Experimental Mathematics: Computational Paths to Discovery

Dalhousie Distributed Research Institute and Virtual Environment

hpcs 2005

What is HIGH PERFORMANCE MATHEMATICS?

Jonathan Borwein, FRSC www.cs.dal.ca/~jborwein

Canada Research Chair in Collaborative Technology

"I feel so strongly about the wrongness of reading a lecture that my language may seem immoderate .... The spoken word and the written word are quite different arts .... I feel that to collect an audience and then read one's material is like inviting a friend to go for a walk and asking him not to mind if you go alongside him in your car."

Sir Lawrence Bragg

What would he say about .ppt?

Atlantic Computational Excellence Network

ACEnet

DALHOUSIE UNIVERSITY *Inspiring Minds*

WestGrid

Revised15/06/2005

"It says it's sick of doing things like inventories and payrolls, and it wants to make some break-throughs in astrophysics."

Outline. What is HIGH PERFORMANCE MATHEMATICS?

1. Visual Data Mining in Mathematics.

- ✓ **Fractals, Polynomials, Continued Fractions, Pseudospectra**

2. High Precision Mathematics.

3. Integer Relation Methods.

- ✓ **Chaos, Zeta and the Riemann Hypothesis, HexPi and Normality**

4. Inverse Symbolic Computation.

- ✓ **A problem of Knuth, $\pi/8$, Extreme Quadrature**

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5. The Future is Here.

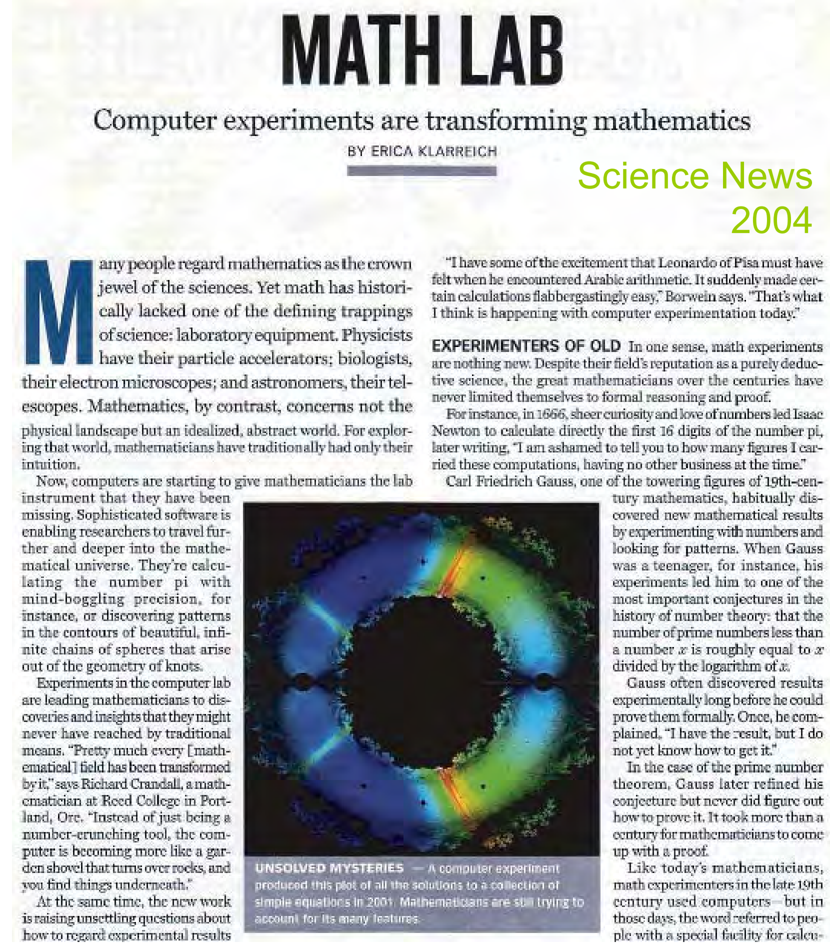
- ✓ **D-DRIVE: Examples and Issues**

6. Conclusion.

- ✓ **Engines of Discovery. The 21st Century Revolution**
  - ✓ **Long Range Plan for HPC in Canada**

Experimental Mathodology

1. Gaining insight and intuition
2. Discovering new relationships
3. Visualizing math principles
4. Testing and especially falsifying conjectures
5. Exploring a possible result to see if it merits formal proof
6. Suggesting approaches for formal proof
7. Computing replacing lengthy hand derivations
8. Confirming analytically derived results



MATH LAB

Computer experiments are transforming mathematics

BY ERICA KLARREICH

Many people regard mathematics as the crown jewel of the sciences. Yet math has historically lacked one of the defining trappings of science: laboratory equipment. Physicists have their particle accelerators; biologists, their electron microscopes; and astronomers, their telescopes. Mathematics, by contrast, concerns not the physical landscape but an idealized, abstract world. For exploring that world, mathematicians have traditionally had only their intuition.

