Exploratory Experimentation and Computation

Fields and IRMACS Workshop on
Discovery and Experimentation in Number Theory
(IRMACS, September 23, 2008)

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"*[I]ntuition comes to us much earlier and with much less outside influence than formal arguments which we cannot really understand unless we have reached a relatively high level of logical experience and sophistication.*"

…

"In the first place, the beginner must be convinced that proofs deserve to be studied, that they have a purpose, that they are interesting." George Polya (1887-1985)

THE UNIVERSITY OF NEWCASTLE AUSTRALIA

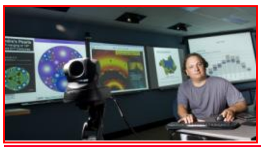
Revised 23/09/09

Where I now live

wine

home

ABSTRACT

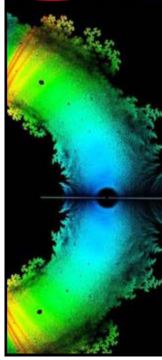


Jonathan M. Borwein
Newcastle

Abstract: The mathematical research community is facing a great challenge to re-evaluate the role of proof in light of the growing power of current computer systems, of modern mathematical computing packages, and of the growing capacity to data-mine on the Internet. Add to that the enormous complexity of many modern capstone results such as the Poincaré conjecture, Fermat's last theorem, and the Classification of finite simple groups. As the need and prospects for inductive mathematics blossom, the requirement to ensure the role of proof is properly founded remains undiminished. I shall look at the philosophical context and then offer five bench-marking examples of the opportunities and challenges we face, along with some interactive demonstrations. (Related paper)

"***The object of mathematical rigor is to sanction and legitimize the conquests of intuition, and there was never any other object for it.***" – Jacques Hadamard (1865-1963)

COMPUTER ASSISTED RESEARCH MATHEMATICS AND ITS APPLICATIONS (CARMA)



NEWCASTLE RESEARCH CENTRE
(9 core members)

OVERVIEW

Two decades ago, few mathematicians used computations in serious research work. There was a wide-spread view that "real mathematicians don't compute." In the ensuing years, computer hardware has skyrocketed in power and plunged in cost, thanks to the remarkable persistent phenomenon of Moore's Law. And many powerful mathematical software products have emerged. Just as importantly, a new generation of mathematicians is eager to use these tools. Thus, many new results are being discovered, and use of mathematics in society is expanding rapidly.

Experimental methodology provides a compelling way to build insight, to find and confirm or confront conjectures; to make mathematics more tangible, lively and fun for a researcher, a practitioner, or a novice. Experimental approaches also broaden the interdisciplinary nature of research: a chemist, physicist, engineer, and mathematician may not understand each others' motivation or jargon, but often share underlying computational tools, usually to the benefit of all parties.

Advanced mathematical computation is equally essential to solution of real-world problems; sophisticated mathematics is core to software used by decision-makers, engineers, scientists, managers, and who design, plan and control the products and systems key to present day life.

OBJECTIVES

- To perform research and development relating to the informed use of computers as an adjunct to mathematical discovery (including current advances in cognitive science, in information technology, operations research and theoretical computer science).
- To perform research and development of mathematics underlying computer-based decision support systems, particularly in automation and optimization of scheduling, planning and design activities, and to undertake mathematical modelling of such activities.
- To promote and advise on the use of appropriate tools (hardware, software, databases, learning object repositories, mathematical knowledge management, collaborative technology) in academia, education and industry.
- To make University of Newcastle a world-leading institution for Computer Assisted Research Mathematics and its Applications

OUTLINE

I. Working Definitions of:

- Discovery
- Proof (and of Mathematics)
- Digital-Assistance
- Experimentation (in Maths and in Science)

II. Five Core Examples:

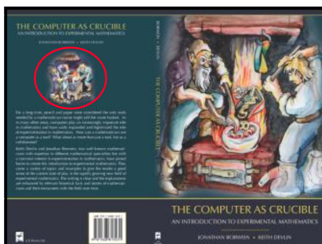
- p(n)
- π
- $\phi(n)$
- $\zeta(3)$
- $1/\pi$

"Keynes distrusted intellectual rigour of the Ricardian type as likely to get in the way of original thinking and saw that it was not uncommon to hit on a valid conclusion before finding a logical path to it." - Sir Alec Cairncross, 1996

III. A Cautionary Finale

IV. Making Some Tacit Conclusions Explicit

"Mathematical proofs like diamonds should be hard and clear, and will be touched with nothing but strict reasoning." - John Locke



Jonathan Borwein
Keith Devlin
with illustrations by Karl H. Hofmann

Contents

PART I. PHILOSOPHY, PSYCHOLOGY, ETC

"This is the essence of science. Even though I do not understand quantum mechanics or the nerve cell membrane, I trust those who do. Most scientists are quite ignorant about most sciences but all use a shared grammar that allows them to recognize their craft when they see it. The motto of the Royal Society of London is 'Nullius in verba' *: trust not in words. Observation and experiment are what count, not opinion and introspection. Few working scientists have much respect for those who try to interpret nature in metaphysical terms.* ***For most wearers of white coats, philosophy is to science as pornography is to sex: it is cheaper, easier, and some people seem, bafflingly, to prefer it.*** *Outside of psychology it plays almost no part in the functions of the research machine."* - Steve Jones

- From his 1997 NYT BR review of Steve Pinker's *How the Mind Works*.

WHAT is a DISCOVERY?

"discovering a truth has three components. First, there is the independence requirement, which is just that one comes to believe the proposition concerned by one's own lights, without reading it or being told. Secondly, there is the requirement that one comes to believe it in a reliable way. Finally, there is the requirement that one's coming to believe it involves no violation of one's epistemic state. …
In short , discovering a truth is coming to believe it in an independent, reliable, and rational way."

