^n r(n)^5(1+8n+20n^2)\left(\frac{1}{2}\right)^{2n+1} \tag{2.11}$$

$$\frac{4}{\pi^3} \;\stackrel{?}{=}\; \sum_{n=0}^{\infty} r(n)^7(1+14n+76n^2+168n^3)\left(\frac{1}{8}\right)^{2n+1}, \tag{2.12}$$

where $r(n) := (1/2\cdot 3/2\cdot\,\cdots\,\cdot(2n-1)/2)/n!$.

Guillera proved (2.10) and (2.11) in tandem, by very ingeniously using the Wilf-Zeilberger algorithm [28, 22] for formally proving hypergeometric-like identities [9, 3, 18, 29]. No other proof is known, and there seem to be no like formulae for $1/\pi^N$ with $N \geq 4$. The third, (2.12), is almost certainly true. Guillera ascribes (2.12) to Gourevich, who used integer relation methods to find it.

We were able to "discover" (2.12) using 30-digit arithmetic, and we checked it to 500 digits in 10 seconds, to 1200 digits in 6.25 minutes, and to 1500 digits in 25 minutes, all with naive command-line instructions in *Maple*. But it has no proof, nor does anyone have an inkling of how to prove it; especially, as experiment suggests, since it has no 'mate' in analogy to (2.10) and (2.11) [3]. Our intuition is that if a proof exists, it is more a verification than an explication and so we stopped looking. We are happy just to "know" that the beautiful identity is true (although it would be more remarkable were it eventually to fail). It may be true for no good reason—it might just have no proof and be a very concrete Gödel-like statement.

In 2008 Guillera [18] produced another lovely pair of third-millennium identities—discovered with integer relation methods and proved with creative telescoping—this time for π^2 rather than its reciprocal. They are

$$\sum_{n=0}^{\infty} \frac{1}{2^{2n}} \frac{\left(x+\frac{1}{2}\right)_n^3}{(x+1)_n^3} \left(6(n+x)+1\right) = 8x \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2}{(x+1)_n^2}, \tag{2.13}$$

and

$$\sum_{n=0}^{\infty} \frac{1}{2^{6n}} \frac{\left(x+\frac{1}{2}\right)_n^3}{(x+1)_n^3} \left(42(n+x)+5\right) = 32x \sum_{n=0}^{\infty} \frac{\left(x+\frac{1}{2}\right)_n^2}{(2x+1)_n^2}. \tag{2.14}$$

Here $(a)_n = a(a+1)\cdot\cdot(a+n-1)$ is the *rising factorial*. Substituting $x = 1/2$ in (2.13) and (2.14), he obtained respectively the formulae

$$\sum_{n=0}^{\infty} \frac{1}{2^{2n}} \frac{(1)_n^3}{\left(\frac{3}{2}\right)_n^3} (3n+2) = \frac{\pi^2}{4}, \qquad \sum_{n=0}^{\infty} \frac{1}{2^{6n}} \frac{(1)_n^3}{\left(\frac{3}{2}\right)_n^3} (21n+13) = 4\,\frac{\pi^2}{3}.$$

3. Formal Verification of Proof

In 1611, Kepler described the stacking of equal-sized spheres into the familiar arrangement we see for oranges in the grocery store. He asserted that this packing is the tightest possible. This assertion is now known as the Kepler conjecture, and has persisted for centuries without rigorous proof. Hilbert implicitly included the irregular case of the Kepler conjecture in problem 18 of his famous list of unsolved problems in 1900: *whether there exist non-regular space-filling polyhedra?* the regular case having been disposed of by Gauss in 1831.

In 1994, Thomas Hales, now at the University of Pittsburgh, proposed a five-step program that would result in a proof: (a) treat maps that only have triangular faces; (b) show that the face-centered cubic and hexagonal-close packings are local maxima in the strong sense that they have a higher score than any Delaunay star with the same graph; (c) treat maps that contain only triangular and quadrilateral faces (except the pentagonal prism); (d) treat maps that contain something other than a triangular or quadrilateral face; and (e) treat pentagonal prisms.

In 1998, Hales announced that the program was now complete, with Samuel Ferguson (son of mathematician-sculptor Helaman Ferguson) completing the crucial fifth step. This project involved extensive computation, using an interval arithmetic package, a graph generator, and *Mathematica*. The computer files containing the source code and computational results occupy more than three Gbytes of disk space. Additional details, including papers, are available at `http://www.math.pitt.edu/~thales/kepler98`. For a mixture of reasons—some more defensible than others—the *Annals of Mathematics* initially decided to publish Hales' paper with a cautionary note, but this disclaimer was deleted before final publication.

Hales [19] has now embarked on a multi-year program to certify the proof by means of computer-based formal methods, a project he has named the "Flyspeck" project. As these techniques become better understood, we can envision a large number of mathematical results eventually being confirmed by computer, as instanced by other articles in the same issue of the *Notices* as Hales' article.

4. Limits of Computation

A remarkable example is the following:

$$\int_0^\infty \cos(2x) \prod_{n=1}^\infty \cos(x/n)\, \mathrm{d}x \quad = \tag{4.1}$$

$$0.39269908169872415480783042290993786052464\underline{5434187231595926}\ldots$$

The computation of this integral to high precision can be performed using a scheme described in [5]. When we first did this computation, we thought that the result was $\pi/8$, but upon careful checking with the numerical value

$$0.39269908169872415480783042290993786052464\underline{6174921888227621}\ldots,$$

it is clear that the two values disagree beginning with the 43rd digit!

Richard Crandall [14, §7.3] later explained this mystery. Via a physically motivated analysis of *running out of fuel* random walks, he showed that $\pi/8$ is given by the following very rapidly convergent series expansion, of which formula (4.1) above is merely the first term:

$$\frac{\pi}{8} \quad = \quad \sum_{m=0}^\infty \int_0^\infty \cos[2(2m+1)x] \prod_{n=1}^\infty \cos(x/n)\, \mathrm{d}x. \tag{4.2}$$

Two terms of the series above suffice for 500-digit agreement.

As a final sobering example, we offer the following "sophomore's dream" identity

$$\begin{aligned}\sigma_{29} &:= \sum_{n=-\infty}^{\infty} \operatorname{sinc}(n)\operatorname{sinc}(n/3)\operatorname{sinc}(n/5)\cdots\operatorname{sinc}(n/23)\operatorname{sinc}(n/29)\\ &= \int_{-\infty}^{\infty} \operatorname{sinc}(x)\operatorname{sinc}(x/3)\operatorname{sinc}(x/5)\cdots\operatorname{sinc}(x/23)\operatorname{sinc}(x/29)\,dx,\end{aligned} \tag{4.3}$$

where the denominators range over the odd primes, which was first discovered empirically. More generally, consider

$$\begin{aligned}\sigma_{p} &:= \sum_{n=-\infty}^{\infty} \operatorname{sinc}(n)\operatorname{sinc}(n/3)\operatorname{sinc}(n/5)\operatorname{sinc}(n/7)\cdots\operatorname{sinc}(n/p)\\ &\stackrel{?}{=} \int_{-\infty}^{\infty} \operatorname{sinc}(x)\operatorname{sinc}(x/3)\operatorname{sinc}(x/5)\operatorname{sinc}(x/7)\cdots\operatorname{sinc}(x/p)\,dx.\end{aligned} \tag{4.4}$$

Provably, the following is true: The "sum equals integral" identity, for σ_p remains valid at least for p among the first 10176 primes; but stops holding after some larger prime, and thereafter the "sum less the integral" is strictly positive, but *they always differ by much less than one part in a googolplex* $= 10^{100}$. An even stronger estimate is possible assuming the Generalized Riemann Hypothesis (see [14, §7] and [7]).

5. Concluding Remarks

The central issues of how to view experimentally discovered results have been discussed before. In 1993, Arthur Jaffe and Frank Quinn warned of the proliferation of not-fully-rigorous mathematical results and proposed a framework for a "healthy and positive" role for "speculative" mathematics [20]. Numerous well-known mathematicians responded [1]. Morris Hirsch, for instance, countered that even Gauss published incomplete proofs, and the 15,000 combined pages of the proof of the classification of finite groups raises questions as to when we should certify a result. He suggested that we attach a label to each proof – e.g., "computer-aided," "mass collaboration," "constructive," etc. Saunders Mac Lane quipped that "we are not saved by faith alone, but by faith and works," meaning that we need both intuitive work and precision.

At the same time, computational tools now offer remarkable facilities to confirm analytically established results, as in the tools in development to check identities in equation-rich manuscripts, and in Hales' project to establish the Kepler conjecture by formal methods.

The flood of information and tools in our information-soaked world is unlikely to abate. We have to learn and teach judgment when it comes to using what is possible digitally. This means mastering the sorts of techniques we have illustrated and having some idea why a software system does what it does. It requires knowing when a computation is or can—in principle or practice—be made into a rigorous proof and when it is only compelling evidence, or is entirely misleading. For instance, even the best commercial linear programming packages of the sort used by Hales will not certify any solution though the codes are almost assuredly correct. It requires rearranging hierarchies of what we view as hard and as easy.

It also requires developing a curriculum that carefully teaches experimental computer-assisted mathematics. Some efforts along this line are already underway by individuals including Marc Chamberland at Grinnell (`http://www.math.grin.edu/~chamberl/courses/MAT444/syllabus.html`), Victor Moll at Tulane, Jan de Gier in Melbourne, and Ole Warnaar at University of Queensland.

Judith Grabner has noted that a large impetus for the development of modern rigor in mathematics came with the Napoleonic introduction of regular courses: lectures and textbooks force a precision and a codification that apprenticeship obviates. But it will never be the case that quasi-inductive mathematics supplants proof. We need to find a new equilibrium. That said, we are only beginning to tap new ways to enrich mathematics. As Jacques Hadamard said [24]:

> The object of mathematical rigor is to sanction and legitimize the conquests of intuition, and there was never any other object for it.

Never have we had such a cornucopia of ways to generate intuition. The challenge is to learn how to harness them, how to develop and how to transmit the necessary theory and practice. The Priority Research Centre for Computer Assisted Research Mathematics and its Applications (CARMA), `http://www.newcastle.edu.au/research/centres/carmacentre.html`, which one of us directs, hopes to play a lead role in this endeavor: an endeavor which in our view encompasses an exciting mix of exploratory experimentation and rigorous proof.

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Who we are and how we got that way

Revised for MAA volume on **The Mind of a Mathematician**

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Abstract. The typical research mathematician's view of the external world's view of mathematicians is more pessimistic and less nuanced than any objective measure would support. I shall explore some of the reasons why I think this is so. I submit that mathematics is a "science of the artificial" [18] and that we should wholeheartedly embrace such a positioning of our subject.

1 Putting Things in Perspective

> All professions look bad in the movies ... why should scientists expect to be treated differently? —Michael Crichton[1]

I greatly enjoyed Steve Krantz's article in this collection that he showed me when I asked him to elaborate what he had in mind. I guess I am less pessimistic than he is. This may well reflect the different milieus we have occupied. I see the same glass but it is half full.

Some years ago, my brother Peter surveyed other academic disciplines. He discovered that students who complain mightily about calculus professors still prefer the relative certainty of what we teach and assess to the subjectivity of a creative writing course or the rigors of a physics or chemistry laboratory course. Similarly, while I have met my share of micro-managing Deans—who view mathematics with disdain when they look at the size of our research grants or the infrequency of our patents—I have encountered more obstacles to mathematical innovation within than without the discipline.

I do wish to aim my scattered reflections in generally the right direction: I am more interested in issues of creativity á la Hadamard [4] than in Russell and foundations or Piaget and epistemology... and I should like a dash of "goodwill computing" thrown in.

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[1] Addressing the 1999 AAAS Meetings, as quoted in *Science* of Feb. 19, 1999, p.1111.

More seriously, I wish to muse about how we work, what keeps us going, how the mathematics profession has changed and how "*la plus ca change, la plus ca reste la même*",[2] and the like while juxtaposing how we perceive these matters and how we are perceived. Elsewhere, I have discussed at length my own views about the nature of mathematics from both an aesthetic and a philosophical perspective (see, e.g., [10, 19]). I have described my self as 'a computer-assisted quasi-empiricist'. For present more psychological proposes I will quote approvingly from [5, p. 239]:

> ... Like 0l' Man River, mathematics just keeps rolling along and produces at an accelerating rate "*200,000 mathematical theorems of the traditional handcrafted variety ... annually.*" Although sometimes proofs can be mistaken—sometimes spectacularly—and it is a matter of contention as to what exactly a "proof" is—there is absolutely no doubt that the bulk of this output is correct (though probably uninteresting) mathematics.— Richard C. Brown

Why do we produce so many unneeded results? In addition to the obvious pressure to publish and to have something to present at the next conference, I suspect Irving Biederman's observations below plays a significant role.

> While you're trying to understand a difficult theorem, it's not fun," said Biederman, professor of neuroscience in the USC College of Letters, Arts and Sciences. ... "But once you get it, you just feel fabulous." ... The brain's craving for a fix motivates humans to maximize the rate at which they absorb knowledge, he said. ... "I think we're exquisitely tuned to this as if we're junkies, second by second."—Irving Biederman[3]

Take away all success or any positive reinforcement and most mathematicians will happily replace research by adminstration, more and (hopefully better) teaching, or perhaps just a favourite hobby. But given just a little stroking by colleagues or referees and the occasional opiate jolt, and the river rolls on.

The pressure to publish is unlikely to abate and qualitative measurements of performance[4] are for the most part fairer than leaving everything to the whim of one's Head of Department. Thirty years ago my career review consisted of a two-line mimeo "*your salary for next year will be ...*" with the relevant number written in by hand. At the same time, it is a great shame that mathematicians have a hard time finding funds to go to conferences just to listen and interact. Csikszentmihalyi [6] writes:

> [C]reativity results from the interaction of a system composed of three elements: a culture that contains symbolic rules, a person who brings novelty into the symbolic domain, and a field of experts who recognize and validate the innovation. All three are necessary for a creative idea, product, or discovery to take place.—Mihalyy Csikszentmihalyi

[2]For an excellent account of the triumphs and vicissitudes of Oxford mathematics over eight centuries see [8]. The description of Haley's ease in acquiring equipment (telescopes) and how he dealt with inadequate money for personnel is by itself worth the price of the book.,

[3]Discussing his article in the *American Scientist* at `www.physorg.com/news70030587.html`

[4]For an incisive analysis of citation metrics in mathematics I thoroughly recommend the recent IMU report and responses at: `http://openaccess.eprints.org/index.php?/archives/417-Citation-Statistics-International-Mathematical-Union-Report.html`.

We have not paid enough attention to what creativity is and how it is nurtured. Conferences need audiences and researchers need feedback other than the mandatory "*nice talk*" at the end of a special session. We have all heard distinguished colleagues mutter a stream of criticism during a plenary lecture only to proffer "*I really enjoyed that*" as they pass the lecturer on the way out. A communal view of creativity requires more of the audience.

2 Who We Are

As to who we are? Sometimes we sit firmly and comfortably in the sciences. Sometimes we practice—as the *Economist* noted—the most inaccessible of the arts[5] possessed in Russell's terms [17, p. 60] of *"a supreme beauty—a beauty cold and austere.*" And sometimes we sit or feel we sit entirely alone. So forgive me if my categorizations slip and slide a bit. Even when we wish to remove ourselves from the sciences—by dint perhaps of our firm deductive underpinnings—they are often more than welcoming. They largely fail to see the stark deductive/inductive and realist/idealist distinctions which reached their apogee in the past century.

Yet many scientists have strong mathematical backgrounds. A few years ago I had the opportunity to participate as one of a team of seven scientists and one humanist who were mandated to write a national report on Canada's future need for advanced computing [14]. Five of us had at least an honours degree in mathematics. At the time none of us (myself included) lived in a mathematics department. The human genome project, the burgeoning development of financial mathematics, finite element modeling, Google and much else have secured the role of mathematics within modern science and technology research and development as "*the language of high technology*"; the most sophisticated language humanity has ever developed. Indeed, in part this scientific ecumenism reflects what one of my colleagues has called "*an astonishing lack of appreciation for how mathematics is done.*" He went on to remark that in this matter we are closer to the fine arts.

Whenever I have worked on major interdisciplinary committees, my strong sense has been of the substantial respect and slight sense of intimidation that most other quantitative scientists have for mathematics. I was sitting on a multi-science national panel when Wiles' proof of Fermat's last theorem was announced. My confreres wanted to know "*What, why and how?*" 'What' was easy, as always 'why' less so, and I did not attempt 'how'. In [10] I wrote

> While we mathematicians have often separated ourselves from the sciences, they have tended to be more ecumenical. For example, a recent review of *Models. The Third Dimension of Science*[6] chose a mathematical plaster model of a Clebsch diagonal surface as its only illustration. Similarly, authors seeking examples of the aesthetic in science often choose iconic mathematics formulae such as $E = MC^2$.

[5] In "Proof and Beauty," *Economist* article, 31 Mar 2005. "*Why should the non-mathematician care about things of this nature? The foremost reason is that mathematics is beautiful, even if it is, sadly, more inaccessible than other forms of art."*

[6] See Julie K. Brown, "Solid Tools for Visualizing Science," *Science,* November 19, 2004, 1136–37.

‘How’ is not easy even within mathematics. *A Passion for Science* [21] is the written record of thirteen fascinating BBC interviews with scientists including Nobelist Abdus Salam, Stephen Jay Gould, Michael Berry and Christopher Zeeman. The communalities of their scientific experiences far outstrip the differences. Zeeman tells a nice story of how his Centre’s administrator (a non mathematician) in Warwick could tell whether the upcoming summer was dedicated to geometry and topology, to algebra, or to analysis—purely on the basis of their domestic arrangements and logistics. For instance algebraists were very precise in their travel plans, topologists very inclusive in their social group activities and analysts were predictably unpredictable. I won’t spoil the anecdote entirely but it reinforces my sense that the cognitive differences between those three main divisions of pure mathematics are at least as great as those with many cognate fields. In this taxonomy I am definitely an analyst not a geometer or an algebraist.

There do appear to be some cognitive communalities across mathematics. In [4] my brother with Peter Liljedahl and Helen Zhai report on the responses to an updated version of Hadamard’s questionnaire [13] which they circulated to a cross-section of leading living mathematicians. This was clearly a subject the target group wanted to speak about. The response rate was excellent (over 50%) and the answers striking. According to the survey responses, the respondents placed a high premium on serendipity—but as Pasteur observed “*fate favours the prepared mind.*” Judging by where they said they have their best ideas they take frequent showers and like to walk while thinking. They don’t read much mathematics, preferring to have mathematics explained to them in person. They much more resemble theorists throughout the sciences than careful methodical scholars in the humanities.

My academic life started in the short but wonderful infusion of resources for science and mathematics ‘after sputnik’—I started University in 1967— and now includes the *Kindle Reader* (on which I am listening[7] to a fascinating new biography of the *Defense Advanced Research Projects Agency*, DARPA). The tyranny of a Bourbaki-dominated curriculum has been largely replaced by the scary grey-literature world of *Wikipedia* and *Google scholar*.

While typing this paragraph I went out on the web and found the Irving Layton poem, that I quote at the start of Section 4, in entirety within seconds (I merely googled “*And me happiest when I compose poems*” (I know the poem is *somewhere* in my personal library). For the most part this has been a wonderful journey. Not everything has improved from that halcyon pre-post-structuralist period a half-century ago when algebraists could command more attention from funding agencies than could engineers as [5] recalls. But the sense of time for introspection before answering a colleague’s wafer-thin ‘airmail letter’ enquiry, and the smell of mold that accompanied leisurely rummaging in a great library’s stacks are losses in my personal life measure the receding role of the University as the “*last successful medieval institution.*”[11]

2.1 Stereotypes from without looking in

One of the epochal events of my childhood as a faculty brat in St. Andrews, Scotland was when C. P. Snow (1905–1980) delivered an immediately controversial 1959 Rede

[7]They will read to you in a friendly if unnatural voice.

Lecture in Cambridge entitled "The Two Cultures".[8] Snow argued that the breakdown of communication between the "two cultures" of modern society—the sciences and the humanities—was a major obstacle to solving the world's problems—and he had never heard of global warming. In particular, he noted the quality of education was everywhere on the decline. Instancing that many scientists had never read Dickens, while those in the humanities were equally non-conversant with science, he wrote:

> A good many times I have been present at gatherings of people who, by the standards of the traditional culture, are thought highly educated and who have with considerable gusto been expressing their incredulity at the illiteracy of scientists. Once or twice I have been provoked and have asked the company how many of them could describe the Second Law of Thermodynamics, the law of entropy. The response was cold: it was also negative. Yet I was asking something which is about the scientific equivalent of: *'Have you read a work of Shakespeare's?'*

The British musical satirists Michael Flanders and Donald Swann took immediate heed of this for their terrific monologue and song "First and Second Law of Thermodynamics" that I can still recite from memory.

> [Michael:] Snow says that nobody can consider themselves educated who doesn't know at least the basic language of Science. I mean, things like Sir Edward Boyle's Law, for example: the greater the external pressure, the greater the volume of hot air. Or the Second Law of Thermodynamics - this is very important. I was somewhat shocked the other day to discover that my partner not only doesn't know the Second Law, he doesn't even know the First Law of Thermodynamics.
>
> Going back to first principles, very briefly, thermodynamics is of course derived from two Greek words: *thermos*, meaning hot, if you don't drop it, and *dinamiks*, meaning dynamic, work; and thermodynamics is simply the science of heat and work and the relationships between the two, as laid down in the Laws of Thermodynamics, which may be expressed in the following simple terms...
>
> After me...
>
> The First Law of Thermodymamics:
> Heat is work and work is heat
> Heat is work and work is heat
> Very good!
> The Second Law of Thermodymamics:
> Heat cannot of itself pass from one body to a hotter body
> (scat music starts)
> Heat cannot of itself pass from one body to a hotter body
> Heat won't pass from a cooler to a hotter
> Heat won't pass from a cooler to a hotter
> You can try it if you like but you far better notter
> You can try it if you like but you far better notter
> 'Cos the cold in the cooler will get hotter as a ruler
>
> ...

[8] Subsequently republished in [20].

Snow goes on to say:

> I now believe that if I had asked an even simpler question - such as, *What do you mean by mass, or acceleration*, which is the scientific equivalent of saying, *'Can you read?'* - not more than one in ten of the highly educated would have felt that I was speaking the same language. So the great edifice of modern physics goes up, and the majority of the cleverest people in the western world have about as much insight into it as their Neolithic ancestors would have had.

C. P. Snow wrote pre-Kuhn, pre-Foucault, pre-much else [5]; and I submit that a half-century on the situation is worse, knowledge more fragmented, ignorance of science and mathematics more damaging to the public discourse.[9]

In addition, I think the problem was much less symmetric than Snow suggested. I doubt I have ever met a scientist who had not read (or at least watched on BBC) some Dickens, who never went to movies, art galleries or the theatre. It is, however, ever more socially acceptable to be a scientific ignoramus or a mathematical dunce. It is largely allowed to boast "*I was never any good at mathematics at school.*" I was once told exactly that—in soto voce—by the then Canadian Governor General during a formal ceremony at his official residence in Ottawa. Even here we should be heedful not to over-analyse as we are prone to do. Afterwards the 'GG' (as Canadians call their Queen's designate) ruminated apologetically that if he had been a bit better at mathematics he would not have had to become a journalist. Some of this has been 'legitimated' by denigrating science as 'reductionist' and incapable of the deeper verities [5].

As Underwood Dudley has commented, no one apologizes for not being good at geology in school. Most folks understand that failing "Introduction to Rocks" in Grade Nine does not knock you off of a good career path. The outside world knows several truths: mathematics is important, it is hard, it is usually poorly taught in school, and the average middle-class parent is ill-prepared to redress the matter. I have become quite hard-line about this. When a traveling companion on a plane starts telling me that "*Mathematics was my worst subject in school.*" I will reply "*And if you were illiterate would you tell me*?" They usually take the riposte fairly gracefully.

Consider two currently popular TV dramas *Numb3rs* (mathematical) and *House* (medical). A few years ago a then colleague, a distinguished pediatrician, asked me whether I watched Num3rs. I replied "*Do you watch House? Does it sometimes make you cringe?*" He admitted that it did but he still watched it. I said the same was true for me with Numb3rs, that my wife loved it and that I liked lots about it. It made mathematics seem important and was rarely completely off base. The lead character, Charlie, was brilliant and good-looking with a cute smart girl friend. The resident space-cadet on the show was a physicist not a mathematician. What more could one ask for? Sadly for many of our

[9] I can't resist including the following email anecdote:

> This morning Al Gore gave the "keynote" speech at SC09.[10] During the question-answer period, he mentioned a famous talk "The Two Cultures" about lack of communication between science and humanities, by one Chester (??)—he drew a blank as to who it was. Sitting on the third row, I shouted out "Snow" (meaning C. P. Snow). One other person also shouted "Snow", and so Gore acknowledged that it was indeed Snow.

.

colleagues the answer is "*absolute fidelity to mathematical truth in every jot and title.*" No wonder so many of us make a dog's-breakfast of the opportunities given to publicize our work!

> 'Caution, skepticism, scorn, distrust and entitlement seem to be intrinsic to many of us because of our training as scientists.—Stephen Rosen[11]

To "*Caution, skepticism, scorn, distrust and entitlement,*" I'd add "*persistence, intensity, a touch of paranoia, and a certain lack of sartorial elegance*" but I still would not have identified mathematicians within the larger scientific herd. I think we are more inward drawn than theorists in, say, biology or physics. Our terminologies are more speciated between subfields and so we typically graze in smaller groups. But we are still bona fide scientists—contrary to the views of some laboratory scientists and some of our own colleagues.

> This is the essence of science. Even though I do not understand quantum mechanics or the nerve cell membrane, I trust those who do. Most scientists are quite ignorant about most sciences but all use a shared grammar that allows them to recognize their craft when they see it. The motto of the Royal Society of London is 'Nullius in verba' : trust not in words. Observation and experiment are what count, not opinion and introspection. Few working scientists have much respect for those who try to interpret nature in metaphysical terms. For most wearers of white coats, philosophy is to science as pornography is to sex: it is cheaper, easier, and some people seem, bafflingly, to prefer it. Outside of psychology it plays almost no part in the functions of the research machine.—Steve Jones[12]

2.2 Stereotypes from within looking out

Philosophy (not to mention introspection) is arguably more important to, though little more respected by, working mathematicians than it is to experimental scientists.

> Whether we scientists are inspired, bored, or infuriated by philosophy, all our theorizing and experimentation depends on particular philosophical background assumptions. This hidden influence is an acute embarrassment to many researchers, and it is therefore not often acknowledged. Such fundamental notions as reality, space, time, and causality–notions found at the core of the scientific enterprise–all rely on particular metaphysical assumptions about the world. —Christof Koch[13]

As I alluded to above, working mathematicians—by which I mean those of my personal or professional acquaintance—are overinclined by temperament and training to see meaning where none is intended and patterns where none exist. For the most part over

[11]An astrophysicist, turned director of the *Scientific Career Transitions Program* in New York City, giving job-hunting advice in an on-line career counseling session as quoted in *Science*, August 4 1995, p. 637. He continues that these traits hinder career change!

[12]From his review of "How the Mind Works" by Steve Pinker, in the *New York Review of Books*, pp. 13-14, Nov 6, 1997.

[13]In "Thinking About the Conscious Mind," a review of John R. Searle's *Mind. A Brief Introduction*, Oxford University Press, 2004.

the past centuries this somewhat autistic tendency has been a positive adaptation. It has allowed the discipline to develop the most powerful tools and most sophisticated descriptive language possessed by mankind. But as the nature of mathematics changes we should be heedful of Napoleon's adage "*Never ascribe to malice that which is adequately explained by incompetence,*"[14] or as Goethe (1749-1832) put it in [12]:

> Misunderstandings and neglect occasion more mischief in the world than even malice and wickedness. At all events, the two latter are of less frequent occurrence.

Suppose for 'malice/wickedness' we substitute 'meaning/reason' and likewise replace 'incompetence/misunderstandings' by 'chance/randomness'. Then these squibs provide an important caution against seeing mathematical patterns where none exist. They offer equally good advice when dealing with Deans.

3 Changing Modes of Doing Mathematics

Goethe's advice is especially timely as we enter an era of intensive computer-assisted mathematical data-mining; an era in which we will more-and-more encounter unprovable truths and salacious falsehoods. In [10] I wrote

> It is certainly rarer to find a mathematician under thirty who is unfamiliar with at least one of *Maple*, *Mathematica* or MATLAB, than it is to find one over sixty five who is really fluent. As such fluency becomes ubiquitous, I expect a re-balancing of our community's valuing of deductive proof over inductive knowledge.

As we again become comfortable with mathematical *discovery* in Giaquinto's sense of being "*independent, reliable and rational*" [9], assisted by computers, the community sense of a mathematician as a producer of theorems will probably diminish to be replaced by a richer community sense of mathematical understanding. It has been said that Riemann proved very few theorems and even fewer correctly and yet he is inarguably one of the most important mathematical, indeed scientific, thinkers of all time. Similarly most of us were warned off pictorial reasoning:

> A heavy warning used to be given [by lecturers] that pictures are not rigorous; this has never had its bluff called and has permanently frightened its victims into playing for safety. Some pictures, of course, are not rigorous, but I should say most are (and I use them whenever possible myself).—J. E. Littlewood [16, p. 53][15]

Let me indicate how much one can now do with good computer-generated pictures.

3.1 Discovery and proof: Divide-and-concur

In a wide variety of problems such as protein folding, 3SAT, spin glasses, giant Sodoku, etc., we wish to find a point in the intersection of two sets A and B where B is non-convex.

[14] I have collected variants old and new on the theme of over-ratiocination at `www.carma.newcastle.edu/jb616/quotations.html`

[15] Littlewood (1885-1977) published this in 1953 and so long before the current fine graphic, geometric, and other visualization tools were available.

The notion of "divide-and-concur" as described below often works spectacularly—much better than theory can currently explain. Let $P_A(x)$ and $R_A(x) := 2\,P_A(x) - x$ denote respectively the *projector* and *reflector* on a set A as illustrated in Figure 1. Then "divide-and-concur"[16] is the natural geometric iteration "reflect-reflect-average":

$$x_{n+1} \Longrightarrow \frac{x_n + R_A\left(R_B(x_n)\right)}{2}. \tag{1}$$

Consider the simplest case of a line A of height α and the unit circle B [2]. With $z_n := (x_n, y_n)$ we have:

$$x_{n+1} := \cos\theta_n, y_{n+1} := y_n + \alpha - \sin\theta_n, \quad (\theta_n := \arg z_n). \tag{2}$$

This is intended to find a point on the intersection of the unit circle and the line of height α as shown in Figure 2 for $\alpha = .94$.

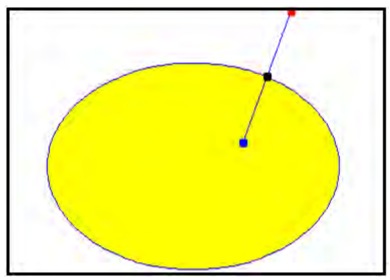

Figure 1: Reflector (interior) and Projector (boundary) of a point external to an ellipse.

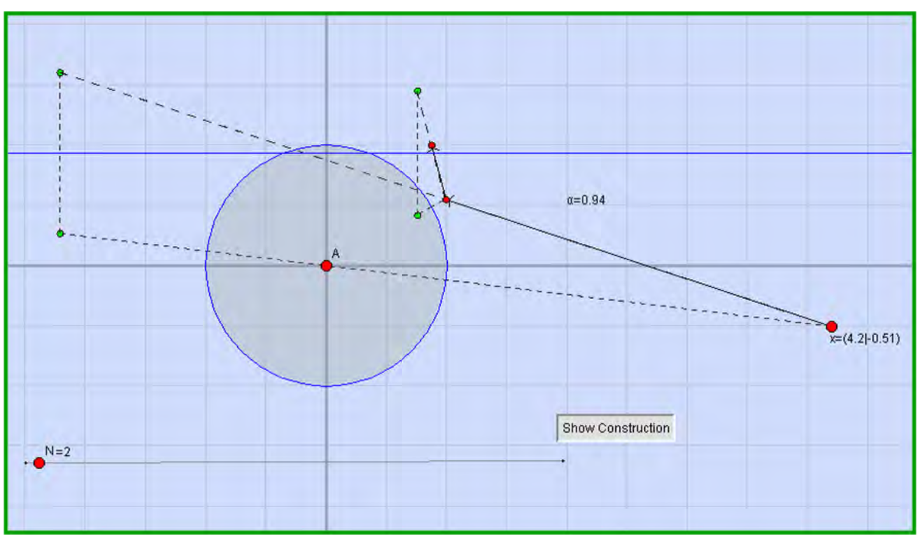

Figure 2: The first three iterates of (2) in *Cinderella*.

[16]This is Cornell physicist Veit Elser's slick term for the algorithm in which the reflection can be performed on separate cpu's (divide) and then averaged (concur).

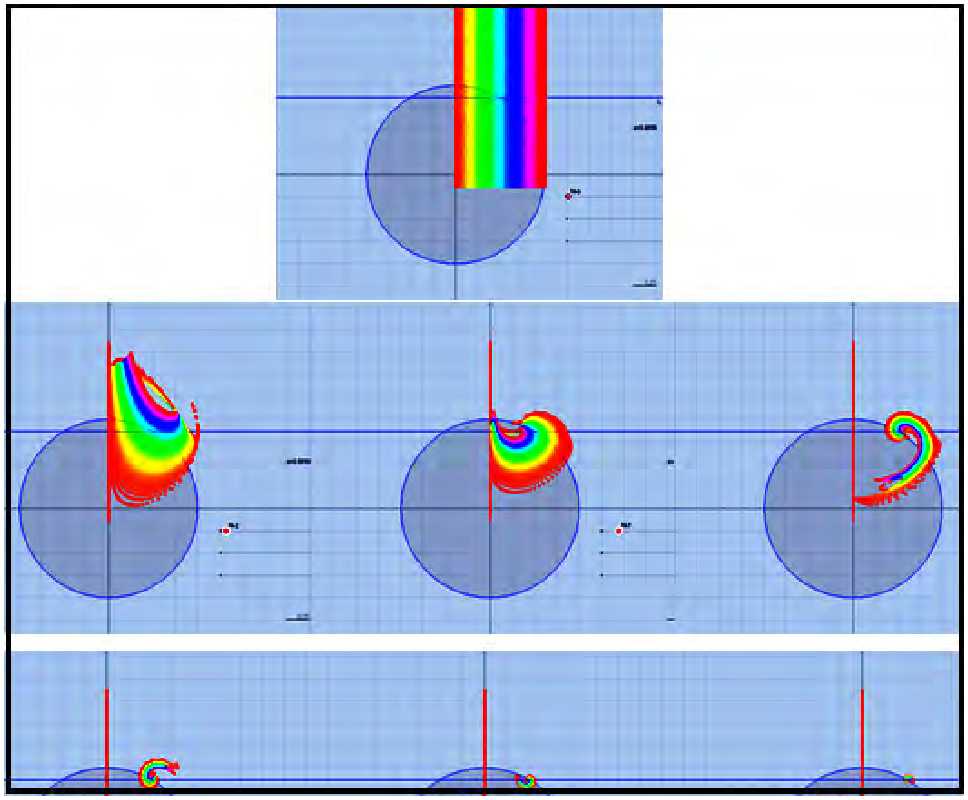

Figure 3: Snapshots of 10,000 points after $0, 2, 7, 13, 16, 21$, and 27 steps of (2).

We have also studied the analogous differential equation since asymptotic techniques for such differential equations are better developed. We decided

$$(3) \qquad \begin{aligned} x'(t) &= \frac{x(t)}{r(t)} - x(t) \text{ where } r(t) := \sqrt{x(t)^2 + y(t)^2}\,] \\ y'(t) &= \alpha - \frac{y(t)}{r(t)} \end{aligned}$$

was a reasonable counterpart to the Cartesian formulation of (2)—we have replaced the difference $x_{n+1} - x_n$ by $x'(t)$, etc.—as shown in Figure 4.

Following Littlewood, I find it hard to persuade myself that the pictures in Figures 3 and 4 do not constitute a *generic proof* of the algorithms they display as implemented in an applet at `http://users.cs.dal.ca/~jborwein/expansion.html`. In Figure 3 we see the iterates spiralling in towards the right-hand point of intersection with those closest to the y-axis lagging behind but being unremittingly reeled in to the point. Brailey Sims and I have now found a conventional proof that the behaviour is as observed [3] but we discovered all the results first graphically and were lead to the appropriate proofs by the dynamic pictures we drew.

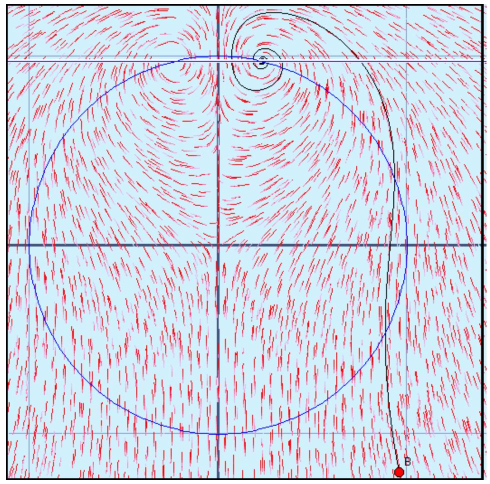

Figure 4: ODE solution and vector field for (3) with $\alpha = 0.97$ in *Cinderella*.

4 The Exceptionalism of Mathematics

And me happiest when I compose poems.
Love, power, the huzza of battle
Are something, are much;
yet a poem includes them like a pool.—Irving Layton [15, p. 189]

This is the first stanza of the Irving Layton (1912-2006) poem "The Birth of Tragedy." Explicitly named after Nietzsche's first book, Layton tussles with Apollonian and Dionysian impulses (reason versus emotion). He calls himself "A quiet madman, never far from tears" and ends "*while someone from afar off blows birthday candles for the world.*" Layton, who was far from a recluse, is one of my favourite Canadian poets.

I often think poetry is a far better sustained metaphor for mathematics than either music or the plastic arts. I do not see poetry making such a good marriage with any other science. Like good poets, good mathematicians are often slightly autistic observers of a somewhat dysphoric universe. Both art forms at their best distill and concentrate beauty like no other and both rely on a delicate balance of form and content, semantics and syntax.

Like all academic disciplines we are (over-)sure of our own specialness.

- *Mathematicians are machines for turning coffee into theorems.* (Renyi)
- *A gregarious mathematician is one who looks at the other person's feet when addressing them.*
- *Mathematics is what mathematicians do late at night.*
- *You want proof. I'll give you proof.* (Harris)
- *There are three kinds of mathematician, those who can count and those who can't.*

Most of these can be—and many have been— used with a word changed here or there about statisticians, computer scientists, chemists, physicists, economists and philosophers.

For instance "*There are 10 kinds of computer scientists, those who understand binary and those who don't.*" It is amusing to ask colleagues in other sciences for their corresponding self-identifying traits. All of the above mentioned groups except the philosophers are pretty much reductionists:

> Harvard evolutionary psychologist Steven Pinker is probed on "*Evolutionary Psychology and the Blank Slate.*" The conversation moves from the structure of the brain to adaptive explanations for music, creationism, and beyond. Stangroom asks Pinker about the accusations that biological explanations of behavior are determinist and reduce human beings to the status of automatons."..."*Most people have no idea what they mean when they level the accusation of determinism,*" "Pinker answers. "*It's a nonspecific "boo" word, intended to make something seem bad without any content.*[17]

Steve Jones is quoted in the same article equating philosophy and pornography and while many of us, myself included, see a current need to rethink the philosophy of mathematics, Pinker and he capture much of the zeitgeist of current science including mathematics.

4.1 Mathematics as a science of the artificial

Pure mathematics, theoretical computer science, and various cognate disciplines are sciences of the artificial in that they study *scientifically* man-made *artificial* concepts. Mathematical experiments and data collection are clearly not taking place in the natural world. They are at best quasi-empirical and yet they subscribe fully to the scientific method. Like other sciences they are increasingly engaged in "exploratory experimentation" [1, 2]. In *The Sciences of the Artificial* [18, p. 16] Herb Simon compellingly wrote about reductionism:

> This skyhook-skyscraper construction of science from the roof down to the yet unconstructed foundations was possible because the behaviour of the system at each level depended only on a very approximate, simplified, abstracted characterization at the level beneath.[1] This is lucky, else the safety of bridges and airplanes might depend on the correctness of the "Eightfold Way" of looking at elementary particles.
>
> > [1] "More than fifty years ago Bertrand Russell made the same point about the architecture of mathematics. See the "Preface" to Principia Mathematica "*... the chief reason in favour of any theory on the principles of mathematics must always be inductive, i.e., it must lie in the fact that the theory in question allows us to deduce ordinary mathematics. In mathematics, the greatest degree of self-evidence is usually not to be found quite at the beginning, but at some later point; hence the early deductions, until they reach this point, give reason rather for believing the premises because true consequences follow from them, than for believing the consequences because they follow from the premises.*"

[17] *The Scientist* of June 20, 2005 describing Jeremy Stangroom's interviews in *What (some) scientists say*, Routledge Press, 2005.

> Contemporary preferences for deductive formalisms frequently blind us to this important fact, which is no less true today than it was in 1910.

I love the fact that Russell the arch-deductivist so clearly describes the fundamental role of inductive reasoning within mathematics. This long-but-rewarding quote leads me to reflect that we mathematicians need more strong-minded and assured critics. I acknowledge that it is easier to challenge a speaker in history or philosophy. One may reasonably disagree in a way that is hard in mathematics.[18] When someone stands up in a mathematics lecture and says she can answer the speaker's hard open question, nine times out of ten the respondent has misunderstood the question or misremembered her own prior work. We do, however, need to develop a culture which encourages spirited debate of such matters as how best to situate our subject within the academy, how important certain areas and approaches are, how to balance research and scholarship, and so on. Moreover, fear and lack of mutual respect for another's discipline makes it hard to venture outside one's own niche. For instance, many physicists fear mathematicians who, in turn, are often most uncomfortable or dismissive of informal reasoning and of 'physical or economic intuition.'

4.2 Pure versus applied mathematics

Mathematics is at once both a set of indispensable tools and a self-motivating discipline; a mind-set and a way of thinking. In consequence there are many research mathematicians working outside mathematics departments and a smaller but still considerable number of non-mathematicians working within. What are the consequences? First, it is no longer possible to assume that all of one's colleagues could in principle—if not with enthusiasm or insight—teach all the mathematics courses in the first two years of the university syllabus. This pushes us in the direction of other disciplines like history or biology in which teaching has always been tightly coupled with core research competence.

At a more fundamental level, I see the discipline boundary as being best determined by answering the question as to whether the mathematics at issue is worth doing in its own right. If the answer is "yes", then it is 'pure'[19] mathematics and belongs in the discipline; if not then, however useful or important the outcome, it does not fit. The later would, for example, be the case of a lot of applied operations research, a good deal of numerical modeling and scientific computation, and most of statistics. All significant mathematics should be nourished within mathematics departments, but there are many important and useful applications that do not by that measure belong.

5 How to Become a Grownup Science

As Darwin [7] ruefully realized rather late in life, we mathematicians have a lot to offer:

[18] Some years ago I persuaded Amazon to remove several unsubstantiated assertions about "errors on every page" in one of my books—by a digital groupy turned stalker—from their website after I pointed out that while one could have an opinion that a Cormac McCarthy novel was dull but assertions of factual error were subject to test.

[19] Which may well be highly applicable.

> During the three years which I spent at Cambridge my time was wasted, as far as the academical studies were concerned, as completely as at Edinburgh and at school. I attempted mathematics, and even went during the summer of 1828 with a private tutor (a very dull man) to Barmouth, but I got on very slowly. The work was repugnant to me, chiefly from my not being able to see any meaning in the early steps in algebra. This impatience was very foolish, and in after years I have deeply regretted that I did not proceed far enough at least to understand something of the great leading principles of mathematics, for men thus endowed seem to have an extra sense.—Charles Darwin

We also have a lot to catch up with. We have too few accolades compared to other sciences: prize lectures, medals, fellowships and the like. We are insufficiently adept at boosting our own cases for tenure, for promotion or for prizes. We are frequently too honest in reference letters. We are often disgracefully terse—unaware of the need to make obvious to others what is for us blindingly obvious. I have seen a Fields medalist recommend a talented colleague for promotion with the one line letter "*Anne has done some quite interesting work.*" Leaving aside the ambiguity of the use of the word "quite" when sent by a European currently based in the United States to a North American promotion committee, this summary is pretty lame when compared to a three page letter for an astrophysicist or chemist—that almost always tells you the candidate is the top whatever-it-is in the field. A little more immodesty in promoting our successes is in order.

I'm not encouraging dishonesty, but it is necessary to understand the ground rules of the enterprise and to make some attempt to adjust to them. When a good candidate for a Rhodes Scholarship turns up at ones office, it should be obvious that a pro forma scrawled note

> Johnny is really smart and got an 'A+' in my advanced algebraic number theory class. You should give him a Rhodes scholarship.

is inadequate. Sadly, the only letters of that kind that I've seen in Rhodes scholarship dossiers have come from mathematicians.

