Mahler Measures, Short Walks and Log-sine Integrals

A case study in hybrid computation

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THIS TALK: `http://carma.newcastle.edu.au/jon/alfcon.pdf`

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COMPANION PAPER AND SOFTWARE (*Th. Comp Sci*) : `http://carma.newcastle.edu.au/jon/wmi-paper.pdf`

Dedication from JB&AS in *J. AustMS*

Remark

We remark that it is fitting given the dedication of this article and volume that Alf van der Poorten [1942–2010] wrote the foreword to Lewin's "bible". In fact, he enthusiastically mentions the [log-sine] evaluation

$$-\operatorname{Ls}_4^{(1)}\left(\frac{\pi}{3}\right) = \frac{17}{6480}\pi^4$$

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Contents. We will cover some of the following:

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  - Very few closed forms are proven but more are conjectured.
- We provide systematic evaluations of various higher and multiple Mahler measures using moments of random walks and values of log-sine integrals.
- We also explore related generating functions for the log-sine integrals and their generalizations.
  - This work would be impossible without very extensive symbolic and numeric computations. It also makes frequent use of the new NIST Handbook of Mathematical Functions.

I intend to show off the interplay between numeric and symbolic computing while exploring the three mathematical topics in my title.

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My Collaborators

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$$\mathrm{Li}_{a_1,\ldots,a_k}(z) := \sum_{n_1>\cdots>n_k>0} \frac{z^{n_1}}{n_1^{a_1}\cdots n_k^{a_k}}.$$

Thus, $\mathrm{Li}_{2,1}(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2} \sum_{j=1}^{k-1} \frac{1}{j}$. Specializing produces:

- The *polylogarithm of order* k: $\mathrm{Li}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k}$.
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$$\zeta(a_1,\ldots,a_k) := \mathrm{Li}_{a_1,\ldots,a_k}(1).$$

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$$\mathrm{Ls}_n(\sigma) := -\int_0^\sigma \log^{n-1}\left|2\sin\frac{\theta}{2}\right| \mathrm{d}\theta \tag{1}$$

and their moments for $k \geq 0$ given by

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Moments of Uniform Random Walks

Definition (Moments)

For complex s the n-th moment function is

$$W_n(s) := \int_{[0,1]^n} \left| \sum_{k=1}^{n} e^{2\pi x_k i} \right|^s \mathrm{d}\boldsymbol{x}$$

Thus, $W_n := W_n(1)$ is the *expectation*.

- The integral for W_n is analytic precisely for $\operatorname{Re} s > -2$.

1905. Originated with Pearson, and Raleigh:

"What is probability at time n that the rambler is within one unit of home?"

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Clearly $W_1 = 1$. What about $W_2(1)$?

$$W_2 = \int_0^1 \int_0^1 \left| e^{2\pi i x} + e^{2\pi i y} \right| \mathrm{d}x\mathrm{d}y = ?$$

– *Mathematica* 7 and *Maple* 14 *think* the answer is 0.

- There is always a 1-dimension reduction

$$\begin{aligned} W_n(s) &= \int_{[0,1]^n} \left| \sum_{k=1}^{n} e^{2\pi x_k i} \right|^s \mathrm{d}\boldsymbol{x} \\ &= \int_{[0,1]^{n-1}} \left| 1 + \sum_{k=1}^{n-1} e^{2\pi x_k i} \right|^s \mathrm{d}(x_1, \ldots, x_{n-1}) \end{aligned}$$

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$$W_2 = 4 \int_0^{1/4} \cos(\pi x)\,\mathrm{d}x = \frac{4}{\pi}.$$

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- Similar problems get *much* more difficult in five or more dimensions — e.g., Bessel moments, Box integrals, Ising integrals (work with Bailey, Broadhurst, Crandall, ...).
  - In fact, $W_5 \approx 2.0081618$ was the best estimate we could compute *directly*, on **256** cores at Lawrence Berkeley National Laboratory.
  - Bailey and I have a general project to develop symbolic numeric techniques for (meaningful) multi-dim integrals.

When the facts change, I change my mind. What do you do, sir?
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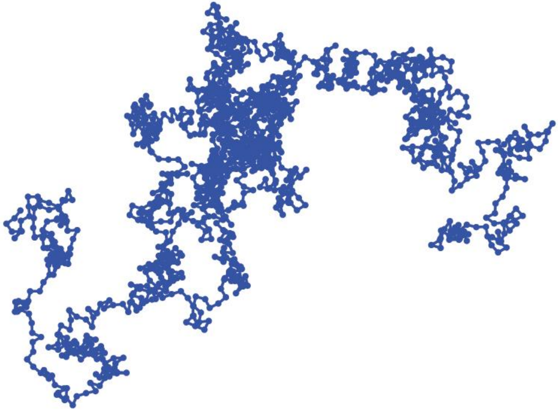
One 1500-step Ramble: a familiar picture



2D and 3D lattice walks are different:

> *A drunk man will find his way home but a drunk bird may get lost forever.*
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- 1D (and 3D) easy. Expectation of RMS distance is easy ($\sqrt{n}$).
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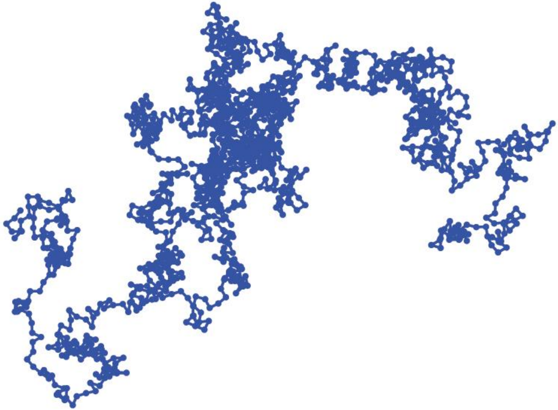

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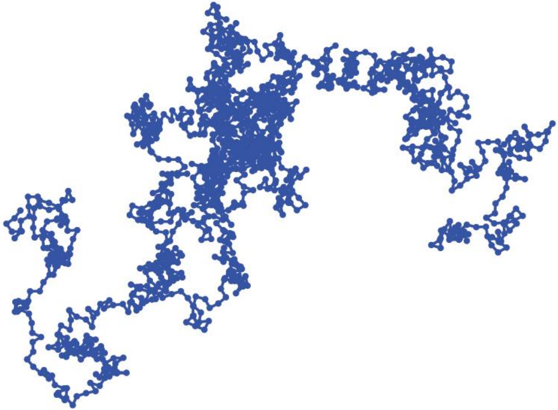

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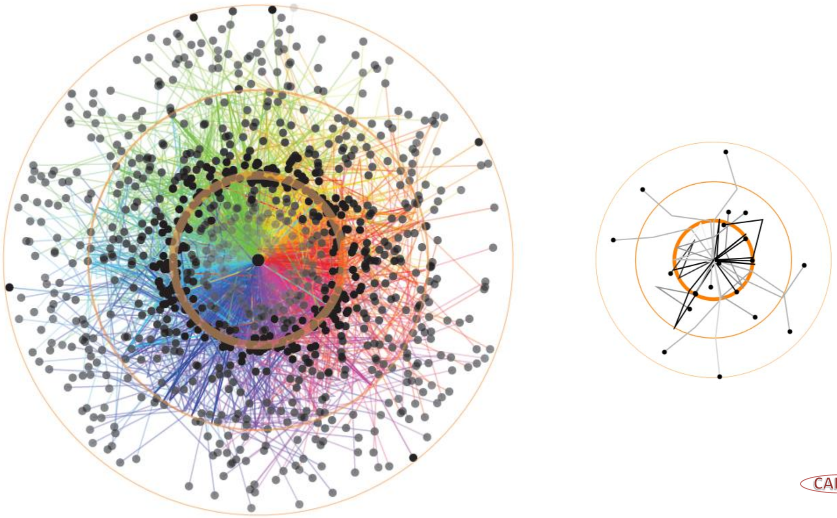