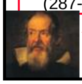
Marcus Giaquinto, Visual Thinking in Mathematics. An Epistemological Study, p. 50, OUP 2007

"***All truths are easy to understand once they are discovered; the point is to discover them.***" – Galileo Galilei

Galileo was not alone in this view

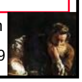
"I will send you the proofs of the theorems in this book. Since, as I said, I know that you are diligent, an excellent teacher of philosophy, and greatly interested in any mathematical investigations that may come your way, I thought it might be appropriate to write down and set forth for you in this same book a certain special method, by means of which you will be enabled to recognize certain mathematical questions with the aid of mechanics. ***I am convinced that this is no less useful for finding proofs of these same theorems.***

For some things, which first became clear to me by the mechanical method, were afterwards proved geometrically, because their investigation by the said method does not furnish an actual demonstration. ***For it is easier to supply the proof when we have previously acquired, by the method, some knowledge of the questions than it is to find it without any previous knowledge****."* - Archimedes (287-212 BCE)



Archimedes to Eratosthenes in the introduction to *The Method* in

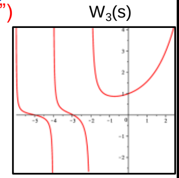
Mario Livio's, *Is God a Mathematician?* Simon and Schuster, 2009



A Very Recent Discovery

("independent, reliable and rational")

$W_3(s)$



The n-dimensional integral

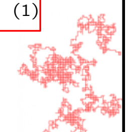
$$W_n(s) := \int_0^1 \int_0^1 \cdots \int_0^1 \left| \sum_{k=1}^n e^{2\pi x_k i} \right|^s dx_1\, dx_2 \cdots dx_n$$

occurs in the study of uniform random walks in the plane.

$W_n(1)$ is the expected distance moved after n steps.

$$W_1(1) = 1 \qquad W_2(1) = \frac{4}{\pi}$$

$$W_3(1) \overset{?}{=} \frac{3}{16}\frac{2^{1/3}}{\pi^4}\Gamma^6\left(\frac{1}{3}\right) + \frac{27}{4}\frac{2^{2/3}}{\pi^4}\Gamma^6\left(\frac{2}{3}\right). \quad (1)$$



(1) has been checked to 170 places on 256 cores in about 15 minutes. It orginates with our proof (JMB-Nuyens-Straub-Wan) that for $k = 0, 1, 2, 3, \ldots$ counts abelian squares

$$W_3(2k) = {}_3F_2\left({\frac{1}{2}, -k, -k \atop 1, 1}\Big|4\right) \text{ and } W_3(1) \overset{?}{=} \mathrm{Re}\, {}_3F_2\left({\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \atop 1, 1}\Big|4\right)$$

WHAT is MATHEMATICS?

MATHEMATICS, n. a group of related subjects, including algebra, geometry, trigonometry and calculus, concerned with the study of number, quantity, shape, and space, and their inter-relationships, applications, generalizations and abstractions.

- This definition, from my *Collins* Dictionary has no mention of proof, nor the means of reasoning to be allowed (vidé Giaquinto). *Webster's* contrasts:

INDUCTION, n. any form of reasoning in which the conclusion, though supported by the premises, does not follow from them necessarily.

and

DEDUCTION, n. **a.** a process of reasoning in which a conclusion follows necessarily from the premises presented, so that the conclusion cannot be false if the premises are true.
b. a conclusion reached by this process.

"***If mathematics describes an objective world just like physics, there is no reason why inductive methods should not be applied in mathematics just the same as in physics***." - Kurt Gödel (in his 1951 Gibbs Lecture) echoes of Quine

WHAT is a PROOF?

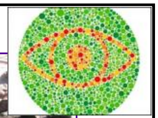
"**PROOF,** *n.* a sequence of statements, each of which is either validly derived from those preceding it or is an axiom or assumption, and the final member of which, the *conclusion* , is the statement of which the truth is thereby established. A *direct proof* proceeds linearly from premises to conclusion; an *indirect proof* (also called reductio ad absurdum) assumes the falsehood of the desired conclusion and shows that to be impossible. See also induction, deduction, valid."

Borowski & JB, Collins Dictionary of Mathematics

INDUCTION, n. 3. (Logic) *a process of reasoning in which a general conclusion is drawn from a set of particular premises, often drawn from experience or from experimental evidence. The conclusion goes beyond the information contained in the premises and does not follow necessarily from them.* ***Thus an inductive argument may be highly probable yet lead to a false conclusion; for example, large numbers of sightings at widely varying times and places provide very strong grounds for the falsehood that all swans are white***.

"***No. I have been teaching it all my life, and I do not want to have my ideas upset***." - Isaac Todhunter (1820-1884) recording Maxwell's response when asked whether he would like to see an experimental demonstration of conical refraction.

Decide for yourself



WHAT is DIGITAL ASSISTANCE?

- Use of Modern Mathematical Computer Packages
  - Symbolic, Numeric, Geometric, Graphical, …
- Use of More Specialist Packages or General Purpose Languages
  - Fortran, C++, CPLEX, GAP, PARI, MAGMA, …
- Use of Web Applications
  - Sloane's Encyclopedia, Inverse Symbolic Calculator, Fractal Explorer, Euclid in Java, Weeks' Topological Games, …
- Use of Web Databases
  - Google, MathSciNet, ArXiv, JSTOR, Wikipedia, MathWorld, Planet Math, DLMF, MacTutor, Amazon, …, Wolfram Alpha (??)
- All entail **data-mining** ["exploratory experimentation" and "widening technology" as in pharmacology, astrophysics, biotech… (Franklin)]
  - Clearly the boundaries are blurred and getting blurrier
  - Judgments of a given source's quality vary and are context dependent

"***Knowing things is very 20th century. You just need to be able to find things***."- Danny Hillis on how Google has already **changed how we think** in Achenblog, July 1 2008

- changing cognitive styles

Exploratory Experimentation

Franklin argues that Steinle's "exploratory experimentation" facilitated by "widening technology", as in pharmacology, astrophysics, medicine, and biotechnology, is leading to a reassessment of what legitimates experiment; in that a "local model" is not now prerequisite. Hendrik Sørenson cogently makes the case that experimental mathematics (as 'defined' below) is following similar tracks:

"*These aspects of exploratory experimentation and wide instrumentation originate from the philosophy of (natural) science and have not been much developed in the context of experimental mathematics. However, I claim that e.g. the importance of wide instrumentation for an exploratory approach to experiments that includes concept formation is also pertinent to mathematics.*"

In consequence, boundaries between mathematics and the natural sciences and between inductive and deductive reasoning are blurred and getting more so.

