# Construction of pathological maximally monotone operators on non-reflexive Banach spaces

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**Abstract**

In this paper, we construct maximally monotone operators that are not of Gossez's dense-type (D) in many nonreflexive spaces. Many of these operators also fail to possess the Brønsted-Rockafellar (BR) property. Using these operators, we show that the partial inf-convolution of two BC–functions will not always be a BC–function. This provides a negative answer to a challenging question posed by Stephen Simons. Among other consequences, we deduce that every Banach space which contains an isomorphic copy of the James space $\mathbf{J}$ or its dual $\mathbf{J}^*$, or $c_0$ or its dual $\ell^1$, admits a non type (D) operator.

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# 1 Preliminaries

Throughout this paper, we assume that $X$ is a real Banach space with norm $\|\cdot\|$, that $X^*$ is the continuous dual of $X$, and that $X$ and $X^*$ are paired by $\langle\cdot,\cdot\rangle$. As usual, we identify $X$ with its canonical image in the bidual space $X^{**}$. Furthermore, $X \times X^*$ and $(X \times X^*)^* := X^* \times X^{**}$ are likewise paired via $\langle (x,x^*),(y^*,y^{**})\rangle := \langle x,y^*\rangle + \langle x^*,y^{**}\rangle$, where $(x,x^*) \in X \times X^*$ and $(y^*,y^{**}) \in X^* \times X^{**}$.

Let $A\colon X \rightrightarrows X^*$ be a *set-valued operator* (also known as a multifunction) from $X$ to $X^*$, i.e., for every $x \in X$, $Ax \subseteq X^*$, and let $\operatorname{gra} A := \big\{(x,x^*) \in X \times X^* \mid x^* \in Ax\big\}$ be the *graph* of $A$. The *domain* of $A$ is $\operatorname{dom} A := \big\{x \in X \mid Ax \neq \varnothing\big\}$, and $\operatorname{ran} A := A(X)$ for the *range* of $A$. Recall that $A$ is *monotone* if

$$(1) \qquad \langle x-y, x^*-y^*\rangle \geq 0, \quad \forall (x,x^*) \in \operatorname{gra} A \; \forall (y,y^*) \in \operatorname{gra} A,$$

and *maximally monotone* if $A$ is monotone and $A$ has no proper monotone extension (in the sense of graph inclusion). Let $A : X \rightrightarrows X^*$ be monotone and $(x,x^*) \in X \times X^*$. We say $(x,x^*)$ is *monotonically related to* $\operatorname{gra} A$ if

$$\langle x-y, x^*-y^*\rangle \geq 0, \quad \forall (y,y^*) \in \operatorname{gra} A.$$

We now recall the three fundamental subclasses of maximally monotone operators.

**Definition 1.1** *Let $A : X \rightrightarrows X^*$ be maximally monotone. Then three key types of monotone operators are defined as follows.*

(i) *$A$ is* of dense type or type (D) *(1971, [19] and [28]) if for every $(x^{**},x^*) \in X^{**} \times X^*$ with*

$$\inf_{(a,a^*)\in \operatorname{gra} A} \langle a - x^{**}, a^* - x^*\rangle \geq 0,$$

*there exist a bounded net $(a_\alpha, a^*_\alpha)_{\alpha\in\Gamma}$ in $\operatorname{gra} A$ such that $(a_\alpha, a^*_\alpha)_{\alpha\in\Gamma}$ weak*×strong converges to $(x^{**},x^*)$.*

(ii) *$A$ is* of type negative infimum (NI) *(1996, [32]) if*

$$\sup_{(a,a^*)\in \operatorname{gra} A} \big(\langle a,x^*\rangle + \langle a^*,x^{**}\rangle - \langle a,a^*\rangle\big) \geq \langle x^{**},x^*\rangle, \quad \forall (x^{**},x^*) \in X^{**} \times X^*.$$

(iii) *$A$ is* of "Brønsted-Rockafellar" (BR) type (1999, [37]) if whenever $(x,x^*) \in X \times X^*$, $\alpha,\beta > 0$ while

$$\inf_{(a,a^*)\in \operatorname{gra} A} \langle x-a, x^*-a^*\rangle > -\alpha\beta$$

then there exists $(b,b^*) \in \operatorname{gra} A$ such that $\|x-b\| < \alpha$, $\|x^*-b^*\| < \beta$.

As we shall see below in Fact 2.7, it is now known that the first two classes coincide. This coincidence is central to many of our proofs. Fact 2.11 also shows us that every maximally monotone operator of type (D) is of type (BR)(The converse fails, see Example 4.1(xiii).). Moreover, in reflexive space every maximally monotone operator is of type (D), as is the subdifferential operator of every closed convex function on a Banach space. While monotone operator theory is rather complete in reflexive space — and for type (D) operators in general space — the general situation is less clear [11, 9]. Hence our continuing interest in operators which are not of type (D).

We shall say a Banach space $X$ is *of type (D)* [9] if every maximally monotone operator on $X$ is of type (D). At present the only known type (D) spaces are the reflexive spaces; and our work here suggests that there are no non-reflexive type (D) spaces. In [11, Exercise 9.6.3] such spaces were called (NI) spaces and some potential non-reflexive examples were conjectured; all of which are ruled out by our current work. In [11, Theorem 9.79] a variety of the pleasant properties of type (D) spaces was listed.

## 1.1 More preliminary technicalities

Maximal monotone operators have proven to be a potent class of objects in modern Optimization and Analysis; see, e.g., [7, 8, 9], the books [6, 11, 13, 27, 33, 35, 31, 42] and the references therein.

We adopt standard notation used in these books especially [11, Chapter 2] and [7, 33, 35]: Given a subset $C$ of $X$, the *indicator function* of $C$, written as $\iota_C$, is defined at $x \in X$ by

$$\iota_C(x) := \begin{cases} 0, & \text{if } x \in C; \\ +\infty, & \text{otherwise.} \end{cases} \tag{2}$$

The *closed unit ball* is $B_X := \big\{x \in X \mid \|x\| \leq 1\big\}$, and $\mathbb{N} := \{1, 2, 3, \ldots\}$.

Let $\alpha, \beta \in \mathbb{R}$. In the sequel it will also be useful to let $\delta_{\alpha,\beta}$ be defined by $\delta_{\alpha,\beta} := 1$, if $\alpha = \beta$; $\delta_{\alpha,\beta} := 0$, otherwise.

For a subset $C^*$ of $X^*$, $\overline{C^*}^{\mathrm{w}^*}$ is the weak$^*$ closure of $C^*$. If $Z$ is a real Banach space with dual $Z^*$ and a set $S \subseteq Z$, we denote $S^\perp$ by $S^\perp := \{z^* \in Z^* \mid \langle z^*, s\rangle = 0, \quad \forall s \in S\}$. Given a subset $D$ of $Z^*$, we define $D_\perp$ [29] by $D_\perp := \{z \in Z \mid \langle z, d^*\rangle = 0, \quad \forall d^* \in D\}$.

The *adjoint* of an operator $A$, written $A^*$, is defined by

$$\operatorname{gra} A^* := \big\{(x^{**}, x^*) \in X^{**} \times X^* \mid (x^*, -x^{**}) \in (\operatorname{gra} A)^\perp\big\}.$$

We say $A$ is a *linear relation* if $\operatorname{gra} A$ is a linear subspace. We say that $A$ is *skew* if $\operatorname{gra} A \subseteq \operatorname{gra}(-A^*)$; equivalently, if $\langle x, x^*\rangle = 0,\ \forall (x, x^*) \in \operatorname{gra} A$. Furthermore, $A$ is *symmetric* if

$\operatorname{gra} A \subseteq \operatorname{gra} A^*$; equivalently, if $\langle x, y^*\rangle = \langle y, x^*\rangle$, $\forall (x,x^*),(y,y^*) \in \operatorname{gra} A$. We define the *symmetric part* and the *skew part* of $A$ via

$$P := \tfrac{1}{2}A + \tfrac{1}{2}A^* \quad \text{and} \quad S := \tfrac{1}{2}A - \tfrac{1}{2}A^*, \tag{3}$$

respectively. It is easy to check that $P$ is symmetric and that $S$ is skew. Let $A : X \rightrightarrows X^*$ be monotone and $S$ be a subspace of $X$. We say $A$ is *$S$–saturated* [35] if

$$Ax + S^\perp = Ax, \quad \forall x \in \operatorname{dom} A.$$

We say a maximally monotone operator $A : X \rightrightarrows X^*$ is *unique* if all maximally monotone extensions of $A$ (in the sense of graph inclusion) in $X^{**} \times X^*$ coincide.

Let $f\colon X \to \left]-\infty,+\infty\right]$. Then $\operatorname{dom} f := f^{-1}(\mathbb{R})$ is the *domain* of $f$, and $f^*\colon X^* \to [-\infty,+\infty] : x^* \mapsto \sup_{x\in X}(\langle x, x^*\rangle - f(x))$ is the *Fenchel conjugate* of $f$. We say $f$ is proper if $\operatorname{dom} f \neq \varnothing$. Let $f$ be proper. The *subdifferential* of $f$ is defined by

$$\partial f\colon X \rightrightarrows X^*\colon x \mapsto \{x^* \in X^* \mid (\forall y \in X)\ \langle y - x, x^*\rangle + f(x) \le f(y)\}.$$

For $\varepsilon \ge 0$, the *$\varepsilon$–subdifferential* of $f$ is defined by

$$\partial_\varepsilon f\colon X \rightrightarrows X^*\colon x \mapsto \big\{x^* \in X^* \mid (\forall y \in X)\ \langle y - x, x^*\rangle + f(x) \le f(y) + \varepsilon\big\}.$$

Note that $\partial f = \partial_0 f$. We denote by $J := J_X$ the duality map, i.e., the subdifferential of the function $\frac{1}{2}\|\cdot\|^2$ mapping $X$ to $X^*$.

Now let $F : X \times X^* \to \left]-\infty,+\infty\right]$. We say $F$ is a *BC–function* (BC stands for "Bigger conjugate") [35] if $F$ is proper and convex with

$$F^*(x^*, x) \ge F(x, x^*) \ge \langle x, x^*\rangle \quad \forall (x,x^*) \in X \times X^*. \tag{4}$$

Let $Y$ be another real Banach space. We set $P_X : X \times Y \to X\colon (x,y) \mapsto x$, and $P_Y : X \times Y \to Y\colon (x,y) \mapsto y$. Let $L : X \to Y$ be linear. We say $L$ is a (linear) *isomorphism* into $Y$ if $L$ is one to one, continuous and $L^{-1}$ is continuous on $\operatorname{ran} L$. We say $L$ is an *isometry* if $\|Lx\| = \|x\|, \forall x \in X$. The spaces $X$, $Y$ are then *isometric* (*isomorphic*) if there exists an isometry (*isomorphism*) from $X$ onto $Y$.

Let $F_1, F_2\colon X \times Y \to \left]-\infty,+\infty\right]$. Then the *partial inf-convolution* $F_1\Box_1 F_2$ is the function defined on $X \times Y$ by

$$F_1\Box_1 F_2\colon (x,y) \mapsto \inf_{u\in X} F_1(u,y) + F_2(x-u,y).$$

Then $F_1\Box_2 F_2$ is the function defined on $X \times Y$ by

$$F_1\Box_2 F_2\colon (x,y) \mapsto \inf_{v\in Y} F_1(x,y-v) + F_2(x,v).$$

In Example 4.1(vi)&(viii) of this paper, we provide a negative answer to the following question posed by S. Simons [35, Problem 22.12]:

*Let* $F_1, F_2 : X \times X^* \to \left]-\infty, +\infty\right]$ *be proper lower semicontinuous and convex. Assume that* $F_1, F_2$ *are BC–functions and that*

$$\bigcup_{\lambda>0} \lambda \left[P_{X^*} \operatorname{dom} F_1 - P_{X^*} \operatorname{dom} F_2\right] \textit{ is a closed subspace of } X^*.$$

*Is* $F_1 \Box_1 F_2$ *necessarily a BC–function?*

We are now ready to set to work. The paper is organized as follows. In Section 2, we collect auxiliary results for future reference and for the reader's convenience. Our main result (Theorem 3.6) is established in Section 3. In Section 4, we provide various applications and extensions including the promised negative answer to Simons' question. Furthermore, we show that every Banach space containing an isomorphic copy of the James space $\mathbf{J}$ or of $\mathbf{J}^*$, of $\ell^1$ or of $c_0$ is not of type (D) (Example 4.1(xi) or Corollary 4.12, Corollary 4.11 and Example 4.13).