Now, computers are starting to give mathematicians the lab instrument that they have been missing. Sophisticated software is enabling researchers to travel further and deeper into the mathematical universe. They're calculating the number pi with mind-boggling precision, for instance, or discovering patterns in the contours of beautiful, infinite chains of spheres that arise out of the geometry of knots.

Experiments in the computer lab are leading mathematicians to discoveries and insights that they might never have reached by traditional means. "Pretty much every [mathematical] field has been transformed by it," says Richard Crandall, a mathematician at Reed College in Portland, Ore. "Instead of just being a number-crunching tool, the computer is becoming more like a garden shovel that turns over rocks, and you find things underneath."

At the same time, the new work is raising unsettling questions about how to regard experimental results

UNSOLVED MYSTERIES — A computer experiment produced this plot of all the solutions to a collection of simple equations in 2001. Mathematicians are still trying to account for its many features.

"I have some of the excitement that Leonardo of Pisa must have felt when he encountered Arabic arithmetic. It suddenly made certain calculations flabbergastingly easy," Borwein says. "That's what I think is happening with computer experimentation today."

EXPERIMENTERS OF OLD In one sense, math experiments are nothing new. Despite their field's reputation as a purely deductive science, the great mathematicians over the centuries have never limited themselves to formal reasoning and proof.

For instance, in 1666, sheer curiosity and love of numbers led Isaac Newton to calculate directly the first 16 digits of the number pi, later writing, "I am ashamed to tell you to how many figures I carried these computations, having no other business at the time."

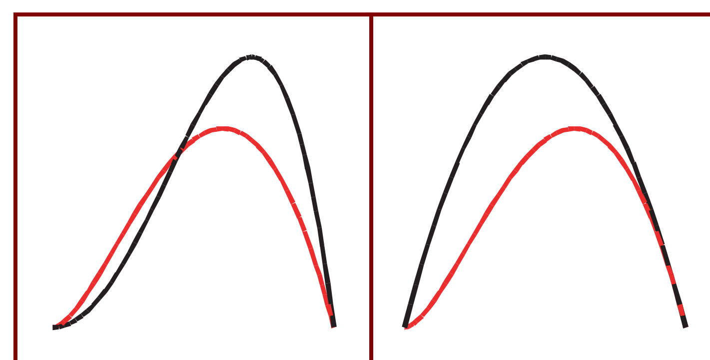
Carl Friedrich Gauss, one of the towering figures of 19th-century mathematics, habitually discovered new mathematical results by experimenting with numbers and looking for patterns. When Gauss was a teenager, for instance, his experiments led him to one of the most important conjectures in the history of number theory: that the number of prime numbers less than a number x is roughly equal to x divided by the logarithm of x.

Gauss often discovered results experimentally long before he could prove them formally. Once, he complained, "I have the result, but I do not yet know how to get it."

In the case of the prime number theorem, Gauss later refined his conjecture but never did figure out how to prove it. It took more than a century for mathematicians to come up with a proof.