Mahler Measures (1923) in several variables

The logarithmic *Mahler measure* of a (Laurent) polynomial P:

$$\mu(P) := \int_0^1 \int_0^1 \cdots \int_0^1 \log \left| P\left(e^{2\pi i\theta_1}, \cdots, e^{2\pi i\theta_n}\right) \right| d\theta_1 \cdots d\theta_n.$$

- $M_1 := P \mapsto \exp(\mu(P))$ is multiplicative.
- $n = 1$: P is a product of cyclotomics $\Leftrightarrow M_1(P) = 1$. Lehmer's conjecture (1931) is: otherwise $M_1(P) \geq M_1(1 - x + x^3 - x^4 + x^5 - x^6 + x^7 - x^9 + x^{10})$.
- $\mu(P)$ turns out to be an example of a period.
- When $n = 1$ and P has integer coefficients $M_1(P)$ is an algebraic integer.
- In several dimensions life is harder.
  - We shall see remarkable recent results — many more discovered than proven — expressing $\mu(P)$ arithmetically.

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- In several dimensions life is harder.
  - We shall see remarkable recent results — many more discovered than proven — expressing $\mu(P)$ arithmetically.

CARMA

Mahler Measures (1923) in several variables

The logarithmic *Mahler measure* of a (Laurent) polynomial P:

$$\mu(P) := \int_0^1 \int_0^1 \cdots \int_0^1 \log \left| P\left(e^{2\pi i\theta_1}, \cdots, e^{2\pi i\theta_n}\right) \right| d\theta_1 \cdots d\theta_n.$$

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Carlson's Theorem: from discrete to continuous

Theorem (Carlson (1914, PhD))

If $f(z)$ *is analytic for* $\mathrm{Re}\,(z) \geq 0$, *its growth on the imaginary axis is bounded by* $e^{cy}, |c| < \pi$, *and*

$$0 = f(0) = f(1) = f(2) = \ldots$$

then $f(z) = 0$ *identically.*

- $\sin(\pi z)$ does not satisfy the conditions of the theorem, as it grows like $e^{\pi y}$ on the imaginary axis.
- $W_n(s)$ satisfies the conditions of the theorem (and is in fact analytic for $\mathrm{Re}\,(s) > -2$ when $n > 2$).
  - There is a lovely **1941** proof by Selberg of the bounded case.
  - The theorem lies under much of what follows.

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CARMA

$W_n(k)$ at even values

Even values are easier (combinatorial – no square roots).

k	0	2	4	6	8	10
$W_2(k)$	1	2	6	20	70	252
$W_3(k)$	1	3	15	93	639	4653
$W_4(k)$	1	4	28	256	2716	31504
$W_5(k)$	1	5	45	545	**7885**	**127905**

- Can get started by *rapidly* computing many values *naively* as symbolic integrals.
  - Observe that $W_2(s) = \binom{s}{s/2}$ for $s > -1$.
  - Entering 1, 5, 45, 545 in the *OIES* now gives *"The function $W_5(2n)$ (see Borwein et al. reference for definition)."*

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$W_n(k)$ at odd integers

n	$k = 1$	$k = 3$	$k = 5$	$k = 7$	$k = 9$
2	1.27324	3.39531	10.8650	37.2514	132.449
3	**1.57460**	6.45168	36.7052	241.544	1714.62
4	1.79909	10.1207	82.6515	822.273	9169.62
5	2.00816	14.2896	152.316	2037.14	31393.1
6	2.19386	18.9133	248.759	4186.19	82718.9

Please, memorize this number!

During the three years which I spent at Cambridge my time was wasted, as far as the academical studies were concerned, as completely as at Edinburgh and at school. I attempted mathematics, and even went during the summer of 1828 with a private tutor (a very dull man) to Barmouth, but I got on very slowly. The work was repugnant to me, chiefly from my not being able to see any meaning in the early steps in algebra. This impatience was very foolish, and in after years I have deeply regretted that I did not proceed far enough at least to understand something of the great leading principles of mathematics, for men thus endowed seem to have an extra sense. —

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CARMA

Resolution at even values

- General even formula counts n-letter abelian squares $x\pi(x)$ of length $2k$.
  - Shallit and Richmond (2008) give asymptotics:

$$W_n(2k) = \sum_{a_1+\ldots+a_n=k} \binom{k}{a_1, \ldots, a_n}^2. \tag{4}$$

- Known to satisfy convolutions:

$$W_{n_1+n_2}(2k) = \sum_{j=0}^{k} \binom{k}{j}^2 W_{n_1}(2j) W_{n_2}(2(k-j)).$$

- Has recursions such as:

$$(k+2)^2 W_3(2k+4) - (10k^2+30k+23) W_3(2k+2) + 9(k+1)^2 W_3(2k) = 0.$$

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CARMA

Analytic continuation: From Carlson's Theorem

- So integer recurrences yield complex functional equations. Viz

$$(s+4)^2W_3(s+4)-2(5s^2+30s+46)W_3(s+2)+9(s+2)^2W_3(s)=0.$$

- This gives analytic continuations of the ramble integrals to the complex plane, with poles at certain negative integers (likewise for all n).
  - $W_3(s)$ has a simple pole at -2 with residue $\frac{2}{\sqrt{3}\pi}$, and other simple poles at $-2k$ with residues a rational multiple of Res_{-2}.

> *"For it is easier to supply the proof when we have previously acquired, by the method [of mechanical theorems], some knowledge of the questions than it is to find it without any previous knowledge.* — Archimedes.

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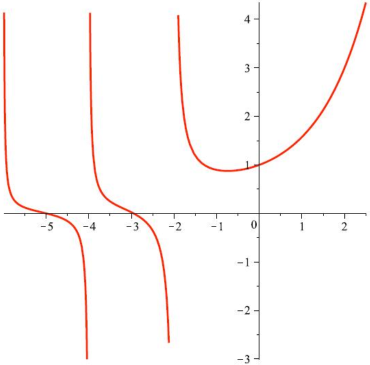
Odd dimensions look like 3

$W_3(s)$ on $[-6, \frac{5}{2}]$



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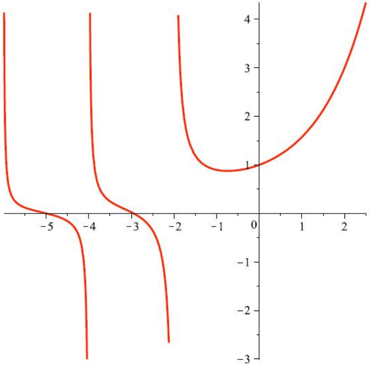

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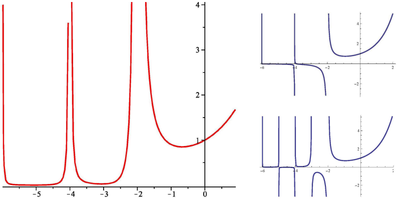
Some even dimensions look more like 4



L: $W_4(s)$ on $[-6, 1/2]$. **R**: W_5 on $[-6, 2]$ (T), W_6 on $[-6, 2]$ (B).