I am a mathematician rather than a computational scientist or a computer scientist primarily because I savour the structures and curiosities (including *spandrels* and *exaptations* in Gould's words) of mathematics. I am never satisfied with my first proof of a result and until I have found limiting counter-examples and adequate corollaries will continue to worry at it. I like attractive generalizations on their own merits. Very often it is the unexpected and unintended consequences of a mathematical argument that when teased out provides the real breakthrough. Such often leads eventually to tangible and dramatic physical consequences: take quantum mechanical tunneling.

A few years ago I had finished a fine piece of work with a frequent collaborator who is a quantum field theorist—and a man of great insight and mathematical power. We had met success by introducing a sixth-root of unity into our considerations. I mooted looking at higher-order analogues. The reply came back "*God in her wisdom is happy to build the universe with sixth-roots. You, a mathematician, can look for generalizations if you wish.*"

6 Conclusion

I became a mathematician largely because mathematics satisfied four criteria. (i) I found it reasonably easy; (ii) I liked understanding or working out how things function; but (iii) I was not much good with my hands and had limited physical intuition; (iv) I really disliked pipettes but I wanted to be a scientist. That left mathematics. Artificial yes, somewhat introspective yes, but informed by many disciplines and clearly an important science.

I have had several students whom I can not imagine following any other life path but I was not one of those. I would I imagine have been happily fulfilled in various careers of the mind; say as an historian or an academic lawyer. But I became a mathematician. It has been and continues to be a pretty wonderful life.

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The Life of Pi: From Archimedes to ENIAC and Beyond [1]

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1 Preamble: Pi and Popular Culture

The desire to understand π, the challenge, and originally the need, to calculate ever more accurate values of π, the ratio of the circumference of a circle to its diameter, has challenged mathematicians–great and less great—for many many centuries and, especially recently, π has provided compelling examples of computational mathematics. Pi, uniquely in mathematics is pervasive in popular culture and the popular imagination.[2]

I shall intersperse this largely chronological account of Pi's mathematical status with examples of its ubiquity. More details will be found in the selected references at the end of the chapter—especially in *Pi: a Source Book* [5]. In [5] all material not otherwise referenced may be followed up upon, as may much other material, both serious and fanciful. Other interesting material is to be found in [16], which includes attractive discussions of topics such as continued fractions and elliptic integrals.

Fascination with π is evidenced by the many recent popular books, television shows, and movies—even perfume—that have mentioned π. In the 1967 *Star Trek* episode "Wolf in the Fold," Kirk asks "*Aren't there some mathematical problems that simply can't be solved?*" And Spock 'fries the brains' of a rogue computer by telling it: "*Compute to the last digit the value of Pi.*" The May 6, 1993 episode of The Simpsons has the character Apu boast "*I can recite pi to 40,000 places. The last digit is one.*" (See Figure 1.)

In November 1996, MSNBC aired a Thanksgiving Day segment about π, including that scene from Star Trek and interviews with the present author and several other mathematicians at Simon Fraser University. The 1997 movie *Contact*, starring Jodie Foster, was based on the 1986 novel by noted astronomer Carl Sagan. In the book, the lead character searched for patterns in the digits of π, and after her mysterious experience found sound confirmation in the base-11 expansion of π. The 1997 book *The Joy of Pi* [7] has sold many thousands of copies and continues to sell well. The 1998 movie entitled *Pi* began with decimal digits of π displayed on the screen. And in the 2003 movie *Matrix Reloaded*, the Key Maker warns that a door will be accessible for exactly 314 seconds, a number that *Time* speculated was a reference to π.

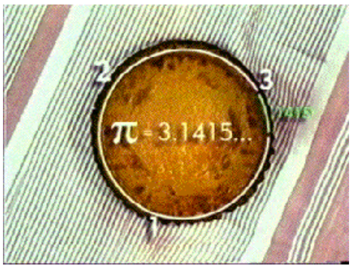
As a forceable example, imagine the following excerpt from Eli Mandel's 2002 Booker Prize winning novel *Life of Pi* being written about another transcendental number:

> "`My name is`
> `Piscine Molitor Patel`
> `known to all as Pi Patel.`
>
> *For good measure I added*
>
> $$\pi = 3.14$$
>
> *and I then drew a large circle which I sliced in two with a diameter, to evoke that basic lesson of geometry.*"

Equally, National Public Radio reported on April 12, 2003 that novelty automatic teller machine withdrawal slips, showing a balance of $314,159.26, were hot in New York City. One could jot a note on the back and,

[1] This paper is an updated and revised version of [9] and is made with permission of the editor.

[2] The *MacTutor* website, `http://www-gap.dcs.st-and.ac.uk/~history`, at the University of St. Andrews—my home town in Scotland—is rather a good accessible source for mathematical history.



Around 250 BCE, Archimedes of Syracuse (287–212 BCE) was the first to show that the “two possible Pi’s” are the same.

> Clearly for a circle of radius r and diameter d, **Area**$= \pi_1 r^2$ while **Perimeter** $= \pi_2 d$, but that $\pi_1 = \pi_2$ is not obvious.

This is often overlooked (Figure 4.).

Figure 1: π**’s original duality**

3 . 1415926535897932384626433832795028841971693993751058209749445923078164062862089986280348253421170679
8214808651328230664709384460955058223172535940812848111745028410270193852110555964462294895493038196
4428810975665933446128475648233786783165271201909145648566923460348610454326648213393607260249141273
7245870066063155881748815209209628292540917153643678925903600113305305488204665213841469519415116094
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9833673362440656643086021394946395224737190702179860943702770539217176293176752384674818467669405132
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4201995611212902196086403441815981362977477130996051870721134999999837297804995105973173281609631859
5024459455346908302642522308253344685035261931188171010003137838752886587533208381420617177669147303
5982534904287554687311595628638823537875937519577818577805321712268066130019278766111959092164201989 3

Figure 2: `1,001` **Decimal Digits of Pi**

apparently innocently, let the intended target be impressed by one’s healthy saving account. Scott Simon, the host, noted the close resemblance to π. Correspondingly, according to the *New York Times* of August 18 2005, Google offered exactly “14, 159, 265 New Slices of Rich Technology” as the number of shares in its then new stock offering. Likewise, March 14 in North America has become π *Day*, since in the USA the month is written before the day (‘314’). In schools throughout North America, it has become a reason for mathematics projects, especially focussing on Pi.

As another sign of true legitimacy, on March 14, 2007 the *New York Times* published a crossword in which to solve the puzzle, one had first to note that the clue for 28 DOWN was “March 14, to Mathematicians,” to which the answer is PIDAY. Moreover, roughly a dozen other characters in the puzzle are PI—for example, the clue for 5 DOWN was “More pleased” with the six character answer HAPπER. The puzzle is reproduced in [10]. Finally, in March 2009,Congress actually made PiDay an official annual national event!

It is hard to imagine e, γ or $\log 2$ playing the same role. A corresponding scientific example [3, p. 11] is

> *“A coded message, for example, might represent gibberish to one person and valuable information to another. Consider the number 14159265... Depending on your prior knowledge, or lack thereof, it is either a meaningless random sequence of digits, or else the fractional part of pi, an important piece of scientific information.”*

For those who know *The Hitchhiker’s Guide to the Galaxy*, it is amusing that `042` occurs at the digits ending at the fifty-billionth decimal place in each of π and $1/\pi$—thereby providing an excellent answer to the ultimate question, “*What is forty two?*” A more intellectual offering is “The Deconstruction of Pi” given by Umberto Eco on page three of his 1988 book *Foucault’s Pendulum*, [5, p. 658].

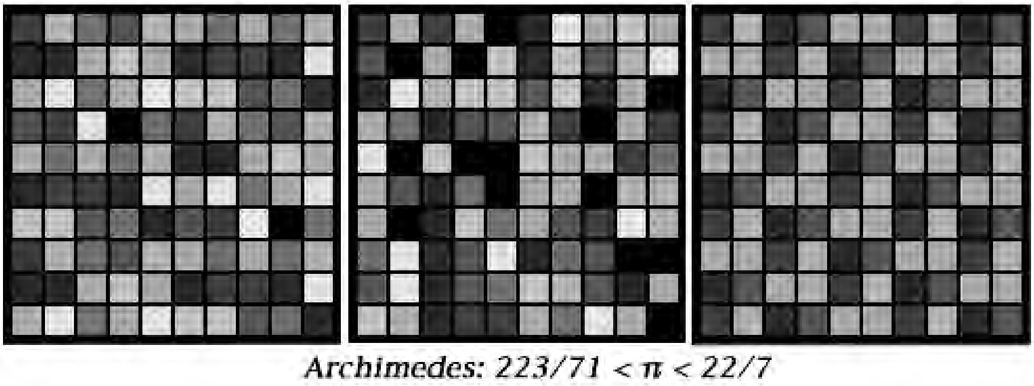


Figure 3: **A pictorial proof of Archimedes**

Pi. Our central character

$$\pi = 3.14159265358979323\ldots$$

is traditionally defined in terms of the area or perimeter of a unit circle, see Figure 1. The notation of π itself was introduced by William Jones in 1737, replacing 'p' and the like, and was popularized by Leonhard Euler who is responsible for much modern nomenclature. A more formal modern definition of π uses the first positive zero of sin defined as a power series. The first thousand decimal digits of Pi are recorded in Figure 2.

Despite continuing rumours to the contrary, π is not equal to 22/7 (see End Note 1). Of course 22/7 is one of the early continued fraction approximations to π. The first six convergents are

$$3, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{103993}{33102}, \frac{104348}{33215}.$$

The convergents are necessarily good rational approximations to π. The sixth differs from π by only $3.31\,10^{-10}$. The corresponding simple continued fraction starts

$$\pi = [3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, 1, 84, 2, 1, 1, \ldots],$$

using the standard concise notation. This continued fraction is still very poorly understood. Compare that for e which starts

$$e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1, 14, 1, 1, 16, 1, 1, 18, \ldots].$$

A proof of this observation shows that e is not a quadratic irrational since such numbers have eventually periodic continued fractions.

Archimedes' famous computation discussed below is:

(1) $$3\frac{10}{71} < \pi < 3\frac{10}{70}.$$

Figure 3 shows this estimate graphically, with the digits shaded modulo ten; one sees structure in 22/7, less obviously in 223/71, and not in π.

2 The Childhood of Pi

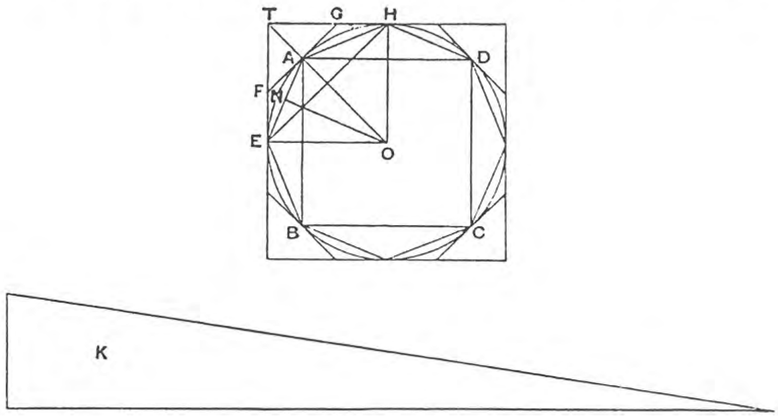
Four thousand years ago, the Babylonians used the approximation $3\frac{1}{8} = 3.125$. Then, or earlier, according to ancient papyri, Egyptians assumed a circle with diameter nine has the same area as a square of side eight, which implies $\pi = 256/81 = 3.1604\ldots$. Some have argued that the ancient Hebrews were satisfied with $\pi = 3$:

MEASUREMENT OF A CIRCLE.

Proposition 1.

The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference, of the circle.

Let $ABCD$ be the given circle, K the triangle described.



Archimedes' construction for the uniqueness of π, taken from his *Measurement of a Circle*

Figure 4: **Pi's uniqueness**

> *"Also, he made a molten sea of ten cubits from brim to brim, round in compass, and five cubits the height thereof; and a line of thirty cubits did compass it round about."* (I Kings 7:23; see also 2 Chronicles 4:2)

One should know that the cubit was a personal not universal measurement. In Judaism's further defense, several millennia later, the great Rabbi Moses ben Maimon Maimonedes (1135–1204) is translated by Langermann, in "The 'true perplexity' [5, p. 753] as fairly clearly asserting the Pi's irrationality.

> "*You ought to know that the ratio of the diameter of the circle to its circumference is unknown, nor will it ever be possible to express it precisely. This is not due to any shortcoming of knowledge on our part, as the ignorant think. Rather, this matter is unknown due to its nature, and its discovery will never be attained.*" (Maimonedes)

In each of these three cases the interest of the civilization in π was primarily in the practical needs of engineering, astronomy, water management and the like. With the Greeks, as with the Hindus, interest was centrally metaphysical and geometric.

Archimedes' Method. The first rigorous mathematical calculation of π was due to Archimedes, who used a brilliant scheme based on **doubling inscribed and circumscribed polygons**

$$6 \mapsto 12 \mapsto 24 \mapsto 48 \mapsto 96$$

and computing the perimeters to obtain the bounds $3\frac{10}{71} < \pi < 3\frac{1}{7}$, that we have recaptured above. The case of 6-gons and 12-gons is shown in Figure 5; for $n = 48$ one already 'sees' near-circles. Arguably no mathematics approached this level of rigour again until the 19th century.

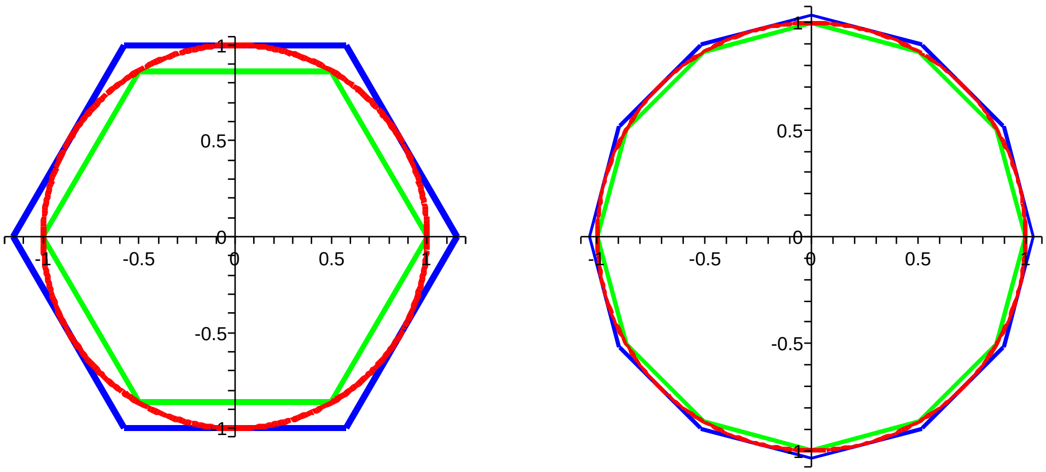


Figure 5: **Archimedes' method of computing π with 6- and 12-gons**

Archimedes' scheme constitutes the first true algorithm for π, in that it is capable of producing an arbitrarily accurate value for π. It also represents the birth of numerical and error analysis—all without positional notation or modern trigonometry. As discovered severally in the 19th century, this scheme can be stated as a simple, numerically stable, recursion, as follows [11].

Archimedean Mean Iteration (Pfaff-Borchardt-Schwab) . Set $a_0 = 2\sqrt{3}$ and $b_0 = 3$—the values for circumscribed and inscribed 6-gons. Then define

$$a_{n+1} = \frac{2a_n b_n}{a_n + b_n} \quad (H) \qquad b_{n+1} = \sqrt{a_{n+1} b_n} \quad (G). \tag{2}$$

This converges to π, with the error decreasing by a factor of four with each iteration. In this case the error is easy to estimate, the limit somewhat less accessible but still reasonably easy [10, 11].

Variations of Archimedes' geometrical scheme were the basis for all high-accuracy calculations of π for the next 1800 years—well beyond its 'best before' date. For example, in fifth century CE China, Tsu Chung-Chih used a variation of this method to get π correct to seven digits. A millennium later, Al-Kashi in Samarkand "*who could calculate as eagles can fly*" obtained 2π in *sexagecimal*:

$$2\pi \approx 6 + \frac{16}{60^1} + \frac{59}{60^2} + \frac{28}{60^3} + \frac{01}{60^4} + \frac{34}{60^5} + \frac{51}{60^6} + \frac{46}{60^7} + \frac{14}{60^8} + \frac{50}{60^9},$$

good to 16 decimal places (using $3 \cdot 2^{28}$-gons). This is a personal favourite, reentering it in my computer centuries later and getting the predicted answer gave me goose-bumps.

3 Pre-calculus Era π Calculations

In Figures 6, 8, and 11 we chronicle the main computational records during the indicated period, only commenting on signal entries.

Name	Year	Digits
Babylonians	2000? BCE	1
Egyptians	2000? BCE	1
Hebrews (1 Kings 7:23)	550? BCE	1
Archimedes	250? BCE	3
Ptolemy	150	3
Liu Hui	263	5
Tsu Ch'ung Chi	480?	7
Al-Kashi	1429	14
Romanus	1593	15
van Ceulen (**Ludolph's number***)	1615	35

Figure 6: **Pre-calculus π Calculations**

Little progress was made in Europe during the 'dark ages', but a significant advance arose in India (450 CE): *modern positional, zero-based decimal arithmetic*—the "Indo-Arabic" system. This greatly enhanced arithmetic in general, and computing π in particular. The Indo-Arabic system arrived with the Moors in Europe around 1000 CE. Resistance ranged from accountants who feared losing their livelihood to clerics who saw the system as 'diabolical'—they incorrectly assumed its origin was Islamic. European commerce resisted into the 18th century, and even in scientific circles usage was limited until the 17th century.

The prior difficulty of doing arithmetic is indicated by college placement advice given a wealthy German merchant in the 16th century:

> *"A wealthy (15th Century) German merchant, seeking to provide his son with a good business education, consulted a learned man as to which European institution offered the best training. 'If you only want him to be able to cope with addition and subtraction,' the expert replied, 'then any French or German university will do. But if you are intent on your son going on to multiplication and division—assuming that he has sufficient gifts—then you will have to send him to Italy.'"* (George Ifrah, [10])

Claude Shannon (1916–2001) had a mechanical calculator wryly called *Throback 1* built to compute in Roman, at Bell Labs in 1953 to show that it was practicable to compute in Roman!

Ludolph van Ceulen (1540–1610). The last great Archimedean calculation, performed by van Ceulen using 2^{62}-gons—to 39 places with 35 correct—was published posthumously. The number is still called Ludolph's number in parts of Europe and was inscribed on his head-stone. This head-stone disappeared centuries ago but was rebuilt, in part from surviving descriptions, recently as shown in Figure 7. It was reconsecrated on July 5th 2000 with Dutch royalty in attendance. Ludolph van Ceulen, a very serious mathematician, was also the discoverer of the cosine formula.

4 Pi's Adolescence

The dawn of modern mathematics appears in *Viéte's* or *Viéta's product* (1579)

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2}\,\frac{\sqrt{2+\sqrt{2}}}{2}\,\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots$$

considered to be the first truly infinite product; and in the *first infinite continued fraction* for $2/\pi$ given by Lord Brouncker (1620–1684), first President of the Royal Society of London:

$$\frac{2}{\pi} = \cfrac{1}{1 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \cdots}}}}.$$

This was based on the following brilliantly 'interpolated' product of John Wallis[3] (1616–1703)

$$\prod_{k=1}^{\infty} \frac{4k^2 - 1}{4k^2} = \frac{2}{\pi}, \tag{3}$$

which led to the discovery of the Gamma function, see End Note 2, and a great deal more.

François Viéte (1540–1603). A flavour of Viéte's writings can be gleaned in this quote from his work, first given in English in [5, p. 759].

> " *Arithmetic is absolutely as much science as geometry [is]. Rational magnitudes are conveniently designated by rational numbers, and irrational [magnitudes] by irrational [numbers]. If someone measures magnitudes with numbers and by his calculation get them different from what they really are, it is not the reckoning's fault but the reckoner's.*
>
> *Rather, says Proclus,* ARITHMETIC IS MORE EXACT THEN GEOMETRY.[4] *To an accurate calculator, if the diameter is set to one unit, the circumference of the inscribed dodecagon will be the side of the binomial [i.e. square root of the difference]* $72 - \sqrt{3888}$. *Whosoever declares any other result, will be mistaken, either the geometer in his measurements or the calculator in his numbers.*" (Viéte)

This fluent rendition is due to Marinus Taisbak, and the full text is worth reading. It certainly underlines how influential an algebraist and geometer Viéte was. Viéte, who was the first to introduce literals ('x' and 'y') into algebra, nonetheless rejected the use of negative numbers.

Equation (3) may be derived from Leonard Euler's (1707–1783) product formula for π, given below in (4), with $x = 1/2$, or by repeatedly integrating $\int_0^{\pi/2} \sin^{2n}(t)\, dt$ by parts. One may divine (4) as Euler did by *considering* $\sin(\pi x)$ *as an 'infinite' polynomial* and obtaining a product in terms of the roots—$0, \{1/n^2 : n = \pm 1, \pm 2, \cdots\}$. It is thus plausible that

$$\frac{\sin(\pi\, x)}{x} \quad = \quad c \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right). \tag{4}$$

Euler, full well knowing that the whole argument was heuristic, argued that, as with a polynomial, c was the value at zero, 1, and the coefficient of x^2 in the Taylor series must be the sum of the roots. Hence, he was able to pick off coefficients to evaluate the *zeta-function* at two:

$$\zeta(2) := \sum_n \frac{1}{n^2} = \frac{\pi^2}{6}.$$

This also leads to the evaluation of $\zeta(2n) := \sum_{k=1}^{\infty} 1/k^{2n}$ as a rational multiple of π^{2n}:

$$\zeta(2) = \frac{\pi^2}{6},\ \zeta(4) = \frac{\pi^4}{90},\ \zeta(6) = \frac{\pi^6}{945},\ \zeta(8) = \frac{\pi^8}{9450}, \ldots$$

[3]One of the few mathematicians whom Newton admitted respecting, and also a calculating prodigy!
[4]This phrase was written in Greek.

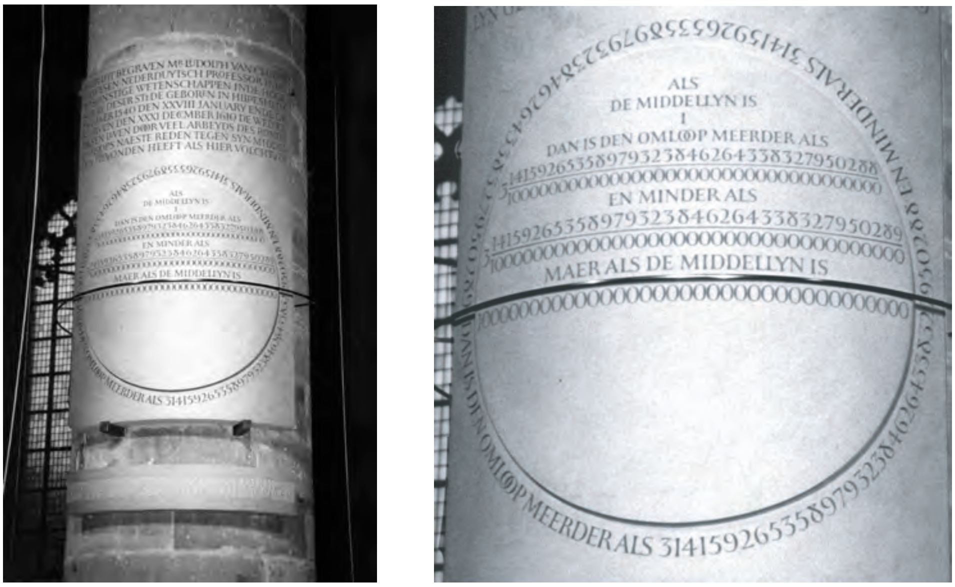


Figure 7: **Ludolph's rebuilt tombstone in Leiden**

in terms of the *Bernoulli numbers*, B_n, where $t/(\exp(t)-1) = \sum_{n\geq 0} B_n t^n/n!$, gives a generating function for the B_n which are perforce rational. The explicit formula which solved the so called *Basel problem* posed by the Bernoullis is

$$\zeta(2m) = (-1)^{m-1}\frac{(2\pi)^{2m}}{2\,(2m)!}\,B_{2m},$$

see also [23].

Much less is known about odd integer values of ζ, though they are almost certainly not rational multiple of powers of π. More than two centuries later, in 1976 Roger Apéry, [5, p. 439], [11], showed $\zeta(3)$ to be irrational, and we now also can prove that *at least one of* $\zeta(5), \zeta(7), \zeta(9)$ or $\zeta(11)$ is irrational, but we can not guarantee which one. All positive integer values are strongly believed to be irrational. Though it is not relevant to our story Euler's work on the zeta-function also lead to the celebrated Riemann hypothesis [10].

5 Pi's Adult Life with Calculus

In the later 17th century, Newton and Leibnitz founded the calculus, and this powerful tool was quickly exploited to find new formulae for π. One early calculus-based formula comes from the integral:

$$\tan^{-1} x \quad = \quad \int_0^x \frac{dt}{1+t^2} \;=\; \int_0^x (1 - t^2 + t^4 - t^6 + \cdots)\,dt = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

Substituting $x = 1$ *formally* proves the well-known *Gregory-Leibnitz formula* (1671–74)

$$\frac{\pi}{4} \quad = \quad 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots \tag{5}$$

James Gregory (1638–75) was the greatest of a large Scottish mathematical family. The point, $x = 1$, however, is on the boundary of the interval of convergence of the series. Justifying substitution requires a careful error estimate for the remainder or Lebesgue's monotone convergence theorem, etc., but most introductory texts ignore the issue.

A Curious Anomaly in the Gregory Series. In 1988, it was observed that Gregory's series for π,

$$\pi \quad = \quad 4\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} \quad = \quad 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots\right) \tag{6}$$

when truncated to 5,000,000 terms, differs strangely from the true value of π:

```
3.14159245358979323846464338327950278419716939938730582097494182230781640...
3.14159265358979323846264338327950288419716939937510582097494459230781640...
         2             -2              10            -122            2770
```

Values differ as expected from truncating an alternating series, in the seventh place—a "4" which should be a "6." But the next 13 digits are correct, and after another blip, for 12 digits. Of the first 46 digits, only four differ from the corresponding digits of π. Further, the "error" digits seemingly occur with a period of 14, as shown above. Such anomalous behavior begs explanation. A great place to start is by using Neil Sloane's Internet-based integer sequence recognition tool, available at `www.research.att.com/~njas/sequences`. This tool has no difficulty recognizing the sequence of errors as twice *Euler numbers*. Even Euler numbers are generated by $\sec x = \sum_{k=0}^{\infty}(-1)^k E_{2k} x^{2k}/(2k)!$. The first few are $1, -1, 5, -61, 1385, -50521, 2702765$. This discovery led to the following asymptotic expansion:

$$\frac{\pi}{2} - 2\sum_{k=1}^{N/2} \frac{(-1)^{k+1}}{2k-1} \quad \approx \quad \sum_{m=0}^{\infty} \frac{E_{2m}}{N^{2m+1}}. \tag{7}$$

Now the genesis of the anomaly is clear: by chance the series had been truncated at 5,000,000 terms—exactly one-half of a fairly large power of ten. Indeed, setting $N = 10,000,000$ in Equation (7) shows that the first hundred or so digits of the truncated series value are small perturbations of the correct decimal expansion for π. And the asymptotic expansions show up on the computer screen, as we observed above. On a hexadecimal computer with $N = 16^7$ the corresponding strings are:

```
3.243F6A8885A308D313198A2E03707344A4093822299F31D0082EFA98EC4E6C89452821E...
3.243F6A6885A308D31319AA2E03707344A3693822299F31D7A82EFA98EC4DBF69452821E...
         2             -2              A             -7A             2AD2
```

with the first being the correct value of π. In hexadecimal or *hex* one uses 'A,B, . . ., F' to write 10 through 15 as single 'hex-digits'. Similar phenomena occur for other constants. (See [5].) Also, knowing the errors means we can correct them and use (7) to make Gregory's formula computationally tractable, despite the following discussion!

6 Calculus Era π Calculations

Used naively, the beautiful formula (5) is computationally useless—so slow that hundreds of terms are needed to compute two digits. Sharp, under the direction of Halley[5], see Figure 8, actually used $\tan^{-1}(1/\sqrt{3})$ which is geometrically convergent.

Moreover, Euler's (1738) trigonometric identity

$$\tan^{-1}(1) = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) \tag{8}$$

[5]The astronomer and mathematician who largely built the Greenwich Observatory and after whom the comet is named.

Name	Year	Correct Digits
Sharp (and Haley)	1699	71
Machin	1706	100
Strassnitzky and Dase	1844	200
Rutherford	1853	440
Shanks	1874	(707) 527
Ferguson (**Calculator**)	1947	808
Reitwiesner et al. (**ENIAC**)	1949	2,037
Genuys	1958	10,000
Shanks and Wrench	1961	100,265
Guilloud and Bouyer	1973	1,001,250

Figure 8: Calculus π Calculations

produces a geometrically convergent rational series

$$\frac{\pi}{4} = \frac{1}{2} - \frac{1}{3\cdot 2^3} + \frac{1}{5\cdot 2^5} - \frac{1}{7\cdot 2^7} + \cdots + \frac{1}{3} - \frac{1}{3\cdot 3^3} + \frac{1}{5\cdot 3^5} - \frac{1}{7\cdot 3^7} + \cdots \tag{9}$$

An even faster formula, found earlier by John Machin, lies similarly in the identity

$$\frac{\pi}{4} = 4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right). \tag{10}$$

This was used in numerous computations of π, given in Figure 8, starting in 1706 and culminating with Shanks' famous computation of π to 707 decimal digits accuracy in 1873 (although it was *found in 1945 to be wrong* after the 527-th decimal place, by Ferguson, during the last adding machine-assisted pre-computer computations.[6]).

Newton's arcsin computation. Newton discovered a different more effective—actually a disguised arcsin—formula. He considering the area A of the left-most region shown in Figure 9. Now, A is the integral

$$A = \int_0^{1/4} \sqrt{x - x^2}\, dx. \tag{11}$$

Also, A is the area of the circular sector, $\pi/24$, less the area of the triangle, $\sqrt{3}/32$. Newton used his newly developed *binomial theorem* in (11):

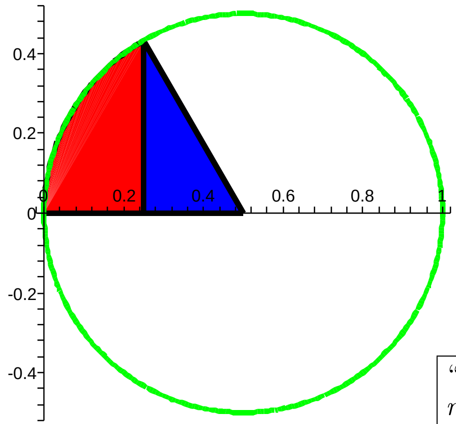
$$\begin{aligned} A &= \int_0^{\frac14} x^{1/2}(1-x)^{1/2}\, dx = \int_0^{\frac14} x^{1/2}\left(1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128} - \cdots\right) dx \\ &= \int_0^{\frac14} \left(x^{1/2} - \frac{x^{3/2}}{2} - \frac{x^{5/2}}{8} - \frac{x^{7/2}}{16} - \frac{5x^{9/2}}{128} \cdots\right) dx \end{aligned}$$

Integrate term-by-term and combining the above produces

$$\pi = \frac{3\sqrt{3}}{4} + 24\left(\frac{1}{3\cdot 8} - \frac{1}{5\cdot 32} - \frac{1}{7\cdot 128} - \frac{1}{9\cdot 512} \cdots\right).$$

Newton used this formula to compute 15 digits of π. As noted, he later 'apologized' for "having no other business at the time." (This was the year of the great plague. It was also directly after the production of Newton's *Principia*.) A standard chronology ([21] and[5, p. 294]) says *"Newton significantly never gave a value for π."* Caveat emptor all users of secondary sources.

[6]This must be some sort a record for the length of time needed to detect a mathematical error.



"I am ashamed to tell you to how many figures I carried these computations, having no other business at the time."
(Isaac Newton, 1666)

The great fire of London ended the plague year in September 1666.

Figure 9: **Newton's method for π**

The Viennese *computer*. Until quite recently—around 1950—a computer was a person. Hence the name of ENIAC discussed later. This computer, one Johann Zacharias Dase (1824–1861), would demonstrate his extraordinary computational skill by, for example, multiplying

$$79532853 \times 93758479 = 7456879327810587$$

in 54 seconds; two 20-digit numbers in six minutes; two 40-digit numbers in 40 minutes; two 100-digit numbers in 8 hours and 45 minutes. In 1844, after being shown

$$\frac{\pi}{4} = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{8}\right)$$

he calculated π to 200 places *in his head* in two months, completing correctly—to my mind—the greatest mental computation ever. Dase later calculated a seven-digit logarithm table, and extended a table of integer factorizations to 10,000,000. Gauss requested that Dase be permitted to assist this project, but Dase died not long afterwards in 1861 by which time Gauss himself already was dead.

An amusing Machin-type identity, that is expressing Pi as linear a combination of arctan's, due to the Oxford logician Charles Dodgson is

$$\tan^{-1}\left(\frac{1}{p}\right) = \tan^{-1}\left(\frac{1}{p+q}\right) + \tan^{-1}\left(\frac{1}{p+r}\right),$$

valid whenever $1 + p^2$ factors as qr. Dodgson is much better known as Lewis Carroll, the author of *Alice in Wonderland.*

7 The Irrationality and Transcendence of π

One motivation for computations of π was very much in the spirit of modern experimental mathematics: to see if the decimal expansion of π repeats, which would mean that π is the ratio of two integers (i.e., rational), or to recognize π as *algebraic*—the root of a polynomial with integer coefficients—and later to look at digit distribution. The question of the *rationality of* π was settled in the late 1700s, when Lambert and Legendre proved (using continued fractions) that the constant is irrational.

The question of whether π was algebraic was settled in 1882, when Lindemann proved that π *is transcendental*. Lindemann's proof also settled, once and for all, the ancient Greek question of whether the circle could be squared with straight-edge and compass. It cannot be, because numbers that are the lengths of lines that can be constructed using ruler and compasses (often called *constructible numbers*) are necessarily algebraic, and squaring the circle is equivalent to constructing the value π. The classical Athenian playwright Aristophanes already 'knew' this and perhaps derided 'circle-squarers' ($\tau\varepsilon\tau\rho\alpha\gamma\omega\sigma\iota\varepsilon\iota\nu$) in his play *The Birds* of 414 BCE. Likewise, the French Academy had stopped accepting proofs of the three great constructions of antiquity—squaring the circle, doubling the cube and trisecting the angle—centuries earlier.

We next give, *in extenso*, Ivan Niven's 1947 short proof of the irrationality of π. It well illustrates the ingredients of more difficult later proofs of irrationality of other constants, and indeed of Lindemann's proof of the transcendence of π building on Hermite's 1873 proof of the transcendence of e.

8 A Proof that π is Irrational

Proof. *Let* $\pi = a/b$, *the quotient of positive integers. We define the polynomials*

$$f(x) = \frac{x^n(a-bx)^n}{n!}$$

$$F(x) = f(x) - f^{(2)}(x) + f^{(4)}(x) - \cdots + (-1)^n f^{(2n)}(x)$$

the positive integer being specified later. Since $n!f(x)$ *has integral coefficients and terms in* x *of degree not less than* n, $f(x)$ *and its derivatives* $f^{(j)}(x)$ *have integral values for* $x = 0$*; also for* $x = \pi = a/b$, *since* $f(x) = f(a/b - x)$. *By elementary calculus we have*

$$\frac{d}{dx}\{F'(x)\sin x - F(x)\cos x\} = F''(x)\sin x + F(x)\sin x = f(x)\sin x$$

and

$$\begin{aligned}\int_0^\pi f(x)\sin x dx &= [F'(x)\sin x - F(x)\cos x]_0^\pi \\ &= F(\pi) + F(0). \qquad (12)\end{aligned}$$

Now $F(\pi) + F(0)$ *is an integer, since* $f^{(j)}(0)$ *and* $f^{(j)}(\pi)$ *are integers. But for* $0 < x < \pi$,

$$0 < f(x)\sin x < \frac{\pi^n a^n}{n!},$$

so that the integral in (12) is positive but arbitrarily small *for n sufficiently large. Thus (12) is false, and so is our assumption that* π *is rational.* **QED**

Irrationality measures. We end this section by touching on the matter of *measures of irrationality*. The infimum $\mu(\alpha)$ of those $\mu > 0$ for which

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{1}{q^\mu}$$

for all integers p, q with sufficiently large q, is called the *Liouville-Roth constant* for α and we say that we have an irrationality measure for α if $\mu(\alpha) < \infty$.

Irrationality measures are difficult. Roth's theorem, [11], implies that $\mu(\alpha) = 2$ for all algebraic irrationals, as is the case for almost all reals. Clearly, $\mu(\alpha) = 1$ for rational α and $\mu(\alpha) = \infty$ iff and only if α is Liouville numbers such as $\sum 1/10^{n!}$. It is known that $\mu(e) = 2$ while in 1993 Hata showed that $\mu(\pi) \leq 8.02$. Similarly, it is known that $\mu(\zeta(2)) \leq 5.45, \mu(\zeta(3)) \leq 4.8$ and $\mu(\log 2) \leq 3.9$.

A consequence of the existence of an irrationality measure μ for π, is the ability to estimate quantities such as $\limsup |\sin(n)|^{1/n} = 1$ for integer n, since for large integer m and n with $m/n \to \pi$, we have eventually

$$|\sin(n)| = |\sin(m\pi) - \sin(n)| \geq \frac{1}{2}|m\pi - n| \geq \frac{1}{2\,m^{\mu-1}}.$$

Related matters are discussed at more length in [1].

9 Pi in the Digital Age

With the substantial development of computer technology in the 1950s, π was computed to thousands and then millions of digits. These computations were greatly facilitated by the discovery soon after of advanced algorithms for the underlying high-precision arithmetic operations. For example, in 1965 it was found that the newly-discovered *fast Fourier transform* (FFT) [11, 10] could be used to perform high-precision multiplications much more rapidly than conventional schemes. Such methods (e.g., for $\div, \sqrt{x}$ see [11, 12, 10]) dramatically lowered the time required for computing π and other constants to high precision. We are now able to compute algebraic values of algebraic functions essentially as fast as we can multiply, $O_B(M(N))$, where $M(N)$ is the cost of multiplication and O_B counts 'bits' or 'flops'. To convert this into practice: a state-of-the-art processor in 2010, such as the latest AMD Opteron, which runs at 2.4 GHz and has four floating-point cores, each of which can do two 64-bit floating-point operations per second, can produce a total of 9.6 billion floating-point operations per second.

In spite of these advances, into the 1970s all computer evaluations of π still employed classical formulae, usually of Machin-type, see Figure 8. We will see below methods that compute N digits of π with time complexity $O_B(M(N)) \log O_B(M(N))$. Showing that the log term is unavoidable, as seems likely, would provide an algorithmic proof that π is not algebraic.

Electronic Numerical Integrator and Calculator. The first computer calculation of Pi was performed on ENIAC—a behemoth with a tiny brain from today's vantage point. The ENIAC was built in Aberdeen Maryland by the US Army:

> **Size/weight.** ENIAC had 18,000 vacuum tubes, 6,000 switches, 10,000 capacitors, 70,000 resistors, 1,500 relays, was 10 feet tall, occupied 1,800 square feet and weighed 30 tons.
> **Speed/memory.** A, now slow, 1.5GHz Pentium does 3 million adds/sec. ENIAC did 5,000, three orders faster than any earlier machine. The first stored-memory computer, ENIAC could hold 200 digits.
> **Input/output.** Data flowed from one accumulator to the next, and after each accumulator finished a calculation, it communicated its results to the next in line. The accumulators were connected to each other manually. The 1949 computation of π to 2,037 places on ENIAC took 70 hours in which output had to be constantly reintroduced as input.

A fascinating description of the ENIAC's technological and commercial travails is to be found in [20].

Ballantine's (1939) Series for π. Another formula of Euler for arccot is

$$x \sum_{n=0}^{\infty} \frac{(n!)^2\,4^n}{(2\,n+1)!\,(x^2+1)^{n+1}} \quad = \quad \arctan\left(\frac{1}{x}\right).$$

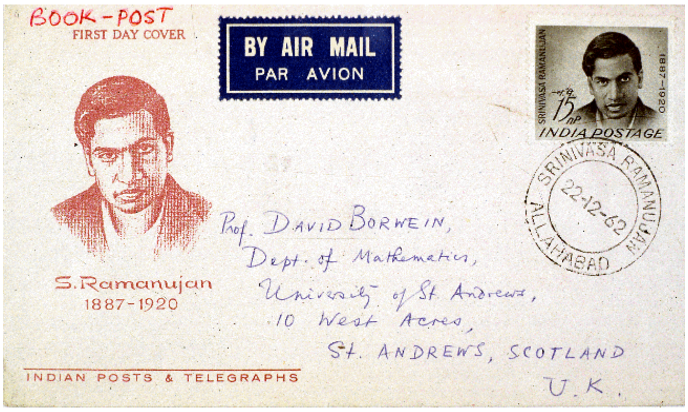


Figure 10: **Ramanujan's seventy-fifth birthday stamp**

G.N. Watson elegantly describes feeling

> *"a thrill which is indistinguishable from the thrill which I feel when I enter the Sagrestia Nuova of the Cappella Medici and see before me the austere beauty of the four statues representing 'Day', 'Night', 'Evening', and 'Dawn' which Michelangelo has set over the tomb of Giuliano de'Medici and Lorenzo de'Medici"*

on viewing formulae of Ramanujan, such as (13).

This, intriguingly and usefully, allowed Guilloud and Boyer to reexpress the formula, used by them in 1973 to compute a million digits of Pi, viz, $\pi/4 = 12\arctan(1/18) + 8\arctan(1/57) - 5\arctan(1/239)$ in the efficient form

$$\pi = 864 \sum_{n=0}^{\infty} \frac{(n!)^2\, 4^n}{(2\,n+1)!\, 325^{n+1}} + 1824 \sum_{n=0}^{\infty} \frac{(n!)^2\, 4^n}{(2\,n+1)!\, 3250^{n+1}} - 20 \arctan\left(\frac{1}{239}\right),$$

where the terms of the second series are now just decimal shifts of the first.

Ramanujan-type elliptic series. Truly new types of infinite series formulae, based on elliptic integral approximations, were discovered by Srinivasa Ramanujan (1887–1920), shown in Figure 10, around 1910, but were not well known (nor fully proven) until quite recently when his writings were widely published. They are based on elliptic functions and are described at length in [5, 11, 10].

One of these series is the remarkable:

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!\,(1103 + 26390k)}{(k!)^4 396^{4k}}. \tag{13}$$

Each term of this series produces an additional *eight* correct digits in the result. When Gosper used this formula to compute 17 million digits of π in 1985, and it agreed to many millions of places with the prior estimates, *this concluded the first proof* of (13), as described in [13]! Actually, Gosper first computed the simple continued fraction for π, hoping to discover some new things in its expansion, but found none.

At about the same time, David and Gregory Chudnovsky found the following rational variation of Ramanujan's formula. It exists because $\sqrt{-163}$ corresponds to an imaginary quadratic field with class number one:

$$\frac{1}{\pi} = 12 \sum_{k=0}^{\infty} \frac{(-1)^k\,(6k)!\,(13591409 + 545140134k)}{(3k)!\,(k!)^3\, 640320^{3k+3/2}} \tag{14}$$

Each term of this series produces an additional 14 correct digits. The Chudnovskys implemented this formula using a clever scheme that enabled them to use the results of an initial level of precision to extend the calculation to even higher precision. They used this in several large calculations of π, culminating with a **then record computation** to over four billion decimal digits in 1994. Their remarkable story was compellingly told by Richard Preston in a prizewinning New Yorker article "The Mountains of Pi" (March 2, 1992).

Name	Year	Correct Digits
Miyoshi and Kanada	1981	2,000,036
Kanada-Yoshino-Tamura	1982	16,777,206
Gosper	1985	17,526,200
Bailey	Jan. 1986	29,360,111
Kanada and Tamura	Sep. 1986	33,554,414
Kanada and Tamura	Oct. 1986	67,108,839
Kanada et. al	Jan. 1987	134,217,700
Kanada and Tamura	Jan. 1988	201,326,551
Chudnovskys	May 1989	480,000,000
Kanada and Tamura	Jul. 1989	536,870,898
Kanada and Tamura	Nov. 1989	1,073,741,799
Chudnovskys	Aug. 1991	2,260,000,000
Chudnovskys	May 1994	4,044,000,000
Kanada and Takahashi	Oct. 1995	6,442,450,938
Kanada and Takahashi	Jul. 1997	51,539,600,000
Kanada and Takahashi	Sep. 1999	206,158,430,000
Kanada-Ushiro-Kuroda	Dec. 2002	1,241,100,000,000
Takahashi	Jan. 2009	1,649,000,000,000
Takahashi	April. 2009	2,576,980,377,524
Bellard	Dec. 2009	2,699,999,990,000

Figure 11: Post-calculus π Calculations

While the Ramanujan and Chudnovsky series are in practice considerably more efficient than classical formulae, they share the property that the number of terms needed increases linearly with the number of digits desired: *if you want to compute twice as many digits of* π*, you must evaluate twice as many terms* of the series.