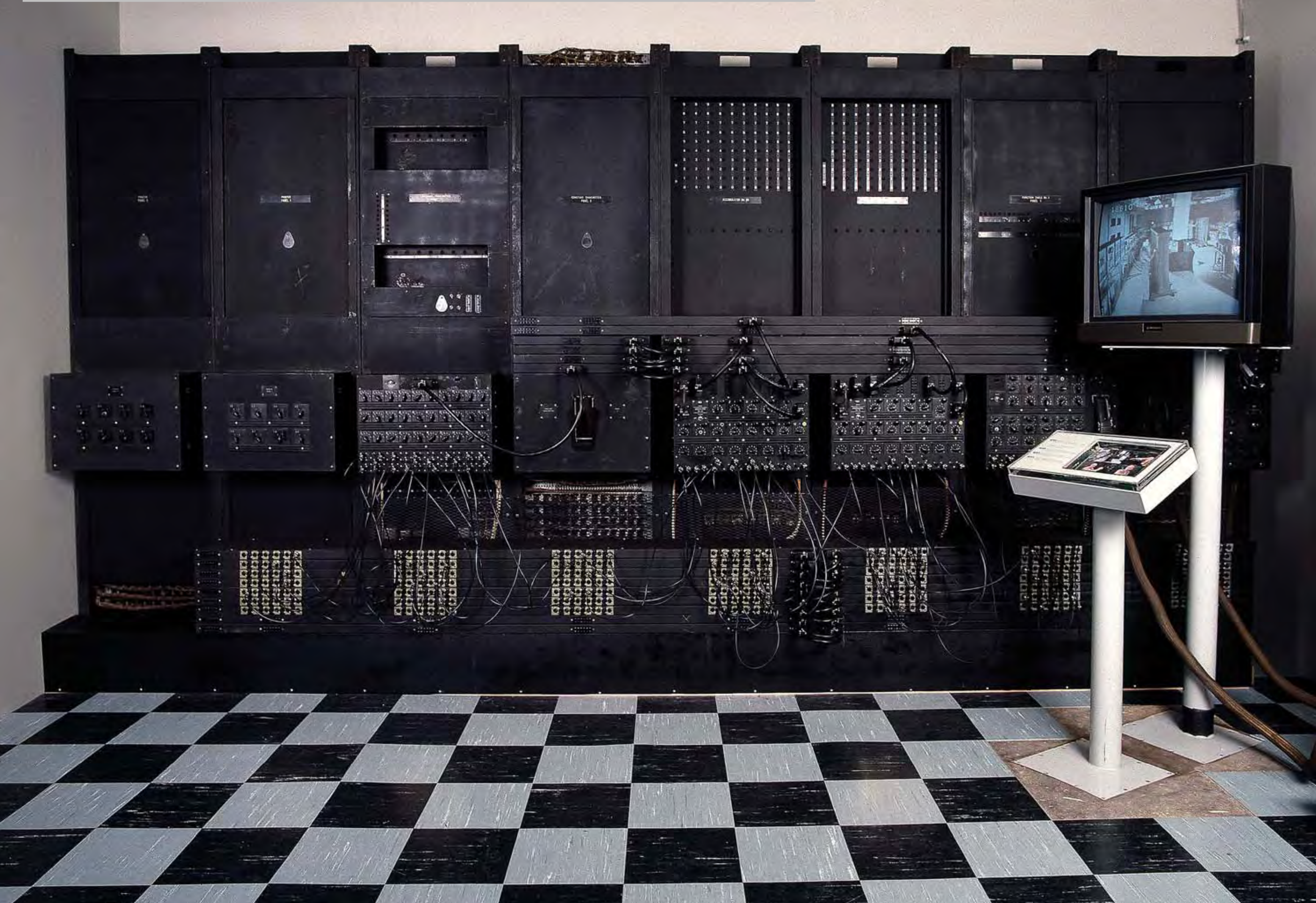
Like today's mathematicians, math experimenters in the late 19th century used computers—but in those days, the word referred to people with a special facility for calcu-

Science News 2004



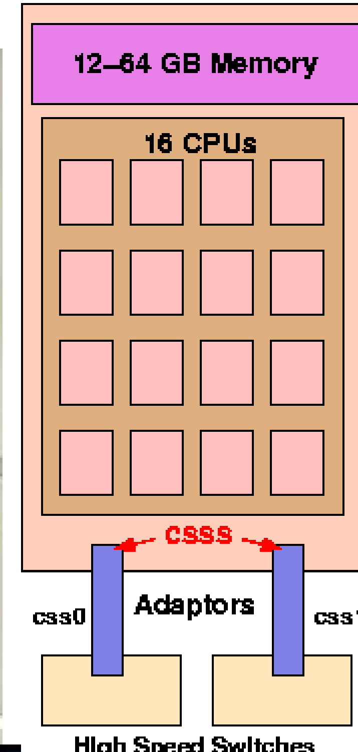
Comparing $-y^2\ln(y)$ (red) to $y-y^2$ and y^2-y^4

This picture is worth 100,000 ENIACs



NERSC's 6000 cpu Seaborg in 2004 (10Tflops/sec)

