PROXIMALITY AND CHEBYSHEV SETS

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ABSTRACT. This paper is a companion to a lecture given at the Prague Spring School in Analysis in April 2006. It highlights four distinct variational methods of proving that a finite dimensional Chebyshev set is convex and hopes to inspire renewed work on the open question of whether every Chebyshev set in Hilbert space is convex.

1. INTRODUCTION

Let us set some notation and definitions which are for the most part consistent with those in [6, 9, 24, 12]. For a nonempty set A in a Banach space $(X, \|\cdot\|)$ we consider the *indicator function* $\iota_A(x) := 0$ if $x \in A$ and $+\infty$ otherwise. The *distance function* $d_A(x) := \inf_{a\in A} \|x - a\|$ and the *radius function* $r_A(x) := \sup_{a\in A} \|x - a\|$ are our main players. Note that r_A is finite if and only if A is bounded and then $r_A = r_{\overline{\mathrm{co}}\, A}$ is a continuous convex function.

The variational problems we consider are to determine when and if d_A and r_A attain their bounds. Specifically

$$P_A(x) := \operatorname{argmin} d_A$$

and

$$F_A(x) := \operatorname{argmax} r_A,$$

define the *nearest point* and *farthest point* operators respectively. When $P_A(x) \neq \emptyset$ we say x admits *best approximations* or *nearest points* and call the elements of $P_A(x)$ *nearest points* or *proximal points. Worst approximation* and *farthest point* are correspondingly defined in terms of F_A. A set is called *proximal* (sometimes *proximinal*) if $D(P_A) = X$ and *Chebyshev* if P_A is both everywhere defined and single-valued. We try to reserve the symbols S for a Chebyshev set and E for a Euclidean space. In that case especially, P_A is often called the *metric projection* on A, and we shall not always distinguish $\{P_A(x)\}$ and $P_A(x)$.

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2. Concepts and Tools

As we shall see, these two problems are wonderful testing grounds for nonlinear and convex analysis. A fine variational tool is:

Theorem 1. (**Basic Ekeland principle**, [9, 6, 15, 17].) *Suppose the function $f : E \mapsto (-\infty, \infty]$ is closed and the point $x \in E$ satisfies $f(x) < \inf f + \epsilon$ for some real $\epsilon > 0$. Then for any real $\lambda > 0$ there is a point $v \in E$ satisfying the conditions*

(a) $\|x - v\| \leq \lambda$,
(b) $f(v) + (\epsilon/\lambda)\|x - v\| \leq f(x)$, *and*
(c) v *minimizes the function* $f(\cdot) + (\epsilon/\lambda)\|\cdot - v\|$.

Usually (b) is decoupled to yield (a) and (b$'$) $f(v) \leq f(x)$, but we shall need the full power of (b). Sadly, the short finite-dimensional proof in [17, 6, 9] does not seem to produce (b).

Fact 2. (**Projection**, [12].) *Let A be a closed set in a Hilbert space. Suppose that $a \in P_A(x)$. Then $P_A(tx + (1-t)a) = \{a\}$ for $0 < t < 1$.*

This clearly holds in any rotund Banach space—that is one with a strictly convex unit ball.

Fact 3. (**Chebyshev**, [12, 15, 9].) *Every Chebyshev set is closed and every closed convex set in a rotund reflexive space is Chebyshev. In particular every non-empty closed convex set in Hilbert space is Chebyshev.*

Uniqueness requires only rotundity. A much deeper result is:

Proposition 4. (**Reflexivity**, ([12, 15].) *A space X is reflexive iff every closed convex set C is proximinal iff every closed convex set has nearest points.*

Proof. In reflexive space every closed convex set is boundedly relatively weakly compact. Since the norm is weakly lower semicontinuous the problem $\min_{c \in C} \|x - c\|$ is attained for all $x \in X$.

If X is not reflexive, then the *James theorem* [14] guarantees the existence of a norm-one linear functional f such that $f(x) < 1$ for all $x \in B_X$, the unit ball. It is an instructive exercise to determine that $d_{f^{-1}(0)}(x)$ is not attained unless $f(x) = 0$. □

We shall see in Corollary 20 that there are non-reflexive spaces in which each bounded closed set admits proximal points. The non-expansiveness of the metric projection on a closed convex set in Hilbert space is standard and follows from the necessary and sufficient condition

$$\langle x - P_C(x), c - x \rangle \leq 0$$

for all $x \in C$.