Relatedly, the Ramanujan-type series

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \left(\frac{\binom{2n}{n}}{16^n} \right)^3 \frac{42\,n+5}{16}. \tag{15}$$

allows one to compute the billionth binary digit of $1/\pi$, or the like, *without computing the first half* of the series, and is a foretaste of our later discussion of Borwein-Bailey-Plouffe (or BBP) formulae.

10 Reduced Operational Complexity Algorithms

In 1976, Eugene Salamin and Richard Brent independently discovered a *reduced complexity* algorithm for π. It is based on the **arithmetic-geometric mean iteration** (AGM) and some other ideas due to Gauss and Legendre around 1800, although Gauss, nor many after him, never directly saw the connection to effectively computing π.

Quadratic Algorithm (Salamin-Brent). Set $a_0 = 1, b_0 = 1/\sqrt{2}$ and $s_0 = 1/2$. Calculate

$$a_k = \frac{a_{k-1} + b_{k-1}}{2} \quad (A) \qquad b_k = \sqrt{a_{k-1} b_{k-1}} \quad (G) \tag{16}$$

$$c_k = a_k^2 - b_k^2, \qquad s_k = s_{k-1} - 2^k c_k \qquad \text{and compute} \quad p_k = \frac{2a_k^2}{s_k}. \tag{17}$$

Then p_k converges *quadratically* to π. Note the similarity between the arithmetic-geometric mean iteration (16), (which for general initial values converges fast to a non-elementary limit) and the out-of-kilter harmonic-geometric mean iteration (2) (which in general converges slowly to an elementary limit), and which is an arithmetic-geometric iteration in the reciprocals (see [11]).

Each iteration of the algorithm *doubles* the correct digits. Successive iterations produce $1, 4, 9, 20, 42, 85, 173, 347$ and 697 good decimal digits of π, and takes $\log N$ operations for N digits. Twenty-five iterations computes π to over 45 million decimal digit accuracy. A disadvantage is that each of these iterations must be performed to the precision of the final result. In 1985, my brother Peter and I discovered families of algorithms of this type. For example, here is a genuinely third-order iteration:

Cubic Algorithm. Set $a_0 = 1/3$ and $s_0 = (\sqrt{3}-1)/2$. Iterate

$$r_{k+1} = \frac{3}{1+2(1-s_k^3)^{1/3}}, \qquad s_{k+1} = \frac{r_{k+1}-1}{2} \quad \text{and} \qquad a_{k+1} = r_{k+1}^2 a_k - 3^k(r_{k+1}^2 - 1).$$

Then $1/a_k$ *converges cubically* to π. Each iteration *triples* the number of correct digits.

Quartic Algorithm. Set $a_0 = 6 - 4\sqrt{2}$ and $y_0 = \sqrt{2} - 1$. Iterate

$$y_{k+1} = \frac{1-(1-y_k^4)^{1/4}}{1+(1-y_k^4)^{1/4}} \qquad \text{and} \qquad a_{k+1} = a_k(1+y_{k+1})^4 - 2^{2k+3} y_{k+1}(1+y_{k+1}+y_{k+1}^2).$$

Then $1/a_k$ *converges quartically* to π. Note that only the power of 2 or 3 used in a_k depends on k.

There are many more and longer mnemonics than the sample given in the inset box—see [5, p. 405, p.560, p. 659] for a fine selection.

Mnemonics for Pi

"Now I , even I, would celebrate
In rhyme inapt, the great
Immortal Syracusan, rivaled nevermore,
Who in his wondrous lore,
Passed on before
Left men for guidance
How to circles mensurate." (30)

"How I want a drink, alcoholic of course, after the heavy lectures involving quantum mechanics." (15)

"See I have a rhyme assisting my feeble brain its tasks ofttimes resisting." (13)

Philosophy of mathematics. In 1997 the first occurrence of the sequence `0123456789` was found (later than expected heuristically) in the decimal expansion of π starting at the `17,387,594,880`-th digit after the decimal point. In consequence the status of several famous *intuitionistic examples* due to Brouwer and Heyting has changed. These challenge the *principle of the excluded middle*—either a predicate holds or it does not— and involve classically well-defined objects that for an intuitionist are ill-founded until one can determine when or if the sequence occurred, [8].

For example, consider the sequence which is '0' except for a '1' in the first place where 0123456789 first begins to appear in order if it ever occurs. Did it converge when first used by Brouwer as an example? Does it now? Was it then and is it now well defined? Classically it always was and converged to '0'. Intuitionistically it converges now. What if we redefine the sequence to have its '1' in the first place that 0123456789101112 first begins?

11 Back to the Future

In December 2002, Kanada computed π to over **1.24 trillion decimal digits**. His team first computed π in hexadecimal (base 16) to 1,030,700,000,000 places, using the following two arctangent relations:

$$\pi = 48\tan^{-1}\frac{1}{49} + 128\tan^{-1}\frac{1}{57} - 20\tan^{-1}\frac{1}{239} + 48\tan^{-1}\frac{1}{110443}$$

$$\pi = 176\tan^{-1}\frac{1}{57} + 28\tan^{-1}\frac{1}{239} - 48\tan^{-1}\frac{1}{682} + 96\tan^{-1}\frac{1}{12943}.$$

The first formula was found in 1982 by K. Takano, a high school teacher and song writer. The second formula was found by F. C. W. Störmer in 1896. Kanada verified the results of these two computations agreed, and then converted the hex digit sequence to decimal. The resulting decimal expansion was checked by converting it back to hex. These conversions are themselves non-trivial, requiring massive computation.

This process is quite different from those of the previous quarter century. One reason is that reduced operational complexity algorithms, require full-scale multiply, divide and square root operations. These in turn require large-scale FFT operations, which demand huge amounts of memory, and massive all-to-all communication between nodes of a large parallel system. For this latest computation, even the very large system available in Tokyo did not have sufficient memory and network bandwidth to perform these operations at reasonable efficiency levels—at least not for trillion-digit computations. Utilizing arctans again meant using many more arithmetic operations, but no system-scale FFTs, and it can be implemented using $\times, \div$ by smallish integer values—additionally, hex is somewhat more efficient!

Kanada and his team evaluated these two formulae using a scheme analogous to that employed by Gosper and by the Chudnovskys in their series computations, in that they were able to avoid explicitly storing the multiprecision numbers involved. This resulted in a scheme that is roughly competitive in *numerical* efficiency with the Salamin-Brent and Borwein quartic algorithms they had previously used, but with a significantly lower total memory requirement. Kanada used a 1 Tbyte main memory system, as with the previous computation, yet got six times as many digits. Hex and decimal evaluations included, it ran 600 hours on a 64-node Hitachi, with the main segment of the program running at a sustained rate of nearly 1 Tflop/sec.

12 Why Pi?

What possible motivation lies behind modern computations of π, given that questions such as the irrationality and transcendence of π were settled more than 100 years ago? One motivation is the raw challenge of harnessing the stupendous power of modern computer systems. Programming such calculations are definitely not trivial, especially on large, distributed memory computer systems.

There have been substantial practical spin-offs. For example, some new techniques for performing the fast Fourier transform (FFT), heavily used in modern science and engineering computing, had their roots in attempts to accelerate computations of π. And always the computations help in road-testing computers—often uncovering subtle hardware and software errors.

Beyond practical considerations lies the abiding interest in the fundamental question of the *normality (digit randomness)* of π. Kanada, for example, has performed detailed statistical analysis of his results to see if there are any statistical abnormalities that suggest π is not normal, so far the answer is "no", see Figures 13 and

Figure 12: **Yasumasa Kanada in his Tokyo office**

14. Indeed the first computer computation of π and e on ENIAC, discussed above, was so motivated by John von Neumann. The digits of π have been studied more than any other single constant, in part because of the widespread fascination with and recognition of π. Kanada reports that the 10 decimal digits ending in position one trillion are `6680122702`, while the 10 hexadecimal digits ending in position one trillion are `3F89341CD5`.

Changing world views. In retrospect, we may wonder why in antiquity π was not *measured* to an accuracy in excess of 22/7? Perhaps it reflects not an inability to do so but a very different mind set to a modern experimental—Baconian or Popperian—one. In the same vein, one reason that Gauss and Ramanujan did not further develop the ideas in their identities for π is that an iterative algorithm, as opposed to explicit results, was not as satisfactory for them (especially Ramanujan). Ramanujan much preferred formulae like

$$\pi \approx \frac{3}{\sqrt{67}} \log(5280), \qquad \frac{3}{\sqrt{163}} \log(640320) \approx \pi$$

correct to *9 and 15 decimal places* both of which rely on deep number theory. Contrastingly, Ramanujan in his famous 1914 paper *Modular Equations and Approximations to Pi* [5, p.253] found

$$\left(9^2 + \frac{19^2}{22}\right)^{1/4} = 3.14159265\overline{2}58\cdots$$

"*empirically, and it has no connection with the preceding theory.*" Only the marked digit is wrong.

Discovering the π Iterations. The genesis of the π algorithms and related material is an illustrative example of experimental mathematics. My brother and I in the early eighties had a family of quadratic algorithms for π, [11], call them $\mathcal{P}_N$, of the kind we saw above. For $N = 1, 2, 3, 4$ we could prove they were correct but and only conjectured for $N = 5, 7$. In each case the algorithm *appeared* to converge quadratically to π. On closer inspection while the provable cases were correct to $5,000$ digits, the empirical versions of agreed with π to roughly 100 places only. Now in many ways to have discovered a "natural" number that agreed with π to that level—and no more—would have been more interesting than the alternative. That seemed unlikely but recoding and rerunning the iterations kept producing identical results.

Decimal Digit	Occurrences
0	99999485134
1	99999945664
2	100000480057
3	99999787805
4	100000357857
5	99999671008
6	99999807503
7	99999818723
8	100000791469
9	99999854780
Total	**1000000000000**

Hex Digit	Occurrences
0	62499881108
1	62500212206
2	62499924780
3	62500188844
4	62499807368
5	62500007205
6	62499925426
7	62499878794
8	62500216752
9	62500120671
A	62500266095
B	62499955595
C	62500188610
D	62499613666
E	62499875079
F	62499937801
Total	**1000000000000**

Figure 13: **Apparently random behaviour of π base 10 and 16**

Two decades ago even moderately high precision calculation was less accessible, and the code was being run remotely over a phone-line in a Berkeley Unix integer package. After about six weeks, it transpired that the package's *square root algorithm was badly flawed*, but *only if run with an odd precision of more than sixty digits*! And for idiosyncratic reasons that had only been the case in the two unproven cases. Needless to say, tracing the bug was a salutary and somewhat chastening experience. And it highlights why one checks computations using different sub-routines and methods.

13 How to Compute the N-th Digits of π

One might be forgiven for thinking that essentially everything of interest with regards to π has been dealt with. This is suggested in the closing chapters of Beckmann's 1971 book *A History of* π. Ironically, the Salamin–Brent quadratically convergent iteration was discovered only five years later, and the higher-order convergent algorithms followed in the 1980s. Then in 1990, Rabinowitz and Wagon discovered a "spigot" algorithm for π—the digits 'drip out' one by one. This permits successive digits of π (in any desired base) to be computed by a relatively simple recursive algorithm based on the *all previously* generated digits.

Even insiders are sometimes surprised by a new discovery. Prior to 1996, most folks thought if you want to determine the d-th digit of π, you had to generate the (order of) the entire first d digits. This is not true, at least for hex (base 16) or binary (base 2) digits of π. In 1996, Peter Borwein, Plouffe, and Bailey found an algorithm for computing individual hex digits of π. It (1) yields a modest-length hex or binary digit string for π, from an arbitrary position, using no prior bits; (2) is implementable on any modern computer; (3) requires no multiple precision software; (4) requires very little memory; and (5) has a computational cost growing only slightly faster than the digit position. For example, the millionth hexadecimal digit (four millionth binary digit) of π could be found in four seconds on a 2005 Apple computer.

This new algorithm is not fundamentally faster than the best known schemes if used for computing *all* digits of π up to some position, but its elegance and simplicity are of considerable interest, and is easy to parallelize.

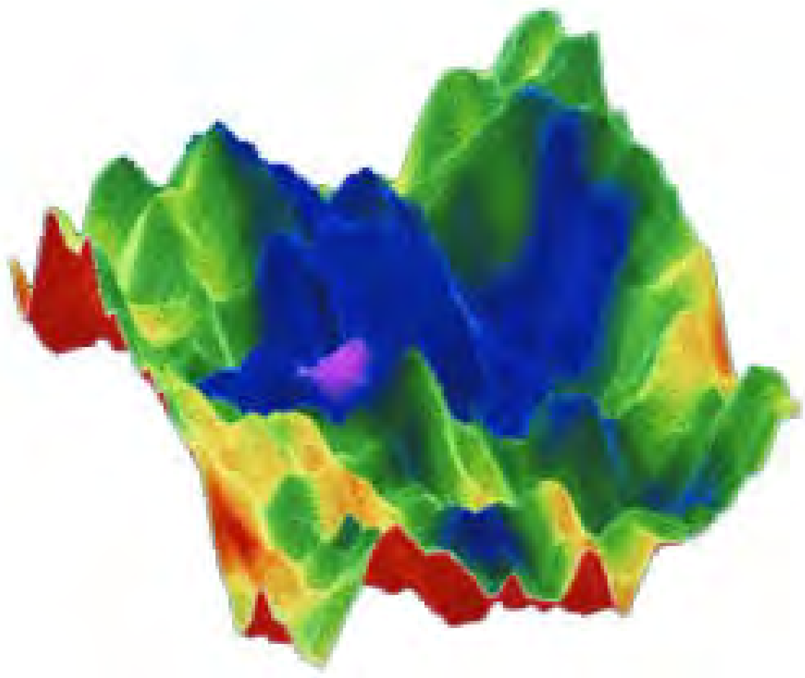


Figure 14: **A 'random walk' on the first one million digits of π (Courtesy D. and G. Chudnovsky)**

It is based on the following at-the-time new formula for π:

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i}\left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6}\right) \tag{18}$$

which was discovered using *integer relation methods* (see [10]), with a computer search that lasted for several months and then produced the (equivalent) relation:

$$\pi = 4\,\mathrm{F}\left(1, \frac{1}{4}; \frac{5}{4}, -\frac{1}{4}\right) + 2\tan^{-1}\left(\frac{1}{2}\right) - \log 5$$

where $\mathrm{F}(1, 1/4; 5/4, -1/4) = 0.955933837\ldots$ is a Gaussian hypergeometric function.

Maple and *Mathematica* can both now prove (18). A human proof may be found in [10].

The algorithm in action. In 1997, Fabrice Bellard at INRIA–whom we shall meet again in Section 15—computed 152 binary digits of π starting at the trillionth position. The computation took 12 days on 20 workstations working in parallel over the Internet. Bellard's scheme is based on the following variant of (18):

$$\pi = 4\sum_{k=0}^{\infty} \frac{(-1)^k}{4^k(2k+1)} - \frac{1}{64}\sum_{k=0}^{\infty} \frac{(-1)^k}{1024^k}\left(\frac{32}{4k+1} + \frac{8}{4k+2} + \frac{1}{4k+3}\right),$$

which permits hex or binary digits of π to be calculated roughly 43% faster than (18).

In 1998 Colin Percival, then a 17-year-old student at Simon Fraser University, utilized 25 machines to calculate first the five trillionth hexadecimal digit, and then the ten trillionth hex digit. In September, 2000, he found the quadrillionth binary digit is **0**, a computation that required *250 CPU-years, using 1734 machines in 56 countries*. We record some computational results in Figure 18.

A last comment for this section is that Kanada was able to confirm his 2002 computation in only 21 hours by computing a 20 hex digit string starting at the trillionth digit, and comparing this string to the hex string he had initially obtained in over 600 hours. Their agreement provided enormously strong confirmation.

Position	Hex strings starting at this Position
10^6	26C65E52CB4593
10^7	17AF5863EFED8D
10^8	ECB840E21926EC
10^9	85895585A0428B
10^{10}	921C73C6838FB2
10^{11}	9C381872D27596
1.25×10^{12}	07E45733CC790B
2.5×10^{14}	E6216B069CB6C1

Borweins and Plouffe (MSNBC, 1996)

Figure 15: **Percival's hexadecimal findings**

14 Further BBP Digit Formulae

Motivated as above, constants α of the form

$$\alpha = \sum_{k=0}^{\infty} \frac{p(k)}{q(k)2^k}, \tag{19}$$

where $p(k)$ and $q(k)$ are integer polynomials, are said to be in the class of *binary (Borwein-Bailey-Plouffe) BBP numbers*. I illustrate for $\log 2$ why this permits one to calculate isolated digits in the binary expansion:

$$\log 2 = \sum_{k=0}^{\infty} \frac{1}{k2^k}. \tag{20}$$

We wish to compute a few binary digits beginning at position $d+1$. This is equivalent to calculating $\{2^d \log 2\}$, where $\{\cdot\}$ denotes fractional part. We can write

$$\{2^d \log 2\} = \left\{\left\{\sum_{k=0}^{d} \frac{2^{d-k}}{k}\right\} + \left\{\sum_{k=d+1}^{\infty} \frac{2^{d-k}}{k}\right\}\right\} = \left\{\left\{\sum_{k=0}^{d} \frac{\mathbf{2^{d-k}} \bmod \mathbf{k}}{k}\right\} + \left\{\sum_{k=d+1}^{\infty} \frac{\mathbf{2^{d-k}}}{\mathbf{k}}\right\}\right\}. \tag{21}$$

The key observation is that the numerator of the first sum in (21), $2^{d-k} \bmod k$, can be calculated rapidly by *binary exponentiation*, performed modulo k. That is, it is economically performed by a factorization based on the binary expansion of the exponent. For example,

$$3^{17} = ((((3^2)^2)^2)^2) \cdot 3$$

uses only five multiplications, not the usual 16. It is important to reduce each product modulo k. Thus, 3^{17} mod 10 is done as

$$3^2 = 9; 9^2 = 1; 1^2 = 1; 1^2 = 1; 1 \times 3 = 3.$$

A natural question in light of (18) is whether there is a formula of this type and an associated computational strategy to compute individual *decimal* digits of π. Searches conducted by numerous researchers have been unfruitful and recently D. Borwein (my father), Galway and I have shown that there are no BBP formulae of the *Machin-type* (as defined in [10]) of (18) for Pi unless the base is a power of two [10].

These 'subtractive' acrylic circles represent the weights $[4, -2, -2, -1]$ in Equation (18)

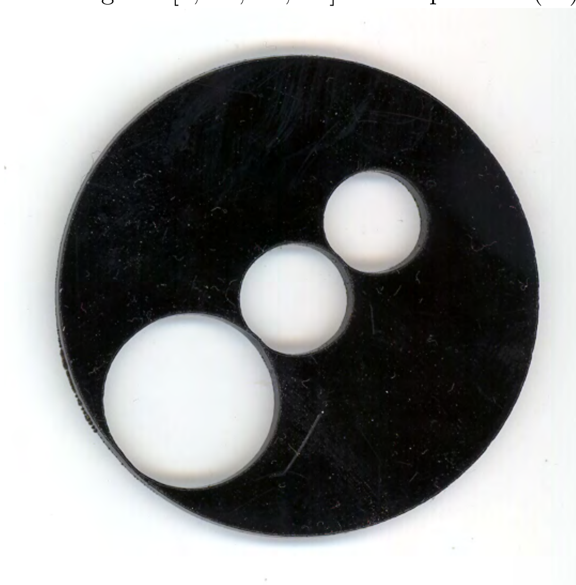


Figure 16: Ferguson's "Eight-Fold Way" and his BBP acrylic circles

Ternary BBP formulae. Yet, BBP formulae exist in other bases for some constants. For example, Broadhurst found this ternary BBP formula for π^2:

$$\pi^2 = \frac{2}{27}\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)^{9k} \times \left\{\frac{243}{(12k+1)^2} - \frac{405}{(12k+2)^2} - \frac{81}{(12k+4)^2} - \frac{27}{(12k+5)^2} - \frac{72}{(12k+6)^2} - \frac{9}{(12k+7)^2} - \frac{9}{(12k+8)^2} - \frac{5}{(12k+10)^2} + \frac{1}{(12k+11)^2}\right\},$$

and π^2 also has a binary BBP formula.

Also, the volume V_8 in *hyperbolic space* of the *figure-eight knot complement* is well known to be

$$V_8 = 2\sqrt{3}\sum_{n=1}^{\infty}\frac{1}{n\binom{2n}{n}}\sum_{k=n}^{2n-1}\frac{1}{k} = 2.029883212819307250042405108549\ldots$$

Surprisingly, it is also expressible as

$$V_8 = \frac{\sqrt{3}}{9}\sum_{n=0}^{\infty}\frac{(-1)^n}{27^n}\left\{\frac{18}{(6n+1)^2} - \frac{18}{(6n+2)^2} - \frac{24}{(6n+3)^2} - \frac{6}{(6n+4)^2} + \frac{2}{(6n+5)^2}\right\},$$

again discovered numerically by Broadhurst, and proved in [10]. A beautiful representation by Helaman Ferguson the mathematical sculptor is given in Figure 19. Ferguson produces art inspired by deep mathematics, but not by a formulaic approach.

Normality and dynamics. Finally, Bailey and Crandall in 2001 made exciting connections between the existence of a b-ary BBP formula for α and its *normality* base b (uniform distribution of base-b digits)[7]. They make a reasonable, hence very hard, conjecture about the *uniform distribution of a related chaotic dynamical*

[7] See www.sciencenews.org/20010901/bob9.asp.

system. This conjecture implies: *Existence of a 'BBP' formula base b for α ensures the normality base b of α.* For log 2, illustratively[8], the dynamical system, base 2, is to set $x_0 = 0$ and compute

$$x_{n+1} \hookleftarrow 2\left(x_n + \frac{1}{n}\right) \mod 1.$$

15 Pi in the Third Millennium

15.1 Reciprocal series

A few years ago Jesús Guillera found various Ramanujan-like identities for π, using integer relation methods. The three most basic are:

$$\frac{4}{\pi^2} = \sum_{n=0}^{\infty}(-1)^n r(n)^5(13+180n+820n^2)\left(\frac{1}{32}\right)^{2n+1} \tag{22}$$

$$\frac{2}{\pi^2} = \sum_{n=0}^{\infty}(-1)^n r(n)^5(1+8n+20n^2)\left(\frac{1}{2}\right)^{2n+1} \tag{23}$$

$$\frac{4}{\pi^3} \stackrel{?}{=} \sum_{n=0}^{\infty} r(n)^7(1+14n+76n^2+168n^3)\left(\frac{1}{8}\right)^{2n+1}, \tag{24}$$

where $r(n) := (1/2 \cdot 3/2 \cdot \cdots \cdot (2n-1)/2)/n!$. Guillera proved (22) and (23) in tandem, using the *Wilf–Zeilberger algorithm* for formally proving hypergeometric-like identities [10, 4, 24] very ingeniously. No other proof is known and there seem to be no like formulae for $1/\pi^d$ with $d \geq 4$. The third (24) is certainly true,[9] but has no proof, nor does anyone have an inkling of how to prove it; especially as experiment suggests that it has no 'mate' unlike (22) and (23) [4]. My intuition is that if a proof exists it is more a verification than an explication and so I stopped looking. I am happy just to know the beautiful identity is true. A very nice account of the current state of knowledge for Ramanujan-type series for $1/\pi$ is to be found in [6].

In 2008 Guillera [17] produced another lovely pair of third millennium identities—discovered with integer relation methods and proved with creative telescoping—this time for π^2 rather than its reciprocal. They are

$$\sum_{n=0}^{\infty}\frac{1}{2^{2n}}\frac{\left(x+\frac{1}{2}\right)_n^3}{(x+1)_n^3}\left(6(n+x)+1\right) = 8x\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^2}{(x+1)_n^2}, \tag{25}$$

and

$$\sum_{n=0}^{\infty}\frac{1}{2^{6n}}\frac{\left(x+\frac{1}{2}\right)_n^3}{(x+1)_n^3}\left(42(n+x)+5\right) = 32x\sum_{n=0}^{\infty}\frac{\left(x+\frac{1}{2}\right)_n^2}{(2x+1)_n^2}. \tag{26}$$

Here $(a)_n = a(a+1)\cdot\cdot(a+n-1)$ is the rising factorial. Substituting $x = 1/2$ in (25) and (26), he obtained respectively the formulae

$$\sum_{n=0}^{\infty}\frac{1}{2^{2n}}\frac{(1)_n^3}{\left(\frac{3}{2}\right)_n^3}(3n+2) = \frac{\pi^2}{4} \qquad \sum_{n=0}^{\infty}\frac{1}{2^{6n}}\frac{(1)_n^3}{\left(\frac{3}{2}\right)_n^3}(21n+13) = 4\,\frac{\pi^2}{3}.$$

[8] In this case it is easy to use Weyl's criterion for equidistribution to establish this equivalence without mention of BBP numbers.

[9] Guillera ascribes (24) to Gourevich, who used integer relation methods. I've 'rediscovered' (24) using integer relation methods with 30 digits. I then checked it to 500 places in 10 seconds, 1200 in 6.25 minutes, and 1500 in 25 minutes: with a naive command-line instruction in *Maple* on a light laptop.

15.2 Computational records

The last decade has seen the record for computation of π broken in some very interesting ways. We have already described Kanada's 2002 computation in Section 11 and noted that he also took advantage of the BBP formula of Section 13. This stood as a record until 2009 when it was broken three times—-twice spectacularly.

Daisuke Takahashi. The record for computation of π went from under 29.37 million decimal digits, by Bailey in 1986, to over 2.649 trillion places by Takahashi in January 2009. Since the same algorithms were used for each computation, it is interesting to review the performance in each case: In 1986 it took 28 hours to compute 29.36 million digits on 1 cpu of the then new CRAY-2 at NASA Ames using (18). Confirmation using the quadratic algorithm 16 took 40 hours. (The computation uncovered hardware and software errors on the CRAY. Success required developing a speedup of the underlying FFT [10].) In comparison, on 1024 cores of a 2592 core *Appro Xtreme-X3* system 2.649 trillion digits via (16) took 64 hours 14 minutes with 6732 GB of main memory, and (18) took 73 hours 28 minutes with 6348 GB of main memory. (The two computations differed only in the last 139 places.) In April Takahashi upped his record to an amazing 2,576,980,377,524 places.

Fabrice Bellard. Near the end of 2009, Bellard magnificently computed nearly 2.7 trillion decimal digits of Pi (first in binary) of Pi using the Chudnovsky series (14). This took 131 days but he only used a single 4-core workstation with a lot of storage and even more human intelligence! For full details of this feat and of Takahashi's most recent computation one can look at

`http://en.wikipedia.org/wiki/Chronology_of_computation_of_pi`

16 . . . Life of Pi.

Paul Churchland writing about the sorry creationist battles of the Kansas school board [15, Kindle ed, loc 1589] observes that:

> *"Even mathematics would not be entirely safe. (Apparently, in the early 1900's, one legislator in a southern state proposed a bill to redefine the value of pi as 3.3 exactly, just to tidy things up.)"*

As we have seen the life of Pi captures a great deal of mathematics—algebraic, geometric and analytic, both pure and applied—along with some history and philosophy. It engages many of the greatest mathematicians and some quite interesting characters along the way. Among the saddest and least-well understood episodes was an abortive 1896 attempt in Indiana to legislate the value(s) of Pi. The bill, reproduced in [5, p. 231-235], is accurately described by David Singmaster, [22] and [5, p. 236-239]. Much life remains in this most central of numbers.

At the end of the novel, Piscine (Pi) Molitor writes

> *"I am a person who believes in form, in harmony of order. Where we can, we must give things a meaningful shape. For example—I wonder—could you tell my jumbled story in exactly one hundred chapters, not one more, not one less? I'll tell you, that's one thing I hate about my nickname, the way that number runs on forever. It's important in life to conclude things properly. Only then can you let go."*

We may well not share the sentiment, but we should celebrate that Pi knows π to be irrational.

17 End Notes

1. Why π is not 22/7. Today, even the computer algebra systems *Maple* or *Mathematica* 'know' this since

$$0 \quad < \quad \int_0^1 \frac{(1-x)^4 x^4}{1+x^2}\, dx \quad = \quad \frac{22}{7} - \pi, \tag{27}$$

though it would be prudent to ask 'why' each can perform the integral and 'whether' to trust it? *Assuming we do trust it*, then the integrand is strictly positive on $(0,1)$, and the answer in (27) is an area and so strictly positive, despite millennia of claims that π is 22/7. In this case, requesting the indefinite integral provides immediate reassurance. We obtain

$$\int_0^t \frac{x^4\,(1-x)^4}{1+x^2}\, dx \quad = \quad \frac{1}{7}\,t^7 - \frac{2}{3}\,t^6 + t^5 - \frac{4}{3}\,t^3 + 4\,t - 4\,\arctan(t)\,,$$

as differentiation easily confirms, and so the Newtonian Fundamental theorem of calculus proves (27).

One can take the idea in Equation (27) a bit further, as in [10]. Note that

$$\int_0^1 x^4\,(1-x)^4\, dx \quad = \quad \frac{1}{630}, \tag{28}$$

and we observe that

$$\frac{1}{2}\int_0^1 x^4\,(1-x)^4\, dx \quad < \quad \int_0^1 \frac{(1-x)^4 x^4}{1+x^2}\, dx < \int_0^1 x^4\,(1-x)^4\, dx. \tag{29}$$

Combine this with (27) and (28) to derive: $223/71 < 22/7 - 1/630 < \pi < 22/7 - 1/1260 < 22/7$ and so re-obtain Archimedes' famous computation

$$\mathbf{3}\frac{\mathbf{10}}{\mathbf{71}} < \pi < \mathbf{3}\frac{\mathbf{10}}{\mathbf{70}}. \tag{30}$$

The derivation above was first popularized in *Eureka*, a Cambridge student journal in 1971.[10] A recent study of related approximations is [19]. (See also [10].)

2. More about Gamma. One may define

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$$

for Re $x > 0$. The starting point is that

$$x\,\Gamma(x) = \Gamma(x+1), \qquad \Gamma(1) = 1. \tag{31}$$

In particular, for integer n, $\Gamma(n+1) = n!$. Also for $0 < x < 1$

$$\Gamma(x)\,\Gamma(1-x) = \frac{\pi}{\sin(\pi x)},$$

since for $x > 0$ we have

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!\, n^x}{\prod_{k=0}^n (x+k)}.$$

[10] Equation (27) was on a Sydney University examination paper in the early sixties and the earliest source I know of dates from the forties [10].

This is a nice consequence of the *Bohr-Mollerup theorem* [11, 10] which shows that Γ is the unique log-convex function on the positive half line satisfying (31). Hence, $\Gamma(1/2) = \sqrt{\pi}$ and equivalently we evaluate the *Gaussian integral*

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi},$$

so central to probability theory. In the same vein, the improper *sinc* function integral evaluates as

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\, dx = \pi.$$

Considerable information about the relationship between Γ and π is to be found in [10, 16].

The Gamma function is as ubiquitous as π. For example, it is shown in [14] that the *expected length*, W_3, of a three-step unit-length random walk in the plane is given by

$$W_3 = \frac{3}{16}\frac{2^{1/3}}{\pi^4}\Gamma^6\left(\frac{1}{3}\right) + \frac{27}{4}\frac{2^{2/3}}{\pi^4}\Gamma^6\left(\frac{2}{3}\right). \tag{32}$$

We recall that $\Gamma(1/2)^2 = \pi$ and that similar algorithms exist for $\Gamma(1/3), \Gamma(1/4)$, and $\Gamma(1/6)$ [11, 10].

2. More about Complexity Reduction. To illustrate the stunning complexity reduction in the elliptic algorithms for Pi, let us write a *complete set of algebraic equations* approximating π to well over a trillion digits.

The number π is transcendental and the number $1/a_{20}$ computed next is algebraic
nonetheless they coincide for over 1.5 trillion places.

Set $a_0 = 6 - 4\sqrt{2}$, $y_0 = \sqrt{2} - 1$ and then solve the following system:

$$y_1 = \frac{1-\sqrt[4]{1-{y_0}^4}}{1+\sqrt[4]{1-{y_0}^4}}, a_1 = a_0\left(1+y_1\right)^4 - 2^3 y_1\left(1+y_1+{y_1}^2\right)$$
$$y_2 = \frac{1-\sqrt[4]{1-{y_1}^4}}{1+\sqrt[4]{1-{y_1}^4}}, a_2 = a_1\left(1+y_2\right)^4 - 2^5 y_2\left(1+y_2+{y_2}^2\right)$$
$$y_3 = \frac{1-\sqrt[4]{1-{y_2}^4}}{1+\sqrt[4]{1-{y_2}^4}}, a_3 = a_2\left(1+y_3\right)^4 - 2^7 y_3\left(1+y_3+{y_3}^2\right)$$
$$y_4 = \frac{1-\sqrt[4]{1-{y_3}^4}}{1+\sqrt[4]{1-{y_3}^4}}, a_4 = a_3\left(1+y_4\right)^4 - 2^9 y_4\left(1+y_4+{y_4}^2\right)$$
$$y_5 = \frac{1-\sqrt[4]{1-{y_4}^4}}{1+\sqrt[4]{1-{y_4}^4}}, a_5 = a_4\left(1+y_5\right)^4 - 2^{11} y_5\left(1+y_5+{y_5}^2\right)$$
$$y_6 = \frac{1-\sqrt[4]{1-{y_5}^4}}{1+\sqrt[4]{1-{y_5}^4}}, a_6 = a_5\left(1+y_6\right)^4 - 2^{13} y_6\left(1+y_6+{y_6}^2\right)$$
$$y_7 = \frac{1-\sqrt[4]{1-{y_6}^4}}{1+\sqrt[4]{1-{y_6}^4}}, a_7 = a_6\left(1+y_7\right)^4 - 2^{15} y_7\left(1+y_7+{y_7}^2\right)$$
$$y_8 = \frac{1-\sqrt[4]{1-{y_7}^4}}{1+\sqrt[4]{1-{y_7}^4}}, a_8 = a_7\left(1+y_8\right)^4 - 2^{17} y_8\left(1+y_8+{y_8}^2\right)$$
$$y_9 = \frac{1-\sqrt[4]{1-{y_8}^4}}{1+\sqrt[4]{1-{y_8}^4}}, a_9 = a_8\left(1+y_9\right)^4 - 2^{19} y_9\left(1+y_9+{y_9}^2\right)$$
$$y_{10} = \frac{1-\sqrt[4]{1-{y_9}^4}}{1+\sqrt[4]{1-{y_9}^4}}, a_{10} = a_9\left(1+y_{10}\right)^4 - 2^{21} y_{10}\left(1+y_{10}+{y_{10}}^2\right)$$
$$y_{11} = \frac{1-\sqrt[4]{1-{y_{10}}^4}}{1+\sqrt[4]{1-{y_{10}}^4}}, a_{11} = a_{10}\left(1+y_{11}\right)^4 - 2^{23} y_{11}\left(1+y_{11}+{y_{11}}^2\right)$$
$$y_{12} = \frac{1-\sqrt[4]{1-{y_{11}}^4}}{1+\sqrt[4]{1-{y_{11}}^4}}, a_{12} = a_{11}\left(1+y_{12}\right)^4 - 2^{25} y_{12}\left(1+y_{12}+{y_{12}}^2\right)$$
$$y_{13} = \frac{1-\sqrt[4]{1-{y_{12}}^4}}{1+\sqrt[4]{1-{y_{12}}^4}}, a_{13} = a_{12}\left(1+y_{13}\right)^4 - 2^{27} y_{13}\left(1+y_{13}+{y_{13}}^2\right)$$
$$y_{14} = \frac{1-\sqrt[4]{1-{y_{13}}^4}}{1+\sqrt[4]{1-{y_{13}}^4}}, a_{14} = a_{13}\left(1+y_{14}\right)^4 - 2^{29} y_{14}\left(1+y_{14}+{y_{14}}^2\right)$$
$$y_{15} = \frac{1-\sqrt[4]{1-{y_{14}}^4}}{1+\sqrt[4]{1-{y_{14}}^4}}, a_{15} = a_{14}\left(1+y_{15}\right)^4 - 2^{31} y_{15}\left(1+y_{15}+{y_{15}}^2\right)$$
$$y_{16} = \frac{1-\sqrt[4]{1-{y_{15}}^4}}{1+\sqrt[4]{1-{y_{15}}^4}}, a_{16} = a_{15}\left(1+y_{16}\right)^4 - 2^{33} y_{16}\left(1+y_{16}+{y_{16}}^2\right)$$
$$y_{17} = \frac{1-\sqrt[4]{1-{y_{16}}^4}}{1+\sqrt[4]{1-{y_{16}}^4}}, a_{17} = a_{16}\left(1+y_{17}\right)^4 - 2^{35} y_{17}\left(1+y_{17}+{y_{17}}^2\right)$$
$$y_{18} = \frac{1-\sqrt[4]{1-{y_{17}}^4}}{1+\sqrt[4]{1-{y_{17}}^4}}, a_{18} = a_{17}\left(1+y_{18}\right)^4 - 2^{37} y_{18}\left(1+y_{18}+{y_{18}}^2\right)$$
$$y_{19} = \frac{1-\sqrt[4]{1-{y_{18}}^4}}{1+\sqrt[4]{1-{y_{18}}^4}}, a_{19} = a_{18}\left(1+y_{19}\right)^4 - 2^{39} y_{19}\left(1+y_{19}+{y_{19}}^2\right)$$
$$y_{20} = \frac{1-\sqrt[4]{1-{y_{19}}^4}}{1+\sqrt[4]{1-{y_{19}}^4}}, \mathbf{a_{20}} = a_{19}\left(1+y_{20}\right)^4 - 2^{41} y_{20}\left(1+y_{20}+{y_{20}}^2\right).$$

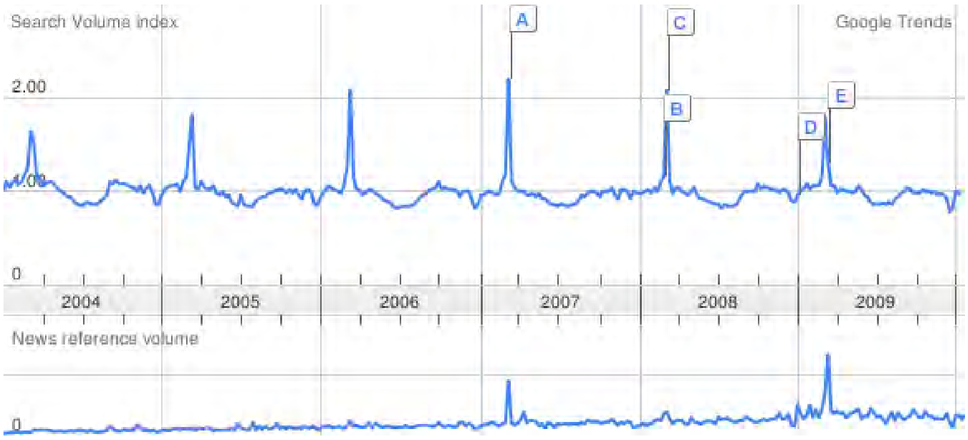


Figure 17: **Google's trend line for 'Pi'**

This quartic algorithm, with the Salamin–Brent scheme, was first used by Bailey in 1986 [13] and was used repeatedly by Yasumasa Kanada, see Figure 12, in Tokyo in computations of π over 15 years or so, culminating in a 200 billion decimal digit computation in 1999. As recorded in Figure 11, it has been used twice very recently by Takahashi. Only thirty five years earlier in 1963, Dan Shanks—a very knowledgeable participant—was confident that computing a billion digits was forever impossible. Today it is 'reasonably easy' on a modest laptop. A fine self-contained study of this quartic algorithm—along with its cubic confrere also described in Section 10—can be read in [18]. The proofs are nicely refined specializations of those in [12].

3. The Difficulty of Popularizing Accurately. Churchland in [15] offers a fascinating set of essays full of interesting anecdotes—which I have no particular reason to doubt—but the brief quote in Section 16 contains four inaccuracies. As noted above: (i) The event took place in 1896/7 and (ii) in Indiana (a northern state); (iii) The prospective bill, #246, offered a geometric construction with inconsistent conclusions and certainly offers no one exact value. Finally, (iv) the intent seems to have been pecuniary not hygienic [22]. As often, this makes me wonder whether mathematics popularization is especially prone to error or if the other disciplines just seem better described because of my relative ignorance. On April 1, 2009, an article entitled "The Changing Value of Pi" appeared in the *New Scientist* with an analysis of how the value of pi has been increasing over time. I hope but am not confident that all readers noted that April First is "April fool's day." (See entry seven of `http://www.museumofhoaxes.com/hoax/aprilfool/`.)

Following Pi on the Web. One can now follow Pi on the web through *Wikipedia*, *MathWorld* or elsewhere, and indeed one may check the performance of π by looking up 'Pi' at `http://www.google.com/trends`. Figure 17 shows very clear seasonal trends.

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There are many other Internet resources on π, a reliable selection is kept at `www.experimentalmath.info`.

CLOSED FORMS: WHAT THEY ARE AND WHY THEY MATTER

JONATHAN M. BORWEIN AND RICHARD E. CRANDALL

Abstract. The term "closed form" is one of those mathematical notions that is commonplace, yet virtually devoid of rigor. And, there is disagreement even on the intuitive side; for example, most everyone would say that $(\pi + \log 2)$ is a closed form, but some of us would think that the Euler constant γ is not closed. Like others before us, we shall try to supply some missing rigor to the notion of closed forms and also to give examples from modern research where the question of closure looms both important and elusive.

1. Closed Forms: What They Are

Mathematics abounds in terms which are in frequent use yet which are rarely made precise. Two such are *rigorous proof* and *closed form* (absent within differential algebra). If a rigorous proof is "that which 'convinces' the appropriate audience" then a closed form is "that which looks 'fundamental' to the requisite consumer." In both cases, this is a community-varying and epoch-dependent notion. What was a compelling proof in 1810 may well not be now; what is a fine closed form in 2010 may have been anathema a century ago.

Let us begin by sampling the Web for various approaches to informal definitions of "closed form."

1.0.1. **First approach to a definition of closed form.** The first comes from MathWorld [43] and so may well be the first and last definition a student or other seeker-after-easy-truth finds.

> An equation is said to be a closed-form solution if it solves a given problem in terms of functions and mathematical operations from a given generally accepted set. For example, an infinite sum would generally not be considered closed-form. However, the choice of what to call closed-form and what not is

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> rather arbitrary since a new "closed-form" function could simply be defined in terms of the infinite sum.—Eric Weisstein

There is not much to disagree with in this but it is far from rigorous.

1.0.2. **Second approach.** The next follows a 16 Sept 1997 question to the long operating "Dr. Math." site[1] and is a good model of what are interested students are told.

> Subject: Closed form solutions
>
> Dear Dr. Math, What is the exact mathematical definition of a closed form solution? Is a solution in "closed form" simply if an expression relating all of the variables can be derived for a problem solution, as opposed to some higher-level problems where there is either no solution, or the problem can only be solved incrementally or numerically?
>
> Sincerely, ...

The answer followed on 22 Sept:

> This is a very good question! This matter has been debated by mathematicians for some time, but without a good resolution.
>
> Some formulas are agreed by all to be "in closed form." Those are the ones which contain only a finite number of symbols, and include only the operators $+, -, *, /$, and a small list of commonly occurring functions such as n-th roots, exponentials, logarithms, trigonometric functions, inverse trigonometric functions, greatest integer functions, factorials, and the like.
>
> More controversial would be formulas that include infinite summations or products, or more exotic functions, such as the Riemann zeta function, functions expressed as integrals of other functions that cannot be performed symbolically, functions that are solutions of differential equations (such as Bessel functions or hypergeometric functions), or some functions defined recursively. Some functions whose values are impossible to compute at some specific points would probably be agreed not to be in closed form (example: $f(x) = 0$ if x is an algebraic number, but $f(x) = 1$ if x is transcendental. For most numbers, we do not know if they are transcendental or not). I hope this is what you wanted.

No more formal, but representative of many dictionary definitions is:

1.0.3. **Third approach.** A coauthor of the current article is at least in part responsible for the following brief definition from a recent mathematics dictionary [16]:

> **closed form** n. an expression for a given function or quantity, especially an integral, in terms of known and well understood quantities, such as the

[1] Available at http://mathforum.org/dr/math/.

evaluation of

$$\int_{\infty}^{\infty} \exp(-x^2)\,\mathrm{d}x$$

as $\sqrt{\pi}$.—Collins Dictionary

With that selection recorded, let us turn to some more formal proposals.