Changing User Experience and Expectations

What is attention? (**Stroop** test, 1935)

red	white	green	brown
green	red	brown	white
white	brown	green	red
red	white	green	brown
brown	green	white	red
white	brown	red	green
green	white	brown	red
red	brown	green	white

1. Say the **color** represented by the **word**.
2. Say the **color** represented by the **font** color.

High (young) multitaskers perform #2 very easily. They are great at suppressing information.

http://www.snre.umich.edu/eplab/demos/st0/stroop_program/stroopgraphicnonshockwave.gif

Acknowledgements: Cliff Nass, CHIME lab, Stanford (interference and twitter?)

Other Cognitive Shifts

Harwell 1951-1973

Science Online August 13, 2009

Strategic Reading, Ontologies, and the Future of Scientific Publishing

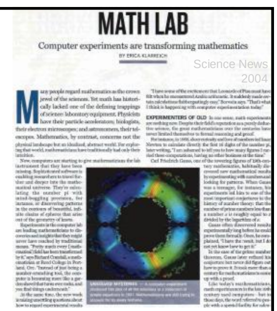
Allen H. Renear* and Carole L. Palmer

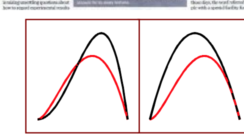
The revolution in scientific publishing that has been promised since the 1980s is about to take place. Scientists have always read strategically, working with many articles simultaneously to search, filter, scan, link, annotate, and analyze fragments of content. An observed recent increase in strategic reading in the online environment will soon be further intensified by two current trends: (i) the widespread use of digital indexing, retrieval, and navigation resources and (ii) the emergence within many scientific disciplines of interoperable ontologies. Accelerated and enhanced by reading tools that take advantage of ontologies, reading practices will become even more rapid and indirect, transforming the ways in which scientists engage the literature and shaping the evolution of scientific publishing.

✓ Potentially hostile to mathematical research patterns

Experimental Mathodology

1. Gaining insight and intuition
2. Discovering new relationships
3. Visualizing math principles
4. Testing and especially falsifying conjectures
5. Exploring a possible result to see if it merits formal proof
6. Suggesting approaches for formal proof
7. Computing replacing lengthy hand derivations
8. Confirming analytically derived results





Comparing $-y^2\ln(y)$ (red) to $y-y^2$ and y^2-y^4

Example 0. What is that number? (1995-2008)

In **1995** or so Andrew Granville emailed me the number

$$\alpha := 1.433127426722312\ldots$$

and challenged me to identify it (our inverse calculator was new in those days).

Changing representations, I asked for its continued fraction? It was

$$[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots] \qquad (1)$$

I reached for a good book on continued fractions and found the answer

$$\alpha = \frac{I_0(2)}{I_1(2)}$$

where I_0 and I_1 are Bessel functions of the first kind. (Actually I knew that all arithmetic continued fractions arise in such fashion).

In 2009 there are at least three other strategies:

- Given (1), type "arithmetic progression", "continued fraction" into Google
- Type "1,4,3,3,1,2,7,4,2" into Sloane's Encyclopaedia of Integer Sequences

I illustrate the results on the next two slides:

"arithmetic progression", "continued fraction"

In Google on October 15 2008 the first three hits were

***Continued Fraction* Constant -- from Wolfram MathWorld**
- 3 visits - 14/09/07Perron (1954-57) discusses *continued fractions* having terms even more general than the *arithmetic progression* and relates them to various special functions. **...**
*mathworld.wolfram.com/**ContinuedFraction**Constant.html - 31k*

HAKMEM -- *CONTINUED FRACTIONS* -- DRAFT, NOT YET PROOFED
The value of a *continued fraction* with partial quotients increasing in *arithmetic progression* is I (2/D) A/D [A+D, A+2D, A+3D, . **...**
www.inwap.com/pdp10/hbaker/hakmem/cf.html - 25k -

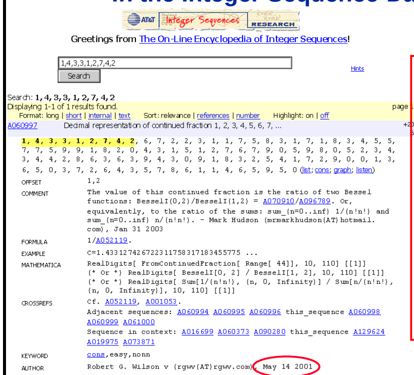
On simple *continued fractions* with partial quotients in *arithmetic* ...
0. This means that the sequence of partial quotients of the *continued fractions* under. investigation consists of finitely many *arithmetic progressions* (with **...**
www.springerlink.com/index/C0VXH713662G1815.pdf - by P Bundschuh – 1998

Moreover the MathWorld entry includes

$$[A+D, A+2D, A+3D, \ldots] = \frac{I_{A/D}\left(\frac{2}{D}\right)}{I_{1+A/D}\left(\frac{2}{D}\right)}$$

(Schroeppel 1972) for real A and $D \neq 0$

In the Integer Sequence Data Base



AT&T Integer Sequences RESEARCH

Greetings from The On-Line Encyclopedia of Integer Sequences!

1,4,3,3,1,2,7,4,2 [Search] Hints

Search: 1,4,3,3,1,2,7,4,2
Displaying 1-1 of 1 results found. page 1
Format: long | short | internal | text Sort: relevance | references | number Highlight: on | off

A060997 Decimal representation of continued fraction 1, 2, 3, 4, 5, 6, 7, ...

1, 4, 3, 3, 1, 2, 7, 4, 2, 6, 7, 2, 2, 3, 1, 1, 7, 5, 8, 3, 1, 7, 1, 8, 3, 4, 5, 5, 7, 7, 5, 9, 9, 1, 8, 2, 0, 4, 3, 1, 5, 1, 2, 7, 6, 7, 9, 0, 5, 9, 8, 0, 5, 2, 3, 4, 3, 4, 4, 2, 8, 6, 3, 6, 3, 9, 4, 3, 0, 9, 1, 8, 3, 2, 5, 4, 1, 7, 2, 9, 0, 0, 1, 3, 6, 5, 0, 3, 7, 2, 6, 4, 3, 5, 7, 8, 6, 1, 1, 4, 6, 5, 9, 5, 0 (list; cons; graph; listen)