- The functional equation (with double poles) for $n = 4$ is

$$\begin{aligned}(s+4)^3 W_4(s+4) \quad &- \quad 4(s+3)(5s^2+30s+48)W_4(s+2) \\ &+ \quad 64(s+2)^3 W_4(s) = 0\end{aligned}$$

- There are (infinitely many) multiple poles if and only if $4|n$.
- Why is W_4 positive on $\mathbb{R}$?

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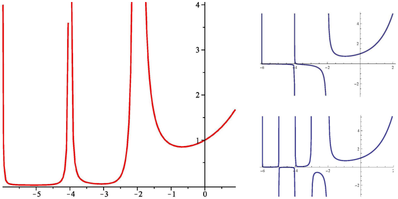

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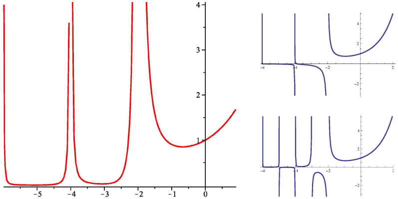

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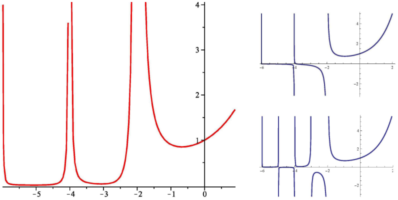

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Meijer-G functions (1936–)

Definition

$$G^{m,n}_{p,q}\left(\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\,\middle|\, x\right) := \frac{1}{2\pi i} \times$$

$$\int_{\mathcal{L}} \frac{\prod_{j=1}^{m}\Gamma(b_j - s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=n+1}^{p}\Gamma(a_j - s)\prod_{j=m+1}^{q}\Gamma(1-b_j+s)} x^s \mathrm{d}s.$$

- Contour $\mathcal{L}$ lies between poles of $\Gamma(1-a_i-s)$ and of $\Gamma(b_i+s)$.
  - A broad generalization of hypergeometric functions — capturing Bessel Y, K and much more.
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Meijer-G forms for W_3

Theorem (Meijer form for W_3)

For s not an odd integer

$$W_3(s) = \frac{\Gamma(1+\frac{s}{2})}{\sqrt{\pi}\,\Gamma(-\frac{s}{2})}\, G_{33}^{21}\left(\begin{matrix} 1,1,1 \\ \frac{1}{2}, -\frac{s}{2}, -\frac{s}{2} \end{matrix} \middle| \frac{1}{4}\right).$$

- First found by Crandall via CAS.
- Proved using residue calculus methods.
- $W_3(s)$ is among few non-trivial Meijer-G with a closed form.

> *The most important aspect in solving a mathematical problem is the conviction of what is the true result. Then it took 2 or 3 years using the techniques that had been developed during the past 20 years or so.*
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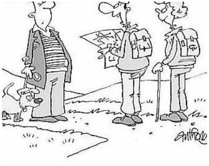
- Not helpful for odd integers. We must again look elsewhere ...



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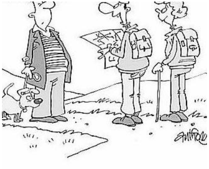

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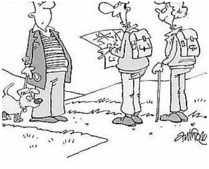

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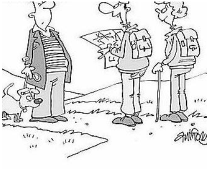
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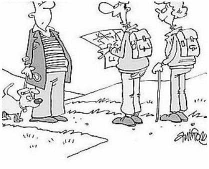

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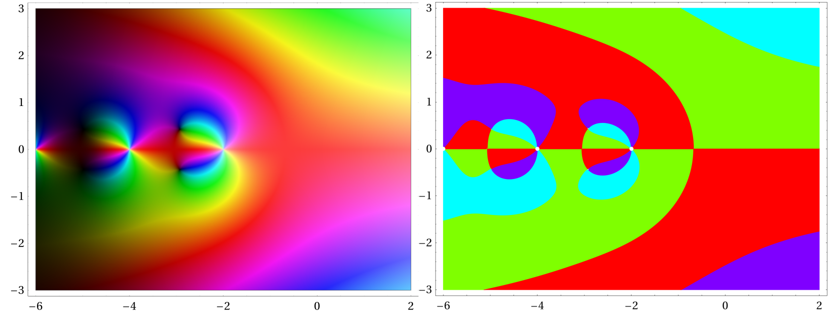
Visualizing W_4 in the complex plane



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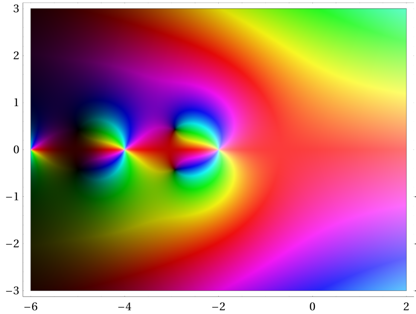

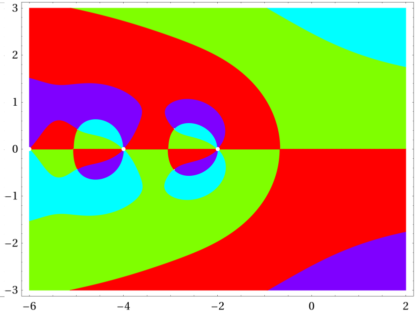

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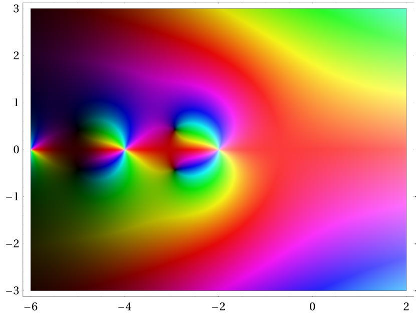

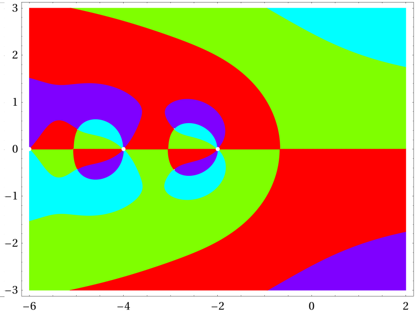

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Simplifying the Meijer integral

Corollary (Hypergeometric forms for noninteger $s > -2$)

$$W_3(s) = \frac{1}{2^{2s+1}} \tan\left(\frac{\pi s}{2}\right) \binom{s}{\frac{s-1}{2}}^2 {}_3F_2\left(\left.{\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \atop \frac{s+3}{2}, \frac{s+3}{2}}\right| \frac{1}{4}\right) + \binom{s}{\frac{s}{2}} {}_3F_2\left(\left.{-\frac{s}{2}, -\frac{s}{2}, -\frac{s}{2} \atop 1, -\frac{s-1}{2}}\right| \frac{1}{4}\right),$$

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$$W_4(-1) = \frac{\pi}{4}\, {}_7F_6\left(\left.{\frac{5}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \atop \frac{1}{4}, 1, 1, 1, 1, 1}\right| 1\right) = \frac{\pi}{4} \sum_{n=0}^{\infty} \frac{(4n+1)\binom{2n}{n}^6}{4^{6n}}$$

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Hypergeometric values of W_3, W_4: from Meijer-G values.