# 2 Auxiliary results

Observation:

**Fact 2.1** (See [26, Proposition 2.6.6(c)]). *Let* $D$ *be a subspace of* $X^*$. *Then* $(D_\perp)^\perp = \overline{D}^{\mathrm{w}^*}$.

We now record a famous Banach space result:

**Fact 2.2 (Banach and Mazur)** (See [16, Theorem 5.8, page 240] or [15, Theorem 5.17, page 144]).) *Every separable Banach space is isometric to a subspace of* $C[0,1]$.

Now we turn to prerequisite results on Fitzpatrick functions, monotone operators, and linear relations.

**Fact 2.3 (Fitzpatrick)** (See [17, Corollary 3.9 and Proposition 4.2] and [7, 11].) *Let* $A\colon X \rightrightarrows X^*$ *be maximally monotone, and set*

$$(5) \qquad F_A\colon X \times X^* \to \left]-\infty, +\infty\right] : (x, x^*) \mapsto \sup_{(a,a^*)\in \operatorname{gra} A} \big(\langle x, a^*\rangle + \langle a, x^*\rangle - \langle a, a^*\rangle\big),$$

*which is the* Fitzpatrick function *associated with* $A$. *Then* $F_A$ *is a BC–function and* $F_A = \langle \cdot, \cdot\rangle$ *on* $\operatorname{gra} A$.

**Fact 2.4 (Simons and Zălinescu)** (See [38, Theorem 4.2] or [35, Theorem 16.4(a)].) *Let $Y$ be a real Banach space and $F_1, F_2\colon X\times Y \to\ ]-\infty,+\infty]$ be proper, lower semicontinuous, and convex. Assume that for every $(x,y)\in X\times Y$,*

$$(F_1\Box_2 F_2)(x,y) > -\infty$$

*and that $\bigcup_{\lambda>0}\lambda\left[P_X \operatorname{dom} F_1 - P_X \operatorname{dom} F_2\right]$ is a closed subspace of $X$. Then for every $(x^*,y^*)\in X^*\times Y^*$,*

$$(F_1\Box_2 F_2)^*(x^*,y^*) = \min_{u^*\in X^*}\left[F_1^*(x^*-u^*,y^*) + F_2^*(u^*,y^*)\right].$$

**Fact 2.5 (Simons and Zălinescu)** (See [35, Theorem 16.4(b)].) *Let $Y$ be a real Banach space and $F_1, F_2\colon X\times Y \to\ ]-\infty,+\infty]$ be proper, lower semicontinuous and convex. Assume that for every $(x,y)\in X\times Y$,*

$$(F_1\Box_1 F_2)(x,y) > -\infty$$

*and that $\bigcup_{\lambda>0}\lambda\left[P_Y \operatorname{dom} F_1 - P_Y \operatorname{dom} F_2\right]$ is a closed subspace of $Y$. Then for every $(x^*,y^*)\in X^*\times Y^*$,*

$$(F_1\Box_1 F_2)^*(x^*,y^*) = \min_{v^*\in Y^*}\left[F_1^*(x^*,v^*) + F_2^*(x^*,y^*-v^*)\right].$$

Phelps and Simons proved the next Fact 2.6 for unbounded linear operators in [29, Proposition 3.2(a)], but their proof can also be adapted for general linear relations. For reader's convenience, we write down their proof.

**Fact 2.6 (Phelps and Simons)** *Let $A : X \rightrightarrows X^*$ be a monotone linear relation. Then $(x,x^*)\in X\times X^*$ is monotonically related to $\operatorname{gra} A$ if and only if*

$$\langle x,x^*\rangle \geq 0 \text{ and } \left[\langle y^*,x\rangle + \langle x^*,y\rangle\right]^2 \leq 4\langle x^*,x\rangle\langle y^*,y\rangle, \quad \forall (y,y^*)\in \operatorname{gra} A.$$

*Proof.* We have the following equivalences:

$(x,x^*)\in X\times X^*$ is monotonically related to $\operatorname{gra} A$

$\Leftrightarrow \lambda^2\langle y,y^*\rangle - \lambda\left[\langle y^*,x\rangle + \langle x^*,y\rangle\right] + \langle x,x^*\rangle = \langle \lambda y^* - x^*, \lambda y - x\rangle \geq 0, \forall \lambda\in\mathbb{R}, \forall (y,y^*)\in \operatorname{gra} A$

$\Leftrightarrow \langle x,x^*\rangle \geq 0$ and $\left[\langle y^*,x\rangle + \langle x^*,y\rangle\right]^2 \leq 4\langle x^*,x\rangle\langle y^*,y\rangle,\ \forall (y,y^*)\in \operatorname{gra} A$ (by [29, Lemma 2.1]).

This completes the proof. ■

**Fact 2.7 (Simons / Marques Alves and Svaiter)** (See [32, Lemma 15] or [35, Theorem 36.3(a)], and [25, Theorem 4.4].) *Let $A : X \rightrightarrows X^*$ be maximally monotone. Then $A$ is of type (D) if and only if it is of type (NI).*

We next cite some properties regarding the *uniqueness* of (maximally) monotone extension of a maximally monotone operator to $X^{**} \times X^*$. Simons showed that every maximally monotone operator of type (NI) is *unique* in [34]. Recently, Marques Alves and Svaiter contributed the following results:

**Fact 2.8 (Marques Alves and Svaiter)** (See [24, Theorem 1.6].) *Let $A : X \rightrightarrows X^*$ be a maximally monotone linear relation that is not of type (D). Assume that $A$ is unique. Then* $\operatorname{gra} A = \operatorname{dom} F_A$.

**Fact 2.9 (Marques Alves and Svaiter)** (See [25, Corollary 4.6].) *Let $A : X \rightrightarrows X^*$ be a maximally monotone operator such that* $\operatorname{gra} A$ *is not affine. Then $A$ is of type (D) if and only if $A$ is unique.*

The Gossez operator defined as in Example 4.1(xii) is a maximally monotone and unique operator that is not of type (D) [20].

The definition of operators of type (BR) directly yields the following result.

**Fact 2.10** *Let $A : X \rightrightarrows X^*$ be maximally monotone and $(x, x^*) \in X \times X^*$. Assume that $A$ is of type (BR) and that* $\inf_{(a,a^*)\in \operatorname{gra} A}\langle x-a, x^*-a^*\rangle > -\infty$. *Then $x \in \overline{\operatorname{dom} A}$ and $x^* \in \overline{\operatorname{ran} A}$.*

Additionally,

**Fact 2.11 (Marques Alves and Svaiter)** (See [24, Theorem 1.4(4)] or [23].) *Let $A : X \rightrightarrows X^*$ be a maximally monotone operator. Assume that $A$ is of type (NI). Then $A$ is of type (BR).*

We shall also need some precise results about linear relations. The first two are elementary.

**Fact 2.12 (Cross)** (See [14, Proposition I.2.8(a)].) *Let $A\colon X \rightrightarrows Y$ be a linear relation. Then* $(\forall (x, x^*) \in \operatorname{gra} A)\ Ax = x^* + A0$.

**Lemma 2.13** *Let $A : X \rightrightarrows X^*$ be a linear relation. Assume that $A^*$ is monotone. Then* $\ker A^* \subseteq (\operatorname{ran} A^*)^{\perp}$.

*Proof.* Let $x^{**} \in \ker A^*$ and then $(\alpha x^{**}, 0) \in \operatorname{gra} A^*, \forall \alpha \in \mathbb{R}$. Then

$$0 \leq \langle \alpha x^{**} + y^{**}, y^*\rangle = \alpha\langle x^{**}, y^*\rangle + \langle y^{**}, y^*\rangle, \quad \forall (y^{**}, y^*) \in \operatorname{gra} A^*, \forall \alpha \in \mathbb{R}.$$

Hence $\langle x^{**}, y^*\rangle = 0, \quad \forall (y^{**}, y^*) \in \operatorname{gra} A^*$ and thus $x^{**} \in (\operatorname{ran} A^*)^{\perp}$. Thus $\ker A^* \subseteq (\operatorname{ran} A^*)^{\perp}$. ■

**Fact 2.14** (See [4, Theorem 3.1].) *Let $A : X \rightrightarrows X^*$ be a maximally monotone linear relation. Then $A$ is of type (D) if and only if $A^*$ is monotone.*

**Fact 2.15** (See [41, Theorem 3.1].) *Let $A : X \rightrightarrows X^*$ be a maximally monotone linear relation, and let $f : X \to \left]-\infty, +\infty\right]$ be a proper lower semicontinuous convex function with* $\operatorname{dom} A \cap \operatorname{int} \operatorname{dom} \partial f \neq \varnothing$. *Then $A + \partial f$ is maximally monotone.*

**Fact 2.16 (Simons)** (See [35, Theorem 28.9].) *Let $Y$ be a Banach space, and $L : Y \to X$ be continuous and linear with* $\operatorname{ran} L$ *closed and* $\operatorname{ran} L^* = Y^*$. *Let $A : X \rightrightarrows X^*$ be monotone with* $\operatorname{dom} A \subseteq \operatorname{ran} L$ *such that* $\operatorname{gra} A \neq \varnothing$. *Then $A$ is maximally monotone if, and only if $A$ is* $\operatorname{ran} L$*–saturated and $L^*AL$ is maximally monotone.*

**Theorem 2.17** *Let $Y$ be a Banach space, and $L : Y \to X$ be an isomorphism into $X$. Let $T : Y \rightrightarrows Y^*$ be monotone. Then $T$ is maximally monotone if, and only if $(L^*)^{-1}TL^{-1}$, mapping $X$ into $X^*$, is maximally monotone.*

*Proof.* Let $A = (L^*)^{-1}TL^{-1}$. Then $\operatorname{dom} A \subseteq \operatorname{ran} L$. Since $L$ is an isomorphism into $X$, $\operatorname{ran} L$ is closed. By [26, Theorem 3.1.22(b)] or [15, Exercise 2.39(i), page 59], $\operatorname{ran} L^* = Y^*$. Hence $\operatorname{gra}(L^*)^{-1}TL^{-1} \neq \varnothing$ if and only if $\operatorname{gra} T \neq \varnothing$. Clearly, $A$ is monotone. Since $\{0\} \times (\operatorname{ran} L)^{\perp} \subseteq \operatorname{gra}(L^*)^{-1}$ and then by Fact 2.12, $A = (L^*)^{-1}TL^{-1}$ is $\operatorname{ran} L$–saturated. By Fact 2.16, $A = (L^*)^{-1}TL^{-1}$ is maximally monotone if and only if $L^*AL = T$ is maximally monotone. ■

The following consequence will allow us to construct maximally monotone operators that are not of type (D) in a variety of non-reflexive Banach spaces.