1.0.4. **Fourth approach.** Various notions of elementary numbers have been proposed.

> **Definition** [28]**.** A subfield F of $\mathbb{C}$ is *closed under* exp and log if (1) $\exp(x) \in F$ for all $x \in F$ and (2) $\log(x) \in F$ for all nonzero $x \in F$, where log is the branch of the natural logarithm function such that $-\pi < \operatorname{Im}(\log x) \le \pi$ for all x. The field $\mathbb{E}$ of EL numbers is the intersection of all subfields of $\mathbb{C}$ that are closed under exp and log.—Tim Chow

Tim Chow explains nicely why he eschews capturing all algebraic numbers in his definition; why he wishes only elementary quantities to have closed forms; whence he prefers $\mathbb{E}$ to Ritt's 1948 definition of *elementary numbers* as the smallest algebraically closed subfield $\mathbb{L}$ of $\mathbb{C}$ that is closed under exp and log. His reasons include that

> *[i]ntuitively, "closed-form" implies "explicit," and most algebraic functions have no simple explicit expression.*

Assuming Shanuel's conjecture [28] then the algebraic members of $\mathbb{E}$ are exactly those solvable in radicals. We may think of Chow's class as the smallest plausible class of closed forms.

1.1. **Special functions.** In an increasingly computational world, an explicit/implicit dichotomy is occasionally useful; but not very frequently. Often we will prefer computationally the numerical implicit value of an algebraic number to its explicit tower of radicals; and it seems increasingly perverse to distinguish the root of $2x^5 - 10x + 5$ to that of $2x^4 - 10x + 5$ or to prefer $\arctan(\pi/7)$ to $\arctan(1)$. We illustrate these issues further in Example 3.1, 3.3 and 4.3.

We would prefer to view all values of classical *special functions* of mathematical physics [41] at algebraic arguments as being closed forms. Again there is no generally accepted class of special functions, but most practitioners would agree that the solutions to the classical second-order algebraic differential equations (linear or say Painlevé) are included. But so are various *hypertranscendental functions* such as Γ, B and ζ which are do not arise in that way.[2]

Hence we can not accept any definition of special function which relies on the underlying functions satisfying natural differential equations. The class must be extensible, new special functions are always being discovered.

[2]Of course a value of an hypertranscendental function at algebraic argument may be very well behaved, see Example 1.2.

A recent *American Mathematical Monthly* review[3] of [37] says:

> There's no rigorous definition of special functions, but the following definition is in line with the general consensus: functions that are commonly used in applications, have many nice properties, and are not typically available on a calculator. Obviously this is a sloppy definition, and yet it works fairly well in practice. Most people would agree, for example, that the gamma function is included in the list of special functions and that trigonometric functions are not.

Once again, there is much to agree with, and much to quibble about, in this reprise. That said, most serious books on the topic are little more specific. For us, special functions are non-elementary functions about which a significant literature has developed because of their importance in either mathematical theory or in practice. We certainly prefer that this literature includes the existence of excellent algorithms for their computation. This is all consonant with—if somewhat more ecumenical than—Temme's description in the preface of his lovely book [41, Preface p. xi]:

> [W]e call a function "special" when the function, just like the logarithm, the exponential and trigonometric functions (the elementary transcendental functions), belongs to the toolbox of the applied mathematician, the physicist or engineer.

Even more economically, Andrews, Askey and Roy start the preface to their important book *Special functions* [1] by writing:

> Paul Turan once remarked that special functions would be more appropriately labeled "useful functions."

With little more ado, they then start to work on the gamma and Beta functions; indeed the term "special function" is not in their index. Near the end of their preface, they also write

> [W]e suggest that the day of formulas may be experiencing a new dawn.

This is a sentiment which we both fully share.

Example 1.1 (Lambert's W). The *Lambert W* function, $W(x)$, is defined by appropriate solution of $y \cdot e^y = x$ [19, pp. 277-279]. This is a function which became a closed form after it was implemented in computer algebra systems (CAS) [36]. It is now embedded as a primitive in *Maple* and *Mathematica* with the same status as any other well studied special or elementary function. (See for example the tome [24].) The CAS know its power series and much more. For instance in *Maple* entering

```
> fsolve(exp(x)*x=1);identify(%);
```

returns

```
0.5671432904, LambertW(1)
```

[3]Available at http://www.maa.org/maa%20reviews/4221.html.

We consider this to be a splendid closed form even though assuming *Shanuel's conjecture* $W(1) \notin \mathbb{E}$ [28]. Additionally, it is only recently rigorously proven that W is not an elementary function in Liouville's precise sense [24]. We also note that successful simplification in a modern CAS [27] requires a great deal of knowledge of special functions. □

1.2. Further approaches.

1.2.1. **Fifth approach.** *PlanetMath*'s offering, as of 15 February 2010[4], is certainly in the elementary number corner.

> **expressible in closed form** (Definition) An expression is expressible in a closed form, if it can be converted (simplified) into an expression containing only elementary functions, combined by a finite amount of rational operations and compositions.—Planet Math

This reflects what is best and worst about 'the mathematical wisdom of crowds'. For the reasons adduced above, we wish to distinguish but admit both those closed forms which give analytic insight from those which are sufficient and prerequisite to effective computation. Our own current preferred class [7] is described next.

1.2.2. **Sixth approach.** We consider *generalized hypergeometric* evaluations (see [7]) as converging sums

$$x = \sum_{n\geq 0} c_n z^n \tag{1.1}$$

where z is any algebraic with $|z| \leq 1$, c_0 is rational, and for $n > 0$, $c_n = \frac{p(n)}{q(n)} c_{n-1}$ for p and q polynomials with integer coefficients, q having no positive integer zeros. The expansion here for x converges absolutely on the open disk $|z| < 1$, while for $|z| = 1$ the convergence is conditional. The ideas herein are readily extended to values x arising from analytic continuation in the z-plane. Let us denote by $\mathbb{X}$ the set of all such evaluations x, with complex-∞ adjoined, in the following:

> **Definition** [7]**.** The *ring of hyperclosure* $\mathbb{H}$ is the smallest subring of $\mathbb{C}$ containing the set $\mathbb{X}$. Elements of $\mathbb{H}$ are deemed *hyperclosed.*

In other words, the ring $\mathbb{H}$ is generated by all general hypergeometric evaluations, under the $\cdot, +$ operators, all symbolized by

$$\mathbb{H} \;=\; \langle X \rangle_{\cdot,+} \;.$$

$\mathbb{H}$ will contain a great many interesting closed forms from modern research. Note that $\mathbb{H}$ contains all closed forms in the sense of [39, Ch. 8] wherein only finite linear combinations of hypergeometric evaluations are allowed.

[4] Available at http://planetmath.org/encyclopedia/ClosedForm4.html.

So what numbers are in the ring $\mathbb{H}$? First off, *almost no* complex numbers belong to this ring! This is easily seen by noting that the set of general hypergeometric evaluations is countable, so the generated ring must also be countable. Still, a great many fundamental numbers are provably hyperclosed. Examples follow, in which we let ω denote an arbitrary algebraic number and n any positive integer:

$$\omega,\ \log\omega,\ e^{\omega},\ \pi$$
$$\text{the dilogarithmic combination } \mathrm{Li}_2(1/\sqrt{5}) + (\log 2)(\log 3),$$
$$\text{the elliptic integral } \mathrm{K}(\omega),$$
$$\text{the zeta function values } \zeta(n),$$
$$\text{special functions such as the Bessel evaluations } J_n(\omega).$$

We are not claiming that hyperclosure is any kind of final definition for "closed forms." But we do believe that any defining paradigm for closed forms must include this ring of hyperclosure $\mathbb{H}$. One way to reach further is to define a *ring of superclosure* as the closure

$$\mathbb{S} \ := \langle \mathbb{H}^{\mathbb{H}} \rangle_{\cdot,+} \ .$$

This ring contains numbers such as

$$e^{\pi} + \pi^{e},\ \frac{1}{\zeta(3)},$$

and of course a vast collection of numbers that may not belong to $\mathbb{H}$ itself. If we say that an element of $\mathbb{S}$ is *superclosed*, we still preserve the countability of all superclosed numbers. Again, any good definition of "closed form" should incorporate whatever is in the ring $\mathbb{S}$.

It is striking how beautiful combinatorial games can be when played under the rubric of hyper- or superclosure.

Example 1.2 (Superclosure of Γ(rational)). Let us begin with the *Beta function*

$$B(r,s) \ := \ \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}.$$

with $\Gamma(s)$ defined if one wishes as $\Gamma(s) := \int_0^\infty t^{s-1}e^{-t}\,\mathrm{d}t$. It turns out that for any rationals r, s the Beta function is hyperclosed. This is immediate from the hypergeometric identities

$$\frac{1}{B(r,s)} \ = \ \frac{rs}{r+s}\ {}_2\mathrm{F}_1(-r,-s,1;1),$$
$$B(r,s) \ = \ \frac{\pi \sin\pi(r+s)}{\sin\pi r\ \sin\pi s}\ \frac{(1-r)_M(1-s)_M}{M!(1-r-s)_M}\ {}_2\mathrm{F}_1(r,s,M+1;1),$$

where M is any integer chosen such that the hypergeometric series converges, say $M = \lceil 1 + r + s \rceil$. (Each of these Beta relations is a variant of the celebrated Gauss evaluation of ${}_2\mathrm{F}_1(\cdot, \cdot, \cdot; 1)$ [1, 41].)

We did not seize upon the Beta function arbitrarily, for, remarkably, the hyperclosure of $B^{\pm 1}(r, s)$ leads to compelling results on the Gamma function itself. Indeed, consider for example this product of four beta-function evaluations:

$$\frac{\Gamma(1/5)\Gamma(1/5)}{\Gamma(2/5)} \cdot \frac{\Gamma(2/5)\Gamma(1/5)}{\Gamma(3/5)} \cdot \frac{\Gamma(3/5)\Gamma(1/5)}{\Gamma(4/5)} \cdot \frac{\Gamma(4/5)\Gamma(1/5)}{\Gamma(5/5)}.$$

We know this product is hyperclosed. But upon inspection we see that the product is just $\Gamma^5(1/5)$. Along such lines one can prove that for any positive rational a/b (in lowest terms), we have hyperclosure of powers of the Gamma-function, in the form:

$$\Gamma^{\pm b}(a/b) \in \mathbb{H}.$$

Perforce, we have therefore a superclosure result for any Γ(rational) and its reciprocal:

$$\Gamma^{\pm 1}(a/b) \in \mathbb{S}.$$

One fundamental observation is thus: $\Gamma^{-2}(1/2) = \frac{1}{\pi}$ is hyperclosed; thus every integer power of π is hyperclosed. Incidentally, deeper combinatorial analysis shows that—in spite of our $\Gamma^5(1/5)$ beta-chain above, it really only takes *logarithmically many* (i.e., $O(\log b)$) hypergeometric evaluations to write Gamma-powers. For example,

$$\Gamma^{-7}\left(\frac{1}{7}\right) = \frac{1}{2^3 7^6}\, {}_2\mathrm{F}_1\left(-\frac{1}{7}, -\frac{1}{7}, 1; 1\right)^4 \, {}_2\mathrm{F}_1\left(-\frac{2}{7}, -\frac{2}{7}, 1; 1\right)^2 \, {}_2\mathrm{F}_1\left(-\frac{4}{7}, -\frac{4}{7}, 1; 1\right).$$

We note also that for $\Gamma(n/24)$ with n integer, elliptic integral algorithms are known which converge as fast as those for π [25, 20]. □

The above remarks on superclosure of $\Gamma(a/b)$ lead to the property of superclosure for special functions such as $J_\nu(\omega)$ for algebraic ω and rational ν; and for many of the mighty Meijer-G functions, as the latter can frequently be written by Slater's theorem [14] as superpositions of hypergeometric evaluations with composite-gamma products as coefficients. (See Example 3.2 below for instances of Meijer-G in current research.)

There is an interesting alternative way to envision hyperclosure, or at least something very close to our above definition. This is an idea of J. Carette [26], to the effect that solutions at algebraic end-points, and algebraic initial points, for *holonomic* ODEs—i.e. differential-equation systems with integer-polynomial coefficients—could be considered closed. One might say "diffeoclosed." An example diffeoclosed number is $J_1(1)$, i.e. from the Bessel differential equation for $J_1(z)$ with $z \in [0, 1]$; it suffices without loss of generality to consider topologically clean trajectories of the variable over $[0, 1]$. There is a formal ring of diffeoclosure, which ring is very similar to our $\mathbb{H}$;

however there is the caution that trajectory solutions can sometimes have nontrivial topology, so precise ring definitions need to be effected carefully.

It is natural to ask "what is the complexity of hypergeometric evaluations?" Certainly for the converging forms with variable z on the open unit disk, convergence is geometric, requiring $O(D^{1+\epsilon})$ operations to achieve D good digits. However in very many cases this can be genuinely enhanced to $O(D^{1/2+\epsilon})$[20].

2. Closed Forms: Why They Matter

> In many optimization problems, *simple, approximate* solutions are more useful than complex exact solutions.—Steve Wright

As Steve Wright observed in a recent lecture on *sparse optimization* it may well be that a complicated analytic solution is practically intractable but a simplifying assumption leads to a very practical closed form approximation (e.g., in compressed sensing). In addition to appealing to Occam's razor, Wright instances that:

(a) the data quality may not justify exactness;
(b) the simple solution may be more robust;
(c) it may be easier to "*explain/ actuate/ implement/ store*";
(d) and it may conform better to prior knowledge.

Example 2.1 (The amplitude of a pendulum). *Wikipedia*[5] after giving the classical small angle (simple harmonic) approximation

$$p \approx 2\pi\sqrt{\frac{L}{g}}$$

for the period p of a pendulum of length L and amplitude α, develops the exact solution in a form equivalent to

$$p = 4\sqrt{\frac{L}{g}}\,\mathrm{K}\left(\sin\frac{\alpha}{2}\right)$$

and writes:

> This integral cannot be evaluated in terms of elementary functions. It can be rewritten in the form of the elliptic function of the first kind (also see Jacobi's elliptic functions), which gives little advantage since that form is also insoluble.

True, an elliptic-integral solution is not elementary, yet the notion of insolubility is misleading for two reasons: First, it is known that for some special angles α, the

[5]Available at http://en.wikipedia.org/wiki/Pendulum_(mathematics).

pendulum period can be given a closed form. As discussed in [29], one exact solution is, for $\alpha = \pi/2$ (so pendulum is released from horizontal-rod position),

$$p \;=\; \left(2\pi\sqrt{\frac{L}{g}}\right)\frac{\sqrt{\pi}}{\Gamma^2(3/4)}.$$

It is readily measurable in even a rudimentary laboratory that the excess factor here, $\sqrt{\pi}\Gamma^{-2}(3/4) \approx 1.18034$ looks just right, i.e., a horizontal-release pendulum takes 18 per cent longer to fall. Moreover, there is an *exact* dynamical solution—covering all motion in the time domain—namely for a pendulum with $\alpha = \pi$, i.e. the bob is released from the vertical-upward position, with the assumption of crossing angle zero at time zero. We have period $p = \infty$, yet the *exact* angle $\alpha(t)$ can be written down in terms of elementary functions!

The second misleading aspect is this: K is—for any α—remarkably tractable in a computational sense. Indeed K admits of a quadratic transformation

$$\mathrm{K}\left(k\right) = \left(1 + k_1\right)\mathrm{K}\left(k_1\right), \quad k_1 := \frac{1-\sqrt{1-k^2}}{1+\sqrt{1-k^2}} \tag{2.1}$$

as was known already to Landen, Legendre and Gauss.

In fact all elementary function to very high precision are well computed via K [20]. So the comment was roughly accurate in the world of slide rules or pocket calculators; it is misleading today—if one has access to any computer package. Nevertheless, both deserve to be called closed forms: one exact and the other an elegant approximate closed form (excellent in its domain of applicability, much as with Newtonian mechanics) which is equivalent to

$$\mathrm{K}\left(\sin\frac{\alpha}{2}\right) \approx \frac{\pi}{2}$$

for small initial amplitude α. To compute $K(\pi/6) = 1.699075885\ldots$ to five places requires using (2.1) only twice and then estimating the resultant integral by $\pi/2$. A third step gives the ten-digit precision shown. □

It is now the case that much mathematical computation is *hybrid*: mixing numeric and symbolic computation. Indeed, which is which may not be clear to the user if, say, numeric techniques have been used to return a symbolic answer or if a symbolic closed form has been used to make possible a numerical integration. Moving from classical to modern physics, both understanding and effectiveness frequently demand hybrid computation.

Example 2.2 (Scattering amplitudes [2]). An international team of physicists, in preparation for the Large Hadron Collider (LHC), is computing scattering amplitudes involving quarks, gluons and gauge vector bosons, in order to predict what results could be expected on the LHC. By default, these computations are performed using conventional double precision (64-bit IEEE) arithmetic. Then if a particular phase space point is deemed numerically unstable, it is recomputed with double-double precision. These researchers expect that further optimization of the procedure for identifying unstable points may be required to arrive at an optimal compromise between numerical accuracy and speed of the code. Thus they plan to incorporate arbitrary precision arithmetic, into these calculations. Their objective is to design a procedure where instead of using fixed double or quadruple precision for unstable points, the number of digits in the higher precision calculation is dynamically set according to the instability of the point. Any subroutine which uses a closed form symbolic solution (exact or approximate) is likely to prove much more robust and efficient. □

3. Detailed Examples

We start with three examples originating in [15].

In the January 2002 issue of *SIAM News*, Nick Trefethen presented ten diverse problems used in teaching *modern* graduate numerical analysis students at Oxford University, the answer to each being a certain real number. Readers were challenged to compute ten digits of each answer, with a $100 prize to the best entrant. Trefethen wrote,

> "*If anyone gets 50 digits in total, I will be impressed.*"

To his surprise, a total of 94 teams, representing 25 different nations, submitted results. Twenty of these teams received a full 100 points (10 correct digits for each problem). The problems and solutions are dissected most entertainingly in [15]. One of the current authors wrote the following in a review [17] of [15].

> Success in solving these problems required a broad knowledge of mathematics and numerical analysis, together with significant computational effort, to obtain solutions and ensure correctness of the results. As described in [15] he strengths and limitations of *Maple*, *Mathematica*, Matlab (The *3Ms*), and other software tools such as PARI or GAP, were strikingly revealed in these ventures. Almost all of the solvers relied in large part on one or more of these three packages, and while most solvers attempted to confirm their results, there was no explicit requirement for proofs to be provided.

Example 3.1 (Trefethen problem #2 [15, 17]).

> A photon moving at speed 1 in the x-y plane starts at $t = 0$ at $(x, y) = (1/2, 1/10)$ heading due east. Around every integer lattice point (i, j) in the plane, a circular mirror of radius $1/3$ has been erected. How far from the origin is the photon at $t = 10$?

Using *interval arithmetic* with starting intervals of size smaller than 10^{-5000}, one can actually find the position of the particle at time 2000—not just time ten. This makes a fine exercise in very high-precision interval computation, but in absence of any closed form one is driven to such numerical gymnastics to deal with error propagation. □

Example 3.2 (Trefethen's problem #9 [15, 17]).

> The integral $I(a) = \int_0^2 [2 + \sin(10\alpha)]x^\alpha \sin(\alpha/(2-x))\,\mathrm{d}x$ depends on the parameter α. What is the value $\alpha \in [0,5]$ at which $I(\alpha)$ achieves its maximum?

The maximum parameter is expressible in terms of a *Meijer-G function* which is a special function with a solid history. While knowledge of this function was not common among the contestants, *Mathematica* and *Maple* both will figure this out [14], and then the help files or a web search will quickly inform the scientist.

This is another measure of the changing environment. It is usually a good idea—and not at all immoral—to data-mine. These Meijer-G functions, first introduced in 1936, also occur in quantum field theory and many other places [8]. For example, the moments of an n-dimensional random walk in the plane are given for $s > 0$ by

$$W_n(s) := \int_{[0,1]^n} \left| \sum_{k=1}^n e^{2\pi x_k i} \right|^s \mathrm{d}x. \tag{3.1}$$

It transpires [22, 32] that for all complex s

$$W_3(s) = \frac{\Gamma(1+s/2)}{\Gamma(1/2)\Gamma(-s/2)} G^{2,1}_{3,3}\left(\begin{matrix} 1,1,1 \\ 1/2, -s/2, -s/2 \end{matrix} \middle| \frac{1}{4} \right), \tag{3.2}$$

$$W_4(s) = \frac{2^s}{\pi} \frac{\Gamma(1+s/2)}{\Gamma(-s/2)} G^{2,2}_{4,4}\left(\begin{matrix} 1, (1-s)/2, 1, 1 \\ 1/2, -s/2, -s/2, -s/2 \end{matrix} \middle| 1 \right). \tag{3.3}$$

Moreover, for s not an odd integer, we have

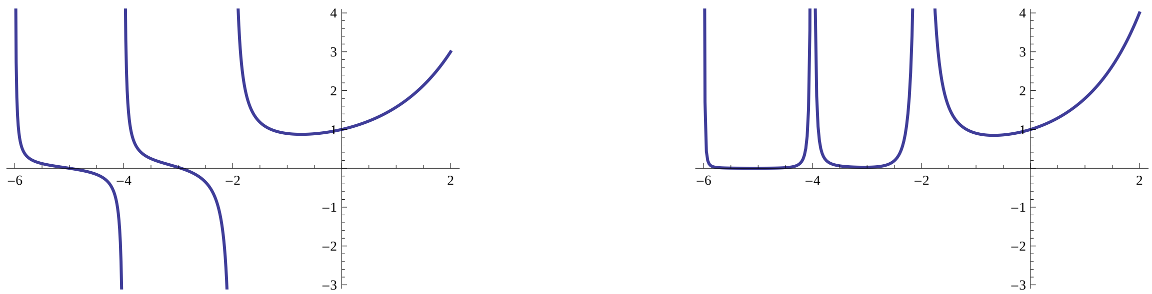
$$W_3(s) = \frac{1}{2^{2s+1}} \tan\left(\frac{\pi s}{2}\right) \binom{s}{\frac{s-1}{2}}^2 {}_3F_2\left(\begin{matrix} \frac12, \frac12, \frac12 \\ \frac{s+3}{2}, \frac{s+3}{2} \end{matrix} \middle| \frac14 \right) + \binom{s}{\frac{s}{2}} {}_3F_2\left(\begin{matrix} -\frac{s}{2}, -\frac{s}{2}, -\frac{s}{2} \\ 1, -\frac{s-1}{2} \end{matrix} \middle| \frac14 \right),$$

and

$$W_4(s) = \frac{1}{2^{2s}} \tan\left(\frac{\pi s}{2}\right) \binom{s}{\frac{s-1}{2}}^3 {}_4F_3\left(\begin{matrix} \frac12, \frac12, \frac12, \frac{s}{2}+1 \\ \frac{s+3}{2}, \frac{s+3}{2}, \frac{s+3}{2} \end{matrix} \middle| 1 \right) + \binom{s}{\frac{s}{2}} {}_4F_3\left(\begin{matrix} \frac12, -\frac{s}{2}, -\frac{s}{2}, -\frac{s}{2} \\ 1, 1, -\frac{s-1}{2} \end{matrix} \middle| 1 \right).$$

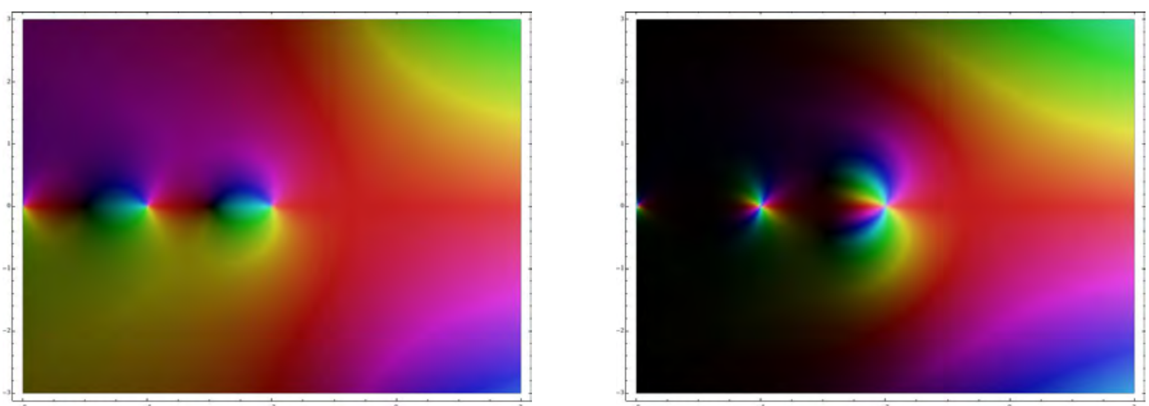
We thus know, from our "Sixth approach" section previous in regard to superclosure of Γ-evaluations, that *both* $W_3(q), W_4(q)$ *are superclosed for rational argument* q for q not an odd integer.

We illustrate by showing graphs of W_3, W_4 on the real line in Figure 1 and in the complex plane in Figure 2. The later highlights the utility of the Meijer-G representations. Note the poles and removable singularities.



(a) W_3 (b) W_4

FIGURE 1. W_3, W_4 analytically continued to the real line.



(a) W_3 (b) W_4

FIGURE 2. W_3 via (3.2) and W_4 via (3.3) in the complex plane.

The Meijer-G functions are now described in the newly completed *Digital Library of Mathematical Functions*[6] and as such are now full, indeed central, members of the family of special functions. □

Example 3.3 (Trefethen's problem #10 [15, 17]).

[6]A massive revision of Abramowitz and Stegun—with the now redundant tables removed, it is available at www.dlmf.nist.gov.

> A particle at the center of a 10×1 rectangle undergoes Brownian motion (i.e., 2-D random walk with infinitesimal step lengths) till it hits the boundary. What is the probability that it hits at one of the ends rather than at one of the sides?

Hitting the Ends. Bornemann [15] starts his remarkable solution by exploring *Monte-Carlo methods*, which are shown to be impracticable. He then reformulates the problem *deterministically* as *the value at the center of a* 10×1 *rectangle of an appropriate* harmonic measure [44] *of the ends*, arising from a 5-point discretization of *Laplace's equation* with Dirichlet boundary conditions. This is then solved by a well chosen *sparse Cholesky* solver. At this point a reliable numerical value of $3.837587979 \cdot 10^{-7}$ is obtained. And the posed problem is solved *numerically* to the requisite ten places.

This is the warm up. We may proceed to develop two analytic solutions, the first using *separation of variables* on the underlying PDE on a general $2a \times 2b$ rectangle. We learn that

$$p(a,b) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \operatorname{sech}\left(\frac{\pi(2n+1)}{2}\,\rho\right) \tag{3.4}$$

where $\rho := a/b$. A second method using *conformal mappings*, yields

$$\operatorname{arccot}\rho = p(a,b)\,\frac{\pi}{2} + \arg \mathrm{K}\left(e^{ip(a,b)\pi}\right) \tag{3.5}$$

where K is again the *complete elliptic integral* of the first kind. It will not be apparent to a reader unfamiliar with inversion of elliptic integrals that (3.4) and (3.5) encode the same solution—though they must as the solution is unique in $(0,1)$—and each can now be used to solve for $\rho = 10$ to arbitrary precision. Bornemann ultimately shows that the answer is

$$p = \frac{2}{\pi}\,\arcsin\left(k_{100}\right), \tag{3.6}$$

where

$$k_{100} := \left(\left(3 - 2\sqrt{2}\right)\left(2+\sqrt{5}\right)\left(-3+\sqrt{10}\right)\left(-\sqrt{2}+\sqrt[4]{5}\right)^2\right)^2.$$

No one (except harmonic analysts perhaps) anticipated a closed form—let alone one like this.

Where does this come from? In fact [20, (3.2.29)] shows that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \operatorname{sech}\left(\frac{\pi(2n+1)}{2}\,\rho\right) = \frac{1}{2}\arcsin k, \tag{3.7}$$

exactly when k_{ρ^2} is parameterized by *theta functions* in terms of the so called *nome*, $q = \exp(-\pi\rho)$, as Jacobi discovered. We have

$$k_{\rho^2} \;=\; \frac{\theta_2^2(q)}{\theta_3^2(q)} = \frac{\sum_{n=-\infty}^{\infty} q^{(n+1/2)^2}}{\sum_{n=-\infty}^{\infty} q^{n^2}}, \qquad q := e^{-\pi\rho}. \tag{3.8}$$

Comparing (3.7) and (3.4) we see that the solution is

$$k_{100} = 6.02806910155971082882540712292\ldots \cdot 10^{-7}$$

as asserted in (3.6).

The explicit form now follows from classical nineteenth century theory as discussed say in [15, 20]. In fact k_{210} is the singular value sent by Ramanujan to Hardy in his famous letter of introduction [19, 20]. If Trefethen had asked for a $\sqrt{210} \times 1$ box, or even better a $\sqrt{15} \times \sqrt{14}$ one, this would have shown up in the answer since in general

$$p(a, b) = \frac{2}{\pi} \arcsin\left(k_{a^2/b^2}\right). \tag{3.9}$$

Alternatively, armed only with the knowledge that the singular values of rational parameters are always algebraic we may finish entirely computationally as described in [17]. □

We finish this section with an attractive example from optics.

Example 3.4 (Mirages [38]). In [38] the authors, using geometric methods, develop an exact but implicit formula for the path followed by a light ray propagating over the earth with radial variations in the refractive index. By suitably simplifying they are able to provide an explicit integral closed form. They then expand it asymptotically. This is done with the knowledge that the approximation is good to six or seven places—more than enough to use it on optically realistic scales. Moreover, in the case of quadratic or linear refractive indices these steps may be done analytically.

In other words, as discussed by Wright, a tractable and elegant approximate closed form is obtained to replace a problematic exact solution. From these forms interesting qualitative consequences follow. With a quadratic index, images are uniformly magnified in the vertical direction; only with higher order indices can nonuniform vertical distortion occur. This sort of knowledge allows one, for example, to correct distortions of photographic images. □

4. Recent Examples from Our Own Work

Example 4.1 (Ising integrals [5, 8]). We recently studied the following classes of integrals [5]. The D_n integrals arise in the Ising model of mathematical physics, and the C_n have tight connections to quantum field theory [8].

$$C_n = \frac{4}{n!}\int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n (u_j + 1/u_j)\right)^2}\frac{\mathrm{d}u_1}{u_1}\cdots\frac{\mathrm{d}u_n}{u_n}$$

$$D_n = \frac{4}{n!}\int_0^\infty \cdots \int_0^\infty \frac{\prod_{i<j}\left(\frac{u_i-u_j}{u_i+u_j}\right)^2}{\left(\sum_{j=1}^n (u_j + 1/u_j)\right)^2}\frac{\mathrm{d}u_1}{u_1}\cdots\frac{\mathrm{d}u_n}{u_n}$$

$$E_n = 2\int_0^1 \cdots \int_0^1 \left(\prod_{1\le j<k\le n}\frac{u_k-u_j}{u_k+u_j}\right)^2 \mathrm{d}t_2\,\mathrm{d}t_3\cdots dt_n,$$

where (in the last line) $u_k = \prod_{i=1}^k t_i$.

Needless to say, evaluating these n-dimensional integrals to high precision presents a daunting computational challenge. Fortunately, in the first case, the C_n integrals can be written as one-dimensional integrals:

$$C_n = \frac{2^n}{n!}\int_0^\infty pK_0^n(p)\,\mathrm{d}p,$$

where K_0 is the *modified Bessel function*. After computing C_n to 1000-digit accuracy for various n, we were able to identify the first few instances of C_n in terms of well-known constants, e.g.,

$$C_3 = \mathrm{L}_{-3}(2) := \sum_{n\ge 0}\left(\frac{1}{(3n+1)^2} - \frac{1}{(3n+2)^2}\right), \qquad C_4 = \frac{7}{12}\zeta(3),$$

where ζ denotes the Riemann zeta function. When we computed C_n for fairly large n, for instance

$$C_{1024} = 0.6304735033743867961220401927108789043545870787127323 4\ldots,$$

we found that these values rather quickly approached a limit. By using the new edition of the *Inverse Symbolic Calculator*[7] this numerical value was identified as

$$\lim_{n\to\infty} C_n = 2e^{-2\gamma},$$

where γ is the *Euler constant*. We later were able to prove this fact—this is merely the first term of an asymptotic expansion—and thus showed that the C_n integrals are fundamental in this context [5].

The integrals D_n and E_n are much more difficult to evaluate, since they are not reducible to one-dimensional integrals (as far as we can tell), but with certain symmetry transformations and symbolic integration we were able to symbolically reduce

[7] Available at http://carma.newcastle.edu.au/isc2/.

the dimension in each case by one or two. In the case of D_5 and E_5, the resulting 3-D integrands are extremely complicated (see Figure 3), but we were nonetheless able to numerically evaluate these to at least 240-digit precision on a highly parallel computer system.

In this way, we produced the following evaluations, all of which except the last we subsequently were able to prove:

$$\begin{aligned} D_2 &= 1/3 \\ D_3 &= 8 + 4\pi^2/3 - 27\,\mathrm{L}_{-3}(2) \\ D_4 &= 4\pi^2/9 - 1/6 - 7\zeta(3)/2 \\ E_2 &= 6 - 8\log 2 \\ E_3 &= 10 - 2\pi^2 - 8\log 2 + 32\log^2 2 \\ E_4 &= 22 - 82\zeta(3) - 24\log 2 + 176\log^2 2 - 256(\log^3 2)/3 + 16\pi^2\log 2 - 22\pi^2/3 \end{aligned}$$

and

$$\begin{aligned} E_5 \stackrel{?}{=}\ & 42 - 1984\,\mathrm{Li}_4(1/2) + 189\pi^4/10 - 74\zeta(3) - 1272\zeta(3)\log 2 + 40\pi^2\log^2 2 \\ & -62\pi^2/3 + 40(\pi^2\log 2)/3 + 88\log^4 2 + 464\log^2 2 - 40\log 2, \end{aligned} \tag{4.1}$$

where Li denotes the polylogarithm function.

In the case of D_2, D_3 and D_4, these are confirmations of known results. We tried but failed to recognize D_5 in terms of similar constants (the 500-digit numerical value is accessible[8] if anyone wishes to try to find a closed form; or in the manner of the hard sciences to confirm our data values). The conjectured identity shown here for E_5 was confirmed to 240-digit accuracy, which is 180 digits beyond the level that could reasonably be ascribed to numerical round-off error; thus we are quite confident in this result even though we do not have a formal proof [5].

Note that every one of the D, E forms above, including the conjectured last one, is hyperclosed in the sense of our "Sixth approach" section. □

Example 4.2 (Weakly coupling oscillators [40, 6]). In an important analysis of coupled *Winfree oscillators*, Quinn, Rand, and Strogatz [40] developed a certain N-oscillator scenario whose bifurcation phase offset ϕ is implicitly defined, with a conjectured asymptotic behavior: $\sin\phi \sim 1 - c_1/N$, with experimental estimate $c_1 = 0.605443657\ldots$. In [6] we were able to derive the exact theoretical value of this "QRS constant" c_1 as the unique zero of the Hurwitz zeta $\zeta(1/2, z/2)$ on $z \in (0, 2)$. In so doing were able to prove the conjectured behaviour. □

It is a frequent experience of ours that the need for high accuracy computation drives the development of effective analytic expressions (closed forms?) which in turn typically shed substantial light on the subject being studied.

[8]Available at http://crd.lbl.gov/~dhbailey/dhbpapers/ising-data.pdf.

$$
\begin{aligned}
E_5 \quad = \quad & \int_0^1 \int_0^1 \int_0^1 \big[2(1-x)^2(1-y)^2(1-xy)^2(1-z)^2(1-yz)^2(1-xyz)^2 \\
& \big(-\big[4(x+1)(xy+1)\log(2)\left(y^5z^3x^7 - y^4z^2(4(y+1)z+3)x^6 - y^3z\left(\left(y^2+1\right)z^2+4(y+\right.\right.\right. \\
& 1)z+5)\,x^5 + y^2\left(4y(y+1)z^3+3\left(y^2+1\right)z^2+4(y+1)z-1\right)x^4 + y\left(z\left(z^2+4z\right.\right. \\
& +5)\,y^2+4\left(z^2+1\right)y+5z+4\big)\,x^3 + \left(\left(-3z^2-4z+1\right)y^2-4zy+1\right)x^2 - (y(5z+4) \\
& +4)x-1)\big]\,/\,\big[(x-1)^3(xy-1)^3(xyz-1)^3\big] + \big[3(y-1)^2y^4(z-1)^2z^2(yz \\
& -1)^2x^6 + 2y^3z\left(3(z-1)^2z^3y^5 + z^2\left(5z^3+3z^2+3z+5\right)y^4 + (z-1)^2z\right. \\
& \left(5z^2+16z+5\right)y^3 + \left(3z^5+3z^4-22z^3-22z^2+3z+3\right)y^2 + 3\left(-2z^4+z^3+2\right. \\
& \left.z^2+z-2\right)y + 3z^3+5z^2+5z+3\big)\,x^5 + y^2\left(7(z-1)^2z^4y^6 - 2z^3\left(z^3+15z^2\right.\right. \\
& +15z+1\big)\,y^5 + 2z^2\left(-21z^4+6z^3+14z^2+6z-21\right)y^4 - 2z\left(z^5-6z^4-27z^3\right. \\
& \left.-27z^2-6z+1\right)y^3 + \left(7z^6-30z^5+28z^4+54z^3+28z^2-30z+7\right)y^2 - 2\left(7z^5\right. \\
& \left.+15z^4-6z^3-6z^2+15z+7\right)y + 7z^4-2z^3-42z^2-2z+7\big)\,x^4 - 2y\left(z^3\left(z^3\right.\right. \\
& \left.-9z^2-9z+1\right)y^6 + z^2\left(7z^4-14z^3-18z^2-14z+7\right)y^5 + z\left(7z^5+14z^4+3\right. \\
& \left.z^3+3z^2+14z+7\right)y^4 + \left(z^6-14z^5+3z^4+84z^3+3z^2-14z+1\right)y^3 - 3\left(3z^5\right. \\
& \left.+6z^4-z^3-z^2+6z+3\right)y^2 - \left(9z^4+14z^3-14z^2+14z+9\right)y + z^3+7z^2+7z \\
& +1\big)\,x^3 + \left(z^2\left(11z^4+6z^3-66z^2+6z+11\right)y^6 + 2z\left(5z^5+13z^4-2z^3-2z^2\right.\right. \\
& +13z+5\big)\,y^5 + \left(11z^6+26z^5+44z^4-66z^3+44z^2+26z+11\right)y^4 + \left(6z^5-4\right. \\
& \left.z^4-66z^3-66z^2-4z+6\right)y^3 - 2\left(33z^4+2z^3-22z^2+2z+33\right)y^2 + \left(6z^3+26\right. \\
& \left.z^2+26z+6\right)y + 11z^2+10z+11\big)\,x^2 - 2\left(z^2\left(5z^3+3z^2+3z+5\right)y^5 + z\left(22z^4\right.\right. \\
& \left.+5z^3-22z^2+5z+22\right)y^4 + \left(5z^5+5z^4-26z^3-26z^2+5z+5\right)y^3 + \left(3z^4-\right. \\
& \left.22z^3-26z^2-22z+3\right)y^2 + \left(3z^3+5z^2+5z+3\right)y + 5z^2+22z+5\big)\,x + 15z^2+2z \\
& +2y(z-1)^2(z+1) + 2y^3(z-1)^2z(z+1) + y^4z^2\left(15z^2+2z+15\right) + y^2\left(15z^4\right. \\
& \left.-2z^3-90z^2-2z+15\right) + 15\big]\,/\,\big[(x-1)^2(y-1)^2(xy-1)^2(z-1)^2(yz-1)^2 \\
& (xyz-1)^2\big] - \big[4(x+1)(y+1)(yz+1)\left(-z^2y^4+4z(z+1)y^3+\left(z^2+1\right)y^2\right. \\
& -4(z+1)y + 4x\left(y^2-1\right)\left(y^2z^2-1\right) + x^2\left(z^2y^4-4z(z+1)y^3-\left(z^2+1\right)y^2\right. \\
& +4(z+1)y+1)-1)\log(x+1)\big]\,/\,\big[(x-1)^3x(y-1)^3(yz-1)^3\big] - [4(y+1)(xy \\
& +1)(z+1)\left(x^2\left(z^2-4z-1\right)y^4 + 4x(x+1)\left(z^2-1\right)y^3 - \left(x^2+1\right)\left(z^2-4z-1\right)\right. \\
& y^2 - 4(x+1)\left(z^2-1\right)y + z^2-4z-1\big)\log(xy+1)\big]\,/\,\big[x(y-1)^3y(xy-1)^3(z- \\
& 1)^3\big] - \big[4(z+1)(yz+1)\left(x^3y^5z^7 + x^2y^4(4x(y+1)+5)z^6 - xy^3\left(\left(y^2+\right.\right.\right. \\
& 1)\,x^2-4(y+1)x-3\big)\,z^5 - y^2\left(4y(y+1)x^3+5\left(y^2+1\right)x^2+4(y+1)x+1\right)z^4 + \\
& y\left(y^2x^3-4y(y+1)x^2-3\left(y^2+1\right)x-4(y+1)\right)z^3 + \left(5x^2y^2+y^2+4x(y+1)\right. \\
& y+1)\,z^2 + ((3x+4)y+4)z-1\big)\log(xyz+1)\big]\,/\,\big[xy(z-1)^3z(yz-1)^3(xyz-1)^3\big]\big)\big] \\
& /\,\big[(x+1)^2(y+1)^2(xy+1)^2(z+1)^2(yz+1)^2(xyz+1)^2\big]\;dx\,dy\,dz
\end{aligned}
$$

FIGURE 3. The reduced multidimensional integral for E_5, which integral has led to the conjectured closed form given in (4.1).

Example 4.3 (Box integrals [3, 7, 21]). There has been recent research on calculation of expected distance of points inside a hypercube to the hypercube. Such expectations are also called "box integrals" [21]. So for example, the expectation $\langle |\vec{r}| \rangle$ for random $\vec{r} \in [0,1]^3$ has the closed form

$$\frac{1}{4}\sqrt{3} - \frac{1}{24}\pi + \frac{1}{2}\log\left(2+\sqrt{3}\right).$$

Incidentally, box integrals are not just a mathematician's curiosity—the integrals have been used recently to assess the randomness of brain synapses positioned within a parallelepiped [34].

A very recent result is that every box integral $\langle |\vec{r}|^n \rangle$ for integer n, and dimensions $1,2,3,4,5$ are *hyperclosed*, in the sense of our "Sixth attempt" section. It turns out that five-dimensional box integrals have been especially difficult, depending on knowledge of a hyperclosed form for a single definite integral $J(3)$, where

$$J(t) \ := \ \int_{[0,1]^2} \frac{\log(t+x^2+y^2)}{(1+x^2)(1+y^2)}\, \mathrm{d}x\, \mathrm{d}y. \tag{4.2}$$

A proof of hyperclosure of $J(t)$ for algebraic $t \geq 0$ is established in [21, Thm. 5.1]. Thus $\langle |\vec{r}|^{-2} \rangle$ for $\vec{r} \in [0,1]^5$ can be written in explicit hyperclosed form involving a 10^5-character symbolic $J(3)$; the authors of [21] were able to reduce the 5-dimensional box integral down to "only" 10^4 characters. A companion integral $J(2)$ also starts out with about 10^5 characters but reduces stunningly to a only a few dozen characters, namely

$$J(2) = \frac{\pi^2}{8}\log 2 - \frac{7}{48}\zeta(3) + \frac{11}{24}\pi\,\mathrm{Cl}_2\left(\frac{\pi}{6}\right) - \frac{29}{24}\pi\,\mathrm{Cl}_2\left(\frac{5\pi}{6}\right), \tag{4.3}$$

where Cl_2 is the *Clausen function* $\mathrm{Cl}_2(\theta) := \sum_{n\geq 1} \sin(n\theta)/n^2$ (Cl_2 is the simplest non-elementary Fourier series).

Automating such reductions will require a sophisticated simplification scheme with a very large and extensible knowledge base. With a current Research Assistant, Alex Kaiser at Berkeley, we have started to design software to refine and automate this process and to run it before submission of any equation-rich paper. This semi-automated integrity checking becomes pressing when—as above—verifiable output from a symbolic manipulation might be the length of a Salinger novel. □

5. Profound curiosities

In our treatment of numbers enjoying hyperclosure or superclosure, we admitted that such numbers are countable, and so almost all complex numbers cannot be given a closed form along such lines. What is stultifying is: *We do not know a single explicit number outside of these countable sets.* The situation is tantamount to the modern bind in regard to *normal numbers*; namely, though almost all numbers are absolutely normal (i.e. have statistically random digit-structure in a certain technical sense), we do not know a single fundamental constant that is provably absolutely normal. (We do know some “artificial” normal numbers, see [13].)