Much work involving moments of elliptic integrals yields:

Theorem (Tractable hypergeometric form for W_3)

(a) *For* $s \neq -3, -5, -7, \ldots$, *we have*

$$W_3(s) = \frac{3^{s+3/2}}{2\pi}\,\beta\left(s+\frac{1}{2}, s+\frac{1}{2}\right)\,{}_3F_2\left(\left.{\frac{s+2}{2}, \frac{s+2}{2}, \frac{s+2}{2} \atop 1, \frac{s+3}{2}}\right|\frac{1}{4}\right). \tag{6}$$

(b) *For every natural number* $k = 1, 2, \ldots$,

$$W_3(-2k-1) = \frac{\sqrt{3}\,\binom{2k}{k}^2}{2^{4k+1}3^{2k}}\,{}_3F_2\left(\left.{\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \atop k+1, k+1}\right|\frac{1}{4}\right).$$

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A Discovery Demystified: on piecing all this together

We first noted that:

$$W_3(2k) = \sum_{a_1+a_2+a_3=k} \binom{k}{a_1, a_2, a_3}^2 = \underbrace{{}_3F_2\left(\left.\begin{matrix} 1/2, -k, -k \\ 1, 1 \end{matrix}\right| 4\right)}_{=:V_3(2k)}.$$

We discovered *numerically* that: $V_3(1) = 1.57459 - .12602652i$

Theorem (Real part)

For all integers k we have $W_3(k) = \mathrm{Re}\,(V_3(k))$.

We have a habit in writing articles published in scientific journals to make the work as finished as possible, to cover up all the tracks, to not worry about the blind alleys or describe how you had the wrong idea first. . . . So there isn't any place to publish, in a dignified manner, what you actually did in order to get to do the work. — Richard Feynman (Nobel acceptance 1966)

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We discovered *numerically* that: $V_3(1) = 1.57459 - .12602652i$

Theorem (Real part)

For all integers k we have $W_3(k) = \mathrm{Re}\,(V_3(k))$.

We have a habit in writing articles published in scientific journals to make the work as finished as possible, to cover up all the tracks, to not worry about the blind alleys or describe how you had the wrong idea first. . . . So there isn't any place to publish, in a dignified manner, what you actually did in order to get to do the work. — Richard Feynman (Nobel acceptance 1966)

Closed Forms for W_3

- We then *confirmed* 175 digits of

$$W_3(1) \approx 1.574597237551893657 49\ldots$$

- Armed with a knowledge of elliptic integrals:

$$W_3(1) = \frac{16\sqrt[3]{4}\,\pi^2}{\Gamma(\frac{1}{3})^6} + \frac{3\Gamma(\frac{1}{3})^6}{8\sqrt[3]{4}\,\pi^4} = W3(-1) + \frac{6/\pi^2}{W3(-1)}, \quad (7)$$

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Here $\beta(s) := B(s,s) = \frac{\Gamma(s)^2}{\Gamma(2s)}$.

- Obtained via singular values of the elliptic integral and Legendre's identity.

CARMA

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CARMA

Probability: Bessel function representations

1906. J.C. Kluyver (1860-1932) derived the cumulative radial distribution function (P_n) and density (p_n) of the n-step distance:

$$P_n(t) = t \int_0^\infty J_1(xt)\, J_0^n(x)\, \mathrm{d}x$$

$$p_n(t) = t \int_0^\infty J_0(xt)\, J_0^n(x) x\, \mathrm{d}x \quad (n \geq 4) \tag{9}$$

where $J_n(x)$ is a Bessel function of the first kind

- See also Watson (1932, §49) – 3-dim walks are *elementary*.
  - From (11) below, we find

$$p_n(1) = \mathrm{Res}_{-2}\,(W_{n+1}) \qquad (n \neq 4). \tag{10}$$

  - As $p_2(\alpha) = \frac{2}{\pi\sqrt{4-\alpha^2}}$, we check in *Maple* that the following code returns $R = 2/(\sqrt{3}\pi)$ *symbolically*:

```
R:=identify(evalf[20](int(BesselJ(0,x)^3*x,x=0..infinity)))
```

CARMA

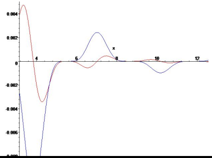
A Bessel Integral for W_n

- Now $P_n(1) = \frac{J_0(0)^{n+1}}{n+1} = \frac{1}{n+1}$ (Pearson's original question).
- Broadhurst used (9) for $2k > s > -\frac{n}{2}$ to write

$$W_n(s) = 2^{s+1-k}\frac{\Gamma(1+\frac{s}{2})}{\Gamma(k-\frac{s}{2})}\int_0^\infty x^{2k-s-1}\left(-\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^k J_0^n(x)\mathrm{d}x, \tag{11}$$

a useful oscillatory 1-dim integral (used below).
- Thence

$$W_n(-1) = \int_0^\infty J_0^n(x)\mathrm{d}x, \;\; W_n(1) = n\int_0^\infty J_1(x)J_0(x)^{n-1}\frac{\mathrm{d}x}{x}. \tag{12}$$



Integrands for $W_4(-1)$ (blue) and $W_4(1)$ (red) on $[\pi, 4\pi]$ from (12).

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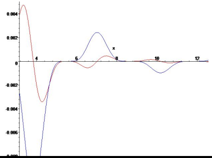

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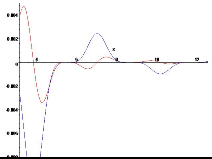

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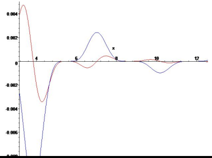

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The Densities for $n = 3, 4$ are Modular

Let $\sigma(x) := \frac{3-x}{1+x}$. Then σ is an involution on $[0, 3]$ sending $[0, 1]$ to $[1, 3]$:

$$p_3(x) = \frac{4x}{(3-x)(x+1)}\, p_3(\sigma(x)).$$

So $\frac{3}{4}p_3'(0) = p_3(3) = \frac{\sqrt{3}}{2\pi}, p(1) = \infty$. We found:

$$p_3(\alpha) = \frac{2\sqrt{3}\alpha}{\pi\,(3+\alpha^2)}\,{}_2F_1\left(\begin{matrix}\frac{1}{3}, \frac{2}{3} \\ 1\end{matrix}\middle|\frac{\alpha^2\left(9-\alpha^2\right)^2}{\left(3+\alpha^2\right)^3}\right) = \frac{2\sqrt{3}}{\pi}\frac{\alpha}{\mathrm{AG}_3(3+\alpha^2, 3\left(1-\alpha^2\right)^{2/3})}$$

where AG_3 is the *cubically convergent* mean iteration (1991):

$$\mathrm{AG}_3(a, b) := \frac{a+2b}{3} \bigotimes \left(b \cdot \frac{a^2+ab+b^2}{3}\right)^{1/3}$$

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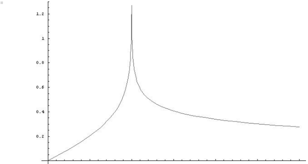