We will now be more precise and interpolate a notion which greatly strengthens the property of Fact 2. We call $S \subset E$ a *sun* if, for each point

$x \in E$, every point on the ray $P_S(x) + \mathbf{R}_+(x - P_S(x))$ has nearest point $P_S(x)$.

Proposition 5. (**Suns**, [6, 12, 15].) *In Hilbert space (i) a closed set C is convex iff (ii) C is a sun iff (iii) the metric projection P_C is nonexpansive.*

Proof. We sketch the proof. It is easy to see that (i) implies (ii); while (iii) implies (i) is usually proved by a mean value argument. It remains to show (ii) implies (iii). Denoting the segment between points $y, z \in E$ by $[y, z]$, one shows that property (ii) implies

$$P_S(x) = P_{[z,P_S(x)]}(x) \text{ for all } x \in E, \ z \in S,$$

which quickly yields (iii), [6, 12]. □

In three(?)–or-more dimensions non-expansivity characterizes Euclidean space [**?**].

A fundamental result of much independent use is:

Proposition 6. (**Characterization of Chebyshev sets**, [6, 12, 15].) *If E is Euclidean then the following are equivalent.*

1. *S is Chebyshev.*
2. *P_S is single-valued and continuous.*
3. *d_S^2 is everywhere Fréchet differentiable with $\nabla_F d_S^2/2 = I - P_S$.*
4. *The Fréchet sub-differential $\partial_F(-d_S)^2(x)$ is never empty.*

Proof. (1) ⇒ (2) follows by a compactness argument. (2) ⇒ (3) is nearly immediate since $I - P_S$ is a continuous selection of $\partial d_S^2/2$. (3) ⇒ (4). We will see a proof of (4) ⇒ (1) in the next section. □

This all remains true assuming only the space to be finite dimensional with a smooth and rotund norm—indeed many of implications remain true in Banach space at least for 'tame' sets. The only really problematic step is (1) ⇒ (2).

A more flexible notion than that of a sun is that of an approximately convex set, [6, 15]. We call $C \subset X$ *approximately convex* if, for any closed norm ball $D \subset X$ disjoint from C, there exists a closed ball $D' \supset D$ disjoint from C with arbitrarily large radius. Immediate from the definitions, as illustrated in Figure 1 we have:

Proposition 7. *Every sun is approximately convex.*

Proposition 8. (**Approximate convexity**, [6, 15].) *Every convex set in a Banach space is approximately convex. When the space is finite dimensional and the dual norm is rotund every approximately convex set is convex.*

Proof. The first assertion follows easily from the Hahn-Banach theorem [15, 9, 24].

Conversely, suppose C is approximately convex but not convex. Then there exist points $a, b \in C$ and a closed ball D centered at the point $c :=$

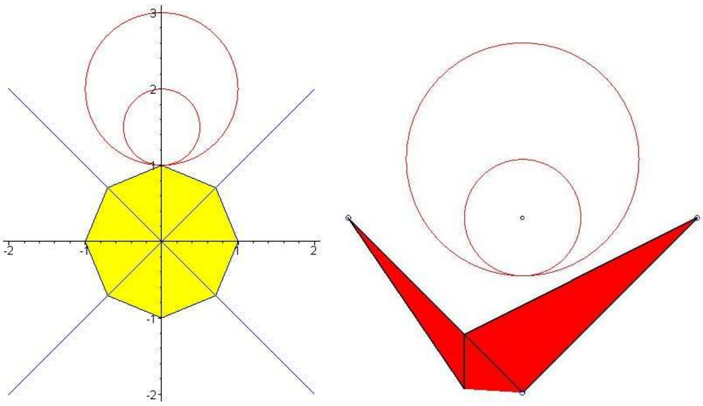

FIGURE 1. Suns and approximate convexity.

$(a+b)/2$ and disjoint from C. Hence, there exists a sequence of points $x_1, x_2, \ldots$ such that the balls $B_r = x_r + rB$ are disjoint from C and satisfy $D \subset B_r \subset B_{r+1}$ for all $r = 1, 2, \ldots$.

The set $H := \mathrm{cl} \cup_r B_r$ is closed and convex, and its interior is disjoint from C but contains c. It remains to confirm that H is a half-space. Suppose the unit vector u lies in the polar set H°. By considering the quantity $\langle u, \|x_r - x\|^{-1}(x_r - x)\rangle$ as $r \uparrow \infty$, we discover H° must be a ray. This means H is a half-space. □

In ℓ^1 or ℓ^∞ norms this clearly fails as the righthand-side of Figure 1 suggests. In the first case consider $\{(x,y)\colon y \le |x|\}$. Vlasov [15, p. 242] shows dual rotundity characterizes the coincidence of convexity and approximate convexity, [15].