**Corollary 2.18 (Subspaces)** *Let $Y$ be a Banach space, and $L : Y \to X$ be an isomorphism into $X$. Let $T : Y \rightrightarrows Y^*$ be maximally monotone. Assume that $T$ is not of type (D). Then $(L^*)^{-1}TL^{-1}$ is a maximally monotone operator mapping $X$ into $X^*$ that is not of type (D). In particular, every Banach subspace of a type (D) space is of type (D).*

*Proof.* By Theorem 2.17, $(L^*)^{-1}TL^{-1}$ is maximally monotone. By Fact 2.7, there exists $(y_0^{**}, y_0^*) \in Y^{**} \times Y^*$ such that

$$\sup_{(b,b^*) \in \operatorname{gra} T} \big\{ \langle y_0^{**}, b^* \rangle + \langle y_0^*, b \rangle - \langle b, b^* \rangle \big\} < \langle y_0^{**}, y_0^* \rangle. \tag{6}$$

By [26, Theorem 3.1.22(b)] or [15, Exercise 2.39(i), page 59], $\operatorname{ran} L^* = Y^*$ and thus there exists $x_0^* \in X^*$ such that $L^* x_0^* = y_0^*$. Let $A = (L^*)^{-1}TL^{-1}$. Then we have

$$\begin{aligned}
&\sup_{(a,a^*) \in \operatorname{gra} A} \big\{ \langle L^{**} y_0^{**}, a^* \rangle + \langle x_0^*, a \rangle - \langle a, a^* \rangle \big\} \\
&= \sup_{(Ly,a^*) \in \operatorname{gra} A} \big\{ \langle y_0^{**}, L^* a^* \rangle + \langle x_0^*, Ly \rangle - \langle Ly, a^* \rangle \big\}
\end{aligned}$$

$$
\begin{aligned}
&= \sup_{(Ly,a^*)\in \operatorname{gra} A} \big\{\langle y_0^{**}, L^*a^*\rangle + \langle L^*x_0^*, y\rangle - \langle y, L^*a^*\rangle\big\} \\
&= \sup_{(Ly,a^*)\in \operatorname{gra} A} \big\{\langle y_0^{**}, L^*a^*\rangle + \langle y_0^*, y\rangle - \langle y, L^*a^*\rangle\big\} \\
&= \sup_{(y,y^*)\in \operatorname{gra} T} \big\{\langle y_0^{**}, y^*\rangle + \langle y_0^*, y\rangle - \langle y, y^*\rangle\big\} \quad (\text{by } (Ly,a^*)\in \operatorname{gra} A \Leftrightarrow (y, L^*a^*)\in \operatorname{gra} T) \\
&< \langle y_0^{**}, y_0^*\rangle \quad (\text{by } (6)) \\
&= \langle L^{**}y_0^{**}, x_0^*\rangle. \qquad (7)
\end{aligned}
$$

Thus $A$ is not of type (NI) and hence $A = (L^*)^{-1}TL^{-1}$ is not of type (D) by Fact 2.7. ■

Note that it follows that $X$ is of type (D) whenever $X^{**}$ is.

# 3 Main result

We start with several technical tools. To relate Fitzpatrick functions and skew operators we have:

**Lemma 3.1** *Let $A\colon X \rightrightarrows X^*$ be a skew linear relation. Then*

$$F_A = \iota_{\operatorname{gra}(-A^*)\cap X\times X^*}. \qquad (8)$$

*Proof.* Let $(x_0, x_0^*) \in X \times X^*$. We have

$$
\begin{aligned}
F_A(x_0, x_0^*) &= \sup_{(x,x^*)\in \operatorname{gra} A} \{\langle (x_0^*, x_0), (x, x^*)\rangle - \langle x, x^*\rangle\} \\
&= \sup_{(x,x^*)\in \operatorname{gra} A} \langle (x_0^*, x_0), (x, x^*)\rangle \\
&= \iota_{(\operatorname{gra} A)^\perp}(x_0^*, x_0) \\
&= \iota_{\operatorname{gra}(-A^*)}(x_0, x_0^*) \\
&= \iota_{\operatorname{gra}(-A^*)\cap X\times X^*}(x_0, x_0^*).
\end{aligned}
$$

Hence (8) holds. ■

To produce operators not of type (D) but that are of (BR) we exploit:

**Lemma 3.2** *Let $A : X \rightrightarrows X^*$ be a maximally monotone and linear skew operator. Assume that* $\operatorname{gra}(-A^*) \cap X \times X^* \subseteq \operatorname{gra} A$*. Then $A$ is of type (BR).*

*Proof.* Let $\alpha, \beta > 0$ and $(x, x^*) \in X \times X^*$ be such that $\inf_{(a,a^*)\in \operatorname{gra} A}\langle x - a, x^* - a^*\rangle > -\alpha\beta$. Since $A$ is skew, we have

$$\inf_{(a,a^*)\in \operatorname{gra} A}\langle x, x^*\rangle - [\langle x, a^*\rangle + \langle a, x^*\rangle] = \inf_{(a,a^*)\in \operatorname{gra} A}\langle x - a, x^* - a^*\rangle > -\alpha\beta. \tag{9}$$

Thus, $\langle x, a^*\rangle + \langle a, x^*\rangle = 0, \forall (a, a^*) \in \operatorname{gra} A$ and hence $(x, x^*) \in \operatorname{gra}(-A^*)$. Then by assumption, $(x, x^*) \in \operatorname{gra} A$. Taking $(b, b^*) = (x, x^*)$, we have $\|b-x\| < \alpha$ and $\|b^* - x^*\| < \beta$. Hence $A$ is of type (BR). ■

**Corollary 3.3** *Let $A : X \rightrightarrows X^*$ be a maximally monotone and linear skew operator that is not of type (D). Assume that $A$ is unique. Then $\operatorname{gra} A = \operatorname{gra}(-A^*) \cap X \times X^*$ and so $A$ is of type (BR).*

*Proof.* Apply Fact 2.8, Lemma 3.1 and Lemma 3.2 directly. ■

**Proposition 3.4** *Let $A\colon X \rightrightarrows X^*$ be maximally monotone. Assume that $A$ is of type (NI) and that there exists $e \in X^*$ such that*

$$\langle x^*, x\rangle \geq \langle e, x\rangle^2, \quad \forall (x, x^*) \in \operatorname{gra} A.$$

*Then $e \in \overline{\operatorname{conv} \operatorname{ran} A}$.*

*Proof.* Suppose $e \notin \overline{\operatorname{conv} \operatorname{ran} A}$. Then by the Separation Theorem, there exists $x_0^{**} \in X^{**}$ such that $\langle e - x^*, x_0^{**}\rangle \geq 1$ for all $x^* \in \operatorname{ran} A$. Then we have

$$\begin{aligned}\langle x^* - e, x - x_0^{**}\rangle &= \langle e - x^*, x_0^{**}\rangle + \langle x^* - e, x\rangle, \quad \forall (x, x^*) \in \operatorname{gra} A\\ &\geq 1 + \langle e, x\rangle^2 - \langle e, x\rangle\\ &\geq \min_{t\in\mathbb{R}} t^2 - t + 1 = \frac{3}{4}.\end{aligned}$$

Thus $A$ is not of type (NI), which contradicts the assumption. ■

The proof of the following result was partially inspired by that [12, Proposition 2.2].

**Proposition 3.5** *Let $A\colon X \rightrightarrows X^*$ be a maximally monotone linear relation. Assume that there exists $e \in X^*$ such that $e \notin \overline{\operatorname{ran} A}$ and that*

$$\langle x^*, x\rangle \geq \langle e, x\rangle^2, \quad \forall (x, x^*) \in \operatorname{gra} A.$$

*Then $A$ is neither of type (D) nor unique.*

*Proof.* By Proposition 3.4, $A$ is not of type (NI) and hence $A$ is not of type (D) by Fact 2.7. Similar to the proof of Proposition 3.4, there exists $x_0^{**} \in X^{**}$ such that $\langle e, x_0^{**}\rangle \geq 1$ and $x_0^{**} \in (\operatorname{ran} A)^{\perp}$. Let $0 < \alpha < 2$. Then we have

$$\begin{aligned}\langle x^* - \alpha e, x - \tfrac{1}{\alpha}x_0^{**}\rangle &= \langle \alpha e - x^*, \tfrac{1}{\alpha}x_0^{**}\rangle + \langle x^* - \alpha e, x\rangle, \quad \forall (x,x^*) \in \operatorname{gra} A\\ &\geq 1 + \langle e, x\rangle^2 - \alpha\langle e, x\rangle\\ &\geq \min_{t\in\mathbb{R}} t^2 - \alpha t + 1\\ &= 1 - \frac{\alpha^2}{4} > 0.\end{aligned}$$

Thus for every $0 < \alpha < 2$, $(\frac{1}{\alpha}x_0^{**}, \alpha e) \in X^{**} \times X^*$ is monotonically related to $\operatorname{gra} A$. Take $0 < \alpha_1 < \alpha_2 < 2$. Then by Zorn's Lemma, we have a maximally monotone extension, $A_1 : X^{**} \rightrightarrows X^*$ such that $\operatorname{gra} A_1 \supseteq \operatorname{gra} A \cup \{(\frac{1}{\alpha_1}x_0^{**}, \alpha_1 e,)\}$, and we can also obtain a maximally monotone extension, $A_2 : X^{**} \rightrightarrows X^*$ such that $\operatorname{gra} A_2 \supseteq \operatorname{gra} A \cup \{(\frac{1}{\alpha_2}x_0^{**}, \alpha_2 e)\}$.

Now we show $\operatorname{gra} A_1 \neq \operatorname{gra} A_2$. Suppose to the contrary that $\operatorname{gra} A_1 = \operatorname{gra} A_2$. Then by the monotonicity of $A_1$, we have

$$\langle \tfrac{1}{\alpha_1}x_0^{**} - \tfrac{1}{\alpha_2}x_0^{**}, \alpha_1 e - \alpha_2 e\rangle \geq 0. \tag{10}$$

On the other hand,

$$\begin{aligned}\langle \tfrac{1}{\alpha_1}x_0^{**} - \tfrac{1}{\alpha_2}x_0^{**}, \alpha_1 e - \alpha_2 e\rangle &= (\alpha_1 - \alpha_2)(\tfrac{1}{\alpha_1} - \tfrac{1}{\alpha_2})\langle x_0^{**}, e\rangle\\ &< (\alpha_1 - \alpha_2)(\tfrac{1}{\alpha_1} - \tfrac{1}{\alpha_2}) < 0,\end{aligned}$$

which contradicts (10). Hence $\operatorname{gra} A_1 \neq \operatorname{gra} A_2$ and thus $A$ is not unique. ■

We are now ready to establish our work-horse Theorem 3.6, which allows us to construct various maximally monotone operators — both linear and nonlinear — that are not of type (D). The idea of constructing the operators in the following fashion is based upon [2, Theorem 5.1] and was stimulated by [12].

**Theorem 3.6 (Predual constructions)** *Let $A : X^* \to X^{**}$ be linear and continuous. Assume that* $\operatorname{ran} A \subseteq X$ *and that there exists $e \in X^{**}\backslash X$ such that*

$$\langle Ax^*, x^*\rangle = \langle e, x^*\rangle^2, \quad \forall x^* \in X^*.$$

*Let $P$ and $S$ respectively be the symmetric part and antisymmetric part of $A$. Let $T : X \rightrightarrows X^*$ be defined by*

$$\begin{aligned}\operatorname{gra} T &:= \big\{(-Sx^*, x^*) \mid x^* \in X^*, \langle e, x^*\rangle = 0\big\}\\ &= \big\{(-Ax^*, x^*) \mid x^* \in X^*, \langle e, x^*\rangle = 0\big\}.\end{aligned} \tag{11}$$

*Let $f : X \to\, ]-\infty, +\infty]$ be a proper lower semicontinuous and convex function. Set $F := f \oplus f^*$ on $X \times X^*$. Then the following hold.*

(i) *$A$ is a maximally monotone operator on $X^*$ that is neither of type (D) nor unique.*

(ii) $Px^* = \langle x^*, e\rangle e, \ \forall x^* \in X^*$.

(iii) *$T$ is maximally monotone and skew on $X$.*

(iv) $\operatorname{gra} T^* = \{(Sx^* + re, x^*) \mid x^* \in X^*, \ r \in \mathbb{R}\}$.