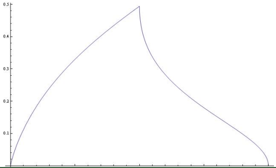

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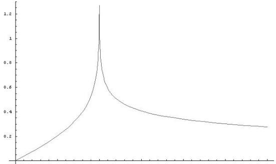
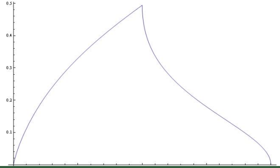
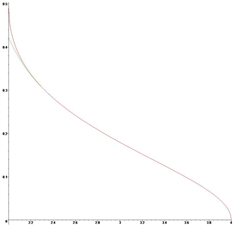




Formula for the 'shark-fin' p_4

We ultimately deduce on $2 \le \alpha \le 4$ a hyper-closed form:

$$p_4(\alpha) = \frac{2}{\pi^2} \frac{\sqrt{16-\alpha^2}}{\alpha} \, {}_3F_2\left(\left. {\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \atop \frac{5}{6}, \frac{7}{6}} \right| \frac{\left(16-\alpha^2\right)^3}{108\,\alpha^4} \right). \tag{13}$$



← p_4 from (13) vs 18-terms of series

✓ Proves $p_4(2) = \frac{2^{7/3}\pi}{3\sqrt{3}} \Gamma\left(\frac{2}{3}\right)^{-6} = \frac{\sqrt{3}}{\pi} W_3(-1) \approx 0.494233 < \frac{1}{2}$

- Marvelously, we found — and proved by a subtle use of distributional Mellin transforms — that on $[0, 2]$ as well:

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(Discovering this Re brought us full circle.)

CARMA

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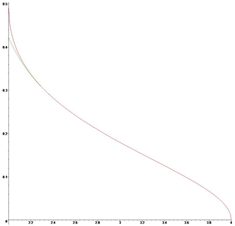

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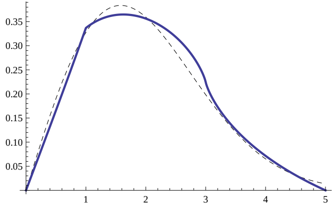
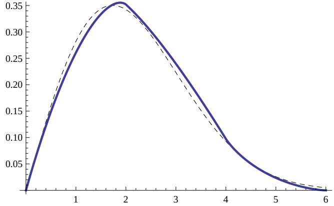
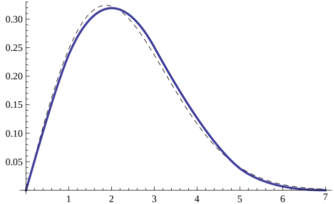
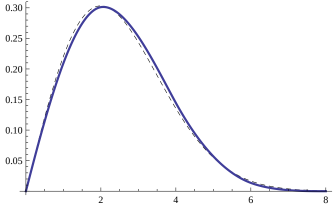
Densities for $5 \le n \le 8$ (and large n approximation)





Both p_{2n+4}, p_{2n+5} are n-times continuously differentiable for $x > 0$ $(p_n(x) \sim \frac{2x}{n} e^{-x^2/n})$. So "four is small" *but* "eight is large."





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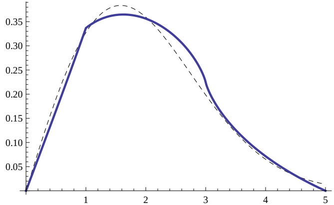

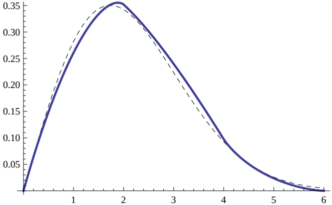

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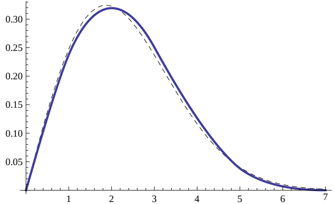

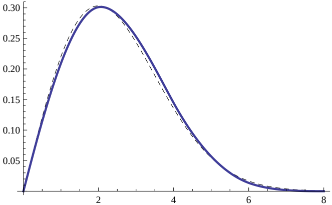

The Five Step Walk

- The functional equation for W_5 is:

$$\begin{aligned} 225(s+4)^2(s+2)^2 W_5(s) &= -(35(s+5)^4+42(s+5)^2+3)W_5(s+4) \\ + \quad & (s+6)^4 W_5(s+6) + (s+4)^2(259(s+4)^2+104)W_5(s+2). \end{aligned}$$

- We deduce *the first two poles* — and so all — *are simple* since

$$\lim_{s\to-2}(s+2)^2 W_5(s) = \frac{4}{225}\left(285\, W_5(0) - 201\, W_5(2) + 16\, W_5(4)\right) = 0$$

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- We *stumbled* upon

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??? Is there a hyper-closed form for $W_5(\mp 1)$ **???**

CARMA

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CARMA

W_5 and p_5: Bessel integrals are hard

- We only knew $\mathrm{Res}_{-4}(W_5)$ *numerically* — but to 500 digits: (Bailey in about **5.5hrs** on 1 MacPro core).
  - Sidi-"mW" method used: i.e., Gaussian quadrature on intervals of $[n\pi, (n+1)\pi]$ plus Richardson-like extrapolation.
  - **July 2011**. $r_5(2)$ was identified (with help from QFT)!

$$r_5(2) \stackrel{?}{=} \frac{13}{225} r_5(1) - \frac{2}{5\pi^4} \frac{1}{r_5(1)}. \tag{14}$$

- Here $r_5(k) := \mathrm{Res}_{(-2k)}(W_5)$. Other residues are then combinations as follows:
- From the W_5-recursion: given $r_5(0) = 0, r_5(1)$ and $r_5(2)$ we have

$$r_5(k+3) = \frac{k^4 r_5(k) - \left(5 + 28\,k + 63\,k^2 + 70\,k^3 + 35\,k^4\right) r_5(k+1)}{225(k+1)^2(k+2)^2} + \frac{\left(285 + 518\,k + 259\,k^2\right) r_5(k+2)}{225(k+2)^2}.$$

W_5 and p_5: Bessel integrals are hard

- We only knew $\mathrm{Res}_{-4}(W_5)$ *numerically* — but to 500 digits: (Bailey in about **5.5hrs** on 1 MacPro core).
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W_5 and p_5: Bessel integrals can be hard

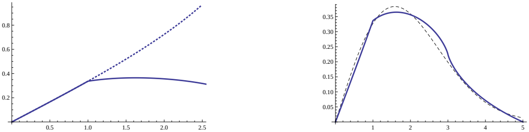

Figure: The series at zero and p_5.

- **1963**. Fettis first rigorously established nonlinearity. A few more residues yield $p_5(x) = 0.329934\,x + 0.00661673\,x^3 + 0.000262333\,x^5 + 0.0000141185\,x^7 + O(x^9)$

Hence the strikingly straight shape of $p_5(x)$ on $[0,1]$:

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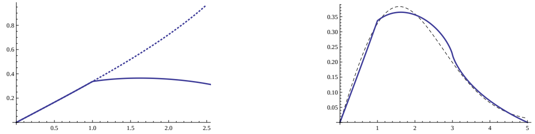

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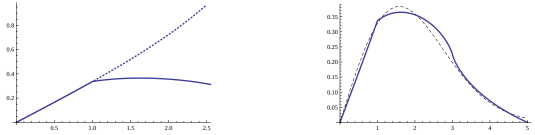

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Short Random Walks: Derivatives of W_3, W_4

From the hypergeometric forms above we get:

$$W_3'(0) = \frac{1}{\pi}\,{}_3F_2\left(\begin{matrix}\frac12,\frac12,\frac12\\ \frac32,\frac32\end{matrix}\,\middle|\,\frac14\right) = \frac{1}{\pi}\,\mathrm{Cl}\left(\frac{\pi}{3}\right). \tag{15}$$