Let us focus on some constants that *might* not be hyperclosed (nor superclosed). One such constant is the celebrated *Euler constant* $\gamma := \lim_{n\to\infty} \sum_{k=1}^{n} 1/k - \log n$. We know of no hypergeometric form for γ; said constant may well lie outside of $\mathbb{H}$ (or even $\mathbb{S}$). There *are* expansions for the Euler constant, such as

$$\gamma \;=\; \log \pi - 4 \log \Gamma(3/4) + \frac{4}{\pi} \sum_{k\ge 1} (-1)^{k+1} \frac{\log(2k+1)}{2k+1},$$

and even more exotic series (see [12]). One certainly cannot call this a closed form, even if the infinite sum be conceptually simple. Relatedly, the classical Bessel expansion is

$$K_0(z) = -\left(\ln\left(\frac{z}{2}\right) + \gamma\right) I_0(z) + \sum_{n=1}^{\infty} \frac{\sum_{k=1}^{n-1} \frac{1}{k}}{(n!)^2} \left(\frac{z^2}{4}\right)^n.$$

Now $K_0(z)$ has a (degenerate) Meijer-G representation—so potentially is superclosed for algebraic z—and $I_0(z)$ is accordingly hyperclosed, but the harmonic series on the right is problematic. Likewise what is the status of the Ψ function?

Example 5.1 (Madelung constant [20, 33, 45]). Another fascinating number is the *Madelung constant* of chemistry and physics [20, Section 9.3]. This is the potential energy at the origin of an oscillating-charge crystal structure (most often said crystal is NaCl (salt)) and is given by the formal sum

$$\mathcal{M} \;:=\; \sum_{x,y,z \neq 0,0,0} \frac{(-1)^{x+y+z}}{\sqrt{x^2+y^2+z^2}} \;=\; -1.747564594633..., \tag{5.1}$$

and has never been given what a reasonable observer would call a closed form. (Nature plays an interesting trick here: There are other crystal structures that *are* tractable, yet somehow this exquisitely symmetrical salt structure remains elusive.) But here we have another example of a constant having no known closed form, yet rapidly calculable. A classical rapid expansion for Madelung's constant is due to Benson:

$$\mathcal{M} = -12\pi \sum_{m,n\ge 0} \operatorname{sech}^2\left(\frac{\pi}{2}((2m+1)^2 + (2n+1)^2)^{1/2}\right), \tag{5.2}$$

in which convergence is exponential. Summing for $m, n \le 3$ produces $-1.747564594\ldots$, correct to 8 digits. There are great many other such formulae for $\mathcal{M}$ (see [20, 31]).

Through the analytic methods of Buhler, Crandall, Tyagi and Zucker since 1999 (see [31, 33, 42, 45]), we now know approximations such as

$$\mathcal{M} \approx \frac{1}{8} - \frac{\log 2}{4\pi} + \frac{8\pi}{3} + \frac{\Gamma(1/8)\Gamma(3/8)}{\pi^{3/2}\sqrt{2}} + \log \frac{k_4^2}{16 k_4 k_4'},$$

where $k_4 := ((2^{1/4} - 1)/(2^{1/4} + 1))^2$. Two remarkable things: First, this approximation is good to the same 13 decimals we give in the display (5.1); the missing $O(10^{-14})$ error here is a rapidly, exponentially converging—but alas infinite—sum in this modern approximation theory. Second: this 5-term approximation itself is indeed hyperclosed, the only problematic term being the Γ-function part, but we did establish in our "Sixth approach" section that $B(1/8, 3/8)$ and also $1/\pi$ are hyperclosed, which is enough. Moreover, the work of Borwein and Zucker [25] also settles hyperclosure for that term. □

Certainly we have nothing like a proof, or even the beginnings of one, that $\mathcal{M}$ (or γ) lies outside $\mathbb{H}$ (or even $\mathbb{S}$), but we ask on an intuitive basis: Is a constant such as the mighty $\mathcal{M}$ telling us that it is not hyperclosed, in that our toil only seems to bring more "closed-form" terms into play, with no exact resolution in sight?

6. Concluding Remarks and Open Problems

- We have posited several approaches to the elusive notion of "closed form." But what are the intersections and interrelations of said approaches? For example, can our "Fourth approach" be absorbed into the evidently more general "Sixth approach" (hyperclosure and superclosure)?
- How do we find a *single* number that is provably not in the ring of hyperclosure $\mathbb{H}$? (Though no such number is yet known, *almost all* numbers are as noted not in said ring!) Same question persists for the ring of hyperclosure, $\mathbb{S}$. Furthermore, how precisely can one create a *field* out of $\mathbb{H}^{\mathbb{H}}$ via appropriate operator extension?
- Though $\mathbb{H}$ is a subset of $\mathbb{S}$, how might one prove that $\mathbb{H} \neq \mathbb{S}$? (Is the inequality even true?) Likewise, is the set of closed forms in the sense of [39, Ch. 8] (only finite linear combinations of hypergeometric evaluations) properly contained in our $\mathbb{H}$? And what about a construct such as $\mathbb{H}^{\mathbb{H}^{\mathbb{H}}}$? Should such an entity be anything really new? Lest one remark on the folly of such constructions, we observe that most everyone would say $\pi^{\pi^{\pi}}$ is a closed form!
- Having established the property of hyperclosure for $\Gamma^b(a/b)$, are there any cases where the power b may be brought down? For example, $1/\pi$ is hyperclosed, but what about $1/\sqrt{\pi}$?

- There is expounded in reference [21] a theory of "expression entropy," whereby some fundamental entropy estimate gives the true complexity of an expression. So for example, an expression having 1000 instances of the polylog token Li_3 might really involve only about 1000 characters, with that polylogarithm token encoded as a single character, say. (In fact, during the research for [21] it was noted that the entropy of *Maple* and *Mathematica* expressions of the *same* entity often had widely varying text-character counts, but similar entropy assessments.)

  On the other hand, one basic notion of "closed form" is that explicitly infinite sums not be allowed. Can these two concepts be reconciled? Meaning: Can we develop a theory of expression entropy by which an explicit, infinite sum is given infinite entropy? This might be difficult, as for example a sum $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ only takes a few characters to symbolize, as we just did hereby! If one can succeed, though, in resolving thus the entropy business for expressions, "closed form" might be rephrased as "finite entropy."

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``a certain impression I had of mathematicians was ... that they spent immoderate amounts of time declaring each other's work trivial.''

(Richard Preston)

- From his prize winning article ***The Mountains of Pi***, *New Yorker*, March 9, 1992.

``It's about as interesting as going to the beach and counting sand. I wouldn't be caught dead doing that kind of work.''

(Professor Take Your Pick)

``The universe contains at most `two to the power fifty' grains of sand.''

(Archimedes)

``Americans are broad-minded people. They'll accept the fact that a person can be alcoholic, a dope fiend or a wife-beater, but if a man doesn't drive a car, everybody thinks that something is wrong with him.''

(Art Buchwald, local newspaper, March 1996)

``Caution, skepticism, scorn, distrust and entitlement seem to be intrinsic to many of us because of our training as scientists... . These qualities hinder your job search and career change.''

- Former astrophysicist **Stephen Rosen**, now director, Scientific Career Transitions Program, New York City, giving job-hunting advice in an on-line career counseling session.

(Quoted in *Science*, 4 August 1995, page 637)

``Consider a precise number that is well known to generations of parents and doctors: the normal human body temperature of 98.6 Farenheit. Recent investigations involving millions of measurements reveal that this number is wrong; normal human body temperature is actually 98.2 Farenheit. The fault, however, lies not iwth Dr. Wunderlich's original measurements - they were averaged and sensibly rounded to the nearest degree: 37 Celsius. When this temperature was converted to Farenheit, however, the rounding was forgotten and 98.6 was taken to be accurate to the nearest tenth of a degree. Had the original interval between 36.5 and 37.5 Celsius been translated, the equivalent Farenheit temperatures would have ranged from 97.7 to 99.5. Apparently, discalculia can even cause fevers.''

- John Allen Paulus, in **`A Mathematician Reads the Newspaper'** (Basic Books)

(Quoted in *Science*, August 18, 1995, page 992)

``When Gladstone was British Prime Minister he visited Michael Faraday's laboratory and asked if some esoteric substance called `Electricity' would ever have practical significance.

"One day, sir, you will tax it."

was the answer.''

(Quoted in *Science*, 1994). As Michael Saunders points out, this can not be correct because Faraday died in 1867 and Gladstone became PM in 1868. A more plausible PM would be Peel as electricity was discovered in 1831. Equally well it may be an `urbane legend'.

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*`` "the proof is left as an exercise" occurred in **`De Triangulis Omnimodis'** by Regiomontanus, written 1464 and published 1533. He is quoted as saying "This is seen to be the converse of the preceding. Moreover, it has a straightforward proof, as did the preceding. Whereupon I leave it to you for homework." ''*

(Quoted in *Science*, 1994)

``As the fading light of a dying day filtered through the window blinds, Roger stood over his victim with a smoking .45, surprised at the serenity that filled him after pumping six slugs into the bloodless tyrant that had mocked him day after day, and then he shuffled out of the office with one last look back at the shattered computer terminal lying there like a silicon armidillo left to rot on the information highway.''

- *From the winner of the 1994 Bulwer-Lytton Fiction contest for lousy literature*

- *Named for the author of **`It was a dark and stormy night.'** in the novel **`Paul Clifford'**, 1830. (Later winners are quoted below.)*

``I imagine most of that stuff on the information highway is roadkill anyway.''

(John Updike, 1994)

``It's going to be about bad news. It's going to be about the future of this country, about foreign policy, about defense policy. There are a lot of issues left. I'm certain something will pop up in November. So we'll be able to put it together.''

- **Robert Dole** on `what is the key issue?' in the '96 Presidential election.

(Quoted in the *Economist*, March 16, 1996, page 23)

``My dearest Miss Dorothea Sankey

My affectionate & most excellent wife is as you are aware still living - and I am proud to say her health is good. Nevertheless it is always well to take time by the forelock and be prepared for all events. Should anything happen to her, will you supply in her place - as soon as the proper period for decent mourning is over.

Till then I am your devoted servant
Anthony Trollope.''

- **Anthony Trollope** taking precautions in 1861.

- Sotheby's at auction in 1942 described this letter as *"one of the most extraordinary letters ever offered for sale".*

(Quoted from **The Oxford Book of Letters** in the *Economist*, March 23, 1996, page 90)

``I believe that the motion picture is destined to revolutionize our educational system and that in a few years it will supplant largely, if not entirely, the use of textbooks.''

(Thomas Alva Edison, 1922)

``Keynes distrusted intellectual rigour of the Ricardian type as likely to get in the way of original thinking and saw that it was not uncommon to hit on a valid conclusion before finding a logical path to it.

`I don't really start', he said, `until I get my proofs back from the printer. Then I can begin serious writing.' ''

- *two excerpts from **Keynes the man** written on the 50th Anniverary of Keynes' death.*

(Sir Alec Cairncross, in the *Economist*, April 20, 1996)

``One major barrier to entry into new markets is the requirement to see the future with clarity. It has been said that to so fortell the future, one has to invent it. To be able to invent the future is the dividend that basic research pays.''

- ***An econonomic case for basic research***, *by Eugen Wong, Hong Kong University of Science and Technology*

(In *Nature*, May 16, 1996, pages 178-9)

`` 'Ace, watch your head!' hissed Wanda urgently, yet somehow provocatively, through red, full, sensuous lips, but he couldn't, you know, since nobody can actually watch more than part of his nose or a little cheek or lips if he really tries, but he appreciated her warning.''

- *Janice Estey of Aspen* ***1996 Bulwer-Lytton Grand Prize Winner***

``Because the Indians of the high Andes were believed to have little sense of humor, Professor Juan Lyner was amazed to hear this knee-slapper that apparently had been around for centuries at all of the Inca spots: `Llama ask you this. Guanaco on a picnic? Alpaca lunch.' ''

- *John Ashman of Houston* ***1995 Bulwer-Lytton Grand Prize Winner***

``We know [smoking is] not good for kids. But a lot of other things aren't good. Drinking's not good. Some would say milk's not good.''

- **Robert Dole** echoing the tobacco companies on smoking?

(Page 27 in the *Economist*, July 6, 1996)

``I feel so strongly about the wrongness of reading a lecture that my language may seem immoderate .... The spoken word and the written word are quite different arts .... I feel that to collect an audience and then read one's material is like inviting a friend to go for a walk and asking him not to mind if you go alongside him in your car.''

- **Sir Lawrence Bragg.** What would he say about overheads?

(Page 76 in *Science*, July 5, 1996)

``I know, it's hard to believe that Microsoft would release a product before it was ready, but there you have it. A Seattle cyberwag says, "At Microsoft, quality is job 1.1." We had him killed. ''

- from *Welcome to Stale*

A take-off of Microsoft's ``webzine'', Slate, **Stale**, August 1996.

``No presidential candidate in the future will be so inept that four of his major speeches can be boiled down to these four historic sentences: Agriculture is

important. Our rivers are full of fish. You cannot have freedom with out liberty. The future lies ahead.''

- the (no doubt partisan) *Louisville Courier-Journal* on **Thomas Dewey** in 1948, quoted in Jack Beatty's review of James Patterson's *Grand Expectations, The United States, 1945-1974*. Beatty goes on to say:

`Tom Dewey, make room for Bob (``like everyone else in this room I was born'') Dole.`

- and lists many other fine quotes from Patterson's book.

(From pp. 107-112 in the **Atlantic Monthly**, September 1996)

``Writers often thank their colleagues for their help. Mine have given none. .. Writers often thank their typists. I thank mine. Mrs George Cook is not a particularly good typist, but her spelling and grammar are good. The responsibility for any mistakes is mine, but the fault is hers. Finally, writers too often thank their wives. I have no wife.''

- Acknowledgement by Edward Ingram in *The Beginning of the Great Game in Asia, 1828-1834.*

(From p. 83 in the **Economist**, September 7th 1996)

``I see some parallels between the shifts of fashion in mathematics and in music. In music, the popular new styles of jazz and rock became fashionable a little earlier than the new mathematical styles of chaos and complexity theory. Jazz and rock were long despised by classical musicians, but have emerged as art-forms more accessible than classical music to a wide section of the public. Jazz and rock are no longer to be despised as passing fads. Neither are chaos and complexity theory. But still, classical music and classical mathematics are not dead. Mozart lives, and so does Euler. When the wheel of fashion turns once more, quantum mechanics and hard analysis will once again be in style.''

- Freeman Dyson's review of *Nature's Numbers* by Ian Stewart (Basic Books, 1995).

(From p. 612 in the **American Mathematical Monthly**, August-September 1996)

``I was sitting by Dr. Franklin, who perceived that I was not insensible to these mutilations. I have made a rule, said he, whenever in my power, to avoid becoming the draughtsman of papers to be reviewed by a public body.''

- Jefferson writing in 1818 of the drafting of the Declaration of Independence.

(From p. 74 of Conor Cruise O'Brien's disturbing article *Thomas Jefferson: Radical and Racist*, in the **Atlantic Monthly**, October 1996)

- ["The tree of liberty must be refreshed from time to time with the blood of patriots and tyrants."
(Jefferson quoted on Oklahoma bomb suspect McVeigh's T-shirt.)]

``My morale has never been higher than since I stopped asking for grants to keep my lab going.''

- Robert Pollack, Columbia Professor of biology. Speaking on "the crisis in scientific morale", September 19, 1996 at GWU's symposium *Science in Crisis at the Millennium*.

(Quoted from p. 1805 in the September 27, 1996 **Science**)

``smugness, brutality, unctuous rectitude and tact"

- Cecil Rhodes own sardonic paraphrase of the criteria for a Rhodes Scholarship:

  - 30% for "literary and scholarly attainments";
  - 20% for "fondness of and success in manly outdoor sports";
  - 30% for "qualities of manhood, truth, courage, devotion to duty...";
  - 20% for "moral force of character and instincts to lead and to take an interest in his school-mates".

(Quoted from *Cecil Rhodes Traduced*, pp. 80-81 in the October 5, 1996 **Economist**)

``The dictum that everything that people do is 'cultural' ... licenses the idea that every cultural critic can meaningfully analyze even the most intricate accomplishments of art and science. ... It is distinctly weird to listen to pronouncements on the nature of mathematics from the lips of someone who cannot tell you what a complex number is!''

- Norman Levitt, from "The flight From Science and Reason," New York Academy of Science.

(Quoted from p. 183 in the October 11, 1996 **Science**)

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*``Church discipline is also somewhat of a remove from the time when the Emperor Henry IV was made to stand in the snow for three days outside the Pope's castle at Canossa, awaiting forgiveness. A French Bishop, Jacques Gaillot, because of his ultra-liberal views was recently transferred from his position at Evreux, in Normandy, and given charge instead of the defunct dioscese of **Partenia**, in Southern Algeria, which has been covered by sand since the Middle Ages. Gaillot has retaliated by creating a virtual dioscese on the Internet, which can be reached at **http://www.partenia.fr** ''*

- Cullen Murphy, "Broken Covenant?"

(Quoted from p. 24 in the November, 1996 **Atlantic Monthly**)

``We were a polite society and I expected to lead a quiet life teaching mechanics and listening to my senior colleagues gently but obliquely poking fun at one another.

This dream of somnolent peace vanished very quickly when Rutherford came to Cambridge. Rutherford was the only person I have met who immediately impressed me as a great man. He was a big man and made a big noise and he seemed to enjoy every minute of his life. I remember that when transatlantic broadcasting first came in, Rutherford told us at a dinner in Hall how he had spoken into a microphone to America and had been heard all over the continent. One of the bolder of our Fellows said "Surely you did not need to use apparatus for that." ''

- **Geoffrey Fellows**, 1952, as quoted by George Batchelor in *The Life and Legacy of G.I. Taylor* (Cambridge University Press).

(Quoted in "Vignettes: Yesteryear in Oxbridge" p. 733 in **Science** November 1, 1996)

``Then, owls and bats,
Cowls and twats,
Monks and nuns, in a cloister's moods,
Adjourn to the oak-stump pantry.''

- From Robert Browning's (1841) *Pippa Passes*, which also contains "God's in his Heaven, all's right with the world."

(Quoted on page 56-66 of **Bill Byerson**, *Mother Tongue: The English Language*, Penguin, 1990.)

- He goes on to say about "this disconcerting quote" that

``Browning had apparently somewhere come across the word twat - which meant precisely the same as it does now - but somehow took it to mean a piece of head gear for nuns. The verse became a source of twittering amusement for generations of schoolboys and a perennial embarrassment to their elders, but the word was never altered and Browning was allowed to live out his life in wholesome ignorance because no one could think of a suitably delicate way of explaining his mistake to him.''

``Two major advances are responsible for both the recent progress and current optimism. First, recombinant DNA technology has made it possible to identify every gene and protein in an organism and to manipulate them in order to explore their functions. Second, it has been discovered that the molecular mechanisms of development have been conserved during animal evolution to a far greater extent than had been imagined. This conservation means that discoveries about the development of worms and files, which come from the kind of powerful genetic studies that are not possible in mammals, greatly accelerate the rate at which we can discover the mechanisms and molecules that operate during our own development.

... ...

It is tempting to think that the main principles of neural development will have been discovered by the end of the century and that the cellular and molecular basis of the *mind will be the main challenge for the next. An alternative view is that this feeling that understanding is just a few steps away is a recurring and necessary delusion that keeps scientists from dwelling on the complexity the face and how much more remains to be discovered.''*

(Editorial on page 1063 of **Science** November 15, 1996)

- *Neural Development: Mysterious No More?* written by Martin Raff (University College, London).

[It is hard to imagine a better case for ``basic science'' than that afforded by this conservation principle -- if worms were good enough for Darwin ... !]

``3. SPACE SYMPOSIUM: THEOLOGIANS JOIN SCIENTISTS AT WHITE HOUSE.

Vice President Gore, who was clearly on top of the technical issues, met on Wednesday with a group of tough-minded scientists, clergy and fuzzy romantics to discuss the questions raised by evidence of extraterrestrial life. For physicist/astronomer John Bahcall, the remarkable thing was not that such questions were being asked, but that we have the tools to answer them.''

(From WHAT'S NEW by Robert L. Park -- Friday, 13 Dec 96)

- *WHAT'S NEW* is published every Friday by the **AMERICAN PHYSICAL SOCIETY**.

``As the test beds begin to prove WDM (`wavelength division multiplexing') networks feasible, telephone company executives will have to judge whether they are wise. If a single glass fiber can carry all the voice, fax, video and data traffic for a large corporation yet costs little more than today's high-speed Internet connections, how much will they be able to charge for telephone service? Peter Cochrane of BT Laboratories in Ipswich, England, predicts that "photonics will transform the telecoms industry by effectively making bandwidth free and distance irrelevant." Joel Birnbaum, director of Hewlett-Packard Laboratories, expects that this will relegate telephone companies to the role of digital utilities. "You will buy computing like you now buy water or power," he says.

Others, such as industry analyst Francis McInerney, believe the double-time march of technology has already doomed them to fall behind. AT&T and its ilk, he claims, "are already dead. When individuals have [megabits per second of bandwidth], telephone service should cost about three cents a month." Having discovered how to offer high-bandwidth service, telephone companies may now need to invent useful things to do with it, just to stay in business. ''

(From BANDWIDTH, UNLIMITED by W. Wayt Gibbs)

- In the January 1997 on-line **Scientific American**.

``Before Canada jeopardizes its scientific future and compromises its scientific community to achieve short-term budgetary solutions, it must recognize that the

funding of university sicence is both a government responsibility and a long-range investment. Without government support, Canada's university science infrastructure will erode, and along with it, the country's competitiveness in a world economy increasingly based on knowledge.''

(Editorial on page 139 of **Science** January 10, 1997)

Canada's Crisis: Can Business Rescue Science? written by Albert Aguyo and Richard A. Murphy (McGill, Montreal Neurological).

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1. SENATOR GRAMM EMERGES AS THE CHAMPION OF BASIC RESEARCH

178 new bills were introduced in the Senate on Tuesday -- one, S.124, is a thing of beauty: "The National Research Investment Act of 1997." It calls for doubling the federal investment in basic science and medical research over a 10-year period (WN 17 Jan 97). Funds must be allocated by a peer review system and can not be used for the commercialization of technologies. A dozen non-defense agencies and programs are covered by the bill, which is the work of Phil Gramm (R-TX). Gramm pointed out that in 1965, 5.7% of the federal budget went for non-defense R&D -- 32 years later, that has dropped to only 1.9%, and real spending on research has declined for four straight years. Ten-year doubling requires an annual increase of 7% -- just what leaders of the science community have been calling for (WN 10 Jan 97)."

(From WHAT'S NEW by Robert L. Park -- Friday, 24 January, 1997)

WHAT'S NEW is published every Friday by the **AMERICAN PHYSICAL SOCIETY**. It is interesting to contrast a conservative US senator (an ex-academic) from a liberal Canadian government.

``a British officer told a sergeant to post four lookouts to watch for the German army which was advancing through Belgium. Later, the officer discovered that the sergeant had posted only three. Asked to explain his lapse, the soldier said he had judged the fourth guard unnecessary. 'The enemy would hardly come from that direction,' he explained, 'it's private property.' ''

(Quoted from **Back to the Front** by Stephen O'Shea)

From page 59 in **MACLEANS Magazine** of February 10, 1997.

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*``Admirers of Thomas Jefferson have long quoted his statement about black men and women that is inscribed on the Jefferson Memorial: 'Nothing is more certainly written in the book of fate than that these people are to be free.' But they and the inscription, as **Conor Cruise O'Brien** pointed out in 'Thomas Jefferson: Radical and Racist' (October, 1996, Atlantic), omit Jefferson's subsequent clause: 'Nor is it less certain that the two races, equally free, cannot live in the same government.'"*

(Quoted from **What Jefferson Helps to Explain** by Benjamin Schwarz)

From page 60 in the *Atlantic Monthly* of March, 1997. [There are well established copyright notions of "paternity" and "integrity" in the use of material -- the later which this clearly violates!]

> ``*A centre of excellence is, by definition, a place where second class people may perform first class work.*''
>
> ``*A truly popular lecture cannot teach, and a lecture that truly teaches cannot be popular.*''
>
> ``*The most prominent requisite to a lecturer, though perhaps not really the most important, is a good delivery; for though to all true philosophers science and nature will have charms innumerably in every dress, yet I am sorry to say that the generality of mankind cannot accompany us one short hour unless the path is strewed with flowers.*''

(Three quotes from *Michael Faraday*)

- Excerpted from "**Michael Faraday -- and the Royal Institution, the genius of man and place**", by J.M. Thomas, Adam Hilger, Bristol, 1991.

> ``*The body of mathematics to which the calculus gives rise embodies a certain swashbuckling style of thinking, at once bold and dramatic, given over to large intellectual gestures and indifferent, in large measure, to any very detailed description of the world. It is a style that has shaped the physical but not the biological sciences, and its success in Newtonian mechanics, general relativity and quantum mechanics is among the miracles of mankind. But the era in thought that the calculus made possible is coming to an end. Everyone feels this is so and everyone is right.*''

(From **Vignettes: Changing Times** in *Science*, 28 February 1997, page 1276)

- From David Berlinski's "**A Tour of the Calculus"** (Pantheon Books, 1995)

> ``***[8] 94m:94015 Beutelspacher, Albrecht*** *Cryptology. An introduction to the art and science of enciphering, encrypting, concealing, hiding and safeguarding described without any arcane skullduggery but not without cunning waggery for the delectation and instruction of the general public. Transformation from German into English succored and abetted by J. Chris Fisher. MAA Spectrum. Mathematical Association of America, Washington, DC, 1994. xvi+156 pp. ISBN: 0-88385-504-6 94A60 (94-01)*''

(From **Math Reviews**)

- A serious "best title" candidate...

> ``*It's generally the way with progress that it looks much greater than it really is.*''

(From **The Wittgenstein Controversy**, by Evelyn Toynton in the *Atlantic Monthly* June 1997, pp. 28-41.)

The epigraph that **Ludwig Wittgenstein** (1889-1951) ("whereof one cannot speak, thereof one must be silent") had wished for an unrealized joint publication of *Tractatus Logico-Philosophicus* (1922) and *Philosophical Investigations* (1953): suggesting the two volumes are not irreconcilable.

Compare the following for which I have no good source:

> *"The world will change. It will probably change for the better. It won't seem better to me."*

(J. B. Priestly)

> ``*In 1965 the Russian mathematician Alexander Konrod said "Chess is the* ***Drosophila*** *of artificial intelligence." However, computer chess has developed as genetics might have if the geneticists had concentrated their efforts starting in 1910 on breeding racing* ***Drosophila****. We would have some science, but mainly we would have very fast fruit flies.''*

(From John McCarthy's review of **Kasparov versus Deep Blue** by Monty Newborn (Springer, 1996) in *Science*, 6 June 1997, page 1518)

He goes on to point out that of three features of human chess play two were used in early programs (forward pruning, identifying parallel moves, and partitioning (never used)). None survives in present programs. Material on **Making computer chess scientific** is available from **John McCarthy's web site**

> ``*A research policy does not consist of programs, but of hiring high-quality scientists. When you hire someone good, you've made your research policy for the next 20 years.''*

Chief CNRS advisor Vincent Courtillot quoted in **New CNRS Chief Gets Marching Orders**, *Science*, 18 July, 1997, page 308)

> ``*Mathematicians are like pilots who maneuver their great lumbering planes into the sky without ever asking how the damn things stay aloft.*
> ...
> *The computer has in turn changed the very nature of mathematical experience, suggesting for the first time that mathematics, like physics, may yet become an empirical discipline, a place where things are discovered because they are seen.*
> ...
> *The existence and nature of mathematics is a more compelling and far deeper problem than any of the problems raised by mathematics itself.''*

(From David Berlinski's somewhat negative review of **The Pleasures of Counting** by T. W. Korner (Cambridge, 1996)
in *The Sciences*, July/August 1997, pages 37-41)

Korner is a careful and stimulating writer/teacher.

``If I can give an abstract proof of something, I'm reasonably happy. But if I can get a concrete, computational proof and actually produce numbers I'm much happier. I'm rather an addict of doing things on the computer, because that gives you an explicit criterion of what's going on. I have a visual way of thinking, and I'm happy if I can see a picture of what I'm working with.''

(John Milnor)

- Page 78 of **Who got Einstein's Office?** by Ed Regis, *Addison-Wesley*, 1986. A history of the **Institute for Advanced Study**. The answer is Arnie Beurling.

``The term "reviewed publication" has an appealing ring for the naive rather than the realistic... Let's face it: (1) in this day and age of specialization, you may not find competent reviewers for certain contributions; (2) older scientists may agree that over the past two decades, the relative decline in research funds has been accompanied by an increasing number of meaningless, often unfair reviews; (3) some people are so desperate to get published that they will comply with the demands of reviewers, no matter how asinine they are.''

(August Epple)

- From **Organizing Scientific Meetings** quoted on page 400 of *Science* October 17, 1997.

``The NYT also has a stunning revelation about the way the Ivy League used to do business. Last Friday, the President of Darmouth used the occasion of dedicating a campus Jewish student center to haul out a 1934 letter between an alumnus of the school and the director of admissions. The alum complained that "the campus seems more Jewish each time I arrive in Hanover. And unfortunately many of them (on quick judgment) seem to be the 'kike' type." And the Dartmouth admissions man wrote back, "I am glad to have your comments on the Jewish problem, and I shall appreciate your help along this line in the future. If we go beyond the 5 percent or 6 percent in the Class of 1938, I shall be grieved beyond words." In reacting to the revelation, Elie Wiesel summons a simple fact that suggests how much times have changed: the current presidents of Harvard, Yale, and Princeton are Jewish.''

(SLATE, Tuesday November 11, 1997)

``This is the essence of science. Even though I do not understand quantum mechanics or the nerve cell membrane, I trust those who do. Most scientists are quite ignorant about most sciences but all use a shared grammar that allows them to recognize their craft when they see it. The motto of the Royal Society of London is 'Nullius in verba' : trust not in words. Observation and experiment are what count, not opinion and introspection. Few working scientists have much respect for those who try to interpret nature in metaphysical terms. For most wearers of white coats, philosophy is to science as pornography is to sex: it is cheaper, easier, and some people seem, bafflingly, to prefer it. Outside of psychology it plays almost no part in

the functions of the research machine.''

(Steve Jones, University College, London)

From his review of **How the Mind Works** (by Steve Pinker) in *The New York Review of Books* (pages 13-14) November 6, 1997. [Two solitudes indeed! **See below**]

``If you have a great idea, solid science, and earthshaking discoveries, you are still only 10% of the way there,''

(David Tomei, LXR Biotechnology Inc.)

Quoted in *Science* page 1039, November 7, 1997. [On the vicissitudes of startup companies.]

``There he received his hardest job of the war - a rush request to convert typewriters to twenty-one different languages of Asia and the South Pacific.

...

The implications of the work and its difficulty brought him to near collapse, but he completed it with only one mistake: on the Burmese typewriter he put a letter upside down. Years later, after he had discovered his error, he told the language professor he had worked with that he would fix that letter on the professor's Burmese typewriter. The professor said not to bother; in the intervening years, as a result of typewriters copied from Martin's original, that upside-down letter had been accepted in Burma as proper typewriter style.''

(Ian Frazier)

Page 88 in **Typewriter Man**, the ***Atlantic Monthly***, November 1997: "For Martin Tytell, the machinery of writing has been a life's work." [A fine example of convergence.]

The T-bone terror proves that ministers have no grasp of science or maths - let alone our liberties

``The giant finger whooshes out of the night sky and points at the dumbstruck face in the window. "It could be you," says a voice. This week the Agriculture Minister Jack Cunningham impersonated the National Lottery advertiser. As the nation's fork was poised with a T-bone steak on its way to the nation's mouth, Dr Cunningham screamed: "Don't touch it." According to the great god science, new variant Creutzfeldt-Jakob disease (nvCJD) could be lurking in that mouthful. There is a small risk, and where there is risk, a government must ban.

Perhaps only mathematicians are aware of the enormity of what the Government did this week. It took a risk that is statistically negligible and exploited it as an act of insufferable nannying. Beef ribs, T-bones and oxtails present a public health risk publicised as "very small" and "a chance of one case per year" (though none of Britain's 22 nvCJD cases has been positively linked to beef). Most newspapers cluelessly converted "a chance" into a certainty, and ridiculed the risk as a tiny one in 56 million. But that is not what the scientists said. They suggested the chance was "5 per cent", so the risk is nearer to one in 1.1 billion, or one in 560 million among *the half of the population that eats beef. There can have been no more tenuous basis for an infringement of personal liberty.''*

(Simon Jenkins)

- Simon Jenkins on **Boneless Wonders** in the ***Times of London***, Dec 6, 1997

``The common situation is this: An experimentalist performs a resolution analysis and finds a limited-range power law with a value of D smaller than the embedding dimension. Without necessarily resorting to special underlying mechanistic arguments, the experimentalist then often chooses to label the object for which she or he finds this power law a ''fractal''. This is the fractal geometry of nature.''

(David Avnir et al, Hebrew University)

- From **Is the geometry of nature fractal?** in *Science* January 2, 1998, 39-40. Their review of all articles from 1990 to 1996 in *Physical Reviews* suggests very little substance for claims of fractility.

``Most nonscientists who like to think of themselves as knowledgeable about modern science really know only about technologies - and specifically those technologies likely to bring economic profits in the short term.''

(Takashi Tachibana, Japanese Journalist)

- From **Closing the Knowledge Gap Between Scientist and Nonscientist** in *Science* August 7, 1998, 778-779.

``Another thing I must point out is that you cannot prove a vague theory wrong. ... Also, if the process of computing the consequences is indefinite, then with a little skill any experimental result can be made to look like the expected consequences.''

(Richard Feynman, 1964)

- Quoted by Gary Taubes in **The (Political) Science of Salt**, *Science* August 14, 1998, 898-907.

``Renyi would become one of Erdos's most important collaborators. ... Their long collaborative sessions were often fueled by endless cups of strong coffee. Caffeine is the drug of choice for most of the world's mathematicians and coffee is the preferred delivery system. Renyi, undoubtedly wired on espresso, summed this up in a famous remark almost always attributed to Erdos: ***"A mathematician is a machine for turning coffee into theorems."*** *... Turan, after scornfully drinking a cup of American coffee, invented the corollary:* ***"Weak coffee is only fit for lemmas."*** *''*

(Bruce Schechter, 1998)

- On page 155 of **My Brain is Open**, Schechter's 1998 Simon and Schuster biography of

Erdos. Schechter's Erdos is recognisable. The book contains interesting material on the Erdos-Selberg controversy (pp. 144-151). For more about the coffee see **Dick Askey**'s recollection.

> *``Once the opening ceremonies were over, the real meat of the Congress was then served up in the form of about 1400 individual talks and posters. I estimated that with luck I might be able to comprehend 2% of them. For two successive weeks in the halls of a single University, ICM'98 perpetuated the myth of the unity of mathematics; which myth is supposedly validated by the repetition of that most weaselly of rhetorical phrases: "Well,* ***in principle****, you could understand all the talks." ''*

(Philip Davis, 1998)

- Describing the Berlin International Congress of Mathematicians in the October 1998 *SIAM News*.

> *``Looking over the past 150 years -- at the tiny garden at Brno, the filthy fly room at Columbia, the labs of the New York Botanical Garden, the basement lab at Stanford, and the sun-drenched early gatherings at Cold Spring Harbor -- it seems that the fringes, not the mainstream, are the most promising places to discover revolutionary advances.''*

(Paul Berg and Maxine Singer, 1998)

- In **Inspired Choices**, *Science* October 30, 1998, 873-874s, on the past 150 years of biological research.

> *``Should we teach mathematical proofs in the high school? In my opinion, the answer is yes...Rigorous proofs are the hallmark of mathematics, they are an essential part of mathematics' contribution to general culture.''*

- George Polya (1981). **Mathematical discovery: On understanding, learning, and teaching problem solving (Combined Edition),** New York, Wiley & Sons (p. 2-126)

> *``A mathematical deduction appears to Descartes as a chain of conclusions, a sequence of successive steps. What is needed for the validity of deduction is intuitive insight at each step which shows that the conclusion attained by that step evidently flows and necessarily follows from formerly acquired knowledge (acquired directly by intuition or indirectly by previous steps) ... I think that in teaching high school age youngsters we should emphasize intuitive insight more than, and long before, deductive reasoning.'' (ibid, p. 2-128)*

- This "quasi-experimental" approach to proof can help to de-emphasis a focus on rigor and formality for its own sake, and to instead support the view expressed by Hadamard when he stated "The object of mathematical rigor is to sanction and legitimize the conquests of intuition, and there was never any other object for it" (J. Hadamard, in E. Borel, Lecons sur la theorie des fonctions, 3rd ed. 1928, quoted in Polya, (1981), (p. 2/127).

> *``intuition comes to us much earlier and with much less outside influence than*

formal arguments which we cannot really understand unless we have reached a relatively high level of logical experience and sophistication. Therefore, I think that in teaching high school age youngsters we should emphasize intuitive insight more than, and long before, deductive reasoning.'' (ibid, p. 2-128)

``In the first place, the beginner must be convinced that proofs deserve to be studied, that they have a purpose, that they are interesting.'' (ibid, p. 2-128)

``The purpose of a legal proof is to remove a doubt, but this is also the most obvious and natural purpose of a mathematical proof. We are in doubt about a clearly stated mathematical assertion, we do not know whether it is true or false. Then we have a problem: to remove the doubt, we should either prove that assertion or disprove it.'' (ibid, p. 2-129)

(Polya quotes are thanks to **Laurie Edwards**)

``The basic difference between playing a human and playing a supermatch against Deep Blue is the eerie and almost empty sensation of not having a human sitting opposite you. With humans, you automatically know a lot about their nationality, gender, mannerisms, and such minor things as a persistent cough or bad breath. Years ago we had to endure chain-smokers who blew smoke our way. But Deep Blue wasn't obnoxious, it was simply nothing at all, an empty chair not an opponent but something empty and relentless.''

(Garry Kasparov, 1998)

- Kasparov writing on *TechMate* in **Forbes** (22/2/98) - a collection on super computing.

``All professions look bad in the movies ... why should scientists expect to be treated differently?''

(Michael Crichton, 1999)

- Addressing the **1999 AAAS Meetings**, and quoted in *Science* February 19, 1999, page 1111.

``the academy was a sort of club for retired Parisian scientists, happy to be able to come together once a week to talk about science for 2 hours after lunch and a little nap.''

(Guy Ourison, January 1999)

- Inaugural speech as President to the **French Academy of Science** quoted in *Science* April 23, 1999, page 580.

``User-interface criticism is a genre to watch. It will probably be more influential and beneficial to the next century than film criticism was to the twentieth century. The twenty-first century will be filled with surprises, but one can safely count on it to bring more complexity to almost everything. Bearing the full brunt of that

complexity, the great user-interface designers of the future will provide people with the means to understand and enrich their own humanity, and to stay human.''

(Jaron Lanier, June 1999)

- From page 43 of **Interface-off** in *The Sciences* May/June 1999, pages 38-43.

``A real number complexity model appropriate for this context is given in the recent landmark work of Blum, Cucker, Shub and Smale . In discussing their motivation for seeking a suitable theoretical foundation for modern scientific computing, where most of the algorithms are `real number algorithms' the authors of this work quote the following illuminating remarks of John von Neumann, made in 1948: ``There exists today a very elaborate system of formal logic, and specifically, of logic applied to mathematics. This is a discipline with many good sides but also serious weaknesses.... Everybody who has worked in formal logic will confirm that it is one of the technically most refactory parts of mathematics. The reason for this is that it deals with rigid, all-or-none concepts, and has very little contact with the continuous concept of the real or the complex number, that is with mathematical analysis. Yet analysis is the technically most successful and best-elaborated part of mathematics. Thus formal logic, by the nature of its approach, is cut off from the best cultivated portions of mathematics, and forced onto the most difficult mathematical terrain, into combinatorics.

The theory of automata, of the digital, all-or-none type as discussed up to now, is certainly a chapter in formal logic. It would, therefore, seem that it will have to share this unattractive property of formal logic. It will have to be, from the mathematical point of view, combinatorial rather than analytical.''

(I. Blum, P. Cucker, M. Shub and S. Smale (1998), Complexity and Real Computation, Springer-Verlag, New York)

- Commentary thanks to Larry Nazareth

``Considerable obstacles generally present themselves to the beginner, in studying the elements of Solid Geometry, from the practice which has hitherto uniformly prevailed in this country, of never submitting to the eye of the student, the figures on whose properties he is reasoning, but of drawing perspective representations of them upon a plane. ...I hope that I shall never be obliged to have recourse to a perspective drawing of any figure whose parts are not in the same plane.''

(Augustus De Morgan)

- Adrian Rice (*What Makes a Great Mathematics Teacher?*) from page 540 of The American Mathematical Monthly, June-July 1999

``In 1831, Fourier's posthumous work on equations showed 33 figures of solution, got with enormous labour. Thinking this is a good opportunity to illustrate the superiority of the method of W. G. Horner, not yet known in France, and not much known in England, I proposed to one of my classes, in 1841, to beat Fourier on this

point, as a Christmas exercise. I received several answers, agreeing with each other, to 50 places of decimals. In 1848, I repeated the proposal, requesting that 50 places might be exceeded: I obtained answers of 75, 65, 63, 58, 57, and 52 places.''

(Augustus De Morgan)

- Adrian Rice from page 542 of The American Mathematical Monthly, June-July 1999

``I think we need more institutes, but then you run into the question, Is it better to spend $2 million and have another institute or to fund another twenty-five or so researchers each year? It's a question of trying to keep the discipline alive and thriving. There's no doubt the really big ideas in mathematics come from maybe 5 percent of the people, but you need a broad base to nourish the 5 percent and to work out all the details as they move on to more adventuresome things. Look at, say, mathematicians at Group III universities. It's a rarity when they get funding. How do you keep them in the system? ... We're under terrific pressure to increase the size of our grants. If we did what the [National Science] board wants us to do, we would fund 800 people instead of 1,400. It's a question of whether DMS did the right thing when they pulled so many people down to one month of summer support. This took some of the pressure off the Foundation to put more money in mathematics. Suppose we funded 800 people. How much noise would it create? Would there be a march on Washington? I often think that's the way to go. See whether mathematicians would stand up for themselves or whether they'd just meekly accept. In chemistry, people get declined, and in two months they turn around with another proposal. Mathematicians --- they get declined twice, and they fold. I think mathematicians have such a personal investment in their problems that if you turn down their proposals, they take it as if you're judging them as mathematicians. They're not as flexible and often don't seem to be able to move to another class of problems. We fund proposals, not individuals.''

(D. J. Lewis)

- Interview with Allyn Jackson from page 669 of *The Notices of The AMS*, **June-July 1999**

``Notices: After your time at the NSF, do you have any advice for the math community about what they should be doing to try to improve the funding for mathematics?

Lewis: I don't think that up to this date they've made a very good case for why they should be funded. The bottom line is, What are you doing for the citizens of the country?

Notices: When you say ``make the case,'' what do you mean concretely? Do groups of mathematicians have to descend on Capitol Hill?

Lewis: They've got to do some demonstrations of what mathematics has accomplished for the good of society. One of the things mathematicians have done is education. For example, if mathematicians took seriously the job of training elementary and middle school teachers, they could make some claim that they really improve things. Also, science is getting so complicated, it can be done only with the *help of mathematics. Is the math community willing to step up and participate?*

If so, they will have nonmathematicians making the case for greater funding of mathematics. It is always best to have outsiders make your case for you. Once upon a time I thought going to Capitol Hill would be effective. I don't think it will get very far if mathematicians go to Capitol Hill without the support of others. These days information technology and biology and medicine are the themes that echo well with the president and Congress.''

(D. J. Lewis)

Interview with Allyn Jackson from page 672 of *The Notices of The AMS*, June-July 1999

``*The work then proceeds in a manner unique to science. Because practitioners publish their work electronically, through the e-print archives at the Los Alamos National Laboratory in New Mexico, the entire community can read a paper hours after its authors finish typing the last footnote. As a result, no one theorist or even a collaboration does definitive work.* ***Instead, the field progresses like a jazz performance: A few theorists develop a theme, which others quickly take up and elaborate.*** *By the time it's fully developed, a few dozen physicists, working anywhere from Princeton to Bombay to the beaches of Santa Barbara, may have played important parts.''*

(Gary Taubes)

From *String Theorists Find a Rosetta Stone* on page 513 of *Science*, 23rd July, 1999

'where almost one quarter hour was spent, each beholding the other with admiration before one word was spoken: at last Mr. Briggs began "My Lord, I have undertaken this long journey purposely to see your person, and to know by what wit or ingenuity you first came to think of this most excellent help unto Astronomy, viz. the Logarithms: but my Lord, being by you found out, I wonder nobody else found it out before, when now being known it appears so easy." '

(**Henry Briggs**, 1617)

Briggs, later the first Savelian Professor of Geometry in Oxford, is describing his first meeting with Napier whom he had traveled from London to Edinburgh to meet. From H.W. Turnbull's *The Great Mathematicians*, Methuen, 1929.

