A Counterexample to the Bishop-Phelps Theorem in Complex Spaces.

Victor Lomonosov

Abstract

The Bishop-Phelps Theorem asserts that the set of functionals which attain the maximum value on a closed bounded convex subset S of a real Banach space X is norm dense in X^*. We show that this statement cannot be extended to general complex Banach spaces by constructing a closed bounded convex set with no support points.

Introduction.

If S is a subset of a Banach space X, then a nonzero functional $\varphi \in X^*$ is a support functional for S and a point $x \in S$ is a support point of S if $|\varphi(x)| = \sup_{y \in S} |\varphi(y)|$.

In 1958 [4] Victor Klee asked if each closed bounded convex subset of a Banach space must have a support point.

In 1961 E.Bishop and R.R.Phelps in their fundamental paper [1] proved that the set of support functionals for a closed bounded convex subset S of a real Banach space X is norm dense in X^*, thereby verifying Klee's conjecture. They also showed that the same theorem is true if the set S is the unit ball of a complex Banach space. The natural question of whether this statement is true in a complex Banach space for any closed bounded convex subset was left open since then [6]. In 1977 [2]

Jean Bourgain proved the remarkable result that the Bishop-Phelps Theorem is correct if a complex Banach space X has the Radon-Nikodym property.

In this note we construct a complex Banach space X and a closed bounded convex subset S of X such that the set of the support points of S is empty. This shows that in general Banach spaces the complex version of Klee's Conjecture is false. In particular, it means that the Bishop-Phelps Theorem cannot be extended to Complex Spaces. I am indebted to Professor Joe Diestel for helpful conversations concerning the subject of this note.

Let $\mathcal{H}^\infty$ be the algebra of analytical functions bounded on the open unit disk $\mathcal{D}$ with the norm $\|f\| = \sup_{z\in\mathcal{D}} |f(z)|$ and with the identity function E. Each point $z \in \mathcal{D}$ defines a point evaluation functional $\varphi_z(f) = f(z)$. It is well known that $\mathcal{H}^\infty$ may be identified with the dual space of some Banach space X such that each functional φ_z is an element in X [3]. We use the notation $<,>$ to describe a scalar product between a Banach space and its dual. Let S be a convex closed hull generated by elements $\{\varphi_z\}$. Then obviously for each point $s \in S$ and each function $f \in \mathcal{H}^\infty$ we have

$$\|s\| \leq 1, < s, E >= 1 \tag{1}$$

and

$$\sup_{s\in S} | < s, f > | = \|f\| \tag{2}$$

Lemma 1 *Suppose that $f \in \mathcal{H}^\infty$ and $\|f\| \leq 1$. Then either $f = \lambda E, |\lambda| = 1$ or for every point $s \in S$*

$$\lim_{k\to\infty} < s, f^k >= 0. \tag{3}$$

Proof. Suppose that $f \neq \lambda E$. Since finite convex linear combinations of point evaluations are dense in the set S and the sequence $\{f^k\}$ is bounded by norm we need only to consider the case

$$s = \sum_{i=1}^{n} \alpha_i \varphi_{z_i}$$

where $\alpha_i \geq 0$ for all i and $\sum_{i=1}^{n} \alpha_i = 1$. Put $\theta = \sup_{1 \leq i \leq n} |f(z_i)|$. Then

$$| < s, f^k > | \leq \sum_{i=1}^{n} \alpha_i |f^k(z_i)| \leq \theta^k \sum_{i=1}^{n} |\alpha_i| = \theta^k$$

Since $\|f\| \leq 1$ the maximum modulus principle implies that $\theta < 1$ which implies (3).

Lemma 2 *Suppose there exists an element $s \in S$ and a function $f \in \mathcal{H}^\infty$ such that $< s, f >= \|f\| = 1$. Then, for any positive integer n,*

$$< s, f^n >= 1. \tag{4}$$

Proof. Let M be the space of maximal ideals of the algebra $\mathcal{H}^\infty$ and let $C(M)$ be the algebra of all continuous functions on M with the sup-norm. The algebra $\mathcal{H}^\infty$ is a Banach subalgebra of the algebra $C(M)$. Let $\hat{s}$ be a norm preserving extension of the functional s onto the space $C(M)$. By the Riesz theorem there exists a regular Borel measure $d\nu$ on M such that the equality

$$< s, g >= \int_M g \, d\nu$$

holds for any function $g \in \mathcal{H}^\infty$. The conditions (1) imply that $\int_M d|\nu| \leq 1$ and $\int_M d\nu = 1$. It means that the measure $d\nu$ is a nonnegative probability measure on M. This implies that the function f is equal to the identity function on the support of the measure $d\nu$ which implies (4).

Theorem 1 *Suppose that the modulus of the functional $g \in \mathcal{H}^\infty$ attains its maximum on the set S. Then there exists a complex number α such that $g = \alpha E$.*

Proof. There exists an element $s_0 \in S$ such that

$$| < s_0, g > | = \sup_{s \in S} | < s, g > |$$

From (2) we have that $| < s_0, g > | = ||g||$. If g is the zero function then we can put $\alpha = 0$. If g is a nonzero function then there exists a complex number λ such that $< s_0, \lambda g >= \|\lambda g\| = 1$. Put $f = \lambda g$. Then (4) and lemma 1 implies that $f = \gamma E$ and $g = \lambda f = \lambda \gamma E$.

So the line αE in $\mathcal{H}^\infty$ is the set of all support functionals for the closed bounded convex subset S in the predual space X of $\mathcal{H}^\infty$.

Let φ_0 be the point evaluation at 0 and let L be the line in X generated by φ_0. Let X_1 be the quotient space X/L and $\pi_1 : X \to X_1$ be the corresponding quotient map. The dual space X_1^* is the annihilator $\varphi_0^\perp$ of the vector φ_0 in the space X^* which is the hyper-plane H_0^∞ of all functions in H^∞ vanishing at 0. Put $S_1 = \pi_1(S)$. Then obviously the set S_1 is closed bounded and convex.

Theorem 2 *The set of support points of S_1 is empty.*

Proof. Since the equality $< \pi_1(s), f >=< s, f >$ holds for any point $s \in S$ and any functional $f \in \varphi_0^\perp$, Theorem 1 implies that the only possible support functional for the set S_1 is a functional λE. Since $< \varphi_0, E >= 1$, the line λE has zero intersection with the subspace $\varphi_0^\perp$.

For this reason the only one functional which attains a maximum modulus on the set S_1 is the zero functional and the set of support points of the set S_1 in the predual to the space H_0^∞ is empty.

PUBLICATIONS

1. E. Bishop, R.R. Phelps. *A proof that every Banach space is subreflexive*, Bull AMS, 1961, vol 67, 97–98.

2. J.Bourgain. *On dentability and the Bishop-Phelps property*, Israel J. Math. 1977, vol. 28, 268–271.

3. K.Hoffman. *Banach Spaces of Analytic Functions*, Prentice-Hall Inc, 1962.

4. V.Klee. *Extremal structures of convex sets.*, Math. Z. (1958), vol 69, p.98.

5. J.Lindenstrauss. *On operators which attain their norm*, Israel J. Math. 1963, vol 3, 139–148.

6. R.R. Phelps. *The Bishop-Phelps Theorem in Complex Spaces: An Open Problem*, Lecture notes in pure and applied mathematics, 1991, vol 136, 337–340.

Victor Lomonosov, Dept of Mathematics, Kent State University
lomonoso@mcs.kent.edu

August 30, 1998