The last equality follows from setting $\theta = \pi/6$ in the identity

$$2\sin(\theta)\,{}_3F_2\left(\begin{matrix}\frac12,\frac12,\frac12\\ \frac32,\frac32\end{matrix}\,\middle|\,\sin^2\theta\right) = \mathrm{Cl}\,(2\,\theta) + 2\,\theta\log\,(2\sin\theta)\,.$$

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Here $\mathrm{Cl}(\theta) := \sum_{n=1}^{\infty}\frac{\sin(n\theta)}{n^2}$ is *Clausen's function*. Likewise:

$$W_3'(2) = \frac{3}{\pi}\mathrm{Cl}\left(\frac{\pi}{3}\right) - \frac{3\sqrt{3}}{2\pi} + 2\ldots$$

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CARMA

Multiple Mahler Measures: for $P_1, P_2, \ldots, P_m$

toc

$$\mu(P_1, P_2, \ldots, P_m) := \int_0^1 \cdots \int_0^1 \prod_{k=1}^{m} \log \left| P_k \left(e^{2\pi i\theta_1}, \cdots, e^{2\pi i\theta_n}\right)\right| \, d\theta_1 \cdots d\theta_n,$$

was introduced by Sasaki (2010); while

$$\mu_m(P) := \mu(P, P, \ldots, P), \qquad (\mu_1(P) = \mu(P))$$

is a higher Mahler measure as in (KLO) Kurokawa, Lalín and Ochiai (2008). Also

$$\mu_m\left(1 + \sum_{k=1}^{n-1} x_k\right) = W_n^{(m)}(\mathbf{0}), \tag{17}$$

was evaluated in (15), (16) for $n = 3$ and $n = 4$ and $m = 1$:

1. $\mu(1 + x + y) = L_3'(-1) = \frac{1}{\pi} \operatorname{Cl}\left(\frac{\pi}{3}\right)$ (Smyth)
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CARMA

Relations to Dedekind's η

Denninger's **1997** conjecture, proven recently by Rogers and Zudilin (**2011**), is

$$\mu(1+x+y+1/x+1/y) \stackrel{?}{=} \frac{15}{4\,\pi^2}\,L_E(2)$$

– an L-series value for an elliptic curve E with conductor 15.

- For (17) with $n=5,6$ conjectures of Villegas become:

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$\mu(1+x+y)$ and $\mu(1+x+y+z)$ revisited

We recall:

Lemma (Jensen's formula)

$$\int_0^1 \log\left|\alpha + e^{2\pi i\, t}\right| \,\mathrm{d}t = \log\left(\max\{|\alpha|, 1\}\right). \tag{18}$$

We use (18) to reduce to a one dimensional integral:

$$\mu(1+x+y) = \int_{1/6}^{5/6} \log(2\sin(\pi y))\,\mathrm{d}y = \frac{1}{\pi}\,\mathrm{Ls}_2\left(\frac{\pi}{3}\right) = \frac{1}{\pi}\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right),$$

which is (15).

$\mu(1+x+y)$ and $\mu(1+x+y+z)$ revisited

Following Boyd, on applying Jensen's formula, for complex a and b we have $\mu(ax+b) = \log|a| \vee \log|b|$. Let $w := y/z$. We now write

$$\begin{aligned}
\mu(1+x+y+z) &= \mu(1+x+z(1+w)) = \mu(\log|1+w| \vee \log|1+x|) \\
&= \frac{1}{\pi^2}\int_0^\pi \mathrm{d}\theta \int_0^\pi \max\left\{\log\left(2\sin\frac{\theta}{2}\right), \log 2\left(\sin\frac{t}{2}\right)\right\} \mathrm{d}t \\
&= \frac{2}{\pi^2}\int_0^\pi \mathrm{d}\theta \int_0^\theta \log\left(2\sin\frac{\theta}{2}\right) \mathrm{d}t \\
&= \frac{2}{\pi^2}\int_0^\pi \theta \log\left(2\sin\frac{\theta}{2}\right) \mathrm{d}\theta \\
&= -\frac{2}{\pi^2}\,\mathrm{Ls}_3^{(1)}(\pi) = \frac{7}{2}\frac{\zeta(3)}{\pi^2},
\end{aligned}$$

which is (16).

CARMA

Boyd's 1998 Conjectures

Theorem (Two quadratic evaluations)

Below L_{-n} *is a primitive* L*-series and* G *is Catalan's constant.*

$$\begin{aligned}
\mu_3 := \mu(y^2(x+1)^2 + y(x^2 + \mathbf{6}x + 1) + (x+1)^2) &= \frac{16}{3\pi}\,\mathrm{L}_{-4}(2) \\
&= \frac{16}{3\pi}\,\mathrm{G}, \\
\mu_{-5} := \mu(y^2(x+1)^2 + y(x^2 - \mathbf{10}x + 1) + (x+1)^2) &= \frac{5\sqrt{3}}{\pi}\,\mathrm{L}_{-3}(2) \\
&= \frac{20}{3\,\pi}\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right).
\end{aligned}$$

CARMA

Log-sine Integrals are Again Inside

First proven in **2008** using Bloch-Wigner logarithms, we used a variant of Jensen's formula and slick trigonometry to arrive at:

$$\begin{aligned}\mu_3 &= \frac{1}{\pi}\int_0^{\pi} \log(1+4|\cos\theta|+4|\cos^2\theta|)\,\mathrm{d}\theta\\ &= \frac{4}{\pi}\int_0^{\pi/2} \log(1+2\cos\theta)\,\mathrm{d}\theta\\ &= \frac{4}{\pi}\int_0^{\pi/2} \log\left(\frac{2\sin\frac{3\theta}{2}}{2\sin\frac{\theta}{2}}\right)\mathrm{d}\theta\\ &= \frac{4}{3\,\pi}\left(\mathrm{Ls}_2\left(\frac{3\pi}{2}\right)-3\,\mathrm{Ls}_2\left(\frac{\pi}{2}\right)\right)=\frac{16}{3}\,\frac{\mathrm{L}_{-4}(2)}{\pi}\end{aligned}$$

as needed, since $\mathrm{Ls}_2\left(\frac{3\pi}{2}\right)=-\mathrm{Ls}_2\left(\frac{\pi}{2}\right)=\mathrm{L}_{-4}(2)$, which is Catalan's G. (μ_5 is similar.)

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as needed, since $\mathrm{Ls}_2\left(\frac{3\pi}{2}\right) = -\,\mathrm{Ls}_2\left(\frac{\pi}{2}\right) = \mathrm{L}_{-4}(2)$, which is Catalan's G. (μ_5 is similar.)

CARMA

Sasaki's Multiple Mahler Measures

toc

$$\mu_k(1+x+y_*) := \mu(1+x+y_1, 1+x+y_2, \ldots, 1+x+y_k)$$

was studied by Sasaki (2010). He used (18) to observe that

$$\mu_k(1+x+y_*) = -\int_{1/6}^{5/6} \log^k \left|1+e^{2\pi i\, t}\right| \,\mathrm{d}t \tag{19}$$

and so provides an evaluation of $\mu_2(1+x+y_*)$. Immediately from (19) and the definition of the log-sine integrals we have:

Theorem (For $k = 1, 2, \ldots$)

$$\mu_k(1+x+y_*) = \frac{1}{\pi}\left\{\mathrm{Ls}_{k+1}\left(\frac{\pi}{3}\right) - \mathrm{Ls}_{k+1}(\pi)\right\}, \tag{20}$$

where Ls_{k+1} *is as given by* (1).