OFFSET	1,2
COMMENT	The value of this continued fraction is the ratio of two Bessel functions: BesselI(0,2)/BesselI(1,2) = A070910/A096789. Or, equivalently, to the ratio of the sums: sum_{n=0..inf} 1/(n!n!) and sum_{n=0..inf} n/(n!n!). - Mark Hudson (mrmarkhudson(AT)hotmail.com), Jan 31 2003
FORMULA	1/A052119.
EXAMPLE	C=1.4331274267223117583171834557 75 ...
MATHEMATICA	RealDigits[FromContinuedFraction[Range[44]], 10, 110] [[1]] (* Or *) RealDigits[BesselI[0, 2] / BesselI[1, 2], 10, 110] [[1]] (* Or *) RealDigits[Sum[1/(n!n!), {n, 0, Infinity}] / Sum[n/(n!n!), {n, 0, Infinity}], 10, 110] [[1]]
CROSSREFS	Cf. A052119, A001053. Adjacent sequences: A060994 A060995 A060996 this_sequence A060998 A060999 A061000 Sequence in context: A016699 A060373 A090280 this_sequence A129624 A019975 A073871
KEYWORD	cons,easy,nonn
AUTHOR	Robert G. Wilson v (rgwv(AT)rgwv.com), May 14 2001

The **Inverse Calculator** returns

Best guess:
BesI(0,2)/BesI(1,2)

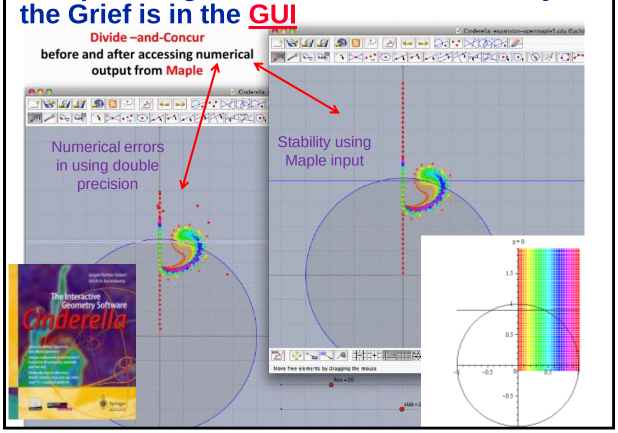
- We show the ISC on another number next
- Most functionality of ISC is built into "identify" in Maple.
- There's also Wolfram α

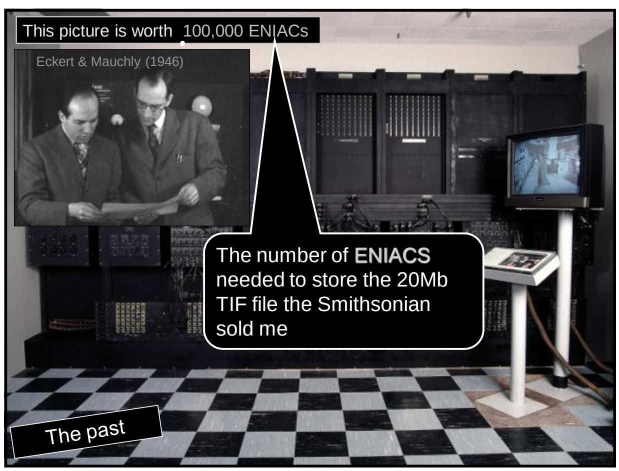
"***The price of metaphor is eternal vigilance***." - Arturo Rosenblueth & Norbert Wiener quoted by R. C. Leowontin, *Science* p.1264, Feb 16, 2001 [Human Genome Issue].

- ISC+ runs on Glooscap
- Less lookup & more algorithms than 1995

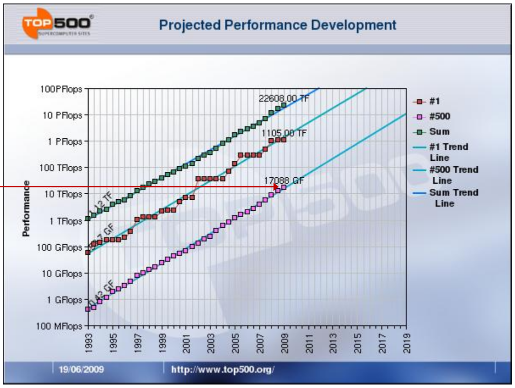
Computer Algebra + Interactive Geometry: the Grief is in the GUI

Divide –and-Concur
before and after accessing numerical output from Maple





Projected Performance



PART II MATHEMATICS

"The question of the ultimate foundations and the ultimate meaning of mathematics remains open: we do not know in what direction it will find its final solution or even whether a final objective answer can be expected at all. 'Mathematizing' may well be a creative activity of man, like language or music, of primary originality, whose historical decisions defy complete objective rationalisation." - Hermann Weyl

In "*Obituary: David Hilbert 1862 – 1943,*" *RSBIOS*, **4**, 1944, pp. 547-553; and *American Philosophical Society Year Book*, 1944, pp. 387-395, p. 392.

IIa. The Partition Function (1991-2009)

Consider the number of *additive* partitions, p(n), of n. Now

5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1

so p(5)=7. The ordinary generating function discovered by Euler is

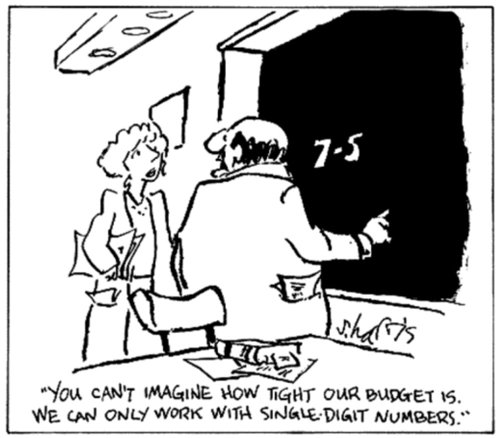
$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{k=1}^{\infty} \left(1-q^k\right)^{-1}. \qquad (1)$$

(Use the geometric formula for $1/(1-q^k)$ and observe how powers of q^n occur.)