``*Far better an approximate answer to the right question, which is often vague, than the exact answer to the wrong question, which can always be made precise.''*

(J. W. Tuckey, 1962)

From the *Annals of Mathematical Statistics* , Volume 33. Compare the 1964 Feynman quote above!

`` One of the beauties of learning is that it admits its provisionality, its imperfections. This scholarly scrupulousness, this willingness to admit that even the best-supported of theories is still a theory, is now being exploited by the unscrupulous. But that we do not know everything does not mean we know nothing. Not all theories are of equal weight. The moon, even the moon over Kansas, is not made of green cheese. Genesis, as a theory, is bunk.

If the overabundant new knowledge of the modern age is, let's say, a tornado, then Oz is the extraordinary, Technicolored new world in which it has landed us, the world from which --- life not being a movie --- there is no way home. In the immortal words of Dorothy Gale, `Toto, something tells me we're not in Kansas any more.' To which one can only add: Thank goodness, baby, and amen.''

(Salman Rushdie)

- From his article "Locking out that disruptive Darwin fellow" in the *Globe and Mail* , September 2, 1999

``The mental maps, gave rise to industries that could not have been predicted, and created a new class of technological workers whom wise societies took pains to nurture. Are we about to go through this process again? A renowned social analyst and management philosopher looks to history for insights.''

(Peter Drucker)

- **Beyond the Information Revolution** in *The Atlantic Monthly* Online November 3, 1999

``When the facts change, I change my mind. What do you do, sir?''

(John Maynard Keynes)

- Quoted in *The Economist*, December 18 1999, page 47

``Look miss, if I disagree with Darwin, he's not going to send me to hell.''

(An anonymous student's "Pascal wager-style" rationale)

- Quoted in *The Globe and Mail*, January 1, 2000, page D22 by Laura Penny describing a first year University class in Buffalo in which one third of the students were creationists.

`` Most working scientists may be naive about the history of their discipline and therefore overly susceptible to the lure of objectivist mythology. But I have never met a pure scientific realist who views social context as entirely irrelevant, or only as an enemy to be expunged by the twin lights of universal reason and incontrovertible observation. And surely, no working scientist can espouse pure relativism at the other pole of the dichotomy. (The public, I suspect, misunderstands the basic reason for such exceptionless denial. In numerous letters and queries, sympathetic and interested nonprofessionals have told me that scientists cannot be relativists because

their commitment to such a grand and glorious goal as the explanation of our vast and mysterious universe must presuppose a genuine reality "out there" to discover. In fact, as all working scientists know in their bones, the incoherence of relativism arises from virtually opposite and much more quotidian motives. Most daily activity in science can only be described as tedious and boring, not to mention expensive and frustrating. Thomas Edison was just about right in his famous formula for invention as 1% inspiration mixed with 99% perspiration. How could scientists ever muster the energy and stamina to clean cages, run gels, calibrate instruments, and replicate experiments, if they did not believe that such exacting, mindless, and repetitious activities can reveal truthful information about a real world? If all science arises as pure social construction, one might as well reside in an armchair and think great thoughts.)

Similarly, and ignoring some self-promoting and cynical rhetoricians, I have never met a serious social critic or historian of science who espoused anything close to a doctrine of pure relativism. The true, insightful, and fundamental statement that science, as a quintessentially human activity, must reflect a surrounding social context does not imply either that no accessible external reality exists, or that science, as a socially embedded and constructed institution, cannot achieve progressively more adequate understanding of nature's facts and mechanisms. ''

(Stephen J. Gould)

- From the article: 'Deconstructing the "Science Wars" by Reconstructing an Old Mold' in *Science*, Jan 14, 2000: 253-261.

`` caused Thorstein Veblen to comment acerbically in 1908 that "business principles" were transforming higher education into "a merchantable commodity, to be produced on a piece-rate plan, rated, bought, and sold by standard units, measured, counted and reduced to staple equivalence by impersonal, mechanical tests. ''

...

``New products and new processes do not appear full-grown," Vannevar Bush, President Franklin Roosevelt's chief science adviser, declared in 1944. "They are founded on new principles and new conceptions, which in turn are painstakingly developed by research in the purest realms of science." ''

(Eyal Press and Jennifer Washburn)

- From **The Kept University** in *The Atlantic Monthly* Online, March 2000. Which quote better reflects Science in 2001?

``Most important to Fox was a young instructor who had arrived at Cornell two years before from Williams and Mary. William Lloyd Garrison Williams had written his Ph.D thesis under Leonard Dickson at Chicago in 1920. Born in Friendship, Kansas, Williams, who was named for the famous abolitionist William LLoyd Garrison, attended a small Quaker school in Indiana, taught school briefly in North Dakota and then attended Haverford College where he earned a B.A. degree. From 1910-13 he attended Oxford University as a Rhodes Scholar, and after receving a B.A. and M.A.,

he took a faculty position at Miami University of Ohio. His Ph.D. work at Chicago was done during the summers. He also taught briefly at Gettysburg College and William and Mary before coming to Cornell.

In 1924, Williams moved to McGill University, where he helped develop the graduate program. He was the founder and organizer of the Canadian Mathematical Congress, the first meeting of which was in Montreal in 1945. Nearly all of the support the Congress was able to acquire was due to William's efforts (see W.L.G. Williams, 1888-1976, G. De B. Robinson, Proc. Royal Society of Canada, 1976). A man of remarkable ability and compassion, Williams took a strong personal interest in his fellowman. A lifelong member of the Society of Friends, he was a tireless worker for Quaker causes.''

(James A. Donaldson and Richard J. Fleming)

- From "Elbert F. Fox: An Early Pioneer", *American Math Monthly* **107** (2000) 105-128.

Gravity Turntable Sets New Record

`LONG BEACH, CALIFORNIA--Scientists have been scrutinizing gravity since the time of Newton, but they've had difficulty measuring the power of its pull. Now, thanks to a clever device, physicists have the most precise measurement yet.

....

"[It] should have been obvious" that previous measures of big G were off, says physicist Randy Newman of the University of California, Irvine. The new result, announced this week at the American Physical Society meeting, sets big G tentatively at 6.67423 0.00009 x 10-11 m3/(kg s2). "It's one of the fundamental constants," Gundlach says. "Mankind should just know it. It's a philosophical thing."'

- From *ScienceNow* May 5, 2000.

``Imagine Dostoyevsky. There are some incidents like this, two boys killing other children, in his famous diary. Imagine what Dostoyevsky would do with that. He would deal with the transcendentally important question of evil in the child. Today the editor would say, "Fyodor, tomorrow, please, your piece. Don't tell me you need ten months for thinking. Fyodor, tomorrow!" "

(George Steiner)

- Quoted in James Gleick's *Faster* (Pantheon 1999), pages 97-88, on instant opinion -- sound bites and 'hurry sickness'.

``So my reaction surprises me. I tell Natalie that math is important and relevant and that I wished I'd made the effort to understand. I wish somebody had found a way of making sense of it all. This revelation comes from reading a stack of magazines about the future, about computers and artificial intelligence, cars and planes, food production and global warming. And I have come to the conclusion that Mr. Kool was

right.

Math has something to do with calculations, formulas, theories and right angles. And everything to do with real life. Mathematicians not only have the language of the future (they didn't send Taming of the Shrew into space, just binary blips) but they can use it to predict when Andromeda will perform a cosmic dance with the Milky Way. It's mathematicians who are designing the intelligent car that knows when you're falling asleep at the wheel or brakes to avoid an accident. It can predict social chaos and the probability of feeding billions. It even explains the stock market and oil prices.''

...

Paulette Bourgeois lives in Toronto where she is calculating the probability of ever balancing her chequebook. She is the author of the *Franklin the Turtle* books for children.

(Paulette Bourgeois)

- Quoted from "The Numbers Game," *The Globe and Mail* July 13, 2000, page A14.

`` Mathematics is the language of high technology. Indeed it is, but I think it is also becoming the eyes of science.''

(Tom Brzustowski, NSERC President)

- Addressing the MITACS NCE annual general meeting June 6, 2000.

``This is fundamentally wrong. We are not entering a time when copyright is more threatened than it is in real space. We are instead entering a time when copyright is more effectively protected than at any time since Gutenberg. The power to regulate access to and use of copyrighted material is about to be perfected. Whatever the mavens of the mid-1990s may have thought, cyberspace is about to give the holders of copyrighted property the biggest gift of protection they have ever known.

*In such an age -- in a time when the protections are being perfected -- the real question for law is not, how can law aid in that protection? but rather, is that protection too great? The mavens were right when they predicted that cyberspace will teach us that everything we thought about copyright was wrong. But the lesson in the future will be that copyright is protected far too well. The problem will center not on copy-right but on copy-**duty** -- the duty of owners of protected property to make that property available.''*

(Lawrence Lessig)

- Quoted from page 127 of his book: "Code and other laws of Cyberspace", Basic Books, 1999.

``An informed list of the most profound scientific developments of the 20th century is likely to include general relativity, quantum mechanics, big bang cosmology, the

unraveling of the genetic code, evolutionary biology, and perhaps a few other topics of the reader's choice. Among these, quantum mechanics is unique because of its profoundly radical quality. Quantum mechanics forced physicists to reshape their ideas of reality, to rethink the nature of things at the deepest level, and to revise their concepts of position and speed, as well as their notions of cause and effect. ''

(Daniel Kleppner and Roman Jackiw)

- Quoted from the article "One Hundred Years of Quantum Physics" in Science August 11, pages 893-898.

http://www.sciencemag.org/cgi/content/full/289/5481/893

``A wealthy (15th Century) German merchant, seeking to provide his son with a good business education, consulted a learned man as to which European institution offered the best training. "If you only want him to be able to cope with addition and subtraction," the expert replied, "then any French or German university will do. But if you are intent on your son going on to multiplication and division -- assuming that he has sufficient gifts -- then you will have to send him to Italy.''

(Georges Ifrah)

- From page 577 of "The Universal History of Numbers: From Prehistory to the Invention of the Computer", translated from French, John Wiley, 2000. (Emphasizing quite how great an advance positional notation was!)

``2000 was a banner year for scientists deciphering the "book of life"; this year saw the completion of the genome sequences of complex organisms ranging from the fruit fly to the human.

*Genomes carry the torch of life from one generation to the next for every organism on Earth. Each genome--physically just molecules of DNA--is a script written in a four-letter alphabet. Not too long ago, determining the precise sequence of those letters was such a slow, tedious process that only the most dedicated geneticist would attempt to read any one "paragraph"--a single gene. But today, genome sequencing is a billion-dollar, worldwide enterprise. Terabytes of sequence data generated **through a melding of biology, chemistry, physics, mathematics, computer science, and engineering** are changing the way biologists work and think. Science marks the production of this torrent of genome data as the Breakthrough of 2000; it might well be the breakthrough of the decade, perhaps even the century, for all its potential to alter our view of the world we live in.''*

(Elizabeth Pennisi)

- From ''BREAKTHROUGH OF THE YEAR: Genomics Comes of Age.'' Cover story in *Science* of December 22, 2000.

"Not until the creation and maintenance of decent conditions of life for all people are recognized and accepted as a common obligation of all people and all countries - not

until then shall we, with a certain degree of justification, be able to speak of humankind as civilized."

(Albert Einstein, 1945)

"Capitalism is the extraordinary belief that the nastiest of men, for the nastiest of reasons, will somehow work for the benefit of us all."

(John Maynard Keynes)

"When we have before us a fine map, in which the line of the coast, now rocky, now sandy, is clearly indicated, together with the winding of the rivers, the elevations of the land, and the distribution of the population, we have the simultaneous suggestion of so many facts, the sense of mastery over so much reality, that we gaze at it with delight, and need no practical motive to keep us studying it, perhaps for hours altogether. A map is not naturally thought of as an aesthetic object... And yet, let the tints of it be a little subtle, let the lines be a little delicate, and the masses of the land and sea somewhat balanced, and we really have a beautiful thing; a thing the charm of which consists almost entirely in its meaning, but which nevertheless pleases us in the same way as a picture or a graphic symbol might please. Give the symbol a little intrinsic worth of form, line and color, and it attracts like a magnet all the values of things it is known to symbolize. It becomes beautiful in its expressiveness."

(George Santayana)

- From "The Sense of Beauty", 1896.

"If my teachers had begun by telling me that mathematics was pure play with presuppositions, and wholly in the air, I might have become a good mathematician, because I am happy enough in the realm of essence. But they were overworked drudges, and I was largely inattentive, and inclined lazily to attribute to incapacity in myself or to a literary temperament that dullness which perhaps was due simply to lack of initiation."

(George Santayana}

- From pp. 238-9 "Persons and Places", 1945.

"He designed and built chess-playing, maze-solving, juggling and mind-reading machines. These activities bear out Shannon's claim that he was more motivated by curiosity than usefulness. In his words `I just wondered how things were put together.' "

(Claude Shannon)

- From Claude Shannon's (1916-2001) **obituary**.

"The price of metaphor is eternal vigilance"

(Arturo Rosenblueth and Norbert Wiener)

- Quoted by R. C. Leowontin, in Science page 1264, Feb 16, 2001 (The Human Genome Issue).

"What is particularly ironic about this is that it follows from the empirical study of numbers as a product of mind that it is natural for people to believe that numbers are not a product of mind!"

(George Lakoff and Rafael E. Nunez)

- On page 81 of *Where Mathematics Comes From,* Basic Books, 2000.

Recent Discoveries about the Nature of Mind. *In recent years, there have been revolutionary advances in cognitive science ---- advances that have a profound bearing on our understanding of mathematics. Perhaps the most profound of these new insights are the following:*
1. The embodiment of mind. *The detailed nature of our bodies, our brains and our everyday functioning in the world structures human concepts and human reason. This includes mathematical concepts and mathematical reason.*
2. The cognitive unconscious. *Most thought is unconscious --- not repressed in the Freudian sense but simply inaccessible to direct conscious introspection. We cannot look directly at our conceptual systems and at our low-level thought processes. This includes most mathematical thought.*
3. Metaphorical thought. *For the most part, human beings conceptualize abstract concepts in concrete terms, using ideas and modes of reasoning grounded in sensory-motor systems. The mechanism by which the abstract is comprehended in terms of the concept is called* conceptual metaphor*. Mathematical thought also makes use of line."*

(George Lakoff and Rafael E. Nunez)

- On page 5 of *Where Mathematics Comes From,* Basic Books, 2000.

"The early study of Euclid made me a hater of geometry."

(James Joseph Sylvester, 1814-97, Second LMS President)

- quoted in D. MacHale, "Comic Sections" (Dublin 1993).

"a thrill which is indistinguishable from the thrill I feel when I enter the Sagrestia Nuovo of the Capella Medici and see before me the austere beauty of the four statues representing 'Day', 'Night', 'Evening', and 'Dawn' which Michelangelo has set over the tomb of Guiliano de'Medici and Lorenzo de'Medici."

(G. N. Watson, 1886-1965)

"All physicists and a good many quite respectable mathematicians are contemptuous about proof."

(G. H. Hardy, 1877-1947)

- A century after biology started to think physically:

"The idea that we could make biology mathematical, I think, perhaps is not working, but what is happening, strangely enough, is that maybe mathematics will become biological,!"

(Greg Chaitin, **Interview**, 2000.)

"The waves of the sea, the little ripples on the shore, the sweeping curve of the sandy bay between the headlands, the outline of the hills, the shape of the clouds, all these are so many riddles of form, so many problems of morphology, and all of them the physicist can more or less easily read and adequately solve."

(D'Arcy Thompson, ``On Growth and Form'' 1917)

- In Philip Ball's "**The Self-Made Tapestry: Pattern Formation in Nature**,''

"A doctorate compels most of us to be detailed and narrow, and to carve out our own specialities, and tenure commitees rarely like boldness. Later, when our jobs are safe we can be synthetic, and generalize."

(Paul Kennedy)

- Writing critically about A.J.P. Taylor (`The Nonconformist') in the *Atlantic Monthly* April 2001, page 114.

``... it is no doubt important to attend to the eternally beautiful and true. But it is more important not to be eaten."

(Jerry Fodor)

Distinguishing effortless early learning of language and social customs from later labourious general purpose concept acquisition, Egan writes:
"The bad news is that our evolution equipped us to live in small, stable, hunter-gatherer societies. We are Pleistocene people, but our languaged brains have created massive, multicultural, technologically sophisticated and rapidly changing societies for us to live in."

"The cement like learning of our early years can accomodate almost anything, then it fixes and becomes almost unmovable."

"we can, as a result, change our earlier beliefs and commitments. We also know this

is difficult for most people."

(Kieran Egan)

- In Kieran Egan's, *Getting it Wrong from the Beginning -- Major Mistakes in the Project to Educate Everybody* (in press).

"EINSTEIN ON SCIENTIFIC TRUTH & ITS TRIUMPH"
This is what Albert Einstein said quoting Max Planck

> *"...a new scientific truth does not triumph by convincing its opponents and making them see the light, but rather because its opponents die and a new generation grows up that's familiar with it."*
> or ...
> *"A new scientific truth usually does not make its way in the sense that its opponents are persuaded and declare themselves enlightened, but rather that the opponents become extinct and the rising generation was made familiar with the truth from the very beginning".*

- Max Planck, in *THE QUANTUM BEAT* by F.G.Major, Springer (1998).

> *"And Max Planck, surveying his own career in his Scientific Autobiography, sadly remarked that 'a new scientific truth does not triumph by convincing its opponents and making them see the light, but rather because its opponents eventually die, and a new generation grows up that is familiar with it.'"*

(Thomas Kuhn)

- On page 151 of T.S. Kuhn, *The Structure of Scientific Revolutions,* 3rd ed., Univ. of Chicago Press, 1996. (Quoting: *Max Planck, Scientific Autobiography and Other Papers,* trans. F. Gaynor (New York, 1949), pp. 33-34. See also "Conversations with a Mathematician" by **Greg Chaitin**.)

> *``the idea that we could make biology mathematical, I think, perhaps is not working, but what is happening, strangely enough, is that maybe mathematics will become biological, not that biology will become mathematical, mathematics may go in that direction!"*

(Interview with Gregory Chaitin by Hans-Ulrich Obrist (Musee d'Art Moderne de la Ville de Paris), Paris/CDG Airport, October 2000.)

- In "**THE CREATIVE LIFE: SCIENCE VS ART**,''

> *"The message is that mathematics is quasi-empirical, that mathematics is not the same as physics, not an empirical science, but I think it's more akin to an empirical science than mathematicians would like to admit."*
>
> *"Mathematicians normally think that they possess absolute truth. They read God's thoughts. They have absolute certainty and all the rest of us have doubts. Even the best physics is uncertain, it is tentative. Newtonian science was replaced by relativity*

theory, and then---wrong!---quantum mechanics showed that relativity theory is incorrect. But mathematicians like to think that mathematics is forever, that it is eternal. Well, there is an element of that. Certainly a mathematical proof gives more certainty than an argument in physics or than experimental evidence, but mathematics is not certain. This is the real message of Godel's famous incompleteness theorem and of Turing's work on uncomputability."

"You see, with Godel and Turing the notion that mathematics has limitations seems very shocking and surprising. But my theory just measures mathematical information. Once you measure mathematical information you see that any mathematical theory can only have a finite amount of information. But the world of mathematics has an infinite amount of information. Therefore it is natural that any given mathematical theory is limited, the same way that as physics progresses you need new laws of physics."

"Mathematicians like to think that they know all the laws. My work suggests that mathematicians also have to add new axioms, simply because there is an infinite amount of mathematical information. This is very controversial. I think mathematicians, in general, hate my ideas. Physicists love my ideas because I am saying that mathematics has some of the uncertainties and some of the characteristics of physics. Another aspect of my work is that I found randomness in the foundations of mathematics. Mathematicians either don't understand that assertion or else it is a nightmare for them... ":

"This skyhook-skyscraper construction of science from the roof down to the yet unconstructed foundations was possible because the behaviour of the system at each level depended only on a very approximate, simplified, abstracted characterization at the level beneath[1]. This is lucky, else the safety of bridges and airplanes might depend on the correctness of the "Eightfold Way" of looking at elementary particles.

[1] ... More than fifty years ago Bertrand Russell made the same point about the architecture of mathematics. See the "Preface" to Principia Mathematica "... the chief reason in favour of any theory on the principles of mathematics must always be inductive, i.e., it must lie in the fact that the theory in question allows us to deduce ordinary mathematics. In mathematics, the greatest degree of self-evidence is usually not to be found quite at the beginning, but at some later point; hence the early deductions, until they reach this point, give reason rather for believing the premises because true consequences follow from them, than for believing the consequences because they follow from the premises." Contemporary preferences for deductive formalisms frequently blind us to this important fact, which is no less true today than it was in 1910."

(Herbert A. Simon)

- On page 16 of ``The Sciences of the Artificial," MIT Press, 1996.

" Hardy `asked `What's your father doing these days. How about that esthetic measure of his?' I replied that my father's book was out. He said, 'Good, now he can get back to real mathematics'."

(Garret Birkoff)

- Quoted in *Towering Figures, 1890-1950*, by David E. Zitarelli on page 618 of *MAA Monthly* Aug-Sept, Vol 108, (2001), 606-635 : regarding G. D. Birkhoff's Aesthetic Measures (1933).

"I DO CONSIDER it appropriate to pay one's tribute to Prof. Subramanyan Chandrasekhar at the outset, before taking a plunge into the aesthetics of macro-causality, based on his book Truth and Beauty: Aesthetics and Motivations in Science. Brought up on the refined diet of music, mathematics and aesthetics, Chandrasekhar's own writing is probably the most appropriate mirror of his personality. I quote: "When Michelson was asked towards the end of his life, why he had devoted such a large fraction of his time to the measurement of the velocity of light, he is said to have replied 'It was so much fun'." Prof. Chandrasekhar goes on to some length to explain the term quoting even the Oxford Dictionary -- "fun" means "drollery", what Michelson really meant, Chandrasekhar asserts is "pleasure" and "enjoyment" - evidently "fun" in the colloquial sense, a concept, so familiar in our so called ordinary life has no place in Chandrasekhar's dictionary..."

(Bikash Sinha)

- In ***AESTHETICS AND MOTIVATIONS IN ARTS AND SCIENCE***.

" `His peculiar gift was the power of holding continuously in his mind a purely mental problem until he had seen straight through it. I fancy his preeminence is due to his muscles of intuition being the strongest and most enduring with which a man has ever been gifted. Anyone who has ever attempted pure scientific or philosophical thought knows how one can hold a problem momentarily in one's mind and apply all one's powers of concentration to piercing through it, and how it will dissolve and escape and you find that what you are surveying is a blank. I believe that Newton could hold a problem in his mind for hours and days and weeks until it surrendered to him its secret. Then being a supreme mathematical technician he could dress it up, how you will, for purposes of exposition, but it was his intuition which was pre-eminently extraordinary---"so happy in his conjectures", said de Morgan, "as to seem to know more than he could possibly have any means of proving."'-- J. M. Keynes 1956

...

If Edison, Fineman, Gauss, and Newton had all been intensely tutored from the age of three by brilliant parents, as J.S. Mill was, then I might at least consider the possibility that my own mental muscles might have been stronger if my own parents had been more demanding. But they were not and I will not. `When you see [Edison's] mind at play in his notebooks, the sheer multitude and richness of his ideas makes you recognize that there is something that can't be understood easily---that we may never be able to understand.' (historian Paul Israel, quoted in McAuliffe 1995). I think what lies at the heart of these mysteries is genetic, probably emergenic. The configuration of traits of intellect, mental energy, and temperament with which, during the plague years of 1665--6, Isaac Newton revolutionized the world of science were, I believe, the consequence of a genetic lottery that occurred about nine months prior to his birth, on Christmas day, in 1642.

...

Gauss's second son, Eugene, emigrated to the United States in 1830, enlisted in the army, and later went into business in Missouri. Eugene is said to have had some of his father's gift for languages and the ability to perform prodigious arithmetic calculations, which he did for recreation after his sight failed him in old age. "

(David T. Lykken)

- In **THE GENETICS OF GENIUS**.

`For Poincare, ignoring the emotional sensibility, even in mathematical demonstrations "would be to forget the feeling of mathematical beauty, of the harmony of numbers and forms, of geometric elegance. This is a true esthetic feeling that all real mathematicians know, and surely it belongs to emotional sensibility" (p. 2047).'

(Nathalie Sinclair)

- Quoting Henri Poincare's "Mathematical creation" (1956). In J. Newman (Ed.), *The World of Mathematics* (pp. 2041-2050). Simon and Schuster.

"The controversy between those who think mathematics is discovered and those who think it is invented may run and run, like many perennial problems of philosophy. Controversies such as those between idealists and realists, and between dogmatists and sceptics, have already lasted more than two and a half thousand years. I do not expect to be able to convert those committed to the discovery view of mathematics to the inventionist view. However what I have shown is that a better case can be put for mathematics being invented than our critics sometimes allow. Just as realists often caricature the relativist views of social constructivists in science, so too the strengths of the fallibilist views are not given enough credit. For although fallibilists believe that mathematics has a contingent, fallible and historically shifting character, they also argue that mathematical knowledge is to a large extent necessary, stable and autonomous. Once humans have invented something by laying down the rules for its existence, like chess, the theory of numbers, or the Mandelbrot set, the implications and patterns that emerge from the underlying constellation of rules may continue to surprise us. But this does not change the fact that we invented the game in the first place. It just shows what a rich invention it was. As the great eighteenth century philosopher Giambattista Vico said, the only truths we can know for certain are those we have invented ourselves. Mathematics is surely the greatest of such inventions."

(Paul Ernst)

- From ***Is Mathematics Discovered or Invented?*** (THES, 1996 and after).

" Who owns the Internet? Until recently, nobody. That's because, although the Internet was "Made in the U.S.A.," its unique design transformed it into a resource for innovation that anyone in the world could use. Today, however, courts and

corporations are attempting to wall off portions of cyberspace. In so doing, they are destroying the Internet's potential to foster democracy and economic growth worldwide. "

(Lawrence Lessig)

- From **Who Owns The Internet?** *Foreign Policy*, November-December 2001.

"Predicting the future is an activity fraught with error. Wilbur Wright, co-inventor of the motorized airplane that successfully completed the first manned flight in 1903, seems to have learned this lesson when he noted: "In 1901, I said to my brother Orville that man would not fly for 50 years. Ever since I have ... avoided predictions." Despite the admonition of Wright, faulty future forecasting seems a favored human pastime, especially among those who would presumably avoid opportunities to so easily put their feet in their mouths.

What follows are some of the more striking exemplars of expert error in forecasting the future of technological innovations.

"Louis Pasteur's theory of germs is ridiculous fiction." -- Piem Pachet, Professor of Physiology, 1872

"The abdomen, the chest, and the brain will forever be shut from the intrusion of the wise and humane surgeon." -- Sir John Eric Erickren, British surgeon to Queen Victoria, 1873

"Radio has no future. Heavier than air flying machines are impossible. X-rays will prove to be a hoax." -- William Thomson (Lord Kelvin), English physicist and inventor, 1899

"There is not the slightest indication that nuclear energy will ever be obtainable. It would mean that the atom would have to be shattered at will." -- Albert Einstein, 1932

"Man will never reach the moon, regardless of all future scientific advances." -- Lee De Forest, Radio pioneer, 1957

Computers and information technologies seem to hold a special place in the forecasters' hall of humiliation, be they predictions from the media, business, politicians, scientists, or technologists. Here are some examples:

"This 'telephone' has too many shortcomings to be seriously considered as a means of communication. The device is inherently of no value to us." -- Western Union internal memo, 1876

"I think there is a world market for maybe five computers." -- Thomas Watson, chair of IBM, 1943

"The problem with television is that people must sit and keep their eyes glued on a screen; the average American family hasn't time for it." -- New York Times, 1949

"Where ... the ENIAC is equipped with 18,000 vacuum tubes and weights 30 tons,

computers in the future may have only 1,000 vacuum tubes and weigh only 1.5 tons." -- Popular Mechanics, 1949

"Folks, the Mac platform is through -- totally." -- John C. Dvorak, PC Magazine, 1998

"There is no reason anyone would want a computer in their home." -- Ken Olson, president, chairman and founder, Digital Equipment Corp, 1977

"640K ought to be enough for anybody." -- Attributed to Bill Gates, Microsoft chair, 1981

"By the turn of this century, we will live in a paperless society." -- Roger Smith, chair of General Motors, 1986

"I predict the Internet ... will go spectacularly supernova and in 1996 catastrophically collapse." -- Bob Metcalfe, 3Com founder and inventor, 1995

"Credit reports are particularly vulnerable ... [as] are billing, payroll, accounting, pension and profit-sharing programs." -- Leon A Kappelman [author of this article] on likely Y2K problems, 1999 "

(Leon A Kappelman)

- From "The Future is Ours," *Communications of the ACM*, March 2001, pg. 46.

" Computation with Roman numerals is certainly algorithmic - it's just that the algorithms are complicated.

In 1953, I had a summer job at Bell Labs in New Jersey (now Lucent), and my supervisor was Claude Shannon (who has died only very recently). On his desk was a mechanical calculator that worked with Roman numerals. Shannon had designed it and had it built in the little shop Bell Labs had put at his disposal. On a name plate, one could read that the machine was to be called: Throback I.

Martin from a foggy morning in Berkeley"

(Martin Davis)

- Martin Davis, Visiting Scholar UC Berkeley, Professor Emeritus, NYU. Following up on queries on the Historia Mathematica list, Jan 12, 2002.

" [1] If nature has made any one thing less susceptible than all others of exclusive property, it is the action of the thinking power called an idea, which an individual may exclusively possess as long as he keeps it to himself; but the moment it is divulged, it forces itself into the possession of everyone, and the receiver cannot dispossess himself of it. [2] Its peculiar character, too, is that no one possesses the less, because every other possesses the whole of it. He who receives an idea from me, receives instruction himself without lessening mine; as he who lites his taper at mine, receives light without darkening me. [3] That ideas should freely spread from one to another over the globe, for the moral and mutual instruction of man, and improvement of his condition, seems to have been peculiarly and benevolently

designed by nature, when she made them, like fire, expansible over all space, without lessening their density at any point, and like the air in which we breathe, move, and have our physical being, incapable of confinement, or exclusive appropriation. [4] Inventions then cannot, in nature, be a subject of property. "

(Thomas Jefferson)

- Letter from Thomas Jefferson to Issac McPherson (August 13, 1813), in *The Writings of Thomas Jefferson 6* quoted from page 94 of **the future of ideas** by Lawrence Lessig, Random House, 2001.

The question of the ultimate foundations and the ultimate meaning of mathematics remains open: we do not know in what direction it will find its final solution or even whether a final objective answer can be expected at all. 'Mathematizing' may well be a creative activity of man, like language or music, of primary originality, whose Historical decisions defy complete objective rationalisation."

(Hermann Weyl)

- In "*Obituary: David Hilbert 1862 - 1943*", *RSBIOS*, **4**, 1944, pp. 547 - 553; and *American Philosophical Society Year Book*, 1944, pp. 387 - 395, p. 392.

Thus mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true. People who have been puzzled by the beginnings of mathematics will, I hope, find comfort in this definition, and will probably agree that it is accurate."

Bertrand Russell)

- From "Recent Work on the Principles of Mathematics in *International Monthly*, **4** (July, 1901), 83-101. (*Collected Papers,* v3, p.366; revised version in Newman's *World of Mathematics*, v3, p. 1577.)

"*The problems of mathematics are not problems in a vacuum. There pulses in them the life of ideas which realize themselves in concreto through our [or throught] human endeavors in our historical existence, but forming an indissoluble whole transcending any particular science."*

(Hermann Weyl)

- In "David Hilbert and his mathematical work," *Bull. Am. Math. Soc.*, **50** (1944), p. 615.

THE FUTURE OF E-PUBLISHING. *Although e-publishing has suffered a series of setbacks this year, Wired magazine still found plenty of optimism about the future of e-books. Michael S. Hart of Project Guttenberg, which offers books in electronic form, says: "The number of e-books available for free download on the Net will pass 20,000. The number of Net users will start heading towards 1 billion." Librarian Cynthia Orr, a co-founder of BookBrowser.com, thinks e-publishers should pay more attention to libraries, and says that if the major publishers worked with librarians or distributors "to figure out how to let libraries purchase or license their e-books, and* *let readers 'check them out' for free," that would help build "a market that otherwise threatens to just collapse for lack of interest. Librarians have been careful defenders of copyright over the years ... and our budgets are far higher than they realize." And Mark Gross, president of Data Conversion Laboratory, thinks that the e-publishing has already won a stealth war: ""What people forget is e-books were going strong before they were called e-books and they went on to sweep into many aspects of business and publishing. Most of this has gone unnoticed by the media. Probably because it has been a kind of backdoor revolution. To cite one example: Print law books are just about gone. People don't use them in law firms anymore. It's all electronic books or online. A revolution has occurred, but no one's noticed."*

(Wired Magazine)

- **Wired,** December 25, 2001.

"*Dear brother;*

I have often been surprised that ***Mathematics, the quintessence of Truth****, should have found admirers so few and so languid. Frequent consideration and minute scrutiny have at length unravelled the cause; viz. that though Reason is feasted, Imagination is starved; while Reason is luxuriating in its proper Paradise, Imagination is wearily travelling on a dreary desert. To assist Reason by the stimulus of Imagination is the design of the following production."*

Samuel Taylor Coleridge then launches into an ode on mathematics, the first verses of which are as follows:

" On a given finite line
Which must no way incline;
To describe an equi -
- lateral tri -
A -N -G -L -E.
Now let AB
Be the given line
Which must no way incline;
The great Mathematician
Makes this requisition,
That we describe an Equi -
- lateral Tri -
- angle on it;
Aid us, Reason - aid us, Wit!

From the centre A at the distance AB,
Describe the circle BCD
At the distance BA from B the centre
The round ACE to describe boldly venture.
(Third postulate see)
And from the point C
In which the circles make a pother
cutting and slashing one another
Bid the straight lines a journeying go.
CACB those lines will show

To the points, which by AB are reckoned
And postulate the second
For authority you know
ABC Triumphant shall be
An equilateral Triangle
No Peter Pindar carp, nor Zoilus can wrangle."

(Samuel Coleridge)

- In a letter to his brother the Reverend George Coleridge.

"*There is a story, no doubt exaggerated, that the Pope once remarked that two types of proposals exist for peace in the Middle East: The realistic and the miraculous. The realistic solution is divine intervention. The miraculous involves a voluntary agreement between the two sides."*

(Paul Adams)

- From his article "Israel, Palestinians now further apart than two years ago" in the *The Globe and Mail*, Monday, April 15,2002

"*Moreover a mathematical problem should be difficult in order to entice us, yet not completely inaccessible, lest it mock our efforts. It should be to us a guidepost on the mazy path to hidden truths, and ultimately a reminder of our pleasure in the successful solution.*

...

Besides it is an error to believe that rigor in the proof is the enemy of simplicity."

(David Hilbert)

- In his `23' *Mathematische Probleme* lecture to the Paris International Congress, 1900 (see Yandell's, fine account in *The Honors Class*, A.K. Peters, 2002).

``... waved his manuscript and confessed his publishing woes. ... "I said, 'I'm afraid no one's going to get to read these words. And I love these words.'"

Ann Sparanese, a librarian in the audience, sent an SOS over the Internet to fellow librarians. Within hours, they inundated HarperCollins with angry e-mails - and orders for Stupid White Men. Some also threatened a boycott.

"Those librarians," says Moore, ... "That's one terrorist group you don't want to mess with."

HarperCollins caved.

...

(Jan Wong)

Quoted from "Lunch with Michael Moore - A smart white guy with attitude," *The Globe and Mail* May 18, 2002, page F2.

> *`` Old ideas give way slowly; for they are more than abstract logical forms and categories. They are habits, predispositions, deeply engrained attitudes of aversion and preference. Moreover, the conviction persists-though history shows it to be a hallucination that all the questions that the human mind has asked are questions that can be answered in terms of the alternatives that the questions themselves present. But in fact intellectual progress usually occurs through sheer abandonment of questions together with both of the alternatives they assume an abandonment that results from their decreasing vitality and a change of urgent interest. We do not solve them: we get over them.*
>
> *Old questions are solved by disappearing, evaporating, while new questions corresponding to the changed attitude of endeavor and preference take their place. Doubtless the greatest dissolvent in contemporary thought of old questions, the greatest precipitant of new methods, new intentions, new problems, is the one effected by the scientific revolution that found its climax in the " Origin of Species." ``*

(John Dewey)

Quoted from ***The Influence of Darwin on Philosophy***, 1910.

> *``The first [axiom] said that when one wrote to the other (they often preferred to exchange thoughts in writing instead of orally), it was completely indifferent whether what they said was right or wrong. As Hardy put it, otherwise they could not write completely as they pleased, but would have to feel a certain responsibility thereby. The second axiom was to the effect that, when one received a letter from the other, he was under no obligation whatsoever to read it, let alone answer it, - because, as they said, it might be that the recipient of the letter would prefer not to work at that particular time, or perhaps that he was just then interested in other problems.... The third axiom was to the effect that, although it did not really matter if they both thought about the same detail, still, it was preferable that they should not do so. And, finally, the fourth, and perhaps most important axiom, stated that it was quite indifferent if one of them had not contributed the least bit to the contents of a paper under their common name; otherwise there would constantly arise quarrels and difficulties in that now one, and now the other, would oppose being named co-author.''*

(Harald Bohr)

Hardy and Littlewood's Four Axioms for Collaboration quoted from the preface of Bella Bollobas' 1988 edition of *Littlewood's Miscellany*. (Other quotes from the **Miscellany**.)

> *"I got into a research project which can be very simply described as concerned with the realization of the "Nash program" (making use of words made conventional by others that refer to suggestions I had originally made in my early works in game*

theory).

In this project a considerable quantity of work in the form of calculations has been done up to now. Much of the value of this work is in developing the methods by which tools like Mathematica can be used with suitable special programs for the solution of problems by successive approximation methods."

(John Nash)

- On page 241 of *"The Essential John Nash"*, edited by Harold W. Kuhn and Sylvia Nasar, Princeton Univ. Press, 2001.

"A proof is a proof. What kind of a proof? It's a proof. A proof is a proof. And when you have a good proof, it's because it's proven."

(Jean Chretien)

- The Canadian Prime Minister **explaining Canada's conditions** for determining if Iraq has complied, September 5, 2002. Sounds a lot like **Bertrand Russell**!

"No man can worthely praise Ptolemye ... yet muste ye and all men take heed, that both in him and in all mennes workes, you be not abused by their autoritye, but evermore attend to their reasons, and examine them well, ever regarding more what is saide, and how it is proved, than who saieth it, for autorite often times deceaveth many menne."

(Robert Record)

- The great textbook writer in his cosmology text `The castle of knowledge' (1556) quoted on page 47 of *Oxford Figures*, Oxford University Press, 2000.

"The future has arrived; it's just not evenly distributed."

(Douglas Gibson)

- On his Vancouver **home page**.

"The plural of 'anecdote' is not 'evidence'."

(Alan L. Leshner)

- *Science*'s publisher speaking at the **Federal S&T Forum**, Oct 2, 2002.

" ... Several years ago I was invited to contemplate being marooned on the proverbial desert island. What book would I most wish to have there, in addition to the Bible and the complete works of Shakespeare? My immediate answer was: Abramowitz and Stegun's Handbook of Mathematical Functions. If I could substitute for the Bible, I would choose Gradsteyn and Ryzhik's Table of Integrals, Series and

Products. Compounding the impiety, I would give up Shakespeare in favor of Prudnikov, Brychkov And Marichev's of Integrals and Series ... On the island, there would be much time to think about waves on the water that carve ridges on the sand beneath and focus sunlight there; shapes of clouds; subtle tints in the sky... With the arrogance that keeps us theorists going, I harbor the delusion that it would be not too difficult to guess the underlying physics and formulate the governing equations. It is when contemplating how to solve these equations - to convert formulations into explanations - that humility sets in. Then, compendia of formulas become indispensable."

(Michael Berry)

- "Why are special functions special?" *Physics Today*, April 2001.

"I will be glad if I have succeeded in impressing the idea that it is not only pleasant to read at times the works of the old mathematical authors , but this may occasionally be of use for the actual advancement of science."

(Constantin Caratheodory)

- Speaking to an MAA meeting in 1936.

"I have myself always thought of a mathematician as in the first instance an observer, a man who gazes at a distant range of mountains and notes down his observations. His object is simply to distinguish clearly and notify to others as many different peaks as he can. There are some peaks which he can distinguish easily, while others are less clear. He sees A sharply, while of B he can obtain only transitory glimpses. At last he makes out a ridge which leads from A, and following it to its end he discovers that it culminates in B. B is now fixed in his vision, and from this point he can proceed to further discoveries. In other cases perhaps he can distinguish a ridge which vanishes in the distance, and conjectures that it leads to a peak in the clouds or below the horizon. But when he sees a peak he believes that it is there simply because he sees it. If he wishes someone else to see it, he points to it, either directly or through the chain of summits which led him to recognize it himself. When his pupil also sees it, the research, the argument, the proof is finished.

The analogy is a rough one, but I am sure that it is not altogether misleading. If we were to push it to its extreme we should be led to a rather paradoxical conclusion; that we can, in the last analysis, do nothing but point; that proofs are what Littlewood and I call gas, rhetorical flourishes designed to affect psychology, pictures on the board in the lecture, devices to stimulate the imagination of pupils. This is plainly not the whole truth, but there is a good deal in it. The image gives us a genuine approximation to the processes of mathematical pedagogy on the one hand and of mathematical discovery on the other; it is only the very unsophisticated outsider who imagines that mathematicians make discoveries by turning the handle of some miraculous machine. Finally the image gives us at any rate a crude picture of Hilbert's metamathematical proof, the sort of proof which is a ground for its conclusion and whose object is to convince ."

(G.H. Hardy)

From the Preface to David Broussoud's recent book **"Proofs and Confirmation: The Story of the Alternating Sign Matrix Conjecture,"** MAA, 1999. Broussoud cites Hardy's "Rouse Ball Lecture of 1928".

> *"[T]o suggest that the normal processes of scholarship work well on the whole and in the long run is in no way contradictory to the view that the processes of selection and sifting which are essential to the scholarly process are filled with error and sometimes prejudice."*

(Kenneth Arrow)

From E. Roy Weintraub and Ted Gayer, "Equilibrium Proofmaking," *Journal of the History of Economic Thought*, 23 (Dec. 2001), 421-442.

> *"Mathematical proofs like diamonds should be hard and clear, and will be touched with nothing but strict reasoning."*

(John Locke)

From *The Mathematical Universe* by William Dunham, John Wiley, 1994.

> *``In his review of Winchester's previous book, The Map That Changed the World (3), Stephen Jay Gould wrote:*
>
> *I don't mean to sound like an academic sourpuss, but I just don't understand the priorities of publishers who spare no expense to produce an elegantly illustrated and beautifully designed book and then permit the text to wallow in simple, straight-out factual errors, all easily corrected for the minimal cost of one scrutiny of the galleys by a reader with professional expertise... (4)*
>
> *With Krakatoa, the publisher clearly spared considerable expense, and this new book also wallows in errors. Perhaps, given our popular culture's appetite for sensationalized disasters, a modern publisher would rather not see all those pesky details corrected.''*

(Tom Simkin and Richard S. Fiske)

Review entitled "Clouded Picture of a Big Bang" from *Science*, July 4, 2003, page 50-51)

> *"Again, I have to repeat the dictum of Harvard's president, Larry Summers: "**In the history of the world, no one has ever washed a rented car.**" Most Iraqis still feel they are renting their own country --- first from Saddam and now from us. They have to be given ownership. If the Bush team is ready to put in the time, energy and money to make that happen --- great. But if not, it's going to have to make the necessary compromises to bring in the U.N. and the international community to help. ''*

(Thomas Freedman)

- *New York Times* August 26, 2003.

> *"The paomnnehil pweor of the hmuan mnid. Aoccdrnig to a rscheearch at Cmabrigde Uinervtisy, it deosn't mttaer in waht oredr the ltteers in a wrod are, the olny iprmoetnt tihng is taht the frist and lsat ltteer be at the rghit pclae. The rset can be a total mses and you can sitll raed it wouthit porbelm. Tihs is bcuseae the huamn mnid deos not raed ervey lteter by istlef, but the wrod as a wlohe."*

- Psased on by *Kevin Hare*, Spetmber 2003.

> *" "The great tragedy of science," the biologist Thomas Henry Huxley lamented, is "the slaying of a beautiful hypothesis by an ugly fact." By that standard, political science is going through a homely phase. It's not even three weeks since the Iowa caucuses, and voters have wiped out several decades' worth of conventional wisdom about presidential primaries."*

- Some columnist in February 2004.