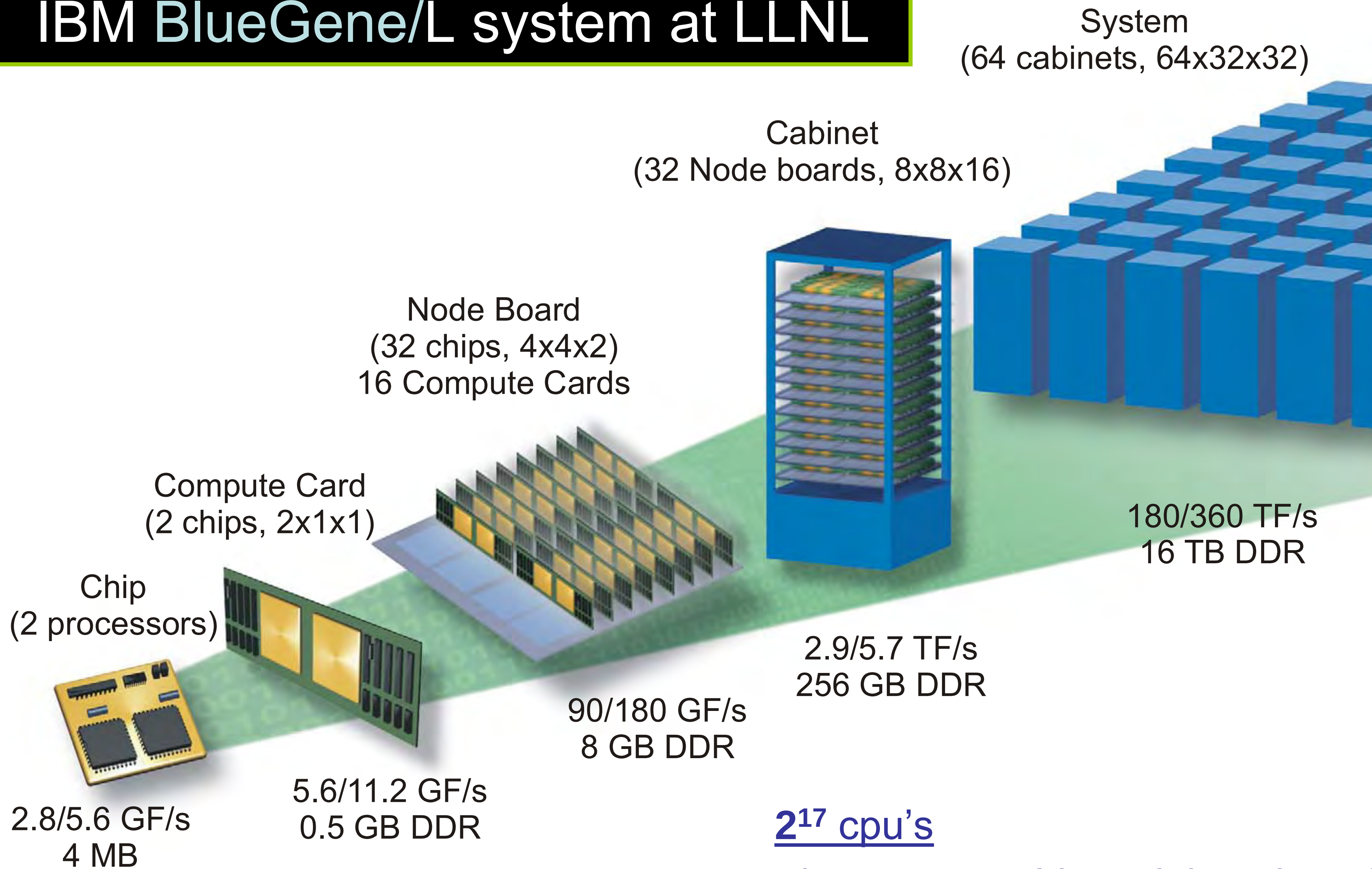
- we need new software paradigms for `bigga-scale' hardware



The present

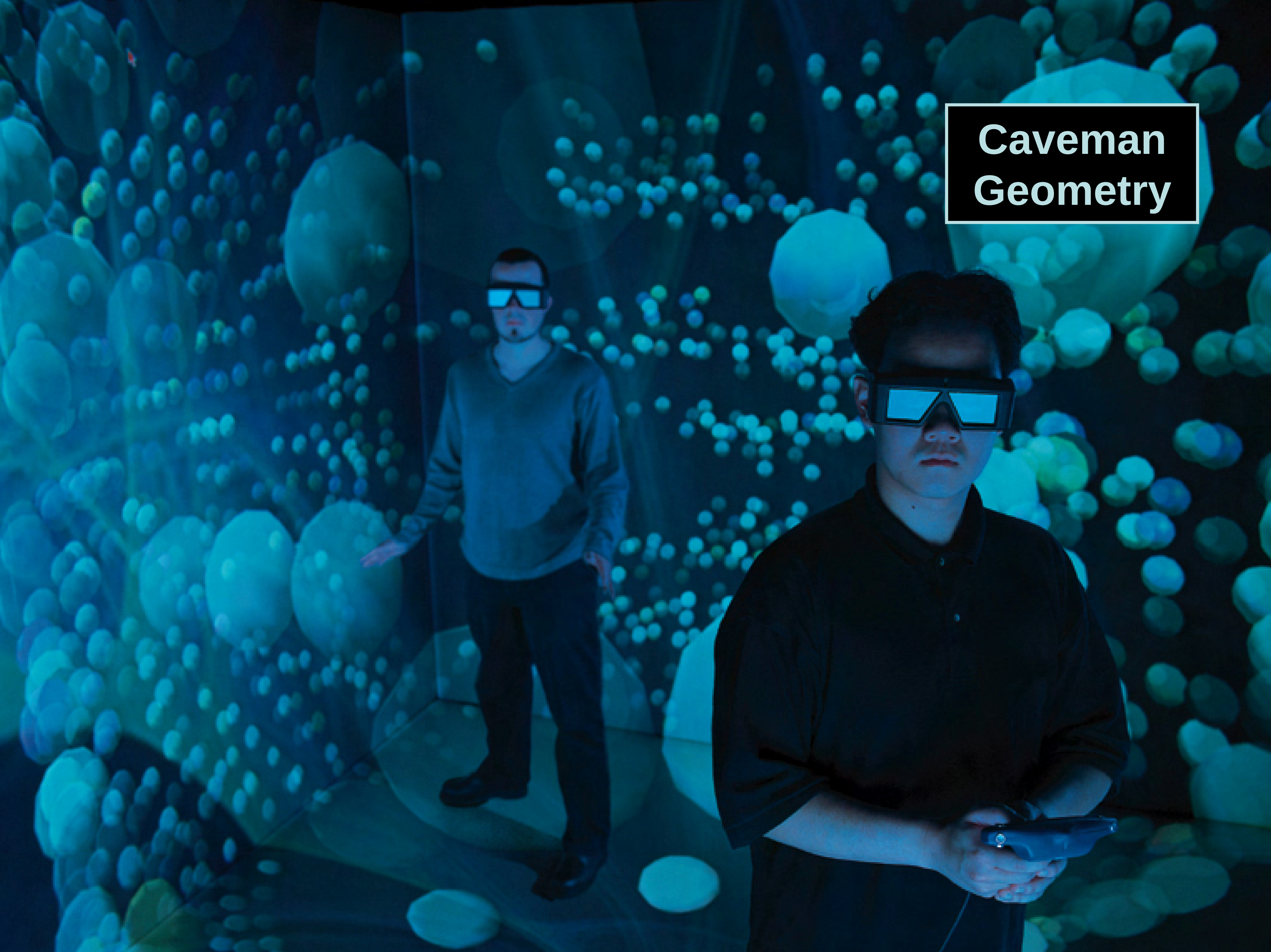
Mathematical Immersive Reality
in Vancouver