We shall also exploit unexpected relationships between convexity and smoothness properties of d_A and r_A. For this we begin with:

Fact 9. (**Fenchel conjugation**,[6, 15].) *The* convex conjugate *of an extended real-valued function f on a Banach space X is defined by*

$$f^*(x^*) := \sup_{x \in X} \{\langle x, x^* \rangle - f(x)\}$$

and is a convex, closed function (possibly infinite). Moreover, the biconjugate *defined on X^* by*

$$f^{**}(x) := \sup_{x^* \in X^*} \{\langle x, x^* \rangle - f^*(x^*)\}$$

agrees with f exactly when f is convex, proper and lower-semicontinuous.

Fact 9 is often a fine way of proving convexity of a function g by showing g arises as a conjugate, see [24, 6, 9], even by computer [2]. A particularly good tool is:

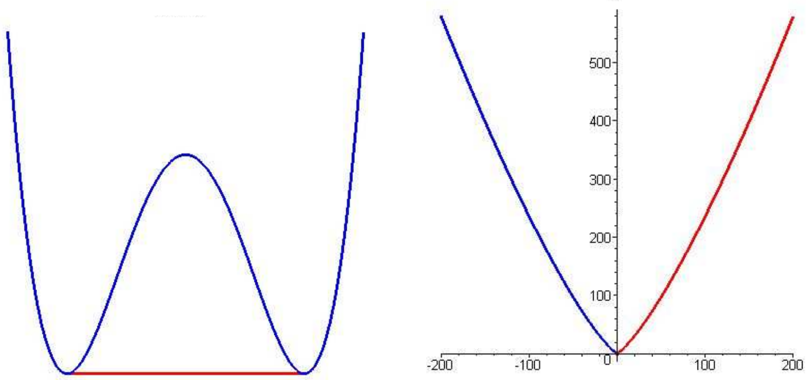

FIGURE 2. A smooth nonconvex 'W' function and its nonsmooth conjugate.

Proposition 10. (**Smoothness and biconjugacy**, [19, 27].) *If f^{**} is proper in a Banach space and f^* is everywhere Fréchet differentiable then f is convex.*

Proof. The general result may be found in [8, 27]. Under stronger conditions in a finite dimensional space E we shall prove more, [6, 18].

We consider an extended real valued function f that is closed and bounded below and satisfies the growth condition

$$\lim_{\|x\|\mapsto\infty} \frac{f(x)}{\|x\|} = +\infty,$$

along with a point $x \in \operatorname{dom} f$. Then *Carathéodory's theorem* [6, §1.2] ensures there exist points $x_1, x_2, \ldots, x_m \in E$ and real $\lambda_1, \lambda_2, \ldots, \lambda_m > 0$ satisfying

$$\sum_i \lambda_i = 1, \ \sum_i \lambda_i x_i = x, \ \sum_i \lambda_i f(x_i) = f^{**}(x).$$

The definitional *Fenchel-Young inequality*, $f(x) + f^*(x^*) \geq \langle x, x^* \rangle$ valid for all x, x^*, implies that

$$\partial(f^{**})(x) = \bigcap_i \partial f(x_i).$$

Suppose now that the conjugate f^* is indeed everywhere differentiable. If $x \in \operatorname{ri}(\operatorname{dom}(f^{**}))$, we argue that $x_i = x$ for each i. We conclude that $\operatorname{ri}(\operatorname{epi}(f^{**})) \subset \operatorname{epi}(f)$, and use the fact that f is closed to deduce $f = f^{**}$; and so f is convex. □

We illustrate the duality for $W := x \mapsto (1-x^2)^2$ in Figure 2. The lefthand picture shows W and W^{**}, the righthand shows W^*.

We record next two lovely Hilbertian duality formulas:

Fact 11. (**Hilbert duality**, [6, 18].) *For any closed set A in a Hilbert space*

$$\left(\frac{\iota_A + \|\cdot\|^2}{2}\right)^* = \frac{\|\cdot\|^2 + d_A^2}{2} \tag{1}$$

$$\left(\frac{\iota_{-A} - \|\cdot\|^2}{2}\right)^* = \frac{r_A^2 - \|\cdot\|^2}{2}. \tag{2}$$

Each identity once known is an easy direct computation from the definitions.