The famous computation by MacMahon of **p(200)=3972999029388** done symbolically and entirely naively using (1) on an Apple laptop took **20 min** in **1991**, and about **0.17 seconds** in **2009.** Now it took **2 min** for **p(2000) = 4720819175619413888601432406799959512200344166**

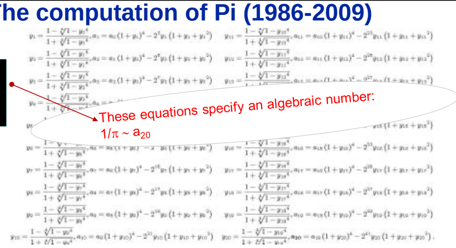
In **2008**, Crandall found **p(10^9)** in **3 seconds** on a laptop, using the Hardy-Ramanujan-Rademacher 'finite' series for p(n) with FFT methods. Such fast partition-number evaluation let Crandall find *probable primes* **p(1000046356)** and **p(1000007396)**. Each has roughly 35,000 digits.

When does easy access to computation discourages innovation: would Hardy and Ramanujan have still discovered their marvellous formula for p(n)?



IIb. The computation of Pi (1986-2009)

BB4: Pi to 1.7 trillion places in 20 steps

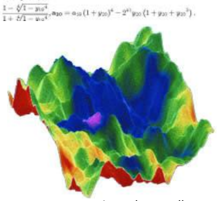


Set $a_0 = 6 - 4\sqrt{2}$ and $y_0 = \sqrt{2} - 1$. Iterate

$$y_{k+1} = \frac{1-(1-y_k^4)^{1/4}}{1+(1-y_k^4)^{1/4}} \quad \text{and}$$

$$a_{k+1} = a_k(1+y_{k+1})^4 - 2^{2k+3}y_{k+1}(1+y_{k+1}+y_{k+1}^2).$$

Then $1/a_k$ converges quartically to π


A random walk on a million digits of Pi

Moore's Law Marches On

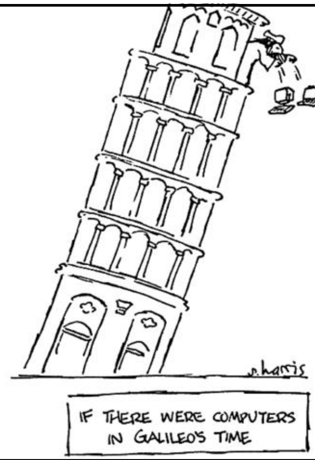
1986: It took Bailey 28 hours to compute **29.36 million digits** on 1 cpu of the then new CRAY-2 at NASA Ames using (BB4). Confirmation using another BB quadratic algorithm took 40 hours.

This uncovered hardware and software errors on the CRAY.

2009 Takahashi on 1024 cores of a 2592 core *Appro Xtreme - X3* system **1.649 trillion digits** via (Salamin-Brent) took 64 hours 14 minutes with 6732 GB of main memory, and (BB4) took 73 hours 28 minutes with 6348 GB of main memory.

The two computations differed only in the last 139 places.

"***The most important aspect in solving a mathematical problem is the conviction of what is the true result. Then it took 2 or 3 years using the techniques that had been developed during the past 20 years or so***." - Leonard Carleson (*Lusin's problem* on p.w. convergence of Fourier series in Hilbert space)



II c. Guiga and Lehmer (1932-2009)

As another measure of what changes over time and what doesn't, consider two conjectures regarding Euler's totient $\phi(n)$ which counts positive numbers less than and relatively prime to n.

Giuga's conjecture (1950) n is prime if and only if

$$\mathcal{G}_n := \sum_{k=1}^{n-1} k^{n-1} \equiv (n-1) \bmod n.$$

Counterexamples are *Carmichael numbers* (rare birds only proven infinite in **1994**) and more: if a number $n = p_1 \cdots p_m$ with m>1 prime factors p_i is a counterexample to Giuga's conjecture then the primes are distinct and satisfy

$$\sum_{i=1}^{m} \frac{1}{p_i} > 1$$

and they form a *normal sequence:* $p_i \neq 1 \bmod p_j$

(3 rules out 7, 13,... and 5 rules out 11, 31, 41,...)

Guiga's Conjecture (1951-2009)

With predictive experimentally-discovered heuristics, we built an efficient algorithm to show (in several months in **1995**) that any counterexample had **3459** prime factors and so exceeded $10^{13886} \rightarrow 10^{14164}$ in a **5 day** desktop **2002** computation.

The method fails after **8135** primes---my goal is to exhaust it.

2009 While preparing this talk, I obtained almost as good a bound of **3050** primes in under **110** minutes on my notebook and a bound of **3486** primes in **14 hours**: using *Maple* not as before C++ which being compiled is faster but in which the coding is much more arduous.

One core of an eight-core *MacPro* obtained **3592** primes and 10^{16673} digits in **13.5 hrs** in *Maple.* (Now running on 8 cores.)

Lehmer's Conjecture (1932-2009)

A tougher related conjecture is

Lehmer's conjecture (1932) n is prime if and only if

$$\phi(n) | (n-1)$$

He called this *"as hard as the existence of odd perfect numbers."*

Again, prime factors of counterexamples form a normal sequence, but now there is little extra structure.

In a 1997 SFU M.Sc. Erick Wong verified this for **14** primes, using normality and a mix of PARI, C++ and Maple to press the bounds of the '*curse of exponentiality*.'

The related $\phi(n) | (n+1)$ is has **8** solutions with at most **7** factors (6 factors is due to Lehmer). Recall $F_n := 2^{2^n}+1$ the *Fermat primes*. The solutions are 2, 3, 3.5, 3.5.17, 3.5.17.257, 3.5.17.257.65537 and a rogue pair: 4919055 and 6992962672132095, but **8** factors seems out of sight**.**

Lehmer "couldn't" factor 6992962672132097= 73×95794009207289. If prime, a 9th would exist: $\phi(n) | (n+1)$ and n+2 prime ⇒ N:=n(n+2) satisfies $\phi(N) | (N+1)$

II d. Apéry-Like Summations

The following formulas for ζ(n) have been known for many decades:

$$(a) \quad \zeta(2) = 3 \sum_{k=1}^{\infty} \frac{1}{k^2 \binom{2k}{k}},$$

$$(b) \quad \zeta(3) = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3 \binom{2k}{k}},$$

$$(c) \quad \zeta(4) = \frac{36}{17} \sum_{k=1}^{\infty} \frac{1}{k^4 \binom{2k}{k}}.$$

The RH in Maple

These results have led many to speculate that

$$Q_5 := \zeta(5) / \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^5 \binom{2k}{k}}$$

might be some nice rational or algebraic value.