IBM BlueGene/L system at LLNL



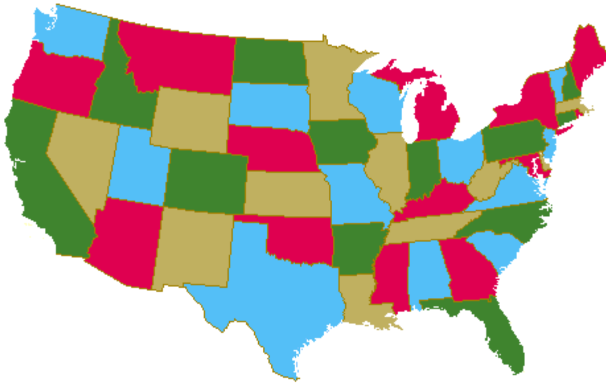
2^{17} cpu's

- has now run Linpack benchmark
- at over **120 Tflop/s**

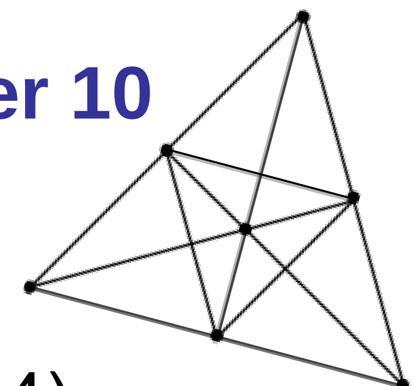

Caveman
Geometry

Grand Challenges in Mathematics (CISE 2000)

Are few and far between



- **Four Colour Theorem** (1976,1997)
- **Kepler's problem** (Hales, 2004-10)
  - next slide
- Nonexistence of **Projective Plane of Order 10**
  - **10^2+10+1** lines and points on each other (n+1 fold)
    - 2000 Cray hrs in 1990
    - next similar case:18 needs10^{12} hours?