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$\mathrm{Ls}_k(\pi)$ and $\mathrm{Ls}_n^{(k)}(\pi)$

$$-\frac{1}{\pi}\sum_{m=0}^{\infty}\mathrm{Ls}_{m+1}(\pi)\,\frac{u^m}{m!}=\frac{\Gamma(1+u)}{\Gamma^2\left(1+\frac{u}{2}\right)}=\binom{u}{u/2}. \qquad (21)$$

Example (Values of $\mathrm{Ls}_n(\pi)$)

For instance, we have $\mathrm{Ls}_2(\pi)=0$ as well as

$$-\mathrm{Ls}_3(\pi)=\frac{1}{12}\pi^3 \qquad \mathrm{Ls}_4(\pi)=\frac{3}{2}\pi\,\zeta(3)$$

$$-\mathrm{Ls}_5(\pi)=\frac{19}{240}\pi^5 \qquad \mathrm{Ls}_6(\pi)=\frac{45}{2}\pi\,\zeta(5)+\frac{5}{4}\pi^3\zeta(3)$$

$$-\mathrm{Ls}_7(\pi)=\frac{275}{1344}\pi^7+\frac{45}{2}\pi\,\zeta^2(3)$$

$\mathrm{Ls}_n(\pi)$ and $\mathrm{Ls}_n^{(k)}(\pi)$

Equation (21) is made for a CAS (Mma, Sage or Maple):

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for k to 7 do
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We studied general log-sine evaluations with an emphasis on automatic provable evaluations. For example:

Theorem (Borwein-Straub)

For $2|\mu| < \lambda < 1$ *we have*

$$-\sum_{n,k\geq 0} \mathrm{Ls}_{n+k+1}^{(k)}(\pi) \frac{\lambda^n}{n!} \frac{(i\mu)^k}{k!} = i \sum_{n\geq 0} \binom{\lambda}{n} \frac{(-1)^n e^{i\pi\frac{\lambda}{2}} - e^{i\pi\mu}}{\mu - \frac{\lambda}{2} + n}.$$

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$\mathrm{Ls}_n^{(k)}(\tau)$ is Made of Sterner Stuff.

- Contour integration and "polylogarithmics" yield an ugly but very efficient result:

Theorem (Reduction Theorem for $0 \le \tau \le 2\pi$)

For n, k such that $n - k \ge 2$, we have

$$\zeta(k, \{1\}^n) - \sum_{j=0}^{k-2} \frac{(-i\tau)^j}{j!} \mathrm{Li}_{k-j,\{1\}^n}(\mathrm{e}^{i\tau})$$

$$= \frac{(-i)^{k-1}}{(k-2)!} \frac{(-1)^n}{(n+1)!} \sum_{r=0}^{n+1} \sum_{m=0}^{r} \binom{n+1}{r} \binom{r}{m} \left(\frac{i}{2}\right)^r (-\pi)^{r-m} \mathrm{Ls}_{n+k-(r-m)}^{(k+m-2)}(\tau).$$

where $\mathrm{Li}_{2+k-j,\{1\}^{n-k-2}}(\mathrm{e}^{i\tau})$ is a harmonic polylogarithm and $\zeta(n-k, \{1\}^k)$ is an Euler-Zagier sum.

CARMA

$\mathrm{Ls}_n^{(k)}\left(\frac{\pi}{3}\right)$: A small miracle occurs: $\mathrm{e}^{-i\frac{\pi}{3}} = \overline{\mathrm{e}^{i\frac{\pi}{3}}}$.

The Reduction Theorem now lets us find all values of $\mathrm{Ls}_n^{(k)}\left(\frac{\pi}{3}\right)$ and so of $\mu_k(1+x+y_*)$:

Example (Values of $\mathrm{Ls}_n\left(\pi/3\right)$)

$$\mathrm{Ls}_2\left(\frac{\pi}{3}\right) = \mathrm{Cl}_2\left(\frac{\pi}{3}\right) \qquad -\mathrm{Ls}_3\left(\frac{\pi}{3}\right) = \frac{7}{108}\,\pi^3$$

$$\mathrm{Ls}_4\left(\frac{\pi}{3}\right) = \frac{1}{2}\pi\,\zeta(3) + \frac{9}{2}\,\mathrm{Cl}_4\left(\frac{\pi}{3}\right)$$

$$-\mathrm{Ls}_5\left(\frac{\pi}{3}\right) = \frac{1543}{19440}\pi^5 - 6\,\mathrm{Gl}_{4,1}\left(\frac{\pi}{3}\right)$$

$$\mathrm{Ls}_6\left(\frac{\pi}{3}\right) = \frac{15}{2}\pi\,\zeta(5) + \frac{35}{36}\,\pi^3\zeta(3) + \frac{135}{2}\,\mathrm{Cl}_6\left(\frac{\pi}{3}\right)$$

$$-\mathrm{Ls}_7\left(\frac{\pi}{3}\right) = \frac{74369}{326592}\pi^7 + \frac{15}{2}\,\pi\zeta(3)^2 - 135\,\mathrm{Gl}_{6,1}\left(\frac{\pi}{3}\right)$$

A Result for General τ

- An illustration of results produced by our programs:

Example (For $0 \le \tau \le 2\pi$)

$$\begin{aligned}
\mathrm{Ls}_4^{(1)}(\tau) &= 2\zeta(3,1) - 2\,\mathrm{Gl}_{3,1}(\tau) - 2\tau\,\mathrm{Gl}_{2,1}(\tau) \\
&+ \frac{1}{4}\mathrm{Ls}_4^{(3)}(\tau) - \frac{1}{2}\pi\,\mathrm{Ls}_3^{(2)}(\tau) + \frac{1}{4}\pi^2\,\mathrm{Ls}_2^{(1)}(\tau) \\
&= \frac{1}{180}\pi^4 - 2\,\mathrm{Gl}_{3,1}(\tau) - 2\tau\,\mathrm{Gl}_{2,1}(\tau) \\
&- \frac{1}{16}\tau^4 + \frac{1}{6}\pi\tau^3 - \frac{1}{8}\pi^2\tau^2.
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CARMA

Irreducibility and Binomial Sums

Example (The first presumably irreducible value for $\pi/3$)

$$\mathrm{Gl}_{4,1}\left(\frac{\pi}{3}\right) = \sum_{n=1}^{\infty} \frac{\sum_{k=1}^{n-1} \frac{1}{k}}{n^4} \sin\left(\frac{n\pi}{3}\right)$$
$$= \frac{3341}{1632960}\pi^5 - \frac{1}{\pi}\zeta^2(3) - \frac{3}{4\pi} \sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n} n^6}$$

while always

$$\mathrm{Ls}_{n+2}^{(1)}\left(\frac{\pi}{3}\right) = \frac{n!(-1)^{n+1}}{2^n} \sum_{k=1}^{\infty} \frac{1}{k^{n+2}\binom{2k}{k}}.$$

- Alternating binomial sums come from imaginary values of τ via $\log\sinh$ integrals at $\rho = \frac{1+\sqrt{5}}{2}$.