> *"By 1948, the Marxist-Leninist ideas about the proletariat and its political capacity seemed more and more to me to disagree with reality ... I pondered my doubts, and for several years the study of mathematics was all that allowed me to preserve my inner equilibrium. Bolshevik ideology was, for me, in ruins. I had to build another life."*

(Jean Van Heijenoort, 1913-1986)

- From his autobiography *With Trotsky in Exile*, quoted in Anita Feferman's *From Trotsky to Godel*

> *"Numbers are not the only thing that computers are good at processing. Indeed, only a cursory familiarity with fractal geometry is needed to see that computers are good at creating and manipulating visual representations of data. There is a story told of the mathematician Claude Chevalley, who, as a true Bourbaki, was extremely opposed to the use of images in geometric reasoning. He is said to have been giving a very abstract and algebraic lecture when he got stuck. After a moment of pondering, he turned to the blackboard, and, trying to hide what he was doing, drew a little diagram, looked at it for a moment, then quickly erased it, and turned back to the audience and proceeded with the lecture. It is perhaps an apocryphal story, but it illustrates the necessary role of images and diagrams in mathematical reasoning-even for the most diehard anti-imagers. The computer offers those less expert, and less stubborn than Chevalley, access to the kinds of images that could only be imagined in the heads of the most gifted mathematicians, images that can be coloured, moved and otherwise manipulated in all sorts of ways. "*

(Nathalie Sinclair, 2004)

- From *Making the Connection: Research and Practice in Undergraduate Mathematics*, M.

Carlson and C. Rasmussen (Eds), MAA Notes, in press.

Greenwood: *It was quite a popular course. There used to be a saying that if Wedderburn says something is true, accept it but don't try to prove it because you won't be able to. If Eisenhart says something is true, get out his book and by using cross references 20 to 30 times you can work up a proof for it. And if Lefschetz says something is true ...*

Tucker*: It is probably false.*

Greenwood: *... my apologies to Professor Lefschetz, look for a proof and for a counterexample at the same time.*

Aspray*: Since you both had close associations with Church, I was wondering if you could tell me something about him. What was his wider mathematical training and interests? What were his research habits? I understood he kept rather unusual working hours. How was he as a lecturer? As a thesis director?*

Rosser: *In his lectures he was painstakingly careful. There was a story that went the rounds. If Church said it's obvious, then everybody saw it a half hour ago. If Weyl says it's obvious, von Neumann can prove it. If Lefschetz says it's obvious, it's false.*

- From the **Princeton Oral History Project**

Excerpts from Google's filing with the SEC

-- Google is not a conventional company. We do not intend to become one.

-- A management team distracted by a series of short-term targets is as pointless as a dieter stepping on the scale every half hour.

-- We will not hesitate to place major bets on promising new opportunities.

-- For example, we would fund projects that have a 10 percent chance of earning a billion dollars over the long term. Do not be surprised if we place smaller bets in areas that seem very speculative or even strange.

-- Our employees, who have named themselves Googlers, are everything.

-- We provide many unusual benefits for our employees, including meals free of charge, doctors and washing machines.

-- Don't be evil. We believe strongly that in the long term, we will be better served

-- as shareholders and in all other ways -- by a company that does good things for the world even if we forgo some short-term gains."

(John Shinal)

- From *San Francisco Chronicle*, Friday, April 30, 2004

"The discussion was going beautifully until I discovered that he was talking about the Peloponnesian War while I was discussing WW II."

(Nicholas Katzenbach)

- Katzenbach writing in the *The American Oxonian*, describing his first meeting with his tutor Lord Lindsay in Balliol around 1948. The subject was the effect of war upon morals.

"A coded message, for example, might represent gibberish to one person and valuable information to another. Consider the number 14159265... Depending on your prior knowledge, or lack thereof, it is either a meaningless random sequence of digits, or else the fractional part of pi, an important piece of scientific information."

(Hans Christian von Baeyer)

- On page 11 of his recent book *Information The New Language of Science*, Weidenfeld and Nicolson, 2003.

The metaphor of shooting naturally became a familiar one in writings about his photography. Cartier-Bresson himself used it often: "approach tenderly, gently . . . on tiptoe even if the subject is a still life," he said. "A velvet hand, a hawk's eye these we should all have." He also said: "I adore shooting photographs. It's like being a hunter. But some hunters are vegetarians which is my relationship to photography." And later, explaining his dislike of the automatic camera, he said, "It's like shooting partridges with a machine gun."

*With a Brownie that he had received as a gift, he began to snap photographs in Africa, but they ended up ruined. Contracting blackwater fever, he nearly died. The way he told the story, a witch doctor got him out of a coma. While still feverish, he wrote a postcard to his grandfather asking that he be buried in Normandy, at the edge of the Eawy forest, with Debussy's string quartet to be played at the funeral. An uncle wrote back: "**Your grandfather finds all that too expensive. It would be preferable that you return first.**"*

(New York Times)

- From Henri Cartier-Bresson's *New York Times* Obituary of August 4, 2004.

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*"Despite the narrative force that the concept of entropy appears to evoke in everyday writing, in scientific writing entropy remains a thermodynamic quantity and a mathematical formula that numerically quantifies disorder. When the American scientist Claude Shannon found that the mathematical formula of Boltzmann defined a useful quantity in information theory, he hesitated to name this newly discovered quantity entropy because of its philosophical baggage. The mathematician John Von Neumann encouraged Shannon to go ahead with the name entropy, however, since "**no one knows what entropy is, so in a debate you will always have the advantage.**"*

From *The American Heritage Book of English Usage*, p. 158.

"The connections between chemical science and technology in the new synthetic-dye industry that began to develop after William Henry Perkin's synthesis of mauve in 1856 are complex. But one contribution of the science of carbon chemistry to the synthetic-dye industry was clearly crucial: chemical theory embodied in chemical formulae. Linear chemical formulae, like H2O for water, had been introduced by the Swedish chemist Jacob Berzelius (1779-1848) in 1813. They presented the composition of chemical compounds according to a theory of definite quantitative units or portions of substances. With atomism, this new quantitative theory shared the assumption of discontinuous composition of substances. But the algebraic form of Berzelian formulae avoided narrow definitions in terms of "atoms," which many chemists rejected as metaphysical entities. Letters, numbers, and additivity were sufficient to represent quantitative units of elements and discontinuous composition of compounds. Different arrangements of letters visually showed how units of elements were combined with each other. The structural formulae of the 1860s displayed chemical and spatial arrangements in an even more pictorial form.

*Beginning in the late 1820s, chemists used chemical formulae as tools on paper to model the constitution of organic compounds. Using chemical formulae as paper tools, chemists reduced the complexity in the "**jungle of organic chemistry**" (F. Wehler). Chemical formulae enabled them, for example, to order organic chemical reactions by formula equations that distinguished between a main reaction, side reactions, and successive reactions.*

***In the 1860s, chemical formulae had become an emblem not only of academic chemistry but also of the synthetic-dye industry.** Quantitative chemical theory was implemented in the new alliance between carbon chemistry and the synthetic-dye industry in the form of paper tools that were subordinated to chemists' experimental and technological goals (6). Compared with the connections between academic chemistry and the arts and crafts in the 18th*

(Ursala Klein)

In "Not a Pure Science: Chemistry in the 18th and 19th Centuries" *Science*, 5 November 2004

"Whether we scientists are inspired, bored, or infuriated by philosophy, all our theorizing and experimentation depends on particular philosophical background assumptions. This hidden influence is an acute embarrassment to many researchers, and it is therefore not often acknowledged. Such fundamental notions as reality, space, time, and causality--notions found at the core of the scientific enterprise--all rely on particular metaphysical assumptions about the world."

(Christof Koch)

In ``Thinking About the Conscious Mind," a review of John R. Searle's *Mind. A Brief Introduction*, OUP 2004.

"And it is one of the ironies of this entire field that were you to write a history of ideas in the whole of DNA, simply from the documented information as it exists in the literature - that is, a kind of Hegelian history of ideas - you would certainly say that Watson and Crick depended on Von Neumann, because von Neumann essentially tells you how it's done. But of course no one knew anything about the other. It's a great paradox to me that this connection was not seen. Of course, all this leads to a real distrust about what historians of science say, especially those of the history of ideas."

(Sidney Brenner)

- 2002 Nobelist Sidney Brenner talking about von Neumann's essay on *The General and Logical Theory of Automata* on pages 35--36 of **My life in Science** as told to Lewis Wolpert.

"Sometime in the 1970s Paul Turan spent part of a summer in Edmonton. I wanted to meet him so went there. He was a few days late so I had arrived a couple of days earlier. A group went to the airport to meet him, and stopped at a coffee shop before going to the university. It was very hot so I offered to stay in the car and keep the windows down. I said I did not drink coffee. Turan then told the joke about mathematicians being machines which turn coffee into theorems, and then added: ***"You prove good theorems. Just think how much better they would be if you drank coffee".*** *I have heard the statement attributed to Renyi by more than one Hungarian, but this was somewhat later. Turan just stated it."*

(Richard Askey)

- The definitive version of **"Erdos and Coffee"?** As told to the *historia mathematica* list on Feb 3, 2005.

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*Elsewhere Kronecker said "****In mathematics, I recognize true scientific value only in concrete mathematical truths, or to put it more pointedly, only in mathematical formulas.****" ... I would rather say "computations" than "formulas", but my view is essentially the same.*

(Harold M. Edwards)

- On page 1 of *Essays on Constructive Mathematics*, Springer 2005. Edwards comments elswhere that his own preference for constructivism was forged by experience of computing in the fifties---"trivial by today's standards".

"One little know piece of Mayr's history, Rubinoff said, was his service on a National Research Council committee, which formed in the late 1960's, to examine the consequences of building a sea-level canal through the Isthmus of Panama. Mayr was accused by one of the committee engineers of "having an elastic collision with reality." But, said Rubinoff, if it weren't for Mayr's tenacity, the proposed canal would have destroyed 3 million years of isolated evolution.

Frank Sulloway, author and former Mayr student, said that his career was influenced by meeting two minds: Darwin's and Mayr's. "The minute you meet one, you sooner *or later meet the other," he said.*

Both were famously persistent. Quoting 19th-century novelist Anthony Trollope, "Darwin once wrote: ***"It's dogged as does it... I have often and often thought that this is the motto for every scientific worker."*** *"The only person I know who's about as dogged is Ernst Mayr," said Sulloway."*

(The Scientist)

February 3, 2005 obituary of Ernst Meyr. (See **www.biomedcentral.com/news/20050204/01**.)

"Dear Friend Wollstein, By the time you receive these lines, we three will have solved the problem in another way - in the way which you have continually attempted to dissuade us. ... What has been done against the Jews in recent months arouses well-founded anxiety that we will no longer be allowed to experience a bearable situation. ... Forgive us, that we still cause you trouble beyond death; I am convinced that you will do what you are able to do (and which perhaps is not very much). Forgive us also our desertion! We wish you and all our friends will experience better times.

Yours faithfully, Felix Hausdorff"

(Felix Hausdorff)

MacTutor gives more of Felix Hausdorff's last letter written on the eve of suicide (January 25, 1942).

About H.E. Smith: In the book "Elementary Number Theory" (Chelsea, New York, 1958. An English translation of vol. 1 of the German book Vorlesungen ueber Zahlentheorie), p.31, the author, Edmund Landau, mentions the question whether the infinite series $\sum \mu(n)/n$ converges (TEX notation; \mu is the Moebius function). After giving a reference to the answer in Part 7 of the same V.u.Z, and without saying what the answer is, Landau writes: **"Gordan used to say something to the effect that 'Number Theory is useful since one can, after all, use it to get a doctorate with.' In 1899 I received my doctorate by answering this question."**

> *He was a brilliant talker and wit. Working in the purely speculative region of the theory of numbers, it was perhaps natural that he should take an anti-utilitarian view of mathematical science, and that he should express it in exaggerated terms as a defiance to the grossly utilitarian views then popular. It is reported that once in a lecture after explaining a new solution of an old problem he said, "It is the peculiar beauty of this method, gentlemen, and one which endears it to the really scientific mind, that under no circumstances can it be of the smallest possible utility." I believe that it was at a banquet of the Red Lions that he proposed the toast* ***"Pure mathematics; may it never be of any use to anyone."***

This is taken from Alexander Macfarlane, _Ten British Mathematicians of the Nineteenth Century_ (1916), 63-4. The text is that of lectures he gave in 1903-

> 1904, and the editors in their introduction say that "His personal acquaintance with British mathematicians of the nineteenth century imparts to many of these lectures a personal touch which greatly adds to their general interest."

A copy of the book is available on the Project Gutenberg website:

> *"By its own count, Wal-Mart has 460 terabytes of data stored on Teradata mainframes, made by NCR, at its Bentonville headquarters.*
>
> *To put that in perspective, the Internet has less than half as much data, according to experts."*

(Constance Hays)

- In "What Wal-Mart Knows About Customers' Habits", NYT November 14, 2004.

> **Just what does it mean to prove something?** *Although the **Annals** will publish Dr Hales's paper, Peter **Annals**, an editor of the Annals, whose own work does not involve the use of computers, says that the paper will be accompanied by an unusual disclaimer, stating that the computer programs accompanying the paper have not undergone peer review. There is a simple reason for that, Dr Sarnak says-it is impossible to find peers who are willing to review the computer code. However, there is a flip-side to the disclaimer as well-Dr Sarnak says that the editors of the **Annals** expect to receive, and publish, more papers of this type-for things, he believes, will change over the next 20-50 years. Dr Sarnak points out that maths may become "a bit like experimental physics" where certain results are taken on trust, and independent duplication of experiments replaces examination of a colleague's paper.*
>
> *Why should the non-mathematician care about things of this nature? The foremost reason is that mathematics is beautiful, even if it is, sadly, more inaccessible than other forms of art. The second is that it is useful, and that its utility depends in part on its certainty, and that that certainty cannot come without a notion of proof. Dr Gonthier, for instance, and his sponsors at Microsoft, hope that the techniques he and his colleagues have developed to formally prove mathematical theorems can be used to "prove" that a computer program is free of bugs-and that would certainly be a useful proposition in today's software society if it does, indeed, turn out to be true.*

- In *Proof and beauty*, the **Economist**, March 31, 2005

> *Writers we admire and re-read are absorbed into the fine print of our consciousness, into the white noise of our thoughts, and in this sense, they can never die. Saul Bellow started publishing in the 1940's, and his work spreads across the century he helped to define. He also redefined the novel, broadened it, liberated it, made it warm with human sense and wit and grand purpose. Henry James once proposed an obvious but helpful truth: **"the deepest quality of a work of art will always be the quality of the mind of the producer."** We are saying farewell to a mind of unrivalled quality. He opened our universe a little more. We owe him everything.*

(Ian McEwan)

Master of the Universe, an obituary for **Saul Bellow (1915-2005) NYT** April 7, 2005.

> *Why should I refuse a good dinner simply because I don't understand the digestive processes involved?*

(Oliver Heaviside)

- Heaviside (1850-1925) when criticized for his daring use of operators before they could be justified formally.

> *Die Mathematiker sind eine Art Franzosen; redet man mit ihnen, so übersetzen sie es in ihre Sprache, und dann ist es alsobald ganz etwas anderes. [Mathematicians are a kind of Frenchman: whatever you say to them they translate into their own language, and right away it is something entirely different.]*

(Johann Wolfgang von Goethe)

- *Maximen und Reflexionen*, no. 1279, on page 160 of the Penguin classic edition.

> *Ask Dr. Edward Witten of the Institute for Advanced Study in Princeton, New Jersey what he does all day, and it's difficult to get a straight answer.*
>
> *"There isn't a clear task," Witten told CNN. "If you are a researcher you are trying to figure out what the question is as well as what the answer is.*
>
> *"You want to find the question that is sufficiently easy that you might be able to answer it, and sufficiently hard that the answer is interesting. You spend a lot of time thinking and you spend a lot of time floundering around." "*

(Ed Witten)

- **CNN** June 27, 2005.

> *"I don't think biochemists are going to be the least bit interested in what philosophers think about genes," Jones replies. "As I've said in the past, philosophy is to science as pornography is to sex: It's cheaper, easier, and some people prefer it.", Moving swiftly along, Jones and Stangroom ponder racial differences in IQ, the debate over genetically modified crops, health insurance, and the future of the human race.*
>
> *In the next chapter, Harvard evolutionary psychologist Steven Pinker is probed on "Evolutionary Psychology and the Blank Slate." The conversation moves from the structure of the brain to adaptive explanations for music, creationism, and beyond. Stangroom asks Pinker about the accusations that biological explanations of behavior are determinist and reduce human beings to the status of automatons. " "Most people have no idea what they mean when they level the accusation of determinism," "Pinker answers. "It's a nonspecific "boo" word, intended to make something seem bad without any content."*

(Jeremy Stangroom's interviews)

- *The Scientist* describing **What (some) scientists say** (Routledge Press). June 20th, 2005. [For earlier quote **See above**]

> *Harald Bohr is reported to have remarked* ***"Most analysts spend half their time hunting through the literature for inequalities they want to use, but cannot prove."***

(D.J.H. Garling)

- On page 575 of his very positive review of Michael Steele's *The Cauchy Schwarz Master Class* in the **MAA Monthly**, June-July 2005, 575-579.

> *"How ridiculous to make evolution the enemy of God. What could be more elegant, more simple, more brilliant, more economical, more creative, indeed more divine than a planet with millions of life forms, distinct and yet interactive, all ultimately derived from accumulated variations in a single double-stranded molecule, pliable and fecund enough to give us mollusks and mice, Newton and Einstein? Even if it did give us the Kansas State Board of Education, too."*

(Charles Krauthammer)

- In "Phony Theory, False Conflict. 'Intelligent Design' Foolishly Pits Evolution Against Faith." The *Washington Post* 18/11/2005

> *"The chief aim of all investigations of the external world should be to discover the rational order and harmony which has been imposed on it by God and which He revealed to us in the language of mathematics. "*

(Johannes Kepler)

- Johannes Kepler (1571 - 1630)

> *"[Maxwell asked whether he would like to see an experimental demonstration of conical refraction]* ***No. I have been teaching it all my life, and I do not want to have my ideas upset."***

(Isaac Todhunter)

- Isaac Todhunter (1820 - 1884)

> *"Rigour is the affair of philosophy, not of mathematics."*

(Bonaventura Cavalieri)

- Bonaventura Cavalieri (1598 - 1647)

"How dreadful are the curses which Mohammedanism lays on its votaries! Besides the fanatical frenzy, which is as dangerous in a man as hydrophobia in a dog, there is this fearful fatalistic apathy. The effects are apparent in many countries. Improvident habits, slovenly systems of agriculture, sluggish methods of commerce, and insecurity of property exist wherever the followers of the Prophet rule or live. A degraded sensualism deprives this life of its grace and refinement; the next of its dignity and sanctity. The fact that in Mohammedan law every woman must belong to some man as his absolute property, either as a child, a wife, or a concubine, must delay the final extinction of slavery until the faith of Islam has ceased to be a great power among men.

Individual Moslems may show splendid qualities, but the influence of the religion paralyses the social development of those who follow it. No stronger retrograde force exists in the world. Far from being moribund, Mohammedanism is a militant and proselytizing faith. It has already spread throughout Central Africa, raising fearless warriors at every step; and were it not that Christianity is sheltered in the strong arms of science, the science against which it had vainly struggled, the civilization of modern Europe might fall, as fell the civilization of ancient Rome." "

(Winston Churchill)

- Speech by Churchill in *The River War,* ed 1, Vol. II, pages 248-50 (London: Longmans, Green & Co., 1899).

"How extremely stupid not to have thought of that!"

(T.H. Huxley)

- Thomas Henry Huxley (1825--1895). Huxley, known as `Darwin's Bulldog' for his tireless defense of Darwin, was initially unconvinced of evolution. Converted by the `Origin of Species', he is recorded (much like **Briggs**) as saying "How extremely stupid not to have thought of that!"

"All truths are easy to understand once they are discovered; the point is to discover them."

(Galileo Galilei, 1564-1642)

- Galileo's view is apparently not a view shared by all. The following *thoughts on quantum theory by various scientists* come from the NYT of Dec 26, 2005.

"On quantum theory, I use up more brain grease than on relativity." (Albert Einstein to Otto Stern in 1911)

"Those are the crazy people who are not working on quantum theory." (Albert Einstein referring to the inmates of an insane asylum near his office in Prague, in 1911)

"I could probably have arrived at something like this myself, but if all this is true then it means the end of physics." (Albert Einstein, referring to a 1913 breakthrough by Niels Bohr)

"Anyone who is not shocked by quantum theory has not understood a single word." (Niels Bohr)

"I don't like it, and I'm sorry I ever had anything to do with it." (Erwin Schrödinger about the probability interpretation of quantum mechanics)

"What we observe is not nature itself, but nature exposed to our method of questioning." (Werner Heisenberg, 1963)

"You know how it always is, every new idea, it takes a generation or two until it becomes obvious that there's no real problem. I cannot define the real problem, therefore I suspect there's no real problem, but I'm not sure there's no real problem." (Richard Feynman, 1982)

"Logic is the hygiene the mathematician practices to keep his ideas healthy and strong."

(Hermann Weyl, 1885 - 1955)

- Weyl brings us full circle back to rigour.

Math Will Rock Your World. **A generation ago, quants turned finance upside down. Now they're mapping out ad campaigns and building new businesses from mountains of personal data.**

"*These slices of our lives now sit in databases, many of them in the public domain. From a business point of view, they're just begging to be analyzed. But even with the most powerful computers and abundant, cheap storage, companies can't sort out their swelling oceans of data, much less build businesses on them, without enlisting skilled mathematicians and computer scientists. The rise of mathematics is heating up the job market for luminary quants, especially at the Internet powerhouses where new math grads land with six-figure salaries and rich stock deals. Tom Leighton, an entrepreneur and applied math professor at Massachusetts Institute of Technology, says: "All of my students have standing offers at Yahoo! and Google."* ***Top mathematicians are becoming a new global elite. It's a force of barely 5,000, by some guesstimates, but every bit as powerful as the armies of Harvard University MBAs who shook up corner suites a generation ago.***"

- **Business Week Cover Story** January 23, 2006.

"The formulas move in advance of thought, while the intuition often lags behind; in the oft-quoted words of d'Alembert, ***"L'algebre est genereuse, elle donne souvent plus qu'on lui demande."***"

(Edward Kasner, 1905)

- Edward Kasner, "The Present Problems of Geometry," *Bulletin of the American Mathematical Society*, (1905) volume XI, p.285.

"Science is a differential equation. Religion is a boundary condition."

(Alan Turing, 1912 - 1954)

- I'm not sure what it means, but I like it!

'Thirst for knowledge' may be opium craving

Neuroscientists have proposed a simple explanation for the pleasure of grasping a new concept: The brain is getting its fix. The "click" of comprehension triggers a biochemical cascade that rewards the brain with a shot of natural opium-like substances, said Irving Biederman of the University of Southern California. He presents his theory in an invited article in the latest issue of American Scientist.

"While you're trying to understand a difficult theorem, it's not fun," *said Biederman, professor of neuroscience in the USC College of Letters, Arts and Sciences.*

"But once you get it, you just feel fabulous."

The brain's craving for a fix motivates humans to maximize the rate at which they absorb knowledge, he said.

I think we're exquisitely tuned to this as if we're junkies, second by second."

(Irving Biederman, 2006)

- From **www.physorg.com/news70030587.html** .

"We [Kaplansky and Halmos] share a philosophy about linear algebra: we think basis-free, we write basis-free, but when the chips are down we close the office door and compute with matrices like fury."

(Irving Kaplansky, 1917-2006)

- Quoted in Paul Halmos' *Celebrating 50 Years of Mathematics.*

---

*"The war became more and more bitter. The Dominican Father Caccini preached a sermon from the text, ``Ye men of Galilee, why stand ye gazing up into heaven?'' and this wretched pun upon the great astronomer's name ushered in sharper weapons; for, before Caccini ended, he insisted that ``**geometry is of the devil,**'' and that ``**mathematicians should be banished as the authors of all heresies.**'' The Church authorities gave Caccini promotion."*

- From *A History of the Warfare of Science with Theology in Christendom* by Andrew Dickson White, Chapter 3, Section 3. An online copy is at: **www.cscs.umich.edu/~crshalizi/White/** .

> *"Equations are more important to me, because politics is for the present, but an equation is something for eternity."*
>
> (Albert Einstein)

• Like so many Einstein quotes, this appears everywhere and seemingly without direct attribution.

- *"Never Ascribe to malice that which is adequately explained by incompetence."* (Napolean Bonaparte?)
- *"Misunderstandings and neglect occasion more mischief in the world than even malice and wickedness. At all events, the two latter are of less frequent occurrence."* (Goethe in *The Sorrows of Young Werther*)
- *"You have attributed conditions to villainy that simply result from stupidity".* (Robert Heinlein in the *Logic of Empire* (1941). He calls this the "devil theory" of sociology.
- *"Many journalists have fallen for the conspiracy theory of government. I do assure you that they would produce more accurate work if they adhered to the cock-up theory."* (Bernard Ingam, 1932- who was Thatcher's press secretary.)

• This is now also called Hanlon's Razor (1980).

> I'm here to help. (With the Poincare conjecture. As for the family, you're on your own.) Poincare conjectured that three-dimensional shapes that share certain easy-to-check properties with spheres actually are spheres. What are these properties? My fellow geometer Christina Sormani describes the setup as follows:
>
> > "The Poincare Conjecture says, Hey, you've got this alien blob that can ooze its way out of the hold of any lasso you tie around it? Then that blob is just an out-of-shape ball. [Grigory] Perelman and [Columbia University's Richard] Hamilton proved this fact by heating the blob up, making it sing, stretching it like hot mozzarella, and chopping it into a million pieces. In short, the alien ain't no bagel you can swing around with a string through his hole."
>
> (Jordan Ellenberg)

• In **Who Cares About Poincare Million-dollar math problem solved. So what?** from **Slate** Posted Friday, Aug. 18, 2006, at 11:59 AM ET

Thank you for your reply. I certainly understand what it means to recall something and have the trail disappear!

The reason I inquired, as in my Tobias conversations with George and his comments re how Tobias influenced him by "feeding" him thousands of geometry problems to solve (see More Mathematical People, Albers et al. (eds.) , Harcourt Brace Jovanovich, 1990), he never indicated that he (George) had any input to Tobias' work. In fact, it went the other way in one important instance. As you may not have encountered it, I cite the following. George wrote in his paper "Reminiscences about the origins of linear programming," 1, 2, Operations Research

Letters, April 1982 (p. 47):

> *""The term Dual is not new. But surprisingly the term Primal, introduced around 1954, is. It came about this way. W. Orchard-Hays, who is responsible for the first commercial grade L.P. software, said to me at RAND one day around 1954: 'We need a word that stands for the original problem of which this is the dual.' I, in turn, asked my father, Tobias Dantzig, mathematician and author, well known for his books popularizing the history of mathematics. He knew his Greek and Latin. Whenever I tried to bring up the subject of linear programming, Toby (as he was affectionately known) became bored and yawned. But on this occasion he did give the matter some thought and several days later suggested Primal as the natural antonym since both primal and dual derive from the Latin. It was Toby's one and only contribution to linear programming: his sole contribution unless, of course, you want to count the training he gave me in classical mathematics or his part in my conception. "*

A lovely story. I heard George recount this a few times and, when he came to the "conception" part, he always had a twinkle in his eyes.

(Saul Gass)

- In a September 2006 **SIAM book review**, I asserted George Dantzig assisted his father Tobias---for reasons I believed but cannot now reconstruct. I also called Lord Chesterfield, Chesterton (gulp!).

> *"Nothing has afforded me so convincing a proof of the unity of the Deity as these purely mental conceptions of numerical and mathematical science which have been by slow degrees vouchsafed to man, and are still granted in these latter times by the Differential Calculus, now superseded by the Higher Algebra, all of which must have existed in that sublimely omniscient Mind from eternity."*

(Mary Somerville, 1780-1872)

- Quoted in *Martha Somerville, Personal Recollections of* ***Mary Somerville*** (Boston, 1874)

> *Today's outcome may end the interest in future chess matches between human champions and computers, according to Monty Newborn, a professor of computer science at McGill University in Montreal. Professor Newborn, who helped organize the match between Mr. Kasparov and Deep Blue, said of future matches:* ***"I don't know what one could get out of it at this point. The science is done."***
>
> *Mr. Newborn said that the development of chess computers had been useful.*
>
> *"If you look back 50 years, that was one thing we thought they couldn't do," he said. "It is one little step, that's all, in the most exciting problem of what can't computers do that we can do."*
>
> *Speculating about where research might go next, Mr. Newborn said,* ***"If you are interested in programming computers so that they compete in games, the two interesting ones are poker and go. That is where the action is."***

(Dylan Loeb McClain)

From a report of the defeat of world champion Vladimir Kramnik by Deep Fritz in **Once Again, Machine Beats Human Champion at Chess** NYT, December 5, 2006.

> *"**Je n'ai fait celle-ci plus longue que parceque je n'ai pas eu le loisir de la faire plus courte.**(I have only made this letter rather long because I have not had time to make it shorter.)"*

(Blaise Pascal)

From Pascal's *Lettres provinciales,* 16, Dec 14, 1656. [Cassell's Book of Quotations, London,1912. P.718.] Similar quotes are due to Goethe and perhaps to Augustine and Cicero.

> *Of course, identifying with one's captors is nothing new. Perhaps the most famous example is the 1973 **Norrmalmstorg bank robbery** in Stockholm. During the five days they were held hostage, the bank employees came to sympathize with the robbers and defended them against the police.*

(Globe and Mail, January 2007)

Describing the origin of the **Stockholm Syndrome**. Try looking up the **Jerusalem Syndrome** and recently named **Paris Syndrome** (2004). What other cities have such an honour?

> *"Bulls don't run reviews. Bulls of 25 don't marry old women of 55 and expect to be invited to dinner. Bulls do not get you cited as co-respondent in Society divorce trials. Bulls don't borrow money. Bulls are edible after they have been killed."*

(Ernest Hemingway, 1925)

From **Napoleon's love letter found in laundry room** (Toronto Star, June 4, 2007). "Another lot of interest is a letter written by Ernest Hemingway to the American poet and critic Ezra Pound in 1925, explaining why bulls are better than literary critics."

Memorable Ends

1. Here lies Ezekial Aikle. Aged 102. The good die young.

2. Here lies an Atheist. All dressed up. And no place to go.

3. A tomb now suffices him for whom the world was not enough. (Alexander the Great)

4. The body of Benjamin Franklin, printer (like the cover of an old book, its *contents worn out, and stript of its lettering and gilding) lies here, food for worms. Yet the work itself shall not be lost, for it will, as he believed appear once more. In a new and more beautiful edition, corrected and amended by its Author. (Benjamin Franklin)*

5. She did it the hard way. (Bette Davis)

6. The best is yet to come. (Frank Sinatra)

7. That's all folks! (Mel Blanc - voice of Bugs Bunny)

8. I told you I was ill. (Spike Milligan)

9. Ope'd my eyes. Took a peep. Didn't like it. Went back to sleep.

10. Called back. (Emily Dickinson)

(Various)

Posted by Joanna Sugdden, July 24th *London Times*. Franklin's epitaph has been banned in Texas school texts (it is clearly anti-Christian).

And Bloomberg can also flash a hard-edged candor. At the breakfast with business leaders, he scoffed at a question about whether the schools' emphasis on math and reading testing was taking away from the "richness" of education in subjects such as art and music. "Well, I don't know about the 'richness of education,' " he said, his voice thick with sarcasm. ***"In my other life, I own a business, and I can tell you, being able to do 2-plus-2 is a lot more important than a lot of other things."***

...

Giuliani seized on it to bolster his campaign's theme, saying, "Today's arrests remind us that we are at war." Bloomberg offered a noticeably milder response: ***"You can't sit there and worry about everything. You have a much greater danger of being hit by lightning than being struck by a terrorist. Get a life."***

(Michael Bloomberg)

Washington Post, August 6, 2007

"This computer, although assigned to me, was being used on board the International Space Station. I was informed that it was tossed overboard to be burned up in the atmosphere when it failed."

(Anonymous)

Science, August 3, 2007, p. 579: "A NASA employee's explanation for the loss of a laptop, recorded in a recent report by the U.S. Government Accountability Office documenting equipment losses of more than $94 million over the past 10 years by the agency."

"Rick Wilson, a Republican consultant based in Florida who has worked for Rudolph W. Giuliani, the former New York mayor, and Katherine Harris, the former Florida congresswoman, among others, said that most states have their own expressions for the circumstances under which open secrets stay secret. In Florida, he said, it's the 'Three County Rule': no girlfriends within three counties of your home district. In New York, it's the 'Bear Mountain Compact': nobody talks about what politicians do with their free time once they've crossed the Bear Mountain Bridge en route to Albany from points south."

(Abby Goodnough)

- From **Oh, everyone knows that (except you)** in the NYT of Sept 2, 2007.

Easy as 1, 2, 3 -- Except for The Maybes. Why No One Can Count On Those Delegates

"The lesson is not to trust the numbers too much. If math were a guy, math would be a pompous guy, the sort who's absolutely always sure about everything and never apologizes when he's wrong. And the fact is, math isn't actually ever wrong, not technically. Math is a perfectly logical and intelligent guy. He just sometimes makes the wrong assumptions. "

(Libby Copeland)

- From **Washington Post** Friday, April 25, 2008.

" the problem of course presents itself already when you are a student and I was thinking about the problem on and off, but the situation was more interesting than that. The great authority in those days was Zygmund and he was completely convinced that what one should produce was not a proof but a counter-example. When I was a young student in the United States, I met Zygmund and I had an idea how to produce some very complicated functions for a counter-example and Zygmund encouraged me very much to do so. I was thinking about it for about 15 years on and off, on how to make these counter-examples work and the interesting thing that happened was that I realised why there should be a counter-example and how you should produce it. I thought I really understood what was the background and then to my amazement I could prove that this "correct" counter-example couldn't exist and I suddenly realised that what you should try to do was the opposite, you should try to prove what was not fashionable, namely to prove convergence. ***The most important aspect in solving a mathematical problem is the conviction of what is the true result.*** *Then it took 2 or 3 years using the techniques that had been developed during the past 20 years or so. .. "*

(Lennart Carleson, 1966)

From 1966 IMU address on his positive solution of Luzin's 1913 conjecture that the Fourier series of every square integrable function converges a.e. to the function.

"In a mathematical conversation, someone suggested to Grothendieck that they should consider a particular prime number. ***"You mean an actual number?"*** *Grothendieck asked. The other person replied, yes, an actual prime number. Grothendieck suggested,* ***"All right, take 57."***

But Grothendieck must have known that 57 is not prime, right? Absolutely not, said David Mumford of Brown University. "He doesn't think concretely." Consider by contrast the Indian mathematician Ramanujan, who was intimately familiar with properties of many numbers, some of them huge. That way of thinking represents a world antipodal to that of Grothendieck. "He really never worked on examples," Mumford observed. ***"I only understand things through examples and then gradually make them more abstract. I don't think it helped Grothendieck in the least to look at an example. He really got control of the situation by thinking of it in absolutely the most abstract possible way. It's just very strange. That's the way his mind worked."****"*

(Allyn Jackson, 2004)

From a two-part biography in the Notices of the AMS.

"The letter was written in German in 1954 to philosopher Eric Gutkind. It is to be auctioned in London, England, on Thursday by Bloomsbury Auctions, and is expected to fetch between $12,000 and $16,000 US. Einstein writes ***"the word God is for me nothing more than the expression and product of human weaknesses, the Bible a collection of honourable but still primitive legends which are nevertheless pretty childish."****"*

From a letter by Einstein auctioned in May 2008 as described on **CBC**.

"It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment. When I have clarified and exhausted a subject, then I turn away from it, in order to go into darkness again; the never-satisfied man is so strange if he has completed a structure, then it is not in order to dwell in it peacefully,but in order to begin another. I imagine the world conqueror must feel thus, who, after one kingdom is scarcely conquered, stretches out his arms for others."

(Carl Friedrich Gauss, 1777-1855)

From an 1808 letter to his friend Farkas Bolyai (the father of Janos Bolyai).

"The difficulty lies, not in the new ideas, but in escaping the old ones, which ramify, for those brought up as most of us have been, into every

corner of our minds"

(John Maynard Keynes, 1883-1946)

Quoted in K E Drexler, ***Engines of Creation:*** *The Coming Era of Nanotechnology*, New York, 1987.

"He is like the fox, who effaces his tracks in the sand with his tail."

(Niels Abel, 1802-1829)

Regarding Gauss' mathematical writing style quoted in G. F. Simmons, *Calculus Gems* New York: Mcgraw Hill, 1992, p. 177.

"We have a habit in writing articles published in scientific journals to make the work as finished as possible, to cover up all the tracks, to not worry about the blind alleys or describe how you had the wrong idea first, and so on. So there isn't any place to publish, in a dignified manner, what you actually did in order to get to do the work."

(Richard Feynman, 1918-1988)

In his 1966 Nobel acceptance lecture.

"Gauss could be a stern, demanding individual, and it is reported that this resulted in friction with two of his sons that caused them to leave Germany and come to the United States; they settled in the midwest and have descendants throughout the plains states. I was living in Greeley, Colorado, when I read this in 1972; looking in the phone book, I found a listing for a Charlotte Gauss living two blocks from my apartment! After considerable internal debate, I called her and found that she was indeed related to Gauss.

My wife, Paulette, and I visited several times with Charlotte and her sister Helen; they were bright, alert, and charming young women, ages 93 and 94, respectively. Their father, Gauss' grandson, had been a Methodist missionary to the region, and he had felt it unseemly to take pride in his famous ancestor (maybe there were some remnants of his father's feelings on leaving Germany); they were nevertheless happy to talk Gauss and their family. They showed us a baby spoon which their father had made out of a gold medal awarded to Gauss, some family papers, and a short biography of Gauss written by an aunt. I vividly remember Helen describing the reaction of one of her math teachers when he discovered he had a real, live, Gauss in his class."

(Jim Kuzmanovichi)

Quoted from **http://www.wfu.edu/~kuz/Stamps/Gauss/Gauss.html**.

"Forget the 'precautionary principle.' The amount of risk to which the public should be exposed is greater than zero."

(Michael Krauss)

- Quoted from "Too cautious" in the **Financial Post**, June 20, 2008.

"Knowing things is very 20th century. You just need to be able to find things."

(Danny Hillis)

- On how Google has changed the way we think as quoted in **Achenblog**, July 1 2008.

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*"McCain would also be wise to study the etymology of his "maverick" image. The term entered the political lexicon because of one Samuel Augustus Maverick, a land owner, legislator, and former mayor of San Antonio who was the grandfather of Maury Maverick, the famous New Dealer who described democracy as "**liberty plus groceries.**" Samuel Maverick stubbornly refused to brand his calves and let them roam wherever they wanted. Other ranchers who encountered these free-spirited yearlings referred to them as "mavericks." Journalists later employed the term to describe politicians who bucked the party line and struck an independent course."*

(John Podesta and John Halpin)

- From **'The Maverick' gets the branding iron** in the *Politico* July 17, 2008.

"Of course I believe in luck. How otherwise to explain the success of those you dislike?"

(Jean Cocteau)

- From **Making His Own Luck**. Eugene Robinson writing about Obama, July 17, 2008.

His ambition to write may have prompted an exchange with T. S. Eliot, then in his late 50s, on the day they met in 1946, when Mr. Giroux, "just past 30," as he recalled the moment in "The Oxford Book of Literary Anecdotes," was an editor at Harcourt, Brace. "His most memorable remark of the day," Mr. Giroux said, "occurred when I asked him if he agreed with the definition that most editors are failed writers, and he replied, 'Perhaps, but so are most writers.'"

(T. S. Elliot)

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From **Robert Giroux, Publisher, Dies at 94** . New York Times, Sept 5, 2008.

" For those who had realized big losses or gains, the mania redistributed wealth. The largest honest fortune was made by Thomas Guy, a stationer turned philanthropist, w ho owned £54,000 of South Sea stock in April 1720 and sold it over the following six weeks for £234,000. Sir Isaac Newton, scientist, master of the mint, and a certifiably rational man, fared less well. He sold his £7,000 of stock in April for a profit of 100 percent. But something induced him to reenter the market at the top, and he lost £20,000. "I can calculate the motions of the heavenly bodies," he said, "but not the madness of people."

(Isaac Newton)

Quoted by Christopher Reed in **"The Damn'd South Sea"**, Harvard Magazine, May-June 1999. See **Newton on Cosmology**.

"When asked about the interruptions to her career caused by three marriages and three divorces, she shrugs. "You can like 'em," she jokes about men, "but it doesn't mean you have to sample every single one."

...

"Toward the end of the writing process, Proulx will often work 16 hours a day. "I love shaping things, pruning out the unnecessary, shaping unshapely sentences. After things are published I never read them again. I never, ever read reviews." (In the case of "Fine Just the Way It Is," that's just as well, since the reviews have been mixed.)."

(Annie Proulx)

Quoted by Susan Renolds in "**Annie Proulx no longer at home on the range**", *LA Times*, October 18, 2008.

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*"**Genetics by second nature** Growing up in Arlington, Virginia, Buckler had unlimited access to a personal computer, on which he designed his own games. To him, genetics is basically life's equivalent of computer programming. "There are not many rules: You get to recombine and to mutate, but you can make incredibly complex things." Buckler laughs, giving his boyish smile: "And it's more rewarding to do genetics than programming."."*

(Edward Buckler)

Quoted by Elizabeth Pennisi in "EDWARD BUCKLER PROFILE: Romping Through Maize Diversity", *Science*, 3 October 2008, pp. 40 – 41.

"Every once in while during a crisis or history-altering event, you run across a quote or an observation that sort of summarizes events on the

ground, in a nutshell. Former U.S. Federal Reserve Chairman Paul Volcker articulated one such observation during a recent chat he had with PBS' Charlie Rose. "It seems to me what our nation needs is more civil engineers and electrical engineers and fewer financial engineers," Volker said."

(Joseph Lazaro)

- Posted Oct 24th 2008 3:56pm at **www.bloggingstocks.com**.

EDITOR'S ENDNOTES "Jeffrey Lagarias (University of Michigan), Colin Mallows (Avaya Labs), and Allan Wilks (AT&T Labs–Research) submitted the following correction to their article *"Beyond the Descartes Circle Theorem,"* which appeared in the April, 2002 issue: We have an historical and a mathematical correction. First, it has been brought to our attention that Frederick Soddy, who won a Nobel Prize in Chemistry (1921) for the discovery of isotopes, did not receive a knighthood (in the English honours list). Davies [loc. cit.] quotes a letter from his nephew, Dr. Kenneth Soddy: *"He suffered a good deal of what might be termed persecution during the first World War . . . It was the recollection of these troubles that made him decline Honours later on."* Besides his scientific work, Soddy loved mathematics and worked on it as a hobby. He also wrote several books setting forth unpopular economic views. Our awarding him a spurious knighthood is an example of the *"Matthew effect" the phenomenon by which famous people become more famous, and less famous people become less famous. Unfortunately this error has propagated to Mumford et al. [Indra's Pearls]"*

(Dan Velleman)

- *American Mathematical Monthly*, Oct 2008, page 769. See also Robert K. Merton, "The Matthew effect in science," *Science* **159** (1968) 56–63.

"Considering that past, perhaps the most incisive comment on Mr. Obama's election actually came long ago. The Rev. Dr. Martin Luther King Jr. addressed the Hawaii Legislature in 1959, two years before Mr. Obama was born in Honolulu, and declared that the civil rights movement aimed not just to free blacks but "to free the soul of America."

Mr. King ended his Hawaii speech by quoting a prayer from a preacher who had once been a slave, and it's an apt description of the idea of America today: "Lord, we ain't what we want to be; we ain't what we ought to be; we ain't what we gonna be, but, thank God, we ain't what we was."

(Nicholas Kristof)

- From "**The Obama Dividend**," NYT, November 5, 2008.

"*"The collapse of communism pushed China to the center and [America] to* *the extreme," said Ben Simpfendorfer, chief China economist at Royal Bank of Scotland.*

The Madoff affair is the cherry on top of a national breakdown in financial propriety, regulations and common sense. Which is why we don't just need a financial bailout; we need an ethical bailout. We need to re-establish the core balance between our markets, ethics and regulations. I don't want to kill the ***animal spirits*** *that necessarily drive capitalism — but I don't want to be eaten by them either."*

(Thomas Friedman)

From "The Great Unravelling," NYT, December 16, 2008.

"The orbit of any one planet depends on the combined motions of all the planets, not to mention the actions of all these on each other. To consider simultaneously all these causes of motion and to define these motions by exact laws allowing of convenient calculation exceeds, unless I am mistaken, the forces of the entire human intellect."

(Isaac Newton, 1687)

Both Cosmology and **Commerce** are complicated. See G. Lake, T. Quinn and D. C. Richardson, "From Sir Isaac to the Sloan Survey: Calculating the Structure and Chaos Due to Gravity in the Universe,"*Proceedings of the Eighth Annual ACM-SIAM Symposium on Discrete Algorithms,* SIAM, Philadelphia, 1997, pg. 1-10.

About TierneyLab

"John Tierney always wanted to be a scientist but went into journalism because its peer-review process was a great deal easier to sneak through. Now a columnist for the Science Times section, Tierney previously wrote columns for the Op-Ed page, the Metro section and the Times Magazine. Before that he covered science for magazines like Discover, Hippocrates and Science 86.