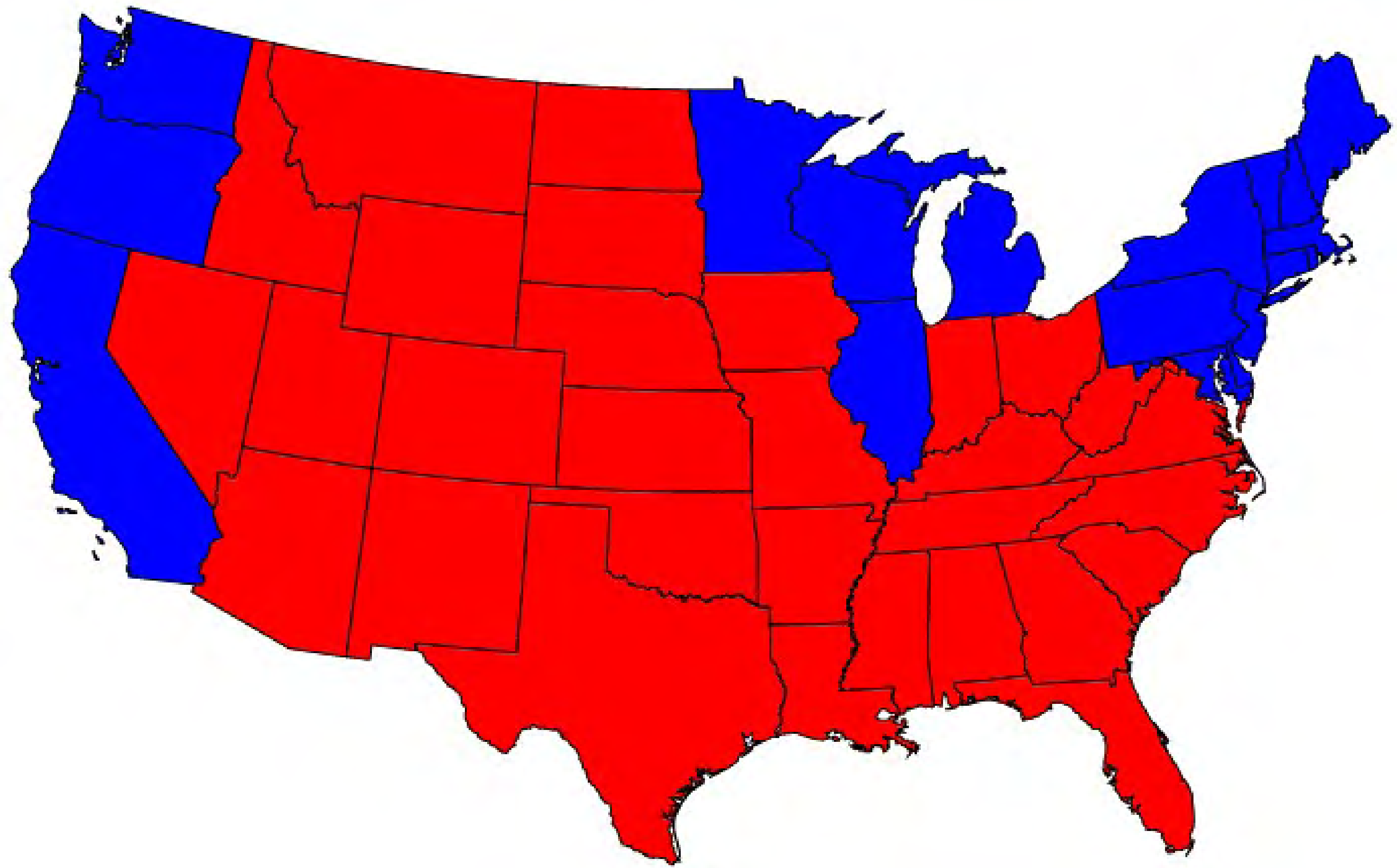
Fano plane of order 2

- **Fermat's Last Theorem** (Wiles 1993, 1994)
  - By contrast, any counterexample was too big to find (1985)