We now turn to our final approach via inversive geometry. The self-inverse map $\iota : E \setminus \{0\} \mapsto E$ defined by $\iota(x) = \|x\|^{-2}x$ is called the *inversion in the unit sphere*. While this is meaningful in any Banach space it is nicest in Hilbert space.

Fact 12. (**Preservation of spheres**, [1].) *If $D \subset E$ is a ball with $0 \in \mathrm{bd}\, D$, then $\iota(D \setminus \{0\})$ is a halfspace disjoint from 0. Otherwise, for any point $x \in E$ and radius $\delta > \|x\|$,*

$$\iota((x + \delta B) \setminus \{0\}) = \frac{1}{\delta^2 - \|x\|^2}\{y \in E : \|y + x\| \geq \delta\}.$$

3. Proximality and Chebyshev sets in Euclidean space

We now describe four approaches to the following classic theorem.

Theorem 13. (**Motzkin-Bunt**, [1, 6, 12, 15, 18].) *A finite dimensional Chebyshev set is convex.*

Proof. (**1, via fixed point theory**, [6, 12].) By Proposition 5 it suffices to show S is a sun. Suppose S is not a sun, so there is a point $x \notin S$ with nearest point $P_S(x) =: \overline{x}$ such that the ray $L := \overline{x} + \mathbf{R}_+(x - \overline{x})$ strictly contains

$$\{z \in L \mid P_S(z) = \overline{x}\}.$$

Hence by Fact 2 and the continuity of P_S, the above set is a nontrivial closed line segment $[\overline{x}, x_0]$ containing x.

Choose a radius $\epsilon > 0$ so that the ball $x_0 + \epsilon B$ is disjoint from S. The continuous self map of this ball

$$z \mapsto x_0 + \epsilon \frac{x_0 - P_S(z)}{\|x_0 - P_S(z)\|}$$

has a fixed point by Brouwer's theorem. We then quickly derive a contradiction to the definition of the point x_0. We illustrate this construction in Figure 3. □

Alternatively, via Proposition 8 it suffices to show S is approximately convex. This method is the least coupled to Hilbert space.

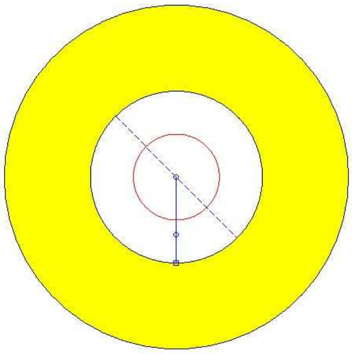

FIGURE 3. Failure of a sun.

Proof. (**2, via the variational principle**, [6, 15].) Suppose S is not approximately convex. We claim that: for each $x \notin S$

$$\limsup_{y \to x} \frac{d_S(y) - d_S(x)}{\|y - x\|} = 1. \tag{3}$$

This is a consequence of the (Lebourg) mean-value for (Lipschitz) functions [6, 11], since all Fréchet (super-)gradients have norm-one off S.

We now appeal to the Basic Ekeland principle of Proposition 1 as follows: Consider any real $\alpha > d_C(x)$. Fix reals $\sigma \in (0, 1)$ and ρ satisfying

$$\frac{\alpha - d_C(x)}{\sigma} < \rho < \alpha - \beta.$$

By applying the Basic Ekeland variational principle to the function $-d_C + \delta_{x+\rho B}$, prove there exists a point $v \in E$ satisfying the conditions

$$\begin{aligned} d_C(x) + \sigma\|x - v\| &\leq d_C(v) \\ d_C(z) - \sigma\|z - v\| &\leq d_C(v) \text{ for all } z \in x + \rho B. \end{aligned}$$

We deduce $\|x - v\| = \rho$, and hence $x + \beta B \subset v + \alpha B$. Thus, C is approximately convex and Proposition 8 concludes this proof. $\square$

We next consider two theorems that exploit conjugate duality.

Proof. (**3, via conjugate duality**, [6, 18].) First, d_S^2 is differentiable by Proposition 6. Now consider formula (1). The righthand side is clearly differentiable and it suffices to appeal to Proposition 10 to deduce that $\iota_S + \|\cdot\|^2$ is convex. A fortiori, so is S. $\square$

We may also deduce a 'dual' result about farthest points that we shall use in our fourth proof.