(v) *$-T$ is not maximally monotone.*

(vi) *$T$ is not of type (D).*

(vii) *$F_T = \iota_C$, where*

$$C := \{(-Ax^*, x^*) \mid x^* \in X^*\}. \tag{12}$$

(viii) *$T$ is not unique.*

(ix) *$T$ is not of type (BR).*

(x) *If $\operatorname{dom} T \cap \operatorname{int}\operatorname{dom} \partial f \neq \varnothing$, then $T + \partial f$ is maximally monotone.*

(xi) *$F$ and $F_T$ are BC–functions on $X \times X^*$.*

(xii) *Moreover,*

$$\bigcup_{\lambda>0} \lambda\big(P_{X^*}(\operatorname{dom} F_T) - P_{X^*}(\operatorname{dom} F)\big) = X^*,$$

*while, assuming that there exists $(v_0, v_0^*) \in X \times X^*$ such that*

$$f^*(v_0^*) + f^{**}(v_0 - A^*v_0^*) < \langle v_0, v_0^*\rangle, \tag{13}$$

*then $F_T \Box_1 F$ is not a BC–function.*

(xiii) *Assume that $\left[\operatorname{ran} A - \bigcup_{\lambda>0} \lambda \operatorname{dom} f\right]$ is a closed subspace of $X$ and that*

$$\varnothing \neq \operatorname{dom} f^{**} \circ A^*|_{X^*} \nsubseteq \{e\}_\perp.$$

*Then $T + \partial f$ is not of type (D).*

(xiv) *Assume that $\operatorname{dom} f^{**} = X^{**}$. Then $T + \partial f$ is a maximally monotone operator that is not of type (D).*

*Proof.* (i): Clearly, $A$ has full domain. Since $A$ is monotone and continuous, $A$ is maximally monotone. By the assumptions that $e \notin X$ and $\overline{\operatorname{ran} A} \subseteq \overline{X} = X$, then by Proposition 3.5, $A$ is neither of type (D) nor unique. See also [1, Theorem 14.2.1 and Theorem 13.2.3] for alternative proof of that $A$ is not of type (D).

(ii): Now we show that

$$Px^* = \langle x^*, e\rangle e, \ \forall x^* \in X^*. \tag{14}$$

Since $\langle \cdot, e\rangle e = \partial(\frac{1}{2}\langle \cdot, e\rangle^2)$ and by [29, Theorem 5.1], $\langle \cdot, e\rangle e$ is a symmetric operator on $X^*$. Clearly, $A - \langle \cdot, e\rangle e$ is skew. Then (14) holds.

(iii): Let $x^* \in X^*$ with $\langle e, x^*\rangle = 0$. Then we have

$$Sx^* = \langle x^*, e\rangle e + Sx^* = Px^* + Sx^* = Ax^* \in \operatorname{ran} A \subseteq X.$$

Thus (11) holds and $T$ is well defined.

We have $S$ is skew and hence $T$ is skew. Let $(z, z^*) \in X \times X^*$ be monotonically related to $\operatorname{gra} T$. By Fact 2.6, we have

$$0 = \langle z, x^*\rangle + \langle -Sx^*, z^*\rangle = \langle z + Sz^*, x^*\rangle, \quad \forall x^* \in \{e\}_\perp.$$

Thus by Fact 2.1, we have $z + Sz^* \in (\{e\}_\perp)^\perp = \operatorname{span}\{e\}$ and then

$$z = -Sz^* + \kappa e, \ \exists \kappa \in \mathbb{R}. \tag{15}$$

By $(0, 0) \in \operatorname{gra} T$,

$$\kappa\langle z^*, e\rangle = \langle -Sz^* + \kappa e, z^*\rangle = \langle z, z^*\rangle \geq 0. \tag{16}$$

Then by (15) and (ii),

$$Az^* = Pz^* + Sz^* = Pz^* + \kappa e - z = [\langle z^*, e\rangle + \kappa]\, e - z. \tag{17}$$

By the assumptions that $z \in X$, $Az^* \in X$ and $e \notin X$, $[\langle z^*, e\rangle + \kappa] = 0$ by (17). Then by (16), we have $\langle z^*, e\rangle = \kappa = 0$ and thus $(z, z^*) \in \operatorname{gra} T$ by (15). Hence $T$ is maximally monotone.

(iv): Let $(x_0^{**}, x_0^*) \in X^{**} \times X^*$. Then we have

$$\begin{aligned}
&(x_0^{**}, x_0^*) \in \operatorname{gra} T^* \Leftrightarrow \langle x_0^*, Sx^*\rangle + \langle x^*, x_0^{**}\rangle = 0, \quad \forall x^* \in \{e\}_\perp \\
&\Leftrightarrow \langle x^*, x_0^{**} - Sx_0^*\rangle = 0, \quad \forall x^* \in \{e\}_\perp \\
&\Leftrightarrow x_0^{**} - Sx_0^* \in (\{e\}_\perp)^\perp = \operatorname{span}\{e\} \quad \text{(by Fact 2.1)} \\
&\Leftrightarrow x_0^{**} - Sx_0^* = re, \quad \exists r \in \mathbb{R}.
\end{aligned}$$

Thus $\operatorname{gra} T^* = \{(Sx^* + re, x^*) \mid x^* \in X^*,\ r \in \mathbb{R}\}$.

(v): Since $e \notin X$, we have $e \neq 0$. Then there exists $z^* \in X^*$ such that $z^* \notin \{e\}_\perp$. Then by (ii)&(iv) and the assumption that $\operatorname{ran} A \subseteq X$, we have

$$(Az^*, z^*) = (Sz^* + \langle e, z^*\rangle e, x^*) \in \operatorname{gra} T^* \cap X \times X^*.$$

Thus we have

$$\begin{aligned}\langle Az^* - x, z^* - x^*\rangle &= \langle Az^*, z^*\rangle - [\langle Az^*, x^*\rangle + \langle x, z^*\rangle] + \langle x, x^*\rangle \\ &= \langle Az^*, z^*\rangle \geq 0, \quad \forall (x, x^*) \in \operatorname{gra}(-T).\end{aligned}$$

Hence $(Az^*, z^*)$ is monotonically related to $\operatorname{gra}(-T)$. Since $z^* \notin \operatorname{ran}(-T)$, $(Az^*, z^*) \notin \operatorname{gra}(-T)$ and then $-T$ is not maximally monotone.

(vi): By (iv), $T^*$ is not monotone. Then by Fact 2.14, $T$ is not of type (D).

(vii): By (iv), we have

$$\begin{aligned}&(z, z^*) \in \operatorname{gra}(-T^*) \cap X \times X^* \\ &\Leftrightarrow (z, z^*) = (-Sz^* - re, z^*), \quad z \in X,\ \exists r \in \mathbb{R},\ z^* \in X^* \\ &\Leftrightarrow (z, z^*) = (-Sz^* - \langle z^*, e\rangle e + [\langle z^*, e\rangle - r]\, e, z^*), \quad z \in X,\ \exists r \in \mathbb{R},\ z^* \in X^* \\ &\Leftrightarrow (z, z^*) = (-Az^* + [\langle z^*, e\rangle - r]\, e, z^*), \quad z \in X,\ \exists r \in \mathbb{R},\ z^* \in X^* \text{ (by (ii))} \\ &\Leftrightarrow (z, z^*) = (-Az^*, z^*), \quad \langle z^*, e\rangle = r \text{ (since } z, Az^* \in X \text{ and } e \notin X), \exists r \in \mathbb{R},\ z^* \in X^* \\ &\Leftrightarrow (z, z^*) \in \{(-Ax^*, x^*) \mid x^* \in X^*\} = C.\end{aligned}$$

Thus by Lemma 3.1, we have $F_T = \iota_C$.

(viii): Since $e \notin X$, we have $e \neq 0$. Then there exists $z^* \in X^*$ such that $z^* \notin \{e\}_\perp$. Thus $z^* \notin \operatorname{ran} T$. By (vii), $z^* \in P_{X^*}[\operatorname{dom} F_T]$. Thus, $\operatorname{gra} T \neq \operatorname{dom} F_T$. Then by (vi) and Fact 2.8, $T$ is not unique.

(ix): Suppose to the contrary that $T$ is of type (BR). Let $z^*$ be as in the proof of (viii). Then by Lemma 3.1 and (vii), we have $(-Az^*, z^*) \in \operatorname{gra}(-T^*) \cap X \times X^*$ and then

$$\inf_{(a,a^*) \in \operatorname{gra} T} \langle -Az^* - a, z^* - a^*\rangle = \langle -Az^*, z^*\rangle > -\infty.$$

Then Fact 2.10 shows $z^* \in \overline{\operatorname{ran} T}$, which contradicts that $z^* \notin \{e\}_\perp = \overline{\operatorname{ran} T}$. Hence $T$ is not of type (BR).

(x): Apply (iii) and Fact 2.15.

(xi): Clearly, $F$ is a BC–function. By (iii) and Fact 2.3, we see that $F_T$ is a BC–function.

(xii): By (vii), we have

$$\bigcup_{\lambda>0} \lambda\big(P_{X^*}(\operatorname{dom} F_T) - P_{X^*}(\operatorname{dom} F)\big) = X^*. \tag{18}$$

Then for every $(x,x^*) \in X \times X^*$ and $u \in X$, by (xi),

$$F_T(x-u,x^*) + F(u,x^*) = F_T(x-u,x^*) + (f \oplus f^*)(u,x^*) \geq \langle x-u, x^*\rangle + \langle u, x^*\rangle = \langle x, x^*\rangle.$$

Hence

$$(F_T \Box_1 F)(x,x^*) \geq \langle x, x^*\rangle > -\infty. \tag{19}$$

Then by (18), (19) and Fact 2.5,

$$\begin{aligned}
(F_T \Box_1 F)^*(v_0^*, v_0) &= \min_{x^{**}\in X^{**}} F_T^*(v_0^*, x^{**}) + F^*(v_0^*, v_0 - x^{**})\\
&\leq F_T^*(v_0^*, A^*v_0^*) + F^*(v_0^*, v_0 - A^*v_0^*)\\
&= 0 + F^*(v_0^*, v_0 - A^*v_0^*) \quad \text{(by (vii))}\\
&= (f \oplus f^*)^*(v_0^*, v_0 - A^*v_0^*) = (f^* \oplus f^{**})(v_0^*, v_0 - A^*v_0^*)\\
&= f^*(v_0^*) + f^{**}(v_0 - A^*v_0^*)\\
&< \langle v_0^*, v_0\rangle \quad \text{(by (13))}.
\end{aligned}$$

Hence $F_T \Box_1 F$ is not a BC–function.