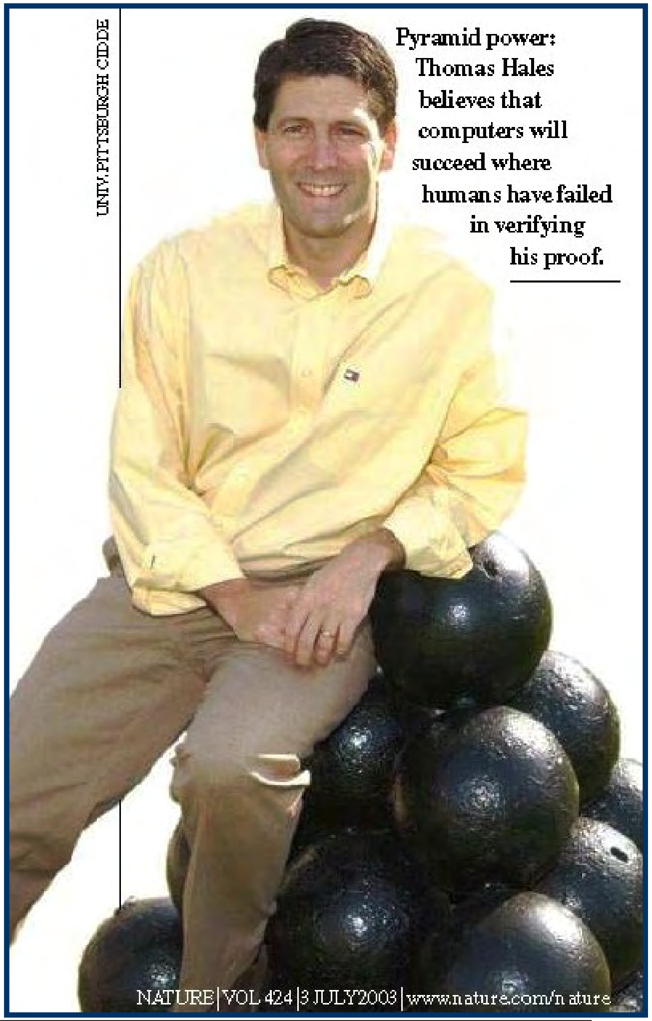
$$x^N + y^N = z^N, N > 2$$

has only trivial integer solutions

An Inadmissible Two-Colouring



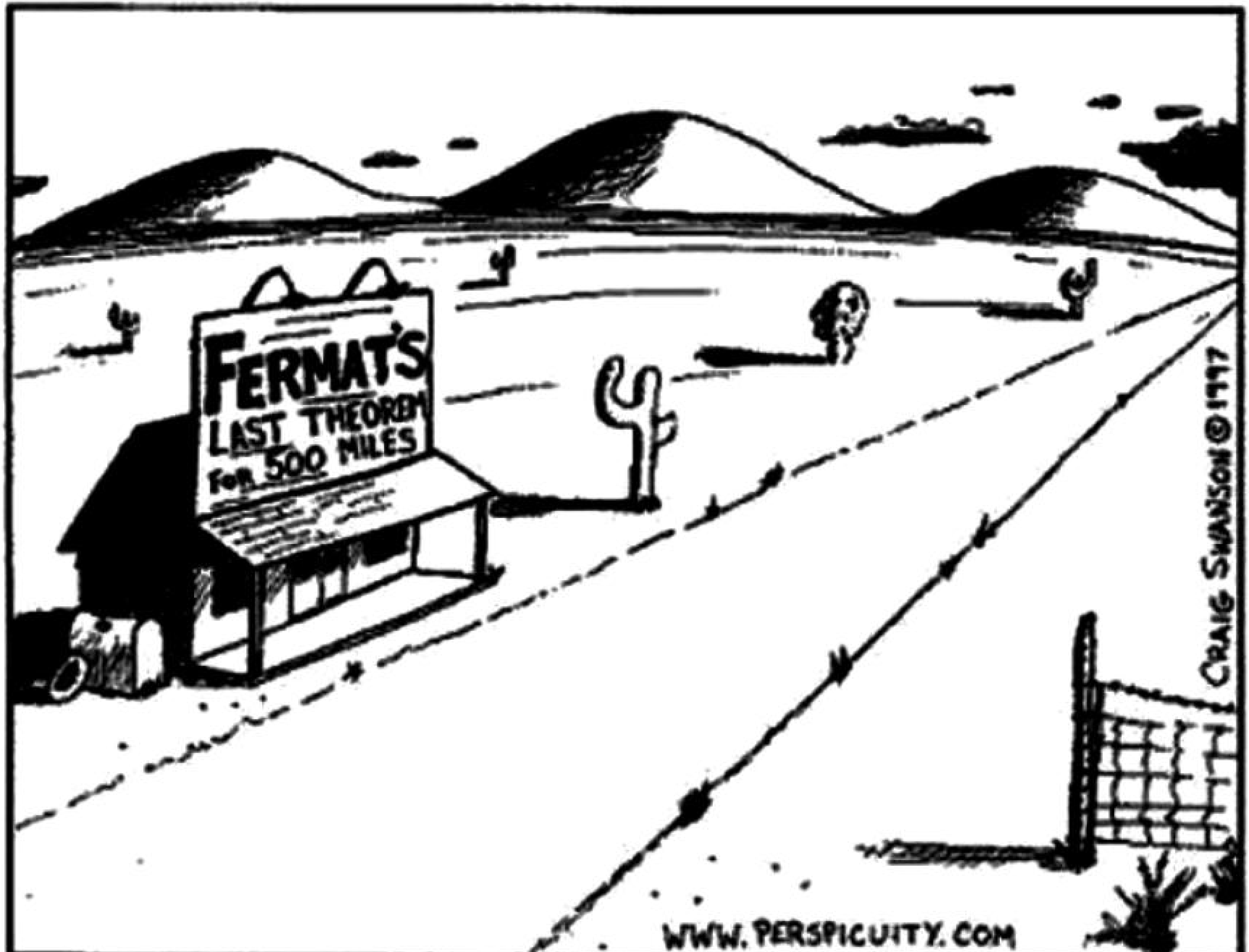
- **Kepler's conjecture**: *the densest way to stack spheres is in a pyramid*
  - oldest problem in discrete geometry
  - most interesting recent example of computer assisted proof
  - published in *Annals of Mathematics* with an ``only 99% checked'' disclaimer
  - Many varied reactions. *In Math, Computers Don't Lie. Or Do They?* (NYT, 6/4/04)



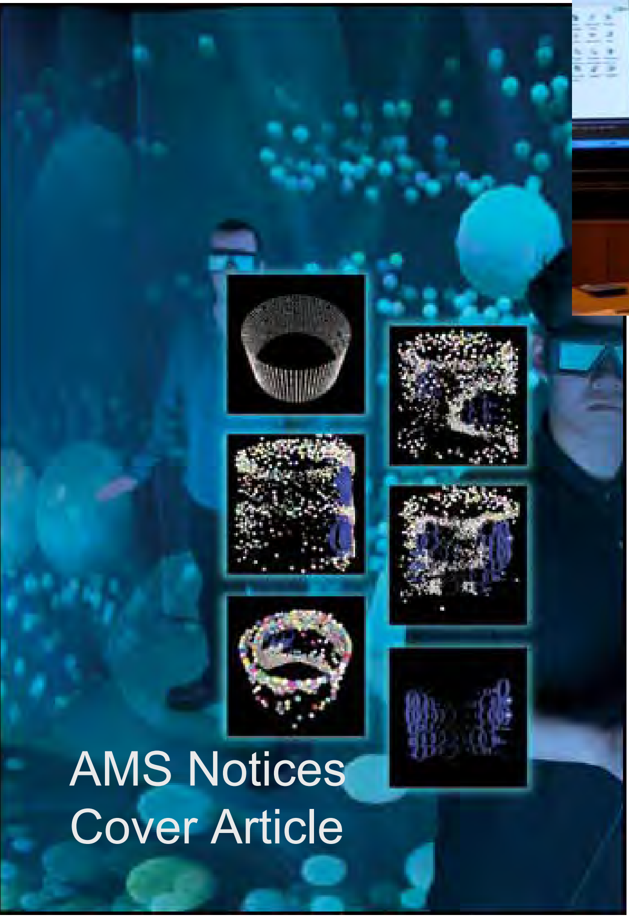
- **Famous earlier examples**: Four Color Theorem and Non-existence of a Projective Plane of Order 10.
  - the three raise quite distinct questions - both real and specious
  - as does status of classification of Finite Simple Groups