Theorem 14. *Suppose that every point in Euclidean space admits a unique farthest point in a set A. Then A is singleton.*

Proof. We leave it to the reader to deduce that r_A^2 is differentiable (and strictly convex), [6, p. 226]. One way is to use the formula for the subgradient of a convex *max-function* over a compact (convex) set [6, p. 129, Exercise 10], or [11, 19, 24, 9]. Uniqueness of the farthest point $F_A(x)$ then implies that

$$\frac{1}{2}\,\partial r_A^2(x) = x - F_A(x) = \frac{1}{2}\,\nabla r_A^2(x).$$

Now consider formula (2). The righthand side is again clearly differentiable and it an to appeal to Proposition 10 to shows that $\iota_{-A} - \|\cdot\|^2$ is convex. As $-\|\cdot\|$ is strictly concave, A can not contain two points. □

Proof. (**4, via inversive geometry**, [1, 6].) Without loss of generality, suppose $0 \notin C$ but $0 \in \operatorname{cl}\operatorname{conv} C$. Consider any point $x \in E$. Fact 12 implies that the quantity

$$\rho := \inf\{\delta > 0 \mid \iota C \subset x + \delta B\}$$

satisfies $\rho > \|x\|$. Now let z denote the unique *nearest* point in C to the point $(-x)/(\rho^2 - \|x\|^2)$. and observe, again via Fact 12, that $\iota(z)$ is the unique *furthest* point in $\iota(C)$ to x. By Theorem 14 the set $\iota(C)$ is a singleton which is not possible. □

4. Proximality and Chebyshev sets in infinite dimensions

In this section we make a discursive look at the subject in infinite dimensions. In 1961, Victor Klee [21] asked whether a Chebyshev set in Hilbert space must be convex? The literature is large but a good start can be made by reading the relevant parts of [12] and [15]. A comprehensive survey up to 1973 is given in [25]. The cleanest partial answer yet known is:

Theorem 15. (**Chebyshev Sets**, [1, 8, 12, 21, 15].) *A weakly closed Chebyshev set in Hilbert space is convex.*

Proof. Once we establish the Fréchet differentiability of d_S^2 the second and third proofs need no change. To do this it suffices to argue that $I - P_S$ is still norm-weak* continuous while $x \mapsto \|x - P_S(x)\| = d_S(x)$ is continuous. We then appeal to the fact that norm and weak convergence agree on spheres in Hilbert space.

Asplund's proof likewise holds—indeed, this was his proof of the theorem, [1]. The first proof also extends as far as boundedly norm-compact sets via *Schauder's fixed point theorem*, albeit with a little more effort [9, p. 219]. □

Remark 16. (**Generalizations**.) Indeed, the second proof actually shows Vlasov's (1970) result that *in a Banach space with a rotund dual norm any Chebyshev set with a continuous projection is convex* as described in [4, 15, 16] since (3) will hold under these hypotheses.

Asplund's method [1] also yields the striking result that if there is a non-convex Chebyshev set in Hilbert space there is also one that is the complement of an open convex body—a so called *Klee cavern*. This is both surprising yet consistent with Figure 3 that we drew for the proof via Brouwer's theorem.

While a sun in a smooth Banach space is known to be convex, [25], the existence in a renorming of $C[0,1]$ of a disconnected non-Chebyshev sun, [22], indicates the limitations of the first approach. □

Remark 17. (**Counter-examples**.) Opinions differ about whether every (norm-closed) Chebyshev set in Hilbert space is convex. Since there are even closed sets of rotund reflexive space with discontinuous projections [10], in that level of generality one must somehow establish the continuity of P_S or avoid the issue to show S is convex.

It is known that any non-convex Chebyshev set in Hilbert space must have a badly discontinuous metric projection [26]. That paper uses monotone operators to show that $H \setminus \{x\colon \nabla_F d_S(x)\,\text{exists}\}$ is the countable union of nonconstant Lipschitz curves. This is based on the fact that P_S is maximal monotone if and only if S is Chebyshev and P_S is continuous. In the separable case Duda [13] shows the the covering can be achieved by *difference-convex* surfaces.

It is also known that there is an example of a bounded non-convex Chebyshev set (actually it can be disconnected *Chebyshev foam*) in an incomplete inner-product space, [20, 12]. □

Recall that a norm is (sequentially) *Kadec-Klee* if weak and norm topologies coincide (sequentially) on norm spheres.