(xiii): By the assumption, there exists $x_0^* \in \operatorname{dom} f^{**} \circ A^*|_{X^*}$ such that $\langle e, x_0^*\rangle \neq 0$. Let $\varepsilon_0 = \frac{\langle e, x_0^*\rangle^2}{2}$. By [42, Theorem 2.4.4(iii)]), there exists $y_0^{***} \in \partial_{\varepsilon_0} f^{**}(A^*x_0^*)$. By [42, Theorem 2.4.2(ii)]),

$$f^{**}(A^*x_0^*) + f^{***}(y_0^{***}) \leq \langle A^*x_0^*, y_0^{***}\rangle + \varepsilon_0. \tag{20}$$

Then by [35, Lemma 45.9] or the proof of [30, Eq.(2.5) in Proposition 1], there exists $y_0^* \in X^*$ such that

$$f^{**}(A^*x_0^*) + f^*(y_0^*) < \langle A^*x_0^*, y_0^*\rangle + 2\varepsilon_0. \tag{21}$$

Let $z_0^* = y_0^* + x_0^*$. Then by (21), we have

$$\begin{aligned}
f^{**}(A^*x_0^*) + f^*(z_0^* - x_0^*) &< \langle A^*x_0^*, z_0^* - x_0^*\rangle + 2\varepsilon_0\\
&= \langle A^*x_0^*, z_0^*\rangle - \langle A^*x_0^*, x_0^*\rangle + 2\varepsilon_0\\
&= \langle A^*x_0^*, z_0^*\rangle - \langle x_0^*, Ax_0^*\rangle + 2\varepsilon_0\\
&= \langle A^*x_0^*, z_0^*\rangle - 2\varepsilon_0 + 2\varepsilon_0\\
&= \langle A^*x_0^*, z_0^*\rangle. \qquad (22)
\end{aligned}$$

Then for every $(x,x^*)\in X\times X^*$ and $u^*\in X$, by (xi),

$$F_T(x,x^*-u^*)+F(x,u^*)=F_T(x,x^*-u^*)+(f\oplus f^*)(x,u^*)\geq\langle x,x^*-u^*\rangle+\langle x,u^*\rangle=\langle x,x^*\rangle.$$

Hence

$$(F_T\Box_2 F)(x,x^*)\geq\langle x,x^*\rangle>-\infty. \tag{23}$$

Then by (23), (vii) and Fact 2.4,

$$\begin{aligned}(F_T\Box_2 F)^*(z_0^*,A^*x_0^*)&=\min_{y^*\in X^*}F_T^*(y^*,A^*x_0^*)+F^*(z_0^*-y^*,A^*x_0^*)\\&\leq F_T^*(x_0^*,A^*x_0^*)+F^*(z_0^*-x_0^*,A^*x_0^*)\\&=0+F^*(z_0^*-x_0^*,A^*x_0^*)\quad\text{(by (vii))}\\&=(f\oplus f^*)^*(z_0^*-x_0^*,A^*x_0^*)\\&=f^*(z_0^*-x_0^*)+f^{**}(A^*x_0^*)\\&<\langle z_0^*,A^*x_0^*\rangle\quad\text{(by (22))}.\end{aligned}\tag{24}$$

Let $F_0:X\times X^*\to\,]-\infty,+\infty]$ be defined by

$$(x,x^*)\mapsto\langle x,x^*\rangle+\iota_{\operatorname{gra}(T+\partial f)}(x,x^*). \tag{25}$$

Clearly, $F_T\Box_2 F\leq F_0$ on $X\times X^*$ and thus $(F_T\Box_2 F)^*\geq F_0^*$ on $X^*\times X^{**}$. By (24), $F_0^*(z_0^*,A^*x_0^*)<\langle z_0^*,A^*x_0^*\rangle$. Hence $T+\partial f$ is not of type (NI) and thus $T+\partial f$ is not of type (D) by Fact 2.7.

(xiv): Since $\operatorname{dom} f^{**}=X^{**}$, $\operatorname{dom} f=X$ by [42, Theorem 2.3.3]. By $\operatorname{dom} f^{**}=X^{**}$ again, $\operatorname{dom} f^{**}\circ A_\alpha^*|_{X^*}=X^*\subsetneqq\{\alpha\}_\perp$. Then apply (x)&(xiii) directly. ■

**Remark 3.7 (Grothendieck spaces [11])** In light of part (xiii) of the previous theorem), we record that for a closed convex function

$$\operatorname{dom} f=X\text{ implies }\operatorname{dom} f^{**}=X^{**}\Leftrightarrow X\text{ is a }\textit{Grothendieck}\text{ space.}$$

All reflexive spaces are Grothendieck spaces while all non-reflexive Grothendieck spaces (such as $L^\infty[0,1]$) contain an isomorphic copy of $c_0$. ◊

We are now ready to exploit Theorem 3.6.

# 4 Examples and applications

We begin with the case of $c_0$ and its dual $\ell^1$.

## 4.1 Applications to $c_0$

**Example 4.1 ($c_0$)** Let $X := c_0$, with norm $\|\cdot\|_\infty$ so that $X^* = \ell^1$ with norm $\|\cdot\|_1$, and $X^{**} = \ell^\infty$ with its second dual norm $\|\cdot\|_*$. Let $\alpha := (\alpha_n)_{n\in\mathbb{N}} \in \ell^\infty$ with $\limsup \alpha_n \neq 0$, and let $A_\alpha : \ell^1 \to \ell^\infty$ be defined by

$$(A_\alpha x^*)_n := \alpha_n^2 x_n^* + 2\sum_{i>n} \alpha_n\alpha_i x_i^*, \quad \forall x^* = (x_n^*)_{n\in\mathbb{N}} \in \ell^1. \tag{26}$$

Now let $P_\alpha$ and $S_\alpha$ respectively be the symmetric part and antisymmetric part of $A_\alpha$. Let $T_\alpha : c_0 \rightrightarrows X^*$ be defined by

$$\begin{aligned}
\operatorname{gra} T_\alpha &:= \big\{(-S_\alpha x^*, x^*) \mid x^* \in X^*, \langle \alpha, x^*\rangle = 0\big\}\\
&= \big\{(-A_\alpha x^*, x^*) \mid x^* \in X^*, \langle \alpha, x^*\rangle = 0\big\}\\
&= \Big\{\big((-\sum_{i>n} \alpha_n\alpha_i x_i^* + \sum_{i<n} \alpha_n\alpha_i x_i^*)_n, x^*\big) \mid x^* \in X^*, \langle \alpha, x^*\rangle = 0\Big\}.
\end{aligned} \tag{27}$$

Then

(i) $\langle A_\alpha x^*, x^*\rangle = \langle \alpha, x^*\rangle^2, \quad \forall x^* = (x_n^*)_{n\in\mathbb{N}} \in \ell^1$ and(27) is well defined.

(ii) $A_\alpha$ is a maximally monotone operator on $\ell^1$ that is neither of type (D) nor unique.

(iii) $T_\alpha$ is a maximally monotone operator on $c_0$ that is not of type (D).

(iv) $-T_\alpha$ is not maximally monotone.

(v) $T_\alpha$ is neither unique nor of type (BR).

(vi) $F_{T_\alpha}\Box_1(\|\cdot\| \oplus \iota_{B_{X^*}})$ is not a BC–function.

(vii) $T_\alpha + \partial\|\cdot\|$ is a maximally monotone operator on $c_0(\mathbb{N})$ that is not of type (D).

(viii) If $\frac{1}{\sqrt{2}} < \|\alpha\|_* \leq 1$, then $F_{T_\alpha}\Box_1(\frac{1}{2}\|\cdot\|^2 \oplus \frac{1}{2}\|\cdot\|_1^2)$ is not a BC–function.

(ix) For $\lambda > 0$, $T_\alpha + \lambda J$ is a maximally monotone operator on $c_0$ that is not of type (D).

(x) Let $\lambda > 0$ and a linear isometry $L$ mapping $c_0$ to a subspace of $C[0,1]$ be given. Then both $(L^*)^{-1}(T_\alpha + \partial\|\cdot\|)L^{-1}$ and $(L^*)^{-1}(T_\alpha + \lambda J)L^{-1}$ are maximally monotone operators that are not of type (D). Hence neither $c_0$ nor $C[0,1]$ is of type (D).

(xi) Every Banach space that contains an isomorphic copy of $c_0$ is not of type (D).

(xii) Let $G : \ell^1 \to \ell^\infty$ be Gossez's operator [20] defined by

$$\big(G(x^*)\big)_n := \sum_{i>n} x_i^* - \sum_{i<n} x_i^*, \quad \forall (x_n^*)_{n\in\mathbb{N}} \in \ell^1.$$

Then $T_e : c_0 \rightrightarrows \ell^1$ as defined by

$$\operatorname{gra} T_e := \{(-G(x^*), x^*) \mid x^* \in \ell^1, \langle x^*, e\rangle = 0\}$$

is a maximally monotone operator that is not of type (D), where $e := (1, 1, \ldots, 1, \ldots)$.

(xiii) Moreover, $G$ is a unique maximally monotone operator that is not of type (D), but $G$ is of type (BR).

*Proof.* We have $\alpha \notin c_0$. Since $\alpha = (\alpha_n)_{n\in\mathbb{N}} \in \ell^\infty$ and $\|A_\alpha\| \leq 2\|\alpha\|^2$, $A_\alpha$ is linear and continuous and $\operatorname{ran} A_\alpha \subseteq c_0 \subseteq \ell^\infty$.

(i): We have

$$\begin{aligned}
\langle A_\alpha x^*, x^*\rangle &= \sum_n x_n^*(\alpha_n^2 x_n^* + 2\sum_{i>n} \alpha_n\alpha_i x_i^*) \\
&= \sum_n \alpha_n^2 {x_n^*}^2 + 2\sum_n \sum_{i>n} \alpha_n\alpha_i x_n^* x_i^* \\
&= \sum_n \alpha_n^2 {x_n^*}^2 + \sum_{n\neq i} \alpha_n\alpha_i x_n^* x_i^* \\
&= (\sum_n \alpha_n x_n^*)^2 = \langle \alpha, x^*\rangle^2, \quad \forall x^* = (x_n^*)_{n\in\mathbb{N}} \in \ell^1. \qquad (28)
\end{aligned}$$

Then Theorem 3.6(ii) shows that the symmetric part $P_\alpha$ of $A_\alpha$ is $P_\alpha x^* = \langle \alpha, x^*\rangle\alpha$ (for every $x^* \in \ell^1$). Thus, the skew part $S_\alpha$ of $A_\alpha$ is

$$\begin{aligned}
(S_\alpha x^*)_n &= (A_\alpha x^*)_n - (P_\alpha x^*)_n \\
&= \alpha_n^2 x_n^* + 2\sum_{i>n} \alpha_n\alpha_i x_i^* - \sum_{i\geq 1} \alpha_n\alpha_i x_i^* \\
&= \sum_{i>n} \alpha_n\alpha_i x_i^* - \sum_{i<n} \alpha_n\alpha_i x_i^*. \qquad (29)
\end{aligned}$$

Then by Theorem 3.6, (27) is well defined.

(ii): Apply (i) and Theorem 3.6(i) directly.

(iii): Combine Theorem 3.6(iii)&(vi).

(iv): Apply Theorem 3.6(v) directly.

(v): Apply Theorem 3.6(viii)&(ix).

(vi) Since $\alpha \neq 0$, there exists $i_0 \in \mathbb{N}$ such that $\alpha_{i_0} \neq 0$. Let $e_{i_0} := (0, \ldots, 0, 1, 0, \ldots)$, i.e., the $i_0$th is 1 and the others are 0. Then by (29), we have

$$S_\alpha e_{i_0} = \alpha_{i_0}(\alpha_1, \ldots, \alpha_{i_0-1}, 0, -\alpha_{i_0+1}, -\alpha_{i_0+2}, \ldots). \tag{30}$$

Then

$$\begin{aligned} A^* e_{i_0} &= P_\alpha e_{i_0} - S_\alpha e_{i_0} \\ &= \alpha_{i_0}(0, \ldots, 0, \alpha_{i_0}, 2\alpha_{i_0+1}, 2\alpha_{i_0+2}, \ldots). \end{aligned} \tag{31}$$

Now set $v_0^* := e_{i_0}$ and $v_0 := 3\|\alpha\|_*^2 e_{i_0}$. Thus by (31),

$$\begin{aligned} v_0 - A^* v_0^* &= 3\|\alpha\|_*^2 e_{i_0} - A^* e_{i_0} \\ &= (0, \ldots, 0, 3\|\alpha\|_*^2 - \alpha_{i_0}^2, -2\alpha_{i_0}\alpha_{i_0+1}, -2\alpha_{i_0}\alpha_{i_0+2}, \ldots). \end{aligned} \tag{32}$$

Let $f := \|\cdot\|$ on $X = c_0$. Then $f^* = \iota_{B_{X^*}}$ by [42, Corollary 2.4.16]. We have

$$\begin{aligned} f^*(v_0^*) + f^{**}(v_0 - A^* e_{i_0}) &= \iota_{B_{X^*}}(e_{i_0}) + \|v_0 - A^* e_{i_0}\|_* \\ &= \|3\|\alpha\|_* e_{i_0} - A^* e_{i_0}\|_* \\ &< 3\|\alpha\|_*^2 \quad \text{(by (32))} \\ &= \langle v_0, v_0^* \rangle. \end{aligned}$$

Hence by Theorem 3.6(xii), $F_{T_\alpha} \Box_1 (\|\cdot\| \oplus \iota_{B_{X^*}})$ is not a BC–function.