With your help, he's using TierneyLab to check out new research and rethink conventional wisdom about science and society. The Lab's work is guided by two founding principles:

- *Just because an idea appeals to a lot of people doesn't mean it's wrong.*
- *But that's a good working theory.*

From **http://tierneylab.blogs.nytimes.com**.

"I don't think of myself as having gone squishy. I think of myself as having grown sober. And my conservative critics? On them, I

think the most apt verdict was delivered by Niccolo Macchiavelli, 500 years ago: "This is the tragedy of man. Circumstances change, and he does not."

(David Frum)

- From "Lies about me, and the lying liars who tell them," National Post, March 28, 2009. Compare various of quotes above by Keynes and those of some of the many bankers and economists who are now suffering buyer's remorse.

"The late Huw Wheldon of the BBC once described to me a series, made in the early days of radio, about celebrated exiles who had lived in London. At one stage, this had involved tracking down an ancient retiree who had toiled in the British Museums reading room during the Victorian epoch. Asked if he could remember a certain Karl Marx, the wheezing old pensioner at first came up empty. But when primed with different prompts about the once-diligent attendee (monopolizing the same seat number, always there between opening and closing time, heavily bearded, suffering from carbuncles, tending to lunch in the Museum Tavern, very much interested in works on political economy), he let the fount of memory be unsealed.Oh Mr. Marx, yes, to be sure. Gave us a lot of work e did, with all is calls for books and papers. His interviewers craned forward eagerly, to hear the man say: And then one day e just stopped coming. And you know whats a funny fing, sir? A pregnant pause. Nobodys ever eard of im since! This, clearly, was one of those stubborn proletarians for the alleviation of whose false consciousness Marx had labored in vain.

(Christopher Hitchens)

- In the *The Revenge of Karl Marx* in **The Atlantic**, April 2009.

Here endeth the Seder.

This year our ceremony still contains some time for reflection, and some ability to remain on the same topic for more than a minute or two. But next year, may our ceremony be faster, divided into bite-sized chunks, and with each utterance no more than 140 characters. And so we say together,

NEXT YEAR IN TWITTER.

(Carl Elkin, 2009)

- From **A Facebook Haggadah**.

"If you're worried that lions are eating too many zebras, you don't say to the lions, 'You're eating too many zebras.' You have to build a fence around the lions. They're not going to build it."

(Judge Richard A. Posner)

- *"One of the most prominent proponents of free-market capitalism is having second thoughts"* in **Huffington Post**, April 20, 2009. A week earlier he wrote in "Shorting Reason" (*The New Republic* of April 15):

> *"They want a pedigree, or a sacred text, to lend authority to their thesis, and they want to champion the liberal Keynes over the conservative Friedman.*
>
> *Hence their appropriation of the term "animal spirits" from a famous passage in The General Theory "Most, probably, of our decisions to do something positive, the full consequences of which will be drawn out over many days to come, can only be taken as a result of animal spirits--of a spontaneous urge to action rather than inaction, and not as the outcome of a weighted average of quantitative benefits multiplied by quantitative probabilities.... Thus if the animal spirits are dimmed and the spontaneous optimism fades, enterprise will fade and die.... It is our innate urge to activity which makes the wheels go round, our rational selves choosing between the alternatives as best we are able, calculating where we can, but often falling back for our motive on whim or sentiment or chance."*

"John Maynard Keynes wrote that ideas, “both when they are right and when they are wrong, are more powerful than is commonly understood. Indeed the world is ruled by little else.” This idea popularized by Professor Singer — that we have ethical obligations that transcend our species — is one whose time appears to have come."

(Nicholas Kristof)

- in **Humanity Even for Nonhumans** (for better or worse) in NYT, April 8 2009.

"Maddox was always a believer in the possibilities of science, reluctant to accept that it could cause problems as well as solve them. When a wave of environmental pessimism swept over the Western world in the early 1970s he was one of the few to resist. He published a book, The Doomsday Syndrome (1972), *denouncing the gloom as overdone.*
...
after retiring as Nature Editor he wrote a scientific tour d’horizon, What Remains to be Discovered, *asserting that far from approaching the end of its glorious run, science was only just*

beginning to tackle a multitude of new problems. The future offered an infinity of possibilities, most of them attractive."

- From the *London Times* obituary of **John Maddox** (1925-2009).

"**"6. We have a patriotic duty to stand up against Washington taxes!"** *Just the opposite. We have a patriotic duty to pay taxes. As multi-billionaire Warren Buffett put it, "If you stick me down in the middle of Bangladesh or Peru or someplace, you'll find out how much this talent is going to product in the wrong kind of soil. I will be struggling thirty years later. President Teddy Roosevelt made the case in 1906 when he argued in favor of continuing the inheritance tax. "The man of great wealth owes a particular obligation to the state because he derives special advantages from the mere existence of government.""*

(Robert Reich)

- From "A Short Citizen's Guide to Kooks, Demagogues, and Right-Wingers," in the **Huffington Post** April 15 (Tax Day).

"The most complete unfolding of his later sense of things can probably be found in a quite astonishing book-length interview published by the magazine Research as the self-standing Research No 8/9 (1984) *but he remained unfailingly eloquent until the end of his life, as the interviews assembled in* Conversations (2005) *attest. "At times", he said in 2004, "I look around the executive housing estates of the Thames Valley and feel that [a vicious and genuinely mindless neo-fascism] is already here, quietly waiting its day, and largely unknown to itself ... What is so disturbing about the 9/11 hijackers is that they had not spent the previous years squatting in the dust on some Afghan hillside ... These were highly educated engineers and architects who had spent years sitting around in shopping malls in Hamburg and London, drinking coffee and listening to the muzak."*

(The Independent)

- April 21 **Obituary of JG Ballard**.

"A heavy warning used to be given [by lecturers] that pictures are not rigorous; this has never had its bluff called and has permanently frightened its victims into playing for safety. Some pictures, of course, are not rigorous, but I should say most are (and I use them whenever possible myself)."

(J. E. Littlewood, 1885-1977)

From *Littlewood's Miscellany* (p 35 in 1953 edition). Said long before the current graphic, visualization and geometric tools were available.

Assorted Americans on Paris (collected in 2003)

"France has neither winter nor summer nor morals. Apart from these drawbacks it is a fine country. France has usually been governed by prostitutes." --Mark Twain.

"I would rather have a German division in front of me than a French one behind me."---General George S. Patton.

"Going to war without France is like going deer hunting without your accordion."---Secretary of Defense Donald Rumsfeld.

"We can stand here like the French, or we can do something about it."---Marge Simpson.

"As far as I'm concerned, war always means failure."---Jacques Chirac, President of France.

"As far as France is concerned, you're right."---Rush Limbaugh.

"The only time France wants us to go to war is when the German Army is sitting in Paris sipping coffee."---Regis Philbin.

"The French are a smallish, monkey-looking bunch and not dressed any better, on average, than the citizens of Baltimore.

True, you can sit outside in Paris and drink little cups of coffee, but why this is more stylish than sitting inside and drinking large glasses of whisky I don't know."---P.J O'Rourke (1989).

"You know, the French remind me a little bit of an aging actress of the 1940s who was still trying to dine out on her looks but doesn't have the face for it."---John McCain, U.S. Senator from Arizona.

"You know why the French don't want to bomb Saddam Hussein? Because he hates America, he loves mistresses, and wears a beret. He is French, people."---Conan O'Brien.

"I don't know why people are surprised that France won't help us get Saddam out of Iraq. After all, France wouldn't help us get Hitler out of France either."---Jay Leno.

"The last time the French asked for 'more proof' it came marching into Paris under a German flag." ---David Letterman

"Only thing worse than a Frenchman is a Frenchman who lives in Canada."---Ted Nugent.

"The favorite bumper sticker in Washington D.C. right now is one that says, 'First Iraq, then France.'"---Tom Brokaw.

"What do you expect from a culture and a nation that exerted more of its national will fighting against Disney World and Big Macs than the Nazis?"---Dennis Miller.

"It is important to remember that the French have always been there when they needed us."---Alan Kent.

"They've taken their own precautions against al-Qa'ida. To prepare for an attack, each Frenchman is urged to keep duct tape, a white flag, and a three-day supply of mistresses in the house."---Argus Hamilton.

"Somebody was telling me about the French Army rifle that was being advertised on eBay the other day--the description was, 'Never shot. Dropped once.'"---Rep. Roy Blunt (MO)).

"The French will only agree to go to war when we've proven we've found truffles in Iraq."---Dennis Miller.

"**Question:** What did the mayor of Paris say to the German army as they entered the city in WWII?

Answer: *Table for cent milles m'sieur?"*

"Do you know how many Frenchmen it takes to defend Paris? It's not known, it's never been tried."---Rep. R. Blount (MO).

"Do you know it only took Germany three days to conquer France in WWII? And that's because it was raining."--John Xereas, Manager, DC Improv.

"The AP and UPI reported that the French Government announced after the London bombings that it has raised its terror alert level from Run to Hide. The only two higher levels in France are Surrender and Collaborate."

" The rise in the alert level was precipitated by a recent fire which destroyed France's white flag factory, effectively disabling their military."

"French Ban Fireworks at Euro Disney (AP), Paris, March 5, 2003, The French government announced today that it is imposing a ban on the use of fireworks at Euro Disney. The decision comes the day after a nightly fireworks display at the park, located just 30 miles outside of Paris, caused the soldiers at a nearby French army garrison to surrender to a group of Czech tourists."

"Roberts's opinion drew an incredulous dissent from Stevens, who said that the Chief Justice's words reminded him of "Anatole France's observation" that the "majestic equality" of the law forbade "rich and poor alike to sleep under bridges, to beg in the streets, and to steal their bread.""

(Jeffrey Toobin)

- Anotole France's famous observation in an incisive if depressing analysis of the Chief Justice: *No More Mr. Nice Guy* in the **New Yorker**.

"*During the three years which I spent at Cambridge my time was wasted, as far as the academical studies were concerned, as completely as at Edinburgh and at school. I attempted mathematics, and even went during the summer of 1828 with a private tutor (a very dull man) to Barmouth, but I got on very slowly. The work was repugnant to me, chiefly from my not being able to see any meaning in the early steps in algebra.This impatience was very foolish, and in after years I have deeply regretted that I did not proceed far enough at least to understand something of the great leading principles of mathematics, for men thus endowed seem to have an extra sense. "*

(Charles Darwin)

- From the **Autobiography of Charles Darwin**.

"*He made little in public of his famous grandfather, Sigmund, who in 1938 followed other members of his family in migrating to Britain beginning in 1933, the year Hitler came to power — "refugees from the Nazis before the habit caught on," as Sir Clement, a secular Jew like many in his family, said many years later. He said he remembered his grandfather, who died in London in 1939, mostly as a faltering old man with oral* *cancer. “He was not, to me, famous,” he said, but rather “a good grandfather in that he didn’t forget my birthdays.”*

"[He] had a testy relationship with his older brother Lucian, the artist, now 86, who is regarded by many critics as one of the greatest Realists of the past century. Late in life Sir Clement told The Observer newspaper he had no interest in reconciling with his brother. “I’m not great at forgiving,” he said.“If I decide I don’t like someone, that’s it.”

(John F. Burns)

- In "Clement Freud, Wit, Politician and Grandson of Famous Psychoanalyst, Dies at 84", **NYT, April 16, 2009**.

Three "laws" of prediction

- *"When a distinguished but elderly scientist states that something is possible, he is almost certainly right. When he states that something is impossible, he is very probably wrong.*
- *"The only way of discovering the limits of the possible is to venture a little way past them into the impossible.*
- *"Any sufficiently advanced technology is indistinguishable from magic."*

(Arthur C. Clarke)

- **From Wikpedia**

"The first of the three laws, previously termed Clarke's Law, was proposed by Arthur C. Clarke in the essay "Hazards of Prophecy: The Failure of Imagination", in Profiles of the Future (1962). *The second law is offered as a simple observation in the same essay; its status as Clarke's Second Law was conferred on it by others. In a 1973 revision of his compendium of essays,* Profiles of the Future, *Clarke acknowledged the Second Law and proposed the Third in order to round out the number, adding "As three laws were good enough for Newton, I have modestly decided to stop there." Of the three, the Third Law is the best known and most widely cited.* [It was used by JPL reporting on gravitational boosting]"

"It was because Hopkins’s superiors in England had so little use for him…that they encouraged him to take a position as Professor of Greek and Examiner in Classics

at the Royal University of Ireland, in Dublin. This prestigious-sounding post actually involved teaching elementary Latin and grading a truly staggering number of tests: six examinations times seven hundred and fifty students, according to Hopkins, for a total of forty-five hundred papers every year.

"Such was the life of Gerard Manley Hopkins, who fortuntely was able to write a little poetry amidst all that grading. His lament about this predicament has its own poetic quality:

"From the college, he issues a series of increasingly desperate cries for help. "The melancholy I have all my life been subject to has become in late years not indeed more intense in its fits but rather more distributed, constant, and crippling."

(Adam Kirsch)

- Review of of **"Gerard Manley Hopkins"** by Paul Mariani in the New Yorker May 11, 2009.

"Such reversals have led the veteran Silicon Valley technology forecaster Paul Saffo to proclaim: "never mistake a clear view for a short distance.""

(John Markoff)

- A look at Strong AI being back in style in **The Coming Superbrain** NYT, May 23, 2009.

"These aspects of exploratory experimentation and wide instrumentation originate from the philosophy of (natural) science and have not been much developed in the context of experimental mathematics. However, I claim that e.g. the importance of wide instrumentation for an exploratory approach to experiments that includes concept formation also pertain to mathematics."

(Hendrik Sorenson)

- From his 2008 preprint "How Experimental is *Experimental* Mathematics?" discussing Franklin's argument that Steinle's notion of ``*exploratory experimentation"* facilitated by ``*widening technology"* (as in pharmacology, astrophysics, medicine, and biotechnology) is leading to a reassessment of what legitimates experiment; in that even a ``*local model"* is not now prerequisite.

Relatedly, as Dave Bailey and I **wrote recently**

> In a provocative 2008 article entitled "**The End of Theory:"** The Data Deluge Makes the Scientific Method Obsolete" Chris Anderson, the Editor-in-Chief of *Wired*, heralds a new mode of scientific inquiry where exploding repositories of data, analyzed using advanced mathematical and statistical techniques in the same manner as Google has analyzed the Internet, are sufficient to render the traditional scientific method (hypothesize, model, test) obsolete:
>
>> *"The new availability of huge amounts of data, along with the statistical tools to crunch these numbers, offers a whole new way of understanding the world. Correlation supersedes causation, and science can advance even without coherent models, unified theories, or really any mechanistic explanation at all. There's no reason to cling to our old ways. It's time to ask: What can science learn from Google?''*
>
> Kevin Kelly, in a **response** to Anderson's article, makes a more modest statement:
>
>> *"My guess is that this emerging method will be one additional tool in the evolution of the scientific method. It will not replace any current methods (sorry, no end of science!) but will complement established theory-driven science. ... The model may be beyond the perception and understanding of the creators of the system, and since it works it is not worth trying to uncover it. But it may still be there. It just operates at a level we don't have access to.''*

And it may not be there in some circumstances; both in mathematics and in what we properly call reality.

> "*The empirical spirit on which the Western democratic societies were founded is currently under attack, and not just by such traditional adversaries as religious fundamentalists and devotees of the occult. Serious scholars claim that there is no such thing as progress and assert that science is but a collection of opinions, as socially conditioned as the weathervane world of Paris couture. "*

(Timothy Ferris)

From *The Whole Shebang: A State of the Universe(s) Report*, Simon and Shuster, 1998, pg. 1.

> "*My larger target is those contemporaries who—in repeated acts of wish-fulfillment—have appropriated conclusions from the philosophy of science and put them to work in aid of a variety of social cum political causes for which those conclusions are ill adapted. Feminists, religious apologists (including "creation scientists"), counterculturalists, neo-conservatives, and a host of other curious fellow-travelers have claimed to find crucial grist for their mills in, for instance, the avowed incommensurability and underdetermination of scientific theories. The displacement of the idea that facts and evidence matter by the idea that everything boils down to subjective interests and perspectives is—second only to American political campaigns—the most prominent and pernicious manifestation of anti-intellectualism in our time.*"

(Larry Laudan)

From *Science and Relativism*, University of Chicago Press, 1990, pg. x.

> "*So to summarise, according to the citation count, in order of descent, the authors are listening to themselves, dead philosophers, other specialists in semiotic work in mathematics education research, other mathematics education research researchers and then just occasionally to social scientists but almost never to other education researchers, including mathematics teacher education researchers, school teachers and teacher educators. The engagement with Peirce is being understood primarily through personal engagements with the original material rather than as a result of working through the filters of history, including those evidenced within mathematics education research reports in the immediate area. The reports, and the hierarchy of power relations implicit in them, marginalise links to education, policy implementation or the broader social sciences.*"

(Tony Brown)

From *"Signifying "students", "teachers" and "mathematics": a reading of a special issue* Published online: 28 May 2008, Springer Science + Business Media B.V. 2008.

Enter Don Tapscott, who is looking at the challenges the digital revolution poses to the fundamental aspects of the University.

"Universities are finally losing their monopoly on higher learning", he writes. *"There is fundamental challenge to the foundational modus operandi of the University — the model of pedagogy. Specifically, there is a widening gap between the model of learning offered by many big universities and the natural way that young people who have grown up digital best learn."*

The old-style lecture, with the professor standing at the podium in front of a large group of students, is still a fixture of university life on many campuses. It's a model that is teacher-focused, one-way, one-size-fits-all and the student is isolated in the learning process. Yet the students, who have grown up in an interactive digital world, learn differently. Schooled on Google and Wikipedia, they want to inquire, not rely on the professor for a detailed roadmap. They want an animated conversation, not a lecture. They want an interactive education, not a broadcast one that might have been perfectly fine for the Industrial Age, or even for boomers. These students are making new demands of universities, and if the universities try to ignore them, they will do so at their peril.

Contrary to Nicholas Carr's proposition that Google is making us stupid, Tapscott counters with the following:

My research suggests these critics are wrong. Growing up digital has changed the way their minds work in a manner that will help them handle the challenges of the digital age. They're used to multi-tasking, and have learned to handle the information overload. They expect a two-way conversation. What's more, growing up digital has encouraged this generation to be active and demanding enquirers. Rather than waiting for a trusted professor to tell them what's going on, they find out on their own on everything from Google to Wikipedia."

(Don Tapscott)

The Edge *describing his article* ***The impending demise of the university***.

Britain pays its penny to poke a stick at Susan Boyle

In one of the few commentaries written by a man, Thomas Sutcliffe at The Independent draws uncomfortable parallels with the treatment of the insane in the 18th century.

“You could pay a penny to visit Bedlam (or Bethlehem mental hospital) and chortle at the deranged. You were even allowed to poke them with a stick if they failed to caper or roar in a satisfactory way
...
and can ease our disquiet about the ethics of such a spectacle by reassuring ourselves that none of these people are under restraint. They choose to take part and, in so choosing, sign up to the loss of dignity that often comes with participation ... The novelty with Susan Boyle was that she sang well enough to get through to the final, elevating her from temporary comic relief into a real person whose health and well-being might arouse our protective sympathy. I doubt very much that she is the first participant to have been left in a state of anxiety by the stress and exposure of such programs, though she is probably the first person whose reaction has had any kind of widespread media coverage.”

(Araminta Wordsworth)

- From a **Financial Post compendium** on June 2, 2009.

Borwein's Five Laws of Travel

- **1. Distance Independence.** "It is an easy 15 minute walk" covers anything from 500 to 5000 metres.
- **2. Time Invariance.** You will learn all the relevant details of how to negotiate your host city and the like adequately, exactly twenty-four hours before your departure. This is independent of the length of your stay.
- **3. Universal Expressions.** Beware of such expressions as they have no fixed meaning. They include: "Free Internet," "Easy Access to Beach," and "Full Continental Breakfast."
- **4. Travel Agents.** Never travel with a travel agent who has never travelled. They will rarely make reasonable bookings and will often make infeasible ones.
- **5a. First Law of Directions.** Never rely on oral directions given in a foreign language. All consonants sound the same after one or two city blocks, while left and right are nearly always wrong and wronger.

5b. Second law of Directions. All directions written or oral given by a host will be missing one salient detail that is so obvious to any local as to be unrememberable. This applies to geography, computer access and much else.

(Jonathan Borwein)

- Based on decades of personal experience.

"And yet since truth will sooner come out of error than from confusion."

(Francis Bacon, 1561-1626)

- From *The New Organon* (1620) in *The Works of Francis Bacon*, James Spedding, Robert Ellis and Douglas Heath (eds.) (1887-1901), Vol. 4, p. 149.

"In closing, I offer two examples from economics of what I hope to have said. Marx said that quantitative differences become qualitative ones, but a dialogue in Paris in the 1920's sums it up even more clearly:
FITZGERALD: The rich are different from us.
HEMINGWAY: Yes, they have more money."

(Phillip Anderson)

- Writing in "More Is Different," *Science*, New Series, Vol. 177, No. 4047. (Aug. 4, 1972), pp. 393-396.

"Who ever became more intelligent," Gödel answered, *"by reading Voltaire?"*

"Only fables," he said, *"present the world as it should be and as if it had meaning."*

(Kurt Gödel)

- In Palle Yourgrau's, *A World Without Time,* Basic Books, 2005, p. 15 and p. 5 respectively.

"In all likelihood, our post-modern habit of viewing science as only a paradigm would evaporate if we developed appendicitis. We should look for a medically trained surgeon who knew what an appendix was, where it was, and how to cut it out without killing us. Likewise, we should be happy to debate the essentially

fictive nature of, let us say, Newton's Laws of Gravity unless and until someone threatened to throw us out of a top-storey window. Then the law of gravity would seem very real indeed."

(A. N. Wilson)

Quoted from *God's Funeral*, Norton, 1999, p. 178, in Richard C. Brown, *Are Science and Mathematics Socially Constructed?* World Scientific, 2009, p 207.

"Philosophical theses may still be churned out about it,

....

but the question of nonconstructive existence proofs or the heinous sins committed with the axiom of choice arouses little interest in the average mathematician. Like Ol' Man River, mathematics just keeps rolling along and produces at an accelerating rate "200,000 mathematical theorems of the traditional handcrafted variety ... annually." Although sometimes proofs can be mistaken---sometimes spectacularly---and it is a matter of contention as to what exactly a "proof" is---there is absolutely no doubt that the bulk of this output is correct (though probably uninteresting) mathematics."

(Richard C. Brown)

Brown is discussing constructivism and intuitionism in *Are Science and Mathematics Socially Constructed?* World Scientific, 2009, p 239.

A QUOTE BY ALBERT EINSTEIN When Paul Newman died, they said how great he was but they failed to mention he considered himself Jewish (born half-Jewish).

When Helen Suzman (who fought apartheid and helped Nelson Mandela) died recently, they said how great she was, but they failed to mention she was Jewish.

On the other side of the equation, when Ivan Boesky or Andrew Fastow or Bernie Madoff committed fraud, almost every article mentioned they were Jewish.

However, when Ken Lay, Jeff Skilling, Martha Stewart, Randy Cunningham, Gov. Edwards, Conrad Black, Senator Keating, Gov Ryan, and Gov Blagojevich messed up; no one reported what religion or denomination they were, because they were not Jewish.

All of this leads to a famous Einstein quote: In 1921, Albert Einstein presented a paper on his then-infant Theory of Relativity at the Sorbonne, the prestigious French university.

> *"If I am proved correct,"* he said,
> *"the Germans will call me a German,*
> *the Swiss will call me a Swiss citizen,*
> *and the French will call me a great scientist".*
>
> *"If relativity is proved wrong,*
> *the French will call me a Swiss,*
> *the Swiss will call me a German,*
> *and the Germans will call me a Jew."*

(anon@anon.com, July 2009)

For another funny-serious reflexion on like matters see Krauthammer's Law (***everyone is Jewish until proven otherwise***) extended to ***we are all Jews now***.

> It was time to leave, but not before raising one last subject. Obama started his campaign in the shadow of the Old State Capitol in Springfield, Ill., where Lincoln had delivered his famous "House Divided" speech, warning that the nation could not survive half-slave and half-free. Now, as he prepared to return to Washington, his transition team had announced plans for him to follow the last part of Lincoln's train ride to Washington before his inauguration. We wondered how Lincoln, an Illinois lawyer with little national experience, affected Obama's thoughts about his own presidency as another young Illinois lawyer with limited national experience soon to take his oath of office.
>
> *"Lincoln's my favorite president and one of my personal heroes,"* he answered. *"I have to be very careful here that in no way am I drawing equivalence between my candidacy, my life experience, or what I face and what he went through. I just want to put that out there so you don't get a bunch of folks saying I'm comparing myself to Lincoln."*
>
> He paused. *"What I admire so deeply about Lincoln -- number one, I think he's the quintessential American because he's self-made. The way Alexander Hamilton was self-made or so many of our great iconic Americans are, that sense that you don't accept limits, that you can shape your own destiny. That obviously has appeal to me, given where I came from. That American spirit is one of the things that is most fundamental to me, and I think he embodies that.*

"But the second thing that I admire most in Lincoln is that there is just a deep-rooted honesty and empathy to the man that allowed him to always be able to see the other person's point of view and always sought to find that truth that is in the gap between you and me. Right? That the truth is out there somewhere and I don't fully possess it and you don't fully possess it and our job then is to listen and learn and imagine enough to be able to get to that truth.

"If you look at his presidency, he never lost that. Most of our other great presidents, there was that sense of working the angles and bending other people to their will. FDR being the classic example. And Lincoln just found a way to shape public opinion and shape people around him and lead them and guide them without tricking them or bullying them, but just through the force of what I just talked about: that way of helping to illuminate the truth. I just find that to be a very compelling style of leadership.

"It's not one that I've mastered, but I think that's when leadership is at its best."

(Barak Obama, 2008)

- From **'The Battle for America 2008: The Story of an Extraordinary Election'** By Dan Balz and Haynes Johnson, Washington Post, Friday, July 31, 2009. How well will this mesh with general perceptions in 2012?

"Progress had always been made, but the nature of the progress could never be divulged."

(Franz Kafka)

- From **The Trial** page 138.

"*Bean, who had said of Monash* ***"We do not want Australia represented by men mainly because of their ability, natural and inborn in Jews, to push themselves",*** *conspired with Keith Murdoch to undermine Monash, and have him removed from the command of the Australian Corps. They misled Prime Minister Billy Hughes into believing that senior officers were opposed to Monash. Hughes arrived at the front before the Battle of Hamel prepared to replace Monash, but after consulting with senior officers, and after seeing the superb power of planning and execution displayed* *by Monash, he changed his mind.*"

(Wikipedia)

- From **Wikipedia** entry on John Monash.

"*Almodóvar was vague, saying,* ***''Everything that isn't autobiographical is plagiarism.''***"

(Lynn Hirschberg)

- From an NYT **interview** with Pedro Almodóvar on Sept 5, 2004.

"*As Aldous Huxley opined,* ***the strict materialist cannot yet derive Shakespeare from the advanced biochemistry of mutton.***"

(Richard Gallagher)

- From letters on Nicholas Wade's review of Richard Dawkin's **Greatest Show on Earth**.

Understanding Human Origins

"*Responding to a question about his soon-to-be-published On the Origin of Species, Charles Darwin wrote in 1857 to Alfred Russel Wallace,* ***"You ask whether I shall discuss 'man'; I think I shall avoid the whole subject, as so surrounded with prejudices, though I freely admit that it is the highest and most interesting problem for the naturalist."*** *Only some 14 years later, in The Descent of Man, did Darwin address this "highest problem" head-on: There, he presciently remarked in his introduction that* ***"It has often and confidently been asserted, that man's origin can never be known: but ... it is those who know little, and not those who know much, who so positively assert that this or that problem will never be solved by science.*** "

(Bruce Alberts)

- Editorial in **Science** 2 October 2009: Vol. 326. no. 5949, p. 17.

"*One mathematician rushes into the office of another and says 'Have you got a minute? I am a bit stuck on this problem. You see ... [goes on for many minutes*

explaining details] ... ah! Thanks very much!' and leaves; the colleague has said nothing, and has not needed to say anything. This behaviour is quite typical.
"

(John Mason)

On page 127 of "Learning from Listening to yourself" in *Listening Figures*, Trentham Books, 2009.

"*In Farrell v. Burke, Sotomayor, resisting the temptation to wax about the First Amendment, chose simply to include the following exchange from the testimony of a police officer who had charged a convicted sex offender for violating the terms of his probation by possessing obscene materials:*

MR. NATHANSON: Ae you saying, for example, that that condition of parole would prohibit Mr. Farrell from possessing, say, Playboy magazine?
P.O. BURKE: Yes.
MR. NATHANSON: Are you saying that that condition of parole would prohibit Mr. Farrell from possessing a photograph of Michelangelo['s] David?
P.O. BURKE: What is that?
MR. NATHANSON: Are you familiar with that sculpture?
P.O. BURKE: No.
MR. NATHANSON: If I tell you it's a large sculpture of a nude youth with his genitals exposed and visible, does that help to refresh your memory of what that is? P.O. BURKE: If he possessed that, yes, he would be locked up for that.

Still, Sotomayor ruled that Farrell had violated his parole. ***"Although a series of strongly worded opinions by this Court and others suggest that the term 'pornography' is unconstitutionally vague, we hold that 'Scum' falls within any reasonable definition of pornography," she wrote.*** "

(Lauren Collins)

From **The Life of Sonia Sotomayor**. The *New Yorker*, Jan 11, 2010.

"And so Einstein and his new wife, Elsa, set sail in late March 1921 for their first visit to America. On the way over, Einstein tried to explain relativity to Weizmann. Asked upon their arrival whether he understood the theory, Weizmann gave a puckish reply: ***"Einstein***

> ***explained his theory to me every day, and by the time we arrived I was fully convinced that he really understands it.”*** "

(Walter Isaacson)

From **How Einstein Divided America's Jews** in the *Atlantic*, December 2009.

> *“That is a brute, cold, hard fact of the universe. When you pull the battery out of your computer, it shuts off. When you end a life, it shuts off. And I think that’s just it.”*

(Brian Greene)

From **The Listener** by Timothy Lavin in the *Atlantic*, January 2010.

> *“Why can't people just have complex views about food without resorting to extremist ideas that both fit as fashions and act as cure-all's for the health of America? Eating and nutrition are complex algorithms to get right! Michael Pollan knows this, because he wrote a great book [In Defense of Food] with a great mantra —**Eat food. Not too much. Mostly plants.**— that stood in the middle. And guess what happened? He heard from every asshole with a fully organic nightshade garden or a meat locker of terror in their brownstone because he wasn't on one side or the other:*
>
> *The adverb "mostly" has been the most controversial. It makes everybody unhappy. The meat people are really upset I'm taking a swipe at meat eating, and the vegetarians are saying, "What's with the ‘mostly?' Why not go all the way?" You can't please everyone. In a way that little word is the most important. It's not all or nothing. Mostly. It's about degree.”*

(Foster Kamer)

A sensible sentiment from **Of Early Birds and Cavemen: The Two Dumbest Hipster Food Trends You'll Read About This Week** in the *Gawker*.

> *“Initiations are welcome, of course, but we do not give children a high school diploma simply for showing up for school on the first day of the first grade. For the same reasons **"born-again" moral characters** should*

probably wait a similar period of time before celebrating their moral achievment or pressing their moral authority."

(Paul Churchland)

- From *Neurophilosophy at Work*, Cambridge University Press 2007 (Locations 10199-25 of the Kindle version).

"Now my mum had no interest whatsoever in science, and I was forever trying to explain to her why, for instance, people in Australia did not fall off the other side of the world. So when I arrived at Caltech, I had an idea: plucking up my courage, I knocked on Feynman's office door and asked, nervously, whether he would write to my mum.

He did. ***"Dear Mrs Chown,""*** *he wrote.* ***"Please ignore your son's attempts to teach you physics. Physics is not the most important thing. Love is. Richard Feynman.""*** *"*

(Marcus Chown)

- From ***Quantum theory via 40-tonne trucks: How science writing became popular*** in the *Independent* January 17, 2010.

Serendipitous Astronomy "Many of the seminal discoveries in astronomy have been unanticipated."

"So our celestial science seems to be primarily instrument-driven, guided by unanticipated discoveries with unique telescopes and novel detection equipment. With our current knowledge, we can be certain that the observed universe is just a modest fraction of what remains to be discovered. Recent evidence for dark, invisible matter and mysterious dark energy indicate that the main ingredients of the universe remain largely unknown, awaiting future, serendipitous discoveries."

(Kenneth R. Lang)

- More evidence for "Exploratory Experimentation and Widening Technology" from *Science*, 1 January 2010: Vol. 327. no. 5961, pp. 39-40. DOI: 10.1126/science.1183653.

"Even mathematics would not be entirely safe. (Apparently, in the early 1900's, one legislator in a southern state proposed a bill to redefine the value of

pi as 3.3 exactly, just to tidy things up.)"

(Paul Churchland)

Writing about the creationist sagas of the Kansas school board in *Neurophilosophy at Work* (Cambridge, 2007); at **location 1589** of the *Kindle edition*. This is a fascinating set of essays and full of interesting anecdotes --- which I have no particular reason to doubt --- but this one quote contains four inaccuracies.

> (i) The event took place in the 1897 (ii) in Indiana (a northern state). (iii) The prospective bill (#246) offers a geometric construction with inconsistent conclusions and certainly offers no exact value. Finally, (iv) the intent seems to have been pecuniary not hygenic. See "The legal values of Pi" by David Singmaster, *Math Intelligencer*, **7**, (1985), 69-72. (Also in *Pi, a Sourcebook*, by Borwein, Borwein and Berggren.)

As often this makes me wonder whether mathematics popularization is especially prone to error or if the other disciplines just seem better described because of my relative ignorance.

> *"The spread of information networks is forming a new nervous system for our planet.*
>
> *Now, in many respects, information has never been so free. There are more ways to spread more ideas to more people than at any moment in history. And even in authoritarian countries, information networks are helping people discover new facts and making governments more accountable.*
>
> *...*
>
> *Because amid this unprecedented surge in connectivity, we must also recognize that these technologies are not an unmitigated blessing. These tools are also being exploited to undermine human progress and political rights. Just as steel can be used to build hospitals or machine guns, or nuclear power can either energize a city or destroy it, modern information networks and the technologies they support can be harnessed for good or for ill. The same networks that help organize movements for freedom also enable al-Qaida to spew hatred and incite violence against the innocent. And technologies with the potential to open up access to government and promote transparency can also be hijacked by governments to crush dissent and deny human rights.*

...

In the last year, we've seen a spike in threats to the free flow of information. China, Tunisia, and Uzbekistan have stepped up their censorship of the internet. In Vietnam, access to popular social networking sites has suddenly disappeared. And last Friday in Egypt, 30 bloggers and activists were detained."

(Hilary Rodham Clinton)

- **Remarks on Internet Freedom** January 20, 2010 at the Newseum. See also James Fallows' **percipient analysis**.

  To Clinton's list of ills one should add "**antisocial networking**" of the kind that allows **Bill Deagle** and thousands of other sociopaths or madmen to exploit the weaknesses of others. Much of this---from newage medicine to "truthers, birthers, teabaggers," Alex Jones and worse---sadly is **now being cultivated** by the "mainstream" right.

"This is not to say that I am not interested in the quest for intelligent machines. My many exhibitions with chess computers stemmed from a desire to participate in this grand experiment. ***It was my luck (perhaps my bad luck) to be the world chess champion during the critical years in which computers challenged, then surpassed, human chess players.*** *Before 1994 and after 2004 these duels held little interest. The computers quickly went from too weak to too strong. But for a span of ten years these contests were fascinating clashes between the computational power of the machines (and, lest we forget, the human wisdom of their programmers) and the intuition and knowledge of the grandmaster."*

...

"Perhaps the current trend of many chess professionals taking up the more lucrative pastime of poker is not a wholly negative one. It may not be too late for humans to relearn how to take risks in order to innovate and thereby maintain the advanced lifestyles we enjoy. And if it takes a poker-playing supercomputer to remind us that we can't enjoy the rewards without taking the risks, so be it."

(Gary Kasparov)

- In **The Chess Master and the Computer** a review of *Chess Metaphors: Artificial Intelligence and the Human Mind* by Diego

Rasskin-Gutman. (In the New York Review of Books Volume 57, Number 2 · February 11, 2010.)

> *"The reference to Tokyo Rose was probably lost on many of Justice Stevens's readers. But the concluding sentence of what may be his last major dissent could not have been clearer.*
>
> *"While American democracy is imperfect," he wrote, "**few outside the majority of this court would have thought its flaws included a dearth of corporate money in politics.**"*
>
> (Adam Liptak)

In **After 34 Years, a Plainspoken Justice Gets Louder** NYT Jan 26, 2010.

> *"Only two years ago, Jobs contemptuously predicted that the Kindle would flop: **"It doesn't matter how good or bad the product is,"** he told The New York Times, because **"the fact is that people don't read anymore. Forty percent of the people in the U.S. read one book or less last year. The whole conception is flawed at the top because people don't read anymore."***
>
> (Alan Deutschman)

In **Steve Jobs: Flip-Flopper**, *Daily Beast* of Jan 26, 2010.

Cut This Story!

> *"There's an old joke about the provincial newspaper that reports a nuclear attack on the nation's largest city under the headline "Local Man Dies in NY Nuclear Holocaust." Something similar happens at the national level, where everything is filtered through politics. (**"In what was widely seen as a setback for Democrats just a year before the midterm elections, nuclear bombs yesterday obliterated seven states, five of which voted for President Obama in the last election ..."**)"*
>
> (Michael Kinsley)

Writing instructively in the *Atlantic* (Feb-March 2010) about the fact that **"Newspaper articles are too long"** and massively formulaic.

“I started drinking the Kool-Aid so long ago that I can no longer taste it. I am sure I will continue my unbroken streak of mindless devotion to Apple and find a way to love the iPad, no matter how expensive and unnecessary it is. Knowledge of self is no fun."

(Sasha Frere-Jones)

- One of many entertaining snippets in the *New Yorker*'s survey of their staffers immediate responses to the **IPAD** (Jan 27, 2010).

---

*“As Garry Trudeau (who is not on Twitter) has his Washington “journotwit” Roland Hedley tweet at the end of “My Shorts R Bunching. Thoughts?,” ... **“The time you spend reading this tweet is gone, lost forever, carrying you closer to death. Am trying not to abuse the privilege.”***

(George Packer)

- From **Neither Luddite nor Biltonite** February 4, 2010. One can google "Biltonite".

---

*“Emerson was a touchstone, and Salinger often quoted him in letters. For instance, ``**A man must have aunts and cousins, must buy carrots and turnips, must have barn and woodshed, must go to market and to the blacksmith’s shop, must saunter and sleep and be inferior and silly.**" Writers, he thought, had trouble abiding by that, and he referred to Flaubert and Kafka as “two other born non-buyers of carrots and turnips."*

(Lillian Ross)

- The distinguished editor, journalist, and author on **My long friendship with J. D. Salinger**.

---

*“Fifth, society is too transparent. Since Watergate, we have tried to make government as open as possible. But as William Galston of the Brookings Institution jokes, **government should sometimes be shrouded for the same reason that middle-aged people should be clothed.** This isn’t Galston’s point, but I’d observe that the more government has become transparent, the less people are inclined to trust it."*

(David Brooks)

- Five observations on the growing insensitivity of *The Power Elite* to societal pressures in the **NYT** February 18, 2010.

> *"My specific aims didn't have `discover telomerase'. I didn't even know I wanted to discover telomerase," she said."*

(Elizabeth Blackburn)

- The Australian 2009 Nobelist discussing research in **The Australian** of February 24, 2010. In a followup piece **on collaboration** she comments:

> *"**My feeling is not to get too cross-disciplinary and shallow and spread all over the place too quick.**" Blackburn tells the HES while visiting Monash University, where she is a distinguished visiting professor.*
>
> *"**One needs to be able to bring something very substantive to the table** because I can see the temptation would be to try to be overly generalised and shallowness would be the consequence."*

This is an opinion I've been expressing: see **Innovation and Creativity**.

> *"**Math came naturally to Martin, and he sought sports with similar elements, anything with angles, geometry, calculations.** He smacked his first pool ball the day he could see over the table. He played billiards for hours at the local senior center, and after the employees there grew tired of unlocking the door at odd times, they made him a key.*
>
> *Martin idolized Ed Lukowich, a champion curler out of Calgary, Alberta. Martin loved the smooth delivery, the flawless mechanics. Lukowich wrapped math into curling's motions.*
>
> *Opponents describe Martin the same way, as a master craftsman, calm, certain, a skip with all the angles, a bald man with a bald eagle's eyesight. **He attacks his sport with a farmer's sensibility and a mathematician's wit.**"*

(Greg Bishop)

-

See **The Pride of Canada, Especially the Grandmas** -- an article about Canadian Olympic curling skipper Kevin Martin (K-Mart) in the *NYT* of Feb 25, 2010.

> *"My name is Odd-Bjoern Hjelmeset. I skied the second lap and I fucked up today. I think I have seen too much porn in the last 14 days. I have the room next to Petter Northhug and every day there is noise in there. So I think that is the reason I fucked up. By the way, Tiger Woods is a really good man."*

(Odd-Bjoern Hjelmeset)

Proving that some athletes are still kids. He did also manage to share a silver relay medal. See **Norwegian skier walks away with Quote of the Games**. (February 25, 2010)

> *"Writing in a 2005 Wired article that "new technologies redefine us," William Gibson hailed audience participation and argued that "an endless, recombinant, and fundamentally social process generates countless hours of creative product." Indeed, he said, "audience is as antique a term as record, the one archaically passive, the other archaically physical. The record, not the remix, is the anomaly today. The remix is the very nature of the digital."*
>
> *To Mr. Lanier, however, the prevalence of mash-ups in today's culture is a sign of "nostalgic malaise." "Online culture," he writes, "is dominated by trivial mash-ups of the culture that existed before the onset of mash-ups, and by fandom responding to the dwindling outposts of centralized mass media. It is a culture of reaction without action."*
>
> *He points out that much of the chatter online today is actually "driven by fan responses to expression that was originally created within the sphere of old media," which many digerati mock as old-fashioned and passé, and which is now being destroyed by the Internet.* ***"Comments about TV shows, major movies, commercial music releases and video games must be responsible for almost as much bit traffic as porn," Mr. Lanier writes. "There is certainly nothing wrong with that, but since the Web is killing the old media, we face a situation in which culture is effectively eating its own seed stock."*** *"*

(Michiko Kakutani)

From **Texts Without Context**, NYT, March 21, 2010.

'Bonkers' Crochet book knits up oddest title prize

"*This year's Diagram Prize for oddest book title has gone to Crocheting Adventures with Hyperbolic Planes, by mathematician Daina Taimina.*

The 32nd annual award, which carries no monetary reward, was announced late Friday by The Bookseller, a U.K. trade magazine.

"I've never won any prizes before. This is my first prize and it's wonderful," said Taimina, who teaches at Cornell University in Ithaca, New York.

The book details how Taimina uses crochet to create hyperbolic planes in which lines curve away from each other instead of running parallel. Her pieces look like complex flowers.

"These are two-dimensional objects which you can see only in three dimensions," explains Taimina.

Philip Stone, an editor with The Bookseller, said the professor's book won because "very simply, the title is completely bonkers."

"On the one hand you have the typically feminine, gentle and woolly world of needlework and on the other, the exciting but incredibly un-woolly world of hyperbolic geometry and negative curvature ... the two worlds collide in a captivating and quite breathtaking way," Stone said in a statement."

(CBC Arts)

The second and third-place finishers were: What Kind of Bean is This Chihuahua? and Collectible Spoons of the Third Reich.

Others in the running include: Afterthoughts of a Worm Hunter. Governing Lethal Behavior in Autonomous Robots. The Changing World of Inflammatory Bowel Disease. Last year's winner was The 2009-2014 World Outlook for 60-Milligram Containers of Fromage Frais by Philip M. Parker. Winners are chosen through a public vote. More than 4,500 people voted online this year, Stone said. (**Read more**.)

"Harold Macmillan, prime minister of Britain from 1957 to 1963, used to quote the opinion of his classics tutor

at Oxford: ***“Nothing you will learn in the course of your studies will be of the slightest possible use to you in after life, save only this: That if you work hard and diligently you should be able to detect when a man is talking rot. And that, in my view, is the main, if not the sole purpose of education.”*"**

(Robert Fulford)

In **The latest from the anti-racism industry** Posted: April 03, 2010, 10:00 AM.

" But the Senate is supposed to be above the game, I tell him [Bob Bennett], at least in the election off-season. Richard Russell, the legendary

“I know,” he said. “My father used to quote it: ‘The Senate allows you two years as a statesman, two years as a politician, and two years as a demagogue.’?” He gave me a wistful look right then, and proceeded to say exactly what I’d been thinking. “And that’s actually changed. You’re now a demagogue the full six years.”

(Jennifer Senior)

From **Mr Woebegone goes to Washington**, NY Magazine, April 4, 2010; a useful article on the US Senate's total disfunction.

- Try the following large collections of **Mathematical Quotes** from: **Platonic Realms** and from **Furman**

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Revised 04/04/2010 by Jonathan Borwein.