Sadly, PSLQ calculations have established that if Q_5 satisfies a polynomial with **degree** at most **25**, then at least **one coefficient** has **380** digits.

> ***"He (Gauss) is like the fox, who effaces his tracks in the sand with his tail."*** - Niels Abel (1802-1829)

Two more things about $\zeta(5)$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^5\binom{2k}{k}} = 2\zeta(5) - \frac{4}{3}L^5 + \frac{8}{3}L^3\zeta(2) + 4L^2\zeta(3) + 80\sum_{n>0}\left(\frac{1}{(2n)^5} - \frac{L}{(2n)^4}\right)\rho^{2n}$$

Here $\rho := \frac{\sqrt{5}-1}{2}$ and $L := \log\rho$ (JMB-Broadhurst-Kamnitzer, 2000). Also, there is a simpler Ramanujan series for $\zeta(4n+1)$. In particular:

$$\zeta(5) = \frac{1}{294}\pi^5 + \frac{2}{35}\sum_{k=1}^{\infty}\frac{1}{(1+e^{2k\pi})k^5} + \frac{72}{35}\sum_{k=1}^{\infty}\frac{1}{(1-e^{2k\pi})k^5},$$

and $\zeta(5) - \pi^5/294 = -0.0039555\ldots$.

Nothing New under the Sun

Margo Kondratieva found a formula of Markov in 1890:

$$\sum_{n=1}^{\infty}\frac{1}{(n+a)^3} = \frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n\,(n!)^6}{(2n+1)!} \times \frac{5\,(n+1)^2 + 6\,(a-1)\,(n+1) + 2\,(a-1)^2}{\prod_{k=0}^{n}(a+k)^4}.$$

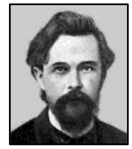
Note: *Maple* establishes this identity as

$$-1/2\,\Psi(2,a) = -1/2\,\Psi(2,a) - \zeta(3) + 5/4\,{}_4F_3([1,1,1,1],[3/2,2,2],-1/4)$$

Hence

$$\zeta(4) = -\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{\binom{2m}{m}m^4} + \frac{10}{3}\sum_{m=1}^{\infty}\frac{(-1)^{m-1}\sum_{k=1}^{m}\frac{1}{k}}{\binom{2m}{m}m^3}.$$

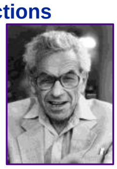
- The case a=0 above is Apéry's formula for $\zeta(3)$!



Andrei Andreyevich Markov (1856-1922)

Two Discoveries: 1995 and 2005

- **Two computer-discovered generating functions**
  - (1) was 'intuited' by Paul Erdös (1913-1996)
- **and (2) was a designed experiment**
  - was proved by the computer (Wilf-Zeilberger)
  - and then by people (Wilf included)
  - What about 4k+1?



$$\sum_{k=0}^{\infty}\zeta(4k+3)\,x^{4k} = \frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3\binom{2k}{k}(1-x^4/k^4)}\prod_{m=1}^{k-1}\left(\frac{1+4x^4/m^4}{1-x^4/m^4}\right) \qquad (1)$$

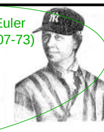
x=0 gives (b) and (a) respectively

$$\sum_{k=0}^{\infty}\zeta(2k+2)\,x^{2k} = 3\sum_{k=1}^{\infty}\frac{1}{k^2\binom{2k}{k}(1-x^2/k^2)}\prod_{m=1}^{k-1}\left(\frac{1-4x^2/m^2}{1-x^2/m^2}\right) \qquad (2)$$

Apéry summary

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$$

$$\zeta(2) = \frac{\pi^2}{6}, \zeta(4) = \frac{\pi^4}{90}, \zeta(6) = \frac{\pi^6}{945}, \ldots$$



♦Euler (1707-73)

1. via PSLQ to **5,000 digits** (120 terms)

$$\mathcal{Z}(x) = 3\sum_{k=1}^{\infty}\frac{1}{\binom{2k}{k}(k^2-x^2)}\prod_{n=1}^{k-1}\frac{4x^2-n^2}{x^2-n^2} = \sum_{k=0}^{\infty}\zeta(2k+2)x^{2k} = \sum_{n=1}^{\infty}\frac{1}{n^2-x^2} = \frac{1-\pi x\cot(\pi x)}{2x^2}$$

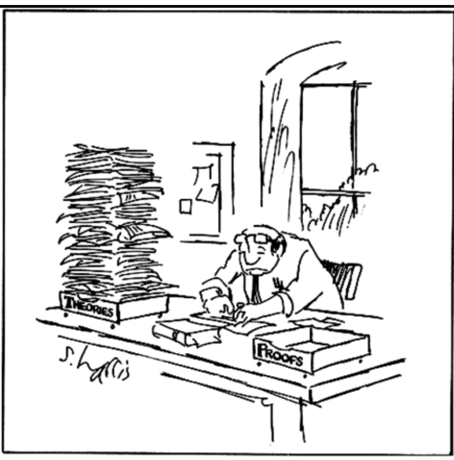
2005 Bailey, Bradley & JMB *discovered and proved* - in 3Ms - three *equivalent* binomial identities

2. reduced as hoped

$$3n^2\sum_{k=n+1}^{2n}\frac{\prod_{m=n+1}^{k-1}\frac{4n^2-m^2}{n^2-m^2}}{\binom{2k}{k}(k^2-n^2)} = \frac{1}{\binom{2n}{n}} - \frac{1}{\binom{3n}{n}}$$

$${}_3F_2\left(\begin{matrix}3n, n+1, -n\\ 2n+1, n+1/2\end{matrix}; \frac{1}{4}\right) = \frac{\binom{2n}{n}}{\binom{3n}{n}}$$

3. was easily **computer proven** (Wilf-Zeilberger) **(now 2 human proofs)**



II e: Ramanujan-Like Identities

Truly novel series for $1/\pi$, based on elliptic integrals, were discovered by Ramanujan around 1910. One is:

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801}\sum_{k=0}^{\infty}\frac{(4k)!\,(1103+26390k)}{(k!)^4 396^{4k}}. \qquad (1)$$

Each term of (1) adds 8 correct digits. Gosper used (1) by the computation of a then-record 17 million digits of the c.f. for π in 1985—completing the first proof of (1).