(vii): Let $f := \|\cdot\|$ on $X$. Since $\operatorname{dom} f^{**} = X^{**}$. Then apply Theorem 3.6(xiv).

(viii): By $\frac{1}{\sqrt{2}} < \|\alpha\|_* \leq 1$, take $|\alpha_{i_0}|^2 > \frac{1}{2}$. Let $e_{i_0}$ be defined as in the proof of (vi). Then take $v_1^* := \frac{1}{2} e_{i_0}$ and $v_1 := \left(1 + \frac{1}{2}\alpha_{i_0}^2\right) e_{i_0}$.

By (31), we have

$$v_1 - A^* v_1^* = (0, \ldots, 0, 1, -\alpha_{i_0}\alpha_{i_0+1}, -\alpha_{i_0}\alpha_{i_0+2}, \ldots). \tag{33}$$

Since $|\alpha_{i_0}\alpha_j| \leq \|\alpha\|_*^2 \leq 1, \ \forall j \in \mathbb{N}$, then

$$\|v_1 - A^* v_1^*\|_* \leq 1. \tag{34}$$

Let $f := \frac{1}{2}\|\cdot\|^2$ on $X = c_0$. Then $f^* = \frac{1}{2}\|\cdot\|_1^2$ and $f^{**} = \frac{1}{2}\|\cdot\|_*^2$. We have

$$f^*(v_1^*) + f^{**}(v_1 - A^* v_1^*) = \tfrac{1}{2}\|v_1^*\|_1^2 + \tfrac{1}{2}\|v_1 - A^* v_1^*\|_*^2$$

$$\leq \tfrac{1}{8} + \tfrac{1}{2} \quad \text{(by (34))}$$
$$< \tfrac{\alpha_{i_0}^2}{4} + \tfrac{1}{2} \quad (\text{since } \alpha_{i_0}^2 > 1/2)$$
$$= \langle v_1^*, v_1 \rangle.$$

Hence by Theorem 3.6(xii), $F_{T_\alpha} \Box_1 (\frac{1}{2}\|\cdot\|^2 \oplus \frac{1}{2}\|\cdot\|_*^2)$ is not a BC–function.

(ix): Let $\lambda > 0$ and $f := \frac{\lambda}{2}\|\cdot\|^2$ on $X = c_0$. Then $f^{**} = \frac{\lambda}{2}\|\cdot\|_*^2$. Then apply Theorem 3.6(xiv).

(x): Since $c_0$ is separable by [26, Example 1.12.6] or [15, Proposition 1.26(ii)], by Fact 2.2, there exists a linear operator $L : c_0 \to C[0,1]$ that is an isometry from $c_0$ to a subspace of $C[0,1]$. Then combine (vii)&(ix) and Corollary 2.18.

(xi) Combine (iii) (or (vii) or (ix)) and Corollary 2.18.

(xii): To obtain the result on $T_e$, directly apply (iii) (or see [2, Example 5.2]).

(xiii) Now $-G$ is type (D) but $G$ is not [2]. To see that $G$ is unique, note that $-G^*$ is monotone by Fact 2.14 and so provides the unique maximal extension. Since $G$ is skew and continuous, clearly, $-G^*x^* = Gx^*, \forall x^* \in \ell^1$. Then Lemma 3.2 implies that $G$ is of type (BR). The uniqueness of $G$ was also verified in [1, Example 14.2.2]. ■

**Remark 4.2** The maximal monotonicity of the operator $T_e$ in Example 4.1(xii) was also verified by Voisei and Zălinescu in [39, Example 19] and later a direct proof given by Bueno and Svaiter in [12, Lemma 2.1]. Herein we have given a more concise proof of above results.

Bueno and Svaiter also showed that $T_e$ is not of type (D) in [12]. They also showed that each Banach space that contains an isometric (isomorphic) copy of $c_0$ is not of type (D) in [12]. Example 4.1(xi) recaptures their result, while Example 4.1(vi)&(viii) provide a negative answer to Simons' [35, Problem 22.12]. ◊

**Remark 4.3 (The continuous case)** We recall that a Banach space $X$ is a *conjugate monotone space* if every continuous linear monotone operator on $X$ has a monotone conjugate. In particular this holds if every continuous linear monotone operator on $X$ is weakly compact. In consequence, a Banach lattice $X$ contains a complemented copy of $\ell^1$ if and only if it admits a non (D) continuous linear monotone operator, on using Fact 2.14 along with [2, Remark 5.5] and [2, Examples. 5.2 and 5.3].

Thus, in lattices such as $c_0$, $c$ and $C[0,1]$ only discontinuous linear monotone operators can fail to be of type (D). This subtlety escaped the current authors for fifteen years. ◊

We now turn to a broader class of spaces:

## 4.2 Applications to more general nonreflexive spaces

Our results below are facilitated by making use of Schauder basis structure [16].

**Definition 4.4** *We say* $(e_n, e_n^*)_{n\in\mathbb{N}}$ *in* $X \times X^*$ *is a* Schauder basis *of* $X$ *if for every* $x \in X$ *there exists a unique sequence* $(\alpha_n)_{n\in\mathbb{N}}$ *in* $\mathbb{R}$ *such that* $x = \sum_{n\geq 1} \alpha_n e_n$*, where* $\alpha_n = \langle x, e_n^*\rangle$ *and* $\langle e_i, e_j^*\rangle = \delta_{i,j}, \forall i, j \in \mathbb{N}$.

**Definition 4.5** *Let* $(e_n, e_n^*)_{n\in\mathbb{N}}$ *in* $X \times X^*$ *be a Schauder basis of* $X$*. We say the basis is* shrinking *if* $\overline{\operatorname{span}\{e_n^* \mid n \in \mathbb{N}\}} = X^*$.

In particular, a Banach space with a shrinking basis has a separable dual and so is an Asplund space [16].

**Fact 4.6** (See [16, Lemma 4.7(iii) and Facts 4.11(ii)&(iii)] or [15, Lemma 6.2(iii) and Facts 6.6(ii)&(iii)] .) *Let* $(e_n, e_n^*)_{n\in\mathbb{N}}$ *in* $X \times X^*$ *be a Schauder basis of* $X$*. Then*

(i) $\lim_n \sum_{i=1}^n \langle x, e_i^*\rangle e_i = x, \quad \forall x \in X$;

(ii) $\sum_{i=1}^n \langle x^*, e_i\rangle e_i^*$ *weak* converges to* $x^*$*, written as,* $\sum_{i=1}^n \langle x^*, e_i\rangle e_i^* \overset{\mathrm{w}^*}{\rightharpoonup} x^*, \quad \forall x^* \in X^*$;

(iii) $(e_n^*, e_n)_{n\in\mathbb{N}}$ *in* $X^* \times X^{**}$ *is a Schauder basis of* $\overline{\operatorname{span}\{e_n^* \mid n \in \mathbb{N}\}}$.

**Lemma 4.7** *Let* $(e_n, e_n^*)_{n\in\mathbb{N}}$ *in* $X \times X^*$ *be a Schauder basis of* $X$*. Then* $e_n^* \overset{\mathrm{w}^*}{\rightharpoonup} 0$ *whenever* $\liminf_{n\in\mathbb{N}} \|e_n\| > 0$.

*Proof.* Let $x \in X$. Since $\|\langle x, e_n^*\rangle e_n\| \to 0$ because of Fact 4.6(i), and since $\liminf_{n\in\mathbb{N}} \|e_n\| > 0$, we have $\langle x, e_n^*\rangle \to 0$. Hence $e_n^* \overset{\mathrm{w}^*}{\rightharpoonup} 0$ as $n \to \infty$. ■

The proof of Example 4.8(i) was inspired by that [3, Proposition 3.5].

**Example 4.8 (Schauder basis)** Let $(e_n, e_n^*)_{n\in\mathbb{N}}$ in $X \times X^*$ be a Schauder basis of $X$. Assume that for some $e \in X^{**}$ we have

$$\sum_{i=1}^n e_i \overset{\mathrm{w}^*}{\rightharpoonup} e \in X^{**}. \tag{35}$$

Let $A : X \rightrightarrows X^*$ be defined by

$$\operatorname{gra} A := \Big\{\Big(\sum_n \big(-\sum_{i>n}\langle e_i, y^*\rangle + \sum_{i<n}\langle e_i, y^*\rangle\big)e_n, y^*\Big) \in X \times X^* \mid y^* \in \{e\}_\perp\Big\}.$$

Assume that $\liminf \|e_n\| > 0$. Then the following hold.

(i) $A$ is a maximally monotone and linear skew operator.

(ii) $A$ is not of type (BR).

(iii) $A$ is not of type (D).

(iv) $A$ is not unique.

(v) Every Banach space containing a copy of $X$ is not of type (D).

*Proof.* (i): First, we show $A$ is skew. Let $(y, y^*) \in \operatorname{gra} A$. Then $\langle e, y^*\rangle = 0$ and $y = \sum_{n=1}^{\infty} \big( -\sum_{i>n} \langle e_i, y^*\rangle + \sum_{i<n} \langle e_i, y^*\rangle \big) e_n$. By the assumption that $\sum_{i=1}^{n} e_i \overset{\mathrm{w}^*}{\rightharpoonup} e \in X^{**}$, we have

$$s := \sum_{i\geq 1} \langle e_i, y^*\rangle = \langle e, y^*\rangle = 0. \tag{36}$$

Thus,

$$\begin{aligned}
\langle y, y^*\rangle &= \langle \sum_n \big( -\sum_{i>n} \langle e_i, y^*\rangle + \sum_{i<n} \langle e_i, y^*\rangle \big) e_n, y^*\rangle \\
&= \lim_k \langle \sum_{n=1}^{k} \big( -\sum_{i>n} \langle e_i, y^*\rangle + \sum_{i<n} \langle e_i, y^*\rangle \big) e_n, y^*\rangle \quad \text{(by Fact 4.6(i))} \\
&= \lim_k \sum_{n=1}^{k} \big( -\sum_{i>n} \langle e_i, y^*\rangle + \sum_{i<n} \langle e_i, y^*\rangle \big) \langle e_n, y^*\rangle \\
&= -\lim_k \sum_{n=1}^{k} \big( \sum_{i>n} \langle e_i, y^*\rangle - \sum_{i<n} \langle e_i, y^*\rangle \big) \langle e_n, y^*\rangle \\
&= -\lim_k \sum_{n=1}^{k} \big( \sum_{i\geq n+1} \langle e_i, y^*\rangle + \sum_{i\geq n} \langle e_i, y^*\rangle \big) \langle e_n, y^*\rangle \quad \text{(by (36))} \qquad (37) \\
&= -\lim_k \Big( \langle e_1, y^*\rangle \sum_{i\geq 1} \langle e_i, y^*\rangle + \langle e_2, y^*\rangle \sum_{i\geq 2} \langle e_i, y^*\rangle + \cdots + \langle e_k, y^*\rangle \sum_{i\geq k} \langle e_i, y^*\rangle \\
&\quad + \langle e_1, y^*\rangle \sum_{i\geq 2} \langle e_i, y^*\rangle + \langle e_2, y^*\rangle \sum_{i\geq 3} \langle e_i, y^*\rangle + \cdots + \langle e_k, y^*\rangle \sum_{i\geq k+1} \langle e_i, y^*\rangle \Big) \\
&= -\lim_k \Big( s\langle e_1, y^*\rangle + (s - \langle e_1, y^*\rangle)\langle e_2, y^*\rangle + \cdots + (s - \sum_{i=1}^{k-1} \langle e_i, y^*\rangle)\langle e_k, y^*\rangle \\
&\quad + (s - \langle e_1, y^*\rangle)\langle e_1, y^*\rangle + \big(s - \sum_{i=1}^{2} \langle e_i, y^*\rangle\big)\langle e_2, y^*\rangle + \cdots + (s - \sum_{i=1}^{k} \langle e_i, y^*\rangle)\langle e_k, y^*\rangle \Big)
\end{aligned}$$

$$= -\lim_k \Big( s\sum_{i=1}^{k}\langle e_i, y^*\rangle - \langle e_1, y^*\rangle\langle e_2, y^*\rangle - \sum_{i=1}^{2}\langle e_i, y^*\rangle\langle e_3, y^*\rangle - \cdots - \sum_{i=1}^{k-1}\langle e_i, y^*\rangle\langle e_k, y^*\rangle$$

$$+ s\sum_{i=1}^{k}\langle e_i, y^*\rangle - \sum_{i=1}^{k}\langle e_i, y^*\rangle^2 - \langle e_1, y^*\rangle\langle e_2, y^*\rangle - \cdots - \sum_{i=1}^{k-1}\langle e_i, y^*\rangle\langle e_k, y^*\rangle \Big)$$

$$= -\lim_k \left[ 2s\sum_{i=1}^{k}\langle e_i, y^*\rangle - (\sum_{i=1}^{k}\langle e_i, y^*\rangle)^2 \right]$$

$$= -(2s^2 - s^2) = -s^2 = 0. \quad \text{(by (36))}$$

Hence $A$ is skew.