Theorem 18. (**Dense and generic proximality.**) *Every closed set A in a Banach space densely (equivalently generically) admits nearest points iff the norm is Kadec-Klee and the space is reflexive.*

Proof. If (originally proved by Lau in [23]). We sketch the proof in [9, 3]. Consider a sub-derivative $\phi \in \partial_F(-d_A)(x)$, which by the smooth variational principle exists for a dense set in $X \setminus A$. Let (a_n) be a bounded minimizing sequence, and use reflexivity to extract a subsequence (we use the same name) converging weakly to $z \in X$. Since $\phi \in \partial_F(-d_A)(x)$ it is easy to show that $\phi\| = 1$ and that $\phi(a_n - x) \to d_A(x)$. Thus, we see that $\|z - x\| \geq \phi(z - x) = d_A(x) \geq \lim \|a_n - x\|$ and by weak lower-semicontinuity of the norm $\|a_n - x\| \to \|z - x\|$. The Kadec-Klee property then implies that $a_n \to z$ in norm and so $z \in A$. As $\|z - a\| = d_A(x)$ we have shown the set of points with nearest points in A is dense. Showing genericity takes a little more effort.

Only if (originally due to Konjagin). We sketch the proof in [3]. We shall construct a norm closed set A and a neighbourhood U within which no point admits a best approximation in A. If the space is not reflexive we appeal to Proposition 4.

In the reflexive setting, failure of the Kadec-Klee property means there must be a weakly-null sequence (x_n) with $\|x_n\| = 1$ and with $\|x_n - x_m\| \geq 3\varepsilon > 0$ (i.e, the sequence is 3ε-*separated*). Let

$$A := \cap_n x_n + \varepsilon B_X.$$

It is routine to verify that in some neighbourhood U of zero there are no points with $P_A(x)$ non-empty. □

Remark 19. (a) An easier version of the 'if' argument exactly proves (4) ⇒ (1) of Proposition 6.

(b) Konjagin's construction produces a distance function d_A which is Fréchet differentiable (even affine) in a neighbourhood of zero but induces no best approximations from that neighbourhood. Thus the geometry of the norm is critical even in the presence of Fréchet derivatives. □

Corollary 20. (**Existence of proximal points.**) *A closed set C in a Banach space X has a nonempty set of proximal points under any of the following conditions.*

1. *X is reflexive and the norm is (sequentially) Kadec-Klee, (Thm. 18).*
2. *X has the* Radon Nikodym property [14] *and C is bounded,* [3].
3. *X is norm closed and boundedly relatively weakly compact,* [7].

This list is far from exhaustive. For instance:

Example 21. (**Norms with dense proximals**, [3].) There is a class of reflexive non-Kadec-Klee norms such that every nonempty closed set A densely possesses proximal points. Explicit examples are given in [3]. The counterexample sketched in Theorem 18 is locally weakly-compact and convex and so admits dense proximals. □

Example 22. (**Multiple caverns**, [3].) Let us call the complement of finitely many disjoint open convex bodies a *multiple cavern*. Using inversive geometry methods as above, one can show that in a reflexive space every multiple Klee cavern admits proximal points. In [3] such sets were called *Swiss cheese*. □

Finally, I discuss two very useful additional properties of the distance function when the norm is *uniformly Gâteaux differentiable* as is the case in Hilbert space and, after renorming, in every super-reflexive and every separable Banach space, [4]. We say that ∂d_A is *minimal* if it contains no smaller *w^*-cusco*—a norm to w*-upper semicontinuous mapping with nonempty w*-compact images.

Remark 23. (**Some additional properties of d_A,** [4].) A Banach space X is uniformly Gâteaux differentiable if and only if ∂d_A is minimal for every closed nonempty set A. This has lovely consequences for proximal normal formulas, [5] (see [9] for the finite dimensional case). It relies on the fact that such norms also characterize those spaces for which

$$\partial_-(-d_A)(x) = \partial_\diamond(-d_A)(x) = \partial_o(-d_A)(x),$$

that is the Dini, Clarke and *Michel-Penot sub-differentials* (see [6]) coincide for all closed sets A, and hence that $-d_A$ is both Clarke and Michel-Penot regular, [4].

□

5. Conclusion

I hope this discussion has whetted some readers' appetites to attempt at least one of the following open questions.

Question 1. *Is every Chebyshev set in Hilbert space convex?*

Question 2. *Is every closed set in Hilbert space with unique farthest points a singleton?*

Question 3. *Is every Chebyshev set in a rotund reflexive Banach space convex?*

Question 4. *Does every closed set in a reflexive Banach space admit a nearest point? What about rotund smooth renormings of Hilbert space?*

Question 5. *Does every closed set in a reflexive Banach space admit proximal normals at a dense set of boundary points?*

And finally, I certainly hope I have made good advertisements for the power of variational and nonsmooth analysis.

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