A little later David and Gregory Chudnovsky found the following variant, which lies in $Q(\sqrt{-163})$ rather than $Q(\sqrt{58})$:

$$\frac{1}{\pi} = 12\sum_{k=0}^{\infty}\frac{(-1)^k\,(6k)!\,(13591409+545140134k)}{(3k)!\,(k!)^3\,640320^{3k+3/2}}. \qquad (2)$$

Each term of (2) adds 14 correct digits.

The brothers used (2) several times --- culminating in a 1994 calculation to over four billion decimal digits. Their remarkable story was told in a Pulitzer-winning New Yorker article.

New Ramanujan-Like Identities

Guillera has recently found Ramanujan-like identities, including:

$$\frac{128}{\pi^2} = \sum_{n=0}^{\infty} (-1)^n r(n)^5 (13 + 180n + 820n^2) \left(\frac{1}{32}\right)^{2n}$$

$$\frac{8}{\pi^2} = \sum_{n=0}^{\infty} (-1)^n r(n)^5 (1 + 8n + 20n^2) \left(\frac{1}{2}\right)^{2n}$$

$$\frac{32}{\pi^3} \stackrel{?}{=} \sum_{n=0}^{\infty} r(n)^7 (1 + 14n + 76n^2 + 168n^3) \left(\frac{1}{8}\right)^{2n}.$$

where

$$r(n) = \frac{(1/2)_n}{n!} = \frac{1/2 \cdot 3/2 \cdot \cdots \cdot (2n-1)/2}{n!} = \frac{\Gamma(n+1/2)}{\sqrt{\pi}\,\Gamma(n+1)}$$

Guillera proved the first two using the Wilf-Zeilberger algorithm. He ascribed the third to Gourevich, who found it using integer relation methods. **It is true but has no proof.**

As far as we can tell there are no higher-order analogues!

Example of Use of Wilf-Zeilberger, I

The first two recent experimentally-discovered identities are

$$\sum_{n=0}^{\infty} \frac{\binom{4n}{2n}\binom{2n}{n}^4}{2^{16n}} \left(120n^2 + 34n + 3\right) = \frac{32}{\pi^2}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n}^5}{2^{20n}} \left(820n^2 + 180n + 13\right) = \frac{128}{\pi^2}$$

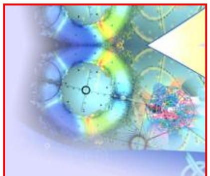
Guillera ***cunningly*** started by defining

$$G(n,k) = \frac{(-1)^k}{2^{16n}2^{4k}} \left(120n^2 + 84nk + 34n + 10k + 3\right) \frac{\binom{2n}{n}^4 \binom{2k}{k}^3 \binom{4n-2k}{2n-k}}{\binom{2n}{k}\binom{n+k}{n}^2}$$

He then used the **EKHAD** software package to obtain the companion

$$F(n,k) = \frac{(-1)^k 512}{2^{16n}2^{4k}} \frac{n^3}{4n-2k-1} \frac{\binom{2n}{n}^4 \binom{2k}{k}^3 \binom{4n-2k}{2n-k}}{\binom{2n}{k}\binom{n+k}{n}^2}$$

Wilf-Zeilberger, II



http://ddrive.cs.dal.ca/~isc/portal

When we define

$$H(n,k) = F(n+1, n+k) + G(n, n+k)$$

Zeilberger's theorem gives the identity

$$\sum_{n=0}^{\infty} G(n,0) = \sum_{n=0}^{\infty} H(n,0)$$

which when written out is

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}^4\binom{4n}{2n}}{2^{16n}} \left(120n^2 + 34n + 3\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{20n+7}} \frac{(n+1)^3}{2n+3} \frac{\binom{2n+2}{n+1}^4 \binom{2n}{n}^3 \binom{2n+4}{n+2}}{\binom{2n+2}{n}\binom{2n+1}{n+1}^2}$$

$$+ \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{20n}} \left(204n^2 + 44n + 3\right) \binom{2n}{n}^5 = \frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^n \binom{2n}{n}^5}{2^{20n}} \left(820n^2 + 180n + 13\right)$$

A limit argument and **Carlson's theorem** completes the proof…

Searches for Additional Formulas

We have no PSLQ over number fields so we searched for additional formulas of either the following forms:

$$\frac{c}{\pi^m} = \sum_{n=0}^{\infty} r(n)^{2m+1} (p_0 + p_1 n + \cdots + p_m n^m) \alpha^{2n}$$

$$\frac{c}{\pi^m} = \sum_{n=0}^{\infty} (-1)^n r(n)^{2m+1} (p_0 + p_1 n + \cdots + p_m n^m) \alpha^{2n}.$$

where c is some linear combination of

$1, 2^{1/2}, 2^{1/3}, 2^{1/4}, 2^{1/6}, 4^{1/3}, 8^{1/4}, 32^{1/6}, 3^{1/2}, 3^{1/3}, 3^{1/4}, 3^{1/6}, 9^{1/3}, 27^{1/4}, 243^{1/6}, 5^{1/2}, 5^{1/4}, 125^{1/4}, 7^{1/2}, 13^{1/2}, 6^{1/2}, 6^{1/3}, 6^{1/4}, 6^{1/6}, 7, 36^{1/3}, 216^{1/4}, 7776^{1/6}, 12^{1/4}, 108^{1/4}, 10^{1/2}, 10^{1/4}, 15^{1/2}$

where each of the coefficients p_i is a linear combination of

$1, 2^{1/2}, 3^{1/2}, 5^{1/2}, 6^{1/2}, 7^{1/2}, 10^{1/2}, 13^{1/2}, 14^{1/2}, 15^{1/2}, 30^{1/2}$

and where α is chosen as one of the following:

$1/2, 1/4, 1/8, 1/16, 1/32, 1/64, 1/128, 1/256, \sqrt{5}-2, (2-\sqrt{3})^2, 5\sqrt{13}-18, (\sqrt{5}-1)^4/128, (\sqrt{5}-2)^4, (2^{1/3}-1)^4/2, 1/(2\sqrt{2}), (\sqrt{2}-1)^2, (\sqrt{5}-2)^2, (\sqrt{3}-\sqrt{2})^4$