To show maximality, let $(x, x^*) \in X \times X^*$ be monotonically related to $\operatorname{gra} A$. By Fact 2.6, we have

$$\langle y^*, x\rangle + \langle x^*, y\rangle = 0, \quad \forall (y, y^*) \in \operatorname{gra} A. \tag{38}$$

By (35), we have

$$\langle e, e_n^*\rangle = \sum_{i\geq 1}\langle e_i, e_n^*\rangle = \delta_{n,n} = 1, \quad \forall n \in \mathbb{N}. \tag{39}$$

Let $y^* := -e_1^* + e_n^*$ $(n \geq 2)$ and $y := -e_1 - 2\sum_{i=2}^{n-1} e_i - e_n$. By (39), we have $\langle e, y^*\rangle = 0$. Hence $y^* \in \{e\}_\perp$ and $(y, y^*) \in \operatorname{gra} A$. Using (38),

$$-\langle x, e_1^*\rangle + \langle x, e_n^*\rangle - \langle x^*, e_1\rangle - \langle x^*, e_n\rangle - 2\sum_{i=2}^{n-1}\langle x^*, e_i\rangle = 0.$$

Thus, we have

$$\langle x, e_n^*\rangle = \langle x, e_1^*\rangle - \langle x^*, e_1\rangle + \langle x^*, e_n\rangle + 2\sum_{i=1}^{n-1}\langle x^*, e_i\rangle. \tag{40}$$

As $\sum_{i\geq 1}\langle e_i, z^*\rangle = \langle e, z^*\rangle (\forall z^* \in X^*)$, we have $\langle x^*, e_n\rangle \to 0$.

Hence, by Lemma 4.7 — since $\liminf \|e_n\| > 0$ — and (40),

$$-2\sum_{i\geq 1}\langle x^*, e_i\rangle = \langle x, e_1^*\rangle - \langle x^*, e_1\rangle. \tag{41}$$

Next we show $-2\sum_{i\geq 1}\langle x^*, e_i\rangle = \langle x, e_1^*\rangle - \langle x^*, e_1\rangle = 0$. Let $t = \sum_{i\geq 1}\langle x^*, e_i\rangle$. Then by (40) and (41),

$$x = \sum_{n\geq 1}\langle x, e_n^*\rangle e_n$$

$$= \sum_{n\geq 1}\Big(-2\sum_{i\geq 1}\langle x^*, e_i\rangle + 2\sum_{i<n}\langle x^*, e_i\rangle + \langle x^*, e_n\rangle\Big)e_n$$
$$= \sum_{n\geq 1}\Big(-2\sum_{i\geq n}\langle x^*, e_i\rangle + \langle x^*, e_n\rangle\Big)e_n$$
$$= \sum_{n\geq 1}\Big(-\sum_{i\geq n}\langle x^*, e_i\rangle - \sum_{i\geq n}\langle x^*, e_i\rangle + \langle x^*, e_n\rangle\Big)e_n$$
$$= \sum_{n\geq 1}\Big(-\sum_{i\geq n}\langle x^*, e_i\rangle - \sum_{i\geq n+1}\langle x^*, e_i\rangle\Big)e_n. \tag{42}$$

Using $(0,0)\in \operatorname{gra} A$, as in the proof of (37), shows

$$0 \geq -\langle x^*, x\rangle = \langle\sum_{n\geq 1}\Big(\sum_{i\geq n}\langle x^*, e_i\rangle + \sum_{i\geq n+1}\langle x^*, e_i\rangle\Big)e_n, x^*\rangle$$
$$= \lim_k\langle\sum_{n=1}^{k}\Big(\sum_{i\geq n}\langle x^*, e_i\rangle + \sum_{i\geq n+1}\langle x^*, e_i\rangle\Big)e_n, x^*\rangle$$
$$= 2t^2 - t^2 = t^2.$$

Hence $t = 0$. By (42),

$$x = \sum_{n\geq 1}\Big(-\sum_{i>n}\langle x^*, e_i\rangle + \sum_{i<n}\langle x^*, e_i\rangle\Big)e_n.$$

Hence $(x, x^*)\in \operatorname{gra} A$. Thus, $A$ is maximally monotone.

(ii): Suppose to the contrary that $A$ is of type (BR). One checks that $(e_1, e_1^*)\in \operatorname{gra} A^*$ and $\langle e, e_1^*\rangle = \lim_n\langle\sum_{i=1}^n e_i, e_1^*\rangle = 1$. Thus, $(e_1, -e_1^*)\in \operatorname{gra}(-A^*)\cap X\times X^*$ and $-e_1^*\notin\{e\}_\perp$. Since $\overline{\operatorname{ran} A}\subseteq\{e\}_\perp$, $-e_1^*\notin\overline{\operatorname{ran} A}$. Then $\inf_{(a,a^*)\in\operatorname{gra} A}\langle e_1 - a, -e_1^* - a^*\rangle = \langle e_1, -e_1^*\rangle = -1 > -\infty$. Then by Fact 2.10, $-e_1^*\in\overline{\operatorname{ran} A}$, which contradicts that $-e_1^*\notin\overline{\operatorname{ran} A}$. Hence $A$ is not of type (BR).

(iii): By Fact 2.11 and (ii), $A$ is not of type (NI) and hence $A$ is not of type (D) by Fact 2.7. *Alternative Proof:* Clearly, $(e, 0)\in\operatorname{gra} A^*$ and thus $e\in\ker A^*$. By the proof of (ii), $(e_1, e_1^*)\in\operatorname{gra} A^*$ and $\langle e, e_1^*\rangle = 1$. Hence $e\notin(\operatorname{ran} A^*)^\perp$. Hence $A^*$ is not monotone by Lemma 2.13. Then Fact 2.14 shows $A$ is not of type (D).

(iv): Apply (iii)&(ii) and Corollary 3.3 directly.

(v): Combine (i)&(iii) and Corollary 2.18. ■

We shall especially exploit the lovely properties of the James space:

**Definition 4.9** *The* James space*,* $\mathbf{J}$*, consists of all the sequences* $x = (x_n)_{n\in\mathbb{N}}$ *in* $c_0$ *with the finite norm*

$$\|x\| := \sup_{n_1<\cdots<n_k} \big((x_{n_1} - x_{n_2})^2 + (x_{n_2} - x_{n_3})^2 + \cdots + (x_{n_{k-1}} - x_{n_k})^2\big)^{\frac{1}{2}}.$$

**Fact 4.10** (See [16, page 205] or [15, Claim, page 185].) *The space* $\mathbf{J}$ *is constructed to be of codimension-one in* $\mathbf{J}^{**}$*. Indeed,* $\mathbf{J}^{**} = \mathbf{J} \oplus \operatorname{span}\{e\}$ *where* $e := (1, 1, \ldots, 1, \ldots)$ *is the constant sequence in* $c(\mathbb{N}) \subset \ell^\infty$*. Thus,* $\mathbf{J}$ *is a separable Asplund space, equivalently* $\mathbf{J}^*$ *is separable [11, 16, 15], and non-reflexive. Inter alia, the basis* $(e_n, e_n^*)_{n\in\mathbb{N}}$ *is a shrinking Schauder basis in* $\mathbf{J}$ *and* $(e_n^*, e_n)_{n\in\mathbb{N}}$ *is a basis for* $\mathbf{J}^*$*, where* $e_n = (0, \ldots, 0, 1, 0, \ldots)$*, i.e., the nth is* 1 *and the others are* 0*.*

**Corollary 4.11 (James space)** *Let* $X$ *be the James space,* $\mathbf{J}$*. Let* $e_n$ *be defined as in Fact 4.10, and let* $A$ *be defined as in Example 4.8. Then* $A$ *is a maximally monotone and skew operator that is neither of type (BR) nor unique and so* $A$ *is not of type (D). Hence, every Banach space that contains an isomorphic copy of* $\boldsymbol{J}$ *is not of type (D).*

*Proof.* To apply Example 4.8 we need only verify that (35) holds. To see this is so, we note that $\big(\sum_{i=1}^n e_i\big)_{n\in\mathbb{N}}$ lies in $B_{\mathbf{J}^{**}}$ — directly from the definition of the norm in $\mathbf{J}$. Now by the Banach-Alaoglu theorem and [16, Proposition 3.103, page 128] or [15, Proposition 3.24, page 72], we have the vector $e = (1, 1, \ldots, 1, \ldots)$ is the unique $w^*$ limit of $\big(\sum_{i=1}^n e_i\big)_{n\in\mathbb{N}}$. ■

An easier version of the same argument leads to a recovery of part of Example 4.1:

**Corollary 4.12 ($c_0$)** *Let* $X = c_0$*. Let* $e_n$ *be defined as in Fact 4.10 and* $e := (1, 1, \ldots, 1, \ldots)$*. Let* $A$ *be defined as in Example 4.8 (thus* $A = T_e$ *in Example 4.1(xii)). Then* $A$ *is a maximally monotone and skew operator that is neither of type (BR) nor unique and so* $A$ *is not of type (D). Hence, every Banach space that contains an isomorphic copy of* $c_0$ *is not of type (D).*

We finish our set of core examples by dealing with the dual space $\mathbf{J}^*$.

**Example 4.13 (Shrinking Schauder basis)** Let $(e_n, e_n^*)_{n\in\mathbb{N}}$ in $X \times X^*$ be a shrinking Schauder basis of $X$. Assume that $\sum_{i=1}^n e_i \overset{\mathrm{w}^*}{\rightharpoonup} e$ for some $e \in X^{**}$. Let $A : X^* \rightrightarrows X^{**}$ be defined by

$$\operatorname{gra} A = \Big\{(y^*, y^{**}) \in X^* \times X^{**} \mid \sum_{n=1}^{k} \big(\sum_{i>n} \langle e_i, y^*\rangle - \sum_{i<n} \langle e_i, y^*\rangle\big) e_n \overset{\mathrm{w}^*}{\rightharpoonup} y^{**}\Big\}. \tag{43}$$

Then $A$ is a maximally monotone and linear skew operator, which is of type (BR).

In particular, let $(e_n)_{n\in\mathbb{N}}$ and $e$ be defined as in Fact 4.10. Then $A + \langle\cdot, e\rangle e$ is a maximally monotone operator that is neither of type (D) nor unique; and every Banach space containing a copy of $\mathbf{J}^*$ is not of type (D).