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CARMA

First Evaluation

Let

$$\mu_k(1+x+y_*+z_*) := \mu(1+x+y_1+z_1,\ldots,1+x+y_k+z_k). \tag{22}$$

Theorem

For all positive integers k, we have

$$\mu_k(1+x+y_*+z_*) = -\frac{1}{\pi^{k+1}}\int_0^\pi \left(\theta\log\left(2\sin\frac{\theta}{2}\right) - \mathrm{Cl}_2\left(\theta\right)\right)^k \mathrm{d}\theta$$

Then

$$\mu_1(1+x+y_*+z_*) = -\frac{2}{\pi^2}\,\mathrm{Ls}_3^{(1)}\left(\pi\right) = \frac{7}{2}\frac{\zeta(3)}{\pi^2},$$

$$\mu_2(1+x+y_*+z_*) = -\frac{1}{\pi^3}\,\mathrm{Ls}_5^{(2)}\left(\pi\right) + \frac{\pi^2}{90} = \frac{4}{\pi^2}\,\mathrm{Li}_{3,1}(-1) + \frac{7}{360}\pi^2.$$

Two More Evaluations: with Kummer-type logarithms

Let

$$\lambda_n(x) := (n-2)! \sum_{k=0}^{n-2} \frac{(-1)^k}{k!} \mathrm{Li}_{n-k}(x) \log^k |x| + \frac{(-1)^n}{n} \log^n |x|,$$

so that

$$\lambda_1\left(\tfrac{1}{2}\right) = \log 2, \quad \lambda_2\left(\tfrac{1}{2}\right) = \frac{1}{2}\zeta(2), \quad \lambda_3\left(\tfrac{1}{2}\right) = \frac{7}{8}\zeta(3),$$

and $\lambda_4\left(\tfrac{1}{2}\right)$ is the first to reveal the presence of $\mathrm{Li}_n\left(\tfrac{1}{2}\right)$. From the value of $W_4''(0)$ we derive:

Theorem

$$\mu_2(1+x+y+z) = \frac{12}{\pi^2}\lambda_4\left(\tfrac{1}{2}\right) - \frac{\pi^2}{5}$$

$$\mu(1+x, 1+x, 1+x+y+z) = \frac{4}{3\pi^2}\lambda_5\left(\tfrac{1}{2}\right) - \frac{3}{4}\zeta(3) + \frac{31}{16\pi^2}\zeta(5).$$

KLO's Mahler Measures

Theorem (Hypergeometric forms for $\mu_n(1+x+y)$)

For complex $|s|<2$, we may write

$$\sum_{n=0}^{\infty}\mu_n(1+x+y)\frac{s^n}{n!}=\frac{\sqrt{3}}{2\pi}\,3^{s+1}\,\frac{\Gamma(1+\frac{s}{2})^2}{\Gamma(s+2)}\,{}_3F_2\left(\left.{\frac{s+2}{2},\frac{s+2}{2},\frac{s+2}{2} \atop 1,\frac{s+3}{2}}\right|\frac{1}{4}\right) \tag{23}$$

$$=\frac{\sqrt{3}}{\pi}\left(\frac{3}{2}\right)^{s+1}\int_0^1\frac{z^{1+s}\,{}_2F_1\left(\left.{1+\frac{s}{2},1+\frac{s}{2} \atop 1}\right|\frac{z^2}{4}\right)}{\sqrt{1-z^2}}\,\mathrm{d}z.$$

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Evaluation of $\mu_n(1+x+y)$ Requires a Taylor Expansion

Consider

$$ {}_3F_2\left(\begin{matrix}\frac{\varepsilon+2}{2}, \frac{\varepsilon+2}{2}, \frac{\varepsilon+2}{2} \\ 1, \frac{\varepsilon+3}{2}\end{matrix}\,\middle|\, \frac{1}{4}\right) = \sum_{n=0}^{\infty} \alpha_n \varepsilon^n. \tag{24} $$

Indeed, from (23) and Leibnitz' rule we have

$$ \mu_n(1+x+y) = \frac{\sqrt{3}}{2\pi} \sum_{k=0}^{n} \binom{n}{k} \alpha_k \beta_{n-k} \tag{25} $$

where β_k is defined by

$$ 3^{\varepsilon+1} \frac{\Gamma(1+\frac{\varepsilon}{2})^2}{\Gamma(\varepsilon+2)} = \sum_{n=0}^{\infty} \beta_n \varepsilon^n. $$

Note, as above, that β_k is easy to compute.

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Faà di Bruno's Formula

We can now read off the terms α_n of the ε-expansion:

Theorem (For $n = 0, 1, 2, \ldots$)

Let $A_{k,j} := \sum_{m=2}^{2j-1} \frac{2(-1)^{m+1}-1}{m^k}$. *Then*

$$[\varepsilon^n]\ {}_3F_2\left(\begin{matrix} \frac{\varepsilon+2}{2}, \frac{\varepsilon+2}{2}, \frac{\varepsilon+2}{2} \\ 1, \frac{\varepsilon+3}{2} \end{matrix} \middle| \frac{1}{4}\right) = (-1)^n \sum_{j=1}^{\infty} \frac{2}{j} \frac{1}{\binom{2j}{j}} \sum \prod_{k=1}^{n} \frac{A_{k,j}^{m_k}}{m_k! k^{m_k}} \tag{26}$$

where we sum over all $m_1, \ldots, m_n$ *with* $m_1 + 2m_2 + \ldots + nm_n = n$.

Proof.

Equation (26) follows from (23) on using Faà di Bruno's formula for the n-th derivative of the composition on two functions via Pochhammer notation. □

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Davydychev and Kalmykov's Binomial Sums Yield:

Example

$$\mu_1(1+x+y) = \frac{3}{2\pi}\,\mathrm{Ls}_2\left(\frac{2\pi}{3}\right)$$

$$\mu_2(1+x+y) = \frac{3}{\pi}\,\mathrm{Ls}_3\left(\frac{2\pi}{3}\right) + \frac{\pi^2}{4}$$

$$\mu_3(1+x+y) \stackrel{?}{=} \frac{6}{\pi}\,\mathrm{Ls}_4\left(\frac{2\pi}{3}\right) - \frac{9}{\pi}\,\mathrm{Cl}_4\left(\frac{\pi}{3}\right) - \frac{\pi}{4}\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{1}{2}\zeta(3).$$

As we had obtained by other methods. Also PSLQ then finds:

$$\pi\mu_4(1+x+y) \stackrel{?}{=} 12\,\mathrm{Ls}_5\left(\frac{2\pi}{3}\right) - \frac{49}{3}\,\mathrm{Ls}_5\left(\frac{\pi}{3}\right) + 81\,\mathrm{Gl}_{4,1}\left(\frac{2\pi}{3}\right)$$

$$+ 3\pi^2\,\mathrm{Gl}_{2,1}\left(\frac{2\pi}{3}\right) + 2\zeta(3)\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \pi\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right)^2 - \frac{29}{90}\pi^5.$$

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Conclusion

We also have generalized arctangent forms, such as:

$$\mu_2(1+x+y) = \frac{24}{5\pi}\mathrm{Ti}_3\left(\frac{1}{\sqrt{3}}\right) + \frac{2\log 3}{\pi}\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{\log^2 3}{10} - \frac{19\pi^2}{180}.$$

1. We still seek for a complete accounting of $\mu_n(1+x+y)$.
2. Our log-sine and MZV algorithms uncovered many errors and gaps (e.g., values of Euler sums such as $\zeta(\overline{2n+1}1)$ in terms of $\mathrm{Ls}_{2n}^{(2n-3)}(\pi)$) in the literature.
3. Automated simplification, validation and correction tools are more and more important.
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