Relations Found by PSLQ

(with Guillera's three we found all known series and no more)

$$\frac{4}{\pi} = \sum_{n=0}^{\infty} r(n)^3 (1+6n) \left(\frac{1}{2}\right)^{2n}$$

$$\frac{16}{\pi} = \sum_{n=0}^{\infty} r(n)^3 (5+42n) \left(\frac{1}{8}\right)^{2n}$$

$$\frac{12^{1/4}}{\pi} = \sum_{n=0}^{\infty} r(n)^3 (-15 + 9\sqrt{3} - 36n + 24\sqrt{3}n) \left(2-\sqrt{3}\right)^{4n}$$

$$\frac{32}{\pi} = \sum_{n=0}^{\infty} r(n)^3 (-1 + 5\sqrt{5} + 30n + 42\sqrt{5}n) \left(\frac{(\sqrt{5}-1)^4}{128}\right)^{2n}$$

$$\frac{5^{1/4}}{\pi} = \sum_{n=0}^{\infty} r(n)^3 (-525 + 235\sqrt{5} - 1200n + 540\sqrt{5}n) \left(\sqrt{5}-2\right)^{8n}$$

$$\frac{2\sqrt{2}}{\pi} = \sum_{n=0}^{\infty} (-1)^n r(n)^3 (1+6n) \left(\frac{1}{2\sqrt{2}}\right)^{2n}$$

$$\frac{2}{\pi} = \sum_{n=0}^{\infty} (-1)^n r(n)^3 (-5 + 4\sqrt{2} - 12n + 12\sqrt{2}n) \left(\sqrt{2}-1\right)^{4n}$$

$$\frac{2}{\pi} = \sum_{n=0}^{\infty} (-1)^n r(n)^3 (23 - 10\sqrt{5} + 60n - 24\sqrt{5}n) \left(\sqrt{5}-2\right)^{4n}$$

$$\frac{2}{\pi} = \sum_{n=0}^{\infty} (-1)^n r(n)^3 (177 - 72\sqrt{6} + 420n - 168\sqrt{6}n) \left(\sqrt{3}-\sqrt{2}\right)^{8n}$$

Baruah, Berndt, Chan, "Ramanujan Series for $1/\pi$. A Survey." Aug 09, MAA *Monthly*

"What I appreciate even more than its remarkable speed and accuracy are the words of understanding and compassion I get from it."

III. A Cautionary Example

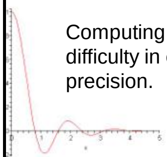
These **constants agree to 42 decimal digits** accuracy, but are **NOT** equal:

$$\int_0^\infty \cos(2x) \prod_{n=0}^{\infty} \cos(x/n)\, dx \;=$$

0.39269908169872415480783042290993786052464543418723...

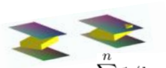
$$\frac{\pi}{8} \;=$$

0.39269908169872415480783042290993786052464617492189...



Computing this integral is (or was) nontrivial, due largely to difficulty in evaluating the integrand function to high precision.

Fourier analysis explains this happens when a hyperplane meets a hypercube (LP) ...



$$\sum_k^n 1/k > 2$$

IV. Some Conclusions

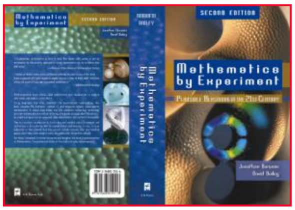
- We like students of **2010** live in an information-rich, judgement-poor world
- The explosion of information is not going to diminish
  - nor is the desire (need?) to collaborate remotely
- So we have to learn and teach judgement (not obsession with plagiarism)
  - that means mastering the sorts of tools I have illustrated
- We also have to acknowledge that most of our classes will contain a very broad variety of skills and interests (few future mathematicians)
  - properly balanced, discovery and proof can live side-by-side and allow for the ordinary and the talented to flourish in their own fashion
- **Impediments** to the assimilation of the tools I have illustrated are myriad
  - as I am only too aware from recent experiences
- These impediments include our own inertia and
  - organizational and technical bottlenecks (IT - not so much dollars)
  - under-prepared or mis-prepared colleagues
  - the dearth of good modern syllabus material and research tools
  - the lack of a compelling business model (societal goods)

"***The plural of 'anecdote' is not 'evidence'.***"
- Alan L. Leshner (*Science's* publisher)

Further Conclusions

New techniques now permit integrals, infinite series sums and other entities to be evaluated to high precision (hundreds or thousands of digits), thus permitting PSLQ-based schemes to discover new identities.

These methods typically do not suggest proofs, but often it is much easier to find a proof (say via WZ) when one "knows" the answer is right.



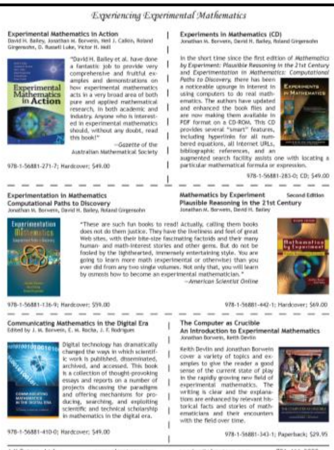
Full details of all the examples are in *Mathematics by Experiment* or its companion volume *Experimentation in Mathematics* written with Roland Girgensohn. A "Reader's Digest" version of these is available at **www.experimentalmath.info** along with much other material.

"*Anyone who is not shocked by quantum theory has not understood a single word*." - Niels Bohr

A Sad Story (UK)

1. **Teaching Maths In 1970** A logger sells a lorry load of timber for £1000. His cost of production is 4/5 of the selling price. What is his profit?
2. **Teaching Maths In 1980** A logger sells a lorry load of timber for £1000. His cost of production is 4/5 of the selling price, or £800. What is his profit?
3. **Teaching Maths In 1990** A logger sells a lorry load of timber for £1000. His cost of production is £800. Did he make a profit?
4. **Teaching Maths In 2000** A logger sells a lorry load of timber for £1000. His cost of production is £800 and his profit is £200. Underline the number 200.
5. **Teaching Maths In 2008** A logger cuts down a beautiful forest because he is a totally selfish and inconsiderate bastard and cares nothing for the habitat of animals or the preservation of our woodlands. He does this so he can make a profit of £200. What do you think of this way of making a living?

Topic for class participation after answering the question: How did the birds and squirrels feel as the logger cut down their homes? (There are no wrong answers. If you are upset about the plight of the animals in question counselling will be available.)



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