*Proof*. Again, we first show $A$ is skew. Let $(y^*, y^{**}) \in \operatorname{gra} A$. Then

$$\sum_{n=1}^{k} \big( \sum_{i>n} \langle e_i, y^* \rangle - \sum_{i<n} \langle e_i, y^* \rangle \big) e_n \overset{\mathrm{w}^*}{\rightharpoonup} y^{**}.$$

By the assumption that $\sum_{i=1}^{n} e_i \overset{\mathrm{w}^*}{\rightharpoonup} e \in X^{**}$, we have

$$s := \sum_{i\geq 1} \langle e_i, y^* \rangle = \langle e, y^* \rangle. \tag{44}$$

Thus,

$$\begin{aligned}
\langle y^{**}, y^* \rangle &= \lim_k \langle \sum_{n=1}^{k} \big( \sum_{i>n} \langle e_i, y^* \rangle - \sum_{i<n} \langle e_i, y^* \rangle \big) e_n, y^* \rangle \\
&= \lim_k \sum_{n=1}^{k} \big( \sum_{i>n} \langle e_i, y^* \rangle - \sum_{i<n} \langle e_i, y^* \rangle \big) \langle e_n, y^* \rangle \\
&= \lim_k \sum_{n=1}^{k} \big( \sum_{i\geq n+1} \langle e_i, y^* \rangle + \sum_{i\geq n} \langle e_i, y^* \rangle - s \big) \langle e_n, y^* \rangle \quad \text{(by (44))} \\
&= -s \lim_k \sum_{n=1}^{k} \langle e_n, y^* \rangle + \lim_k \sum_{n=1}^{k} \big( \sum_{i\geq n+1} \langle e_i, y^* \rangle + \sum_{i\geq n} \langle e_i, y^* \rangle \big) \langle e_n, y^* \rangle \\
&= -s^2 + (2s^2 - s^2) = 0 \quad \text{(as in the proof of (37))}.
\end{aligned}$$

Hence $A$ is skew.

Now we confirm maximality. Let $(x^*, x^{**}) \in X^* \times X^{**}$ be monotonically related to $\operatorname{gra} A$. By Fact 2.6, we have

$$\langle y^*, x^{**} \rangle + \langle x^*, y^{**} \rangle = 0, \quad \forall (y^*, y^{**}) \in \operatorname{gra} A. \tag{45}$$

Fix $n \in \mathbb{N}$ and set $y^* := e_n^*$. Then $\sum_{j=1}^{k} \big( \sum_{i>j} \langle e_i, y^* \rangle - \sum_{i<j} \langle e_i, y^* \rangle \big) e_j = \sum_{j=1}^{n-1} e_j - \sum_{j=n+1}^{k} e_j$. By the assumption that $\sum_{i=1}^{k} e_i \overset{\mathrm{w}^*}{\rightharpoonup} e$, we have

$$\sum_{j=1}^{n-1} e_j - \sum_{j=n+1}^{k} e_j \overset{\mathrm{w}^*}{\rightharpoonup} 2 \sum_{j=1}^{n-1} e_j + e_n - e.$$

Hence $(e_n^*, 2\sum_{j=1}^{n-1} e_j + e_n - e) \in \operatorname{gra} A$. Then by (45),

$$\langle x^{**}, e_n^* \rangle + 2 \sum_{j=1}^{n-1} \langle x^*, e_j \rangle + \langle x^*, e_n \rangle - \langle x^*, e \rangle = 0.$$

Since $\sum_{j\geq 1}\langle x^*, e_j\rangle = \langle x^*, e\rangle$, we have

$$\langle x^{**}, e_n^*\rangle = -2\sum_{j=1}^{n-1}\langle x^*, e_j\rangle - \langle x^*, e_n\rangle + \langle x^*, e\rangle = \sum_{j>n}\langle x^*, e_j\rangle - \sum_{j<n}\langle x^*, e_j\rangle. \tag{46}$$

By Fact 4.6(ii)&(iii), $\sum_{n=1}^{k}\big(\sum_{j>n}\langle x^*, e_j\rangle - \sum_{j<n}\langle x^*, e_j\rangle\big)e_n \overset{\mathrm{w}^*}{\rightharpoonup} x^{**}$. Hence $(x^*, x^{**}) \in \operatorname{gra} A$. Thus, $A$ is maximally monotone.

We next show that $A$ is of type (BR). Let $(z^*, z^{**}) \in \operatorname{gra}(-A^*) \cap X^* \times X^{**}$. Much as in the proof above starting at (45), we have $(z^*, z^{**}) \in \operatorname{gra} A$. Thus, $\operatorname{gra}(-A^*) \cap X \times X^{**} \subseteq \operatorname{gra} A$. Then by Lemma 3.2, $A$ is of type (BR).

We turn to the particularization. By Fact 4.10, $(e_n, e_n^*)_{n\in\mathbb{N}}$ is a shrinking Schauder basis for $\mathbf{J}$. By Fact 2.15 since $A$ is maximal, $T = A + \langle\cdot, e\rangle e = A + \partial\frac{1}{2}\langle\cdot, e\rangle^2$ is maximally monotone. Since $A$ is skew, we have

$$\langle x^*, x^{**}\rangle = \langle x^*, e\rangle^2, \quad \forall (x^*, x^{**}) \in \operatorname{gra} T. \tag{47}$$

Now we claim that

$$e \notin \overline{\operatorname{ran} T}. \tag{48}$$

Let $(y^*, y^{**})$ in $\operatorname{gra} T$. Then

$$\begin{aligned}
&\sum_{j=1}^{k}\big(2\sum_{i>j}\langle e_i, y^*\rangle + \langle e_j, y^*\rangle\big)e_j\\
&= \sum_{j=1}^{k}\big(\langle y^*, e\rangle + \sum_{i>j}\langle e_i, y^*\rangle - \sum_{i<j}\langle e_i, y^*\rangle\big)e_j \quad (\text{by} \sum_{i\geq 1}\langle e_j, y^*\rangle = \langle e, y^*\rangle)\\
&= \langle y^*, e\rangle\sum_{j=1}^{k} e_j + \sum_{j=1}^{k}\big(\sum_{i>j}\langle e_i, y^*\rangle - \sum_{i<j}\langle e_i, y^*\rangle\big)e_j \overset{\mathrm{w}^*}{\rightharpoonup} y^{**}.
\end{aligned} \tag{49}$$

Then by (49),

$$\begin{aligned}
\lim_k\langle y^{**}, e_k^*\rangle &= \lim_k\lim_L\langle\sum_{j=1}^{L}\big(2\sum_{i>j}\langle e_i, y^*\rangle + \langle e_j, y^*\rangle\big)e_j, e_k^*\rangle\\
&= \lim_k\big(2\sum_{i>k}\langle e_i, y^*\rangle + \langle e_k, y^*\rangle\big)\\
&= 0 \quad (\text{by} \sum_{k\geq 1}\langle e_k, y^*\rangle = \langle e, y^*\rangle).
\end{aligned} \tag{50}$$

Then by Fact 4.10, $y^{**} \in \mathbf{J}$ and hence $\operatorname{ran} T \subseteq \mathbf{J}$. Thus

$$\overline{\operatorname{ran} T} \subseteq \mathbf{J}. \tag{51}$$

Since $\langle e, e_k^* \rangle = 1, \ \forall k \in \mathbb{N}$, then by Lemma 4.7, $e \notin \mathbf{J}$. Then by (51), we have (48) holds. Combining (47), (48) and Proposition 3.5, $T = A + \langle \cdot, e \rangle e$ is neither of type (D) nor unique.

This suffices to finish the argument. ■

**Remark 4.14 ($\ell^1$)** A simpler version of the previous result recovers the original result that $\ell^1$ admits Gossez type operators. ◊

# 5 Conclusion

We have provided various tools for the further construction of pathological maximally monotone operators and related Fitzpatrick functions. In particular, we have shown — building on the work of Gossez, Phelps, Simons, Svaiter, Bueno and others, and our own previous work — that every Banach space which contains an isomorphic copy of either the James space $\mathbf{J}$ or its dual $\mathbf{J}^*$, or $c_0$ or its dual $\ell^1$, admits an operator which is not of type (D). We observe that the type (D) property is preserved by direct sums and subspaces. Since every separable space is isometric to a quotient space of $\ell^1$ [16, Theorem 5.1, page 237] or [15, Theorem 5.9, page 140], it is not preserved by quotients.

**Example 5.1 (Summary)** We list some of the salient spaces covered by our work:

(i) Separable Asplund spaces: both $\mathbf{J}$ and $c_0$ afford examples.

(ii) Separable spaces whose dual is nonseparable and contain $\ell^1$: include Ł$^1([0,1])$, $C([0,1])$ and its superspace $L^\infty([0,1])$.

(iii) Separable spaces whose dual is nonseparable but does not contain a copy of $\ell^1$: these include the James tree space **JT** [16, page 233] or [15, page 199] as it contains many copies of $\mathbf{J}$ (and of $\ell^2(\mathbb{N})$).

One remaining potential type (D) space is Gowers' space [21] which is a non-reflexive Banach space containing neither $c_0$, $\ell^1$ or any reflexive subspace. ◊

As we saw, the maximally monotone operators in our examples — with the exception of the Gossez operator — that are not of type (D) are actually not unique. This raises the question of how in generality to construct maximally monotone linear relations that are not of type (D) but that are unique.

### 5.1 Graphic of classes of maximally monotone operators

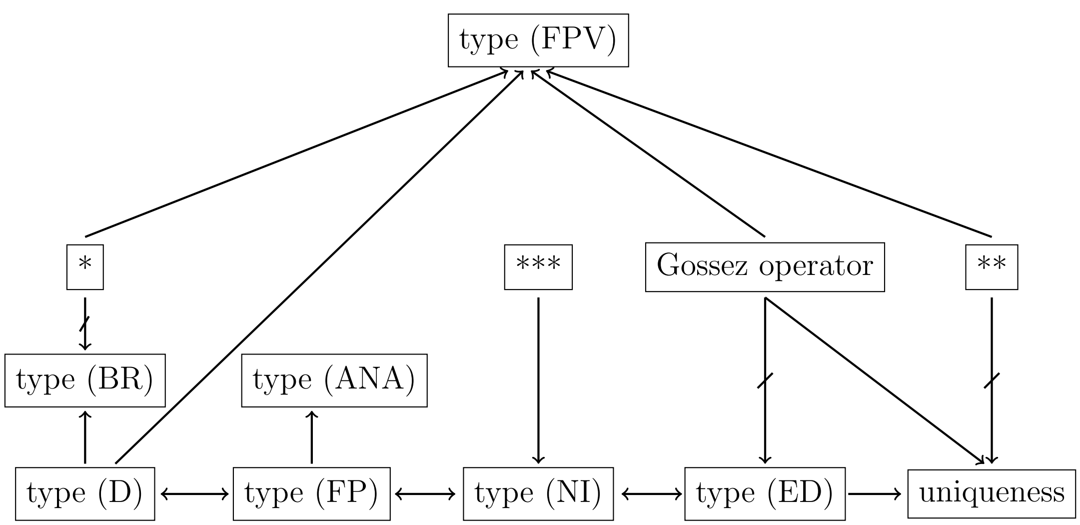
We capture much of the current state of knowledge in the following diagram in which the notation below is used.

"$*$" refers to skew operators such as $T$ in Theorem 3.6, $T_\alpha$ in Example 4.1, $A$ in Example 4.8, $A$ in Corollary 4.11, and $A$ in Corollary 4.12.

"$**$" refers to the operators such as $A\&T$ in Theorem 3.6, $A_\alpha\&T_\alpha$ in Example 4.1, $A$ in Example 4.8, $A$ in Corollary 4.11, $A$ in Corollary 4.12, and $A + \langle\cdot, e\rangle e$ in Example 4.13.

"$***$" denotes maximally monotone and unique operators with non affine graphs.

We let (ANA), (FP) and (FPV) respectively denote the other monotone operator classes "almost negative alignment", "Fitzpatrick-Phelps" and "Fitzpatrick-Phelps-Veronas". Then by [35, 11, 9, 5, 33, 25, 36, 41], we have the following relationships.




The following four questions are left open.

(i) Is every maximally monotone operator necessarily of type (FPV)?

(ii) Is every maximally monotone operator necessarily of type (ANA)?

(iii) Is every maximally monotone linear relation necessarily of type (ANA)?

(iv) Is every maximally monotone operator of type (BR) necessarily of type (ANA)?

The first of these is especially important, being closely related to the sum theorem in general Banach space (see [35, 11, 9, 40]).

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