Formal Proof theory (code validation) has received an unexpected boost: automated proofs *may* now exist of the Four Color Theorem and Prime Number Theorem

- COQ: *When is a proof a proof* ? Economist, April 2005


FERMAT'S
LAST THEOREM
FOR 500 MILES
CRAIG SWANSON © 1997
WWW.PERSPICUITY.COM

5 SMART Touch-sensitive Interwoven Screens



My intention is to show a variety of mathematical uses of high performance computing and communicating as part of

Experimental Inductive Mathematics

Our web site:

www.experimentalmath.info

contains all links and references

"Elsewhere Kronecker said ``In mathematics, I recognize true scientific value only in concrete mathematical truths, or to put it more pointedly, only in mathematical formulas." ... I would rather say ``computations" than ``formulas", but my view is essentially the same."

Harold Edwards, *Essays in Constructive Mathematics*, 2004

May 2005
AMS Notices Cover

About the Cover

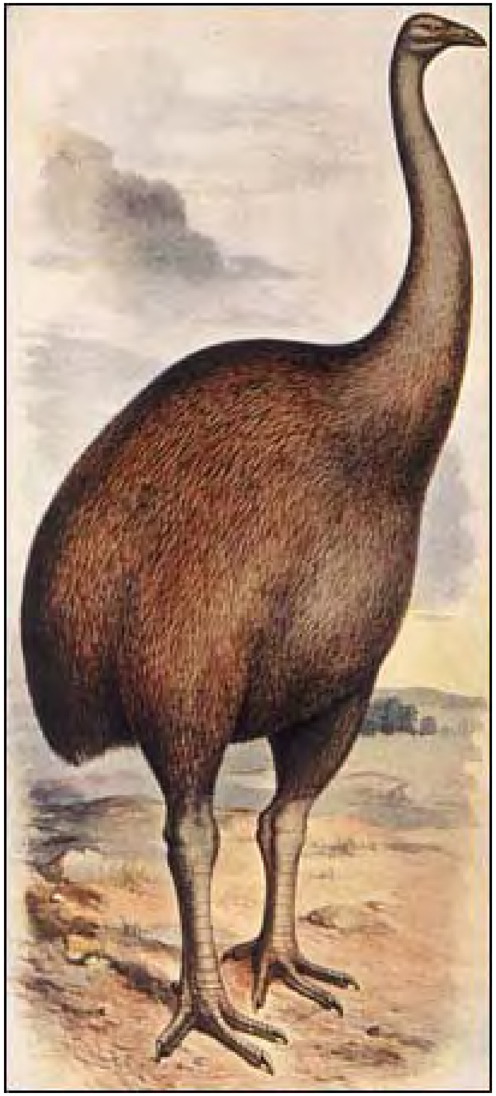
Extreme 3D visualization

The background image of this month's cover is a photograph included by Jonathan Borwein and David Bailey, perhaps somewhat whimsically, in their article on experimental mathematics. The photograph was taken for a publicity brochure for the now defunct New Media Innovation Centre in downtown Vancouver, British Columbia, an organization partially sponsored by Simon Fraser University, to which Borwein is affiliated. The two young men, who are graduate students in the the department of Electrical and Computer Engineering at the University of British Columbia, are in a kind of box with what might be called surround-projection. The approximate spheres are displayed in duplicate at rapidly alternating times in synchronization with the goggles they are wearing, so that what they see is a simulated 3D image, not just the flat projections on the walls on their box. The projections are interactive, controlled by input through a key pad held by Timothy Chen, the student on the right. The project the students are involved in is part of Mr. Chen's student work at U. B. C. What is being projected is a flow field of spheres in a cylinder with various obstacles interactively superimposed into the flow. The inset photographs are screen displays produced by Mr. Chen from the same project.

It's hard to imagine exactly what role such high end visualization technology will play in mathematical research, but not impossible. One likely application for similar, but not quite so sophisticated, display systems might very well be in public presentations. The effects can be spectacular.

Brian Corrie of Simon Fraser University provided us with the digital version of the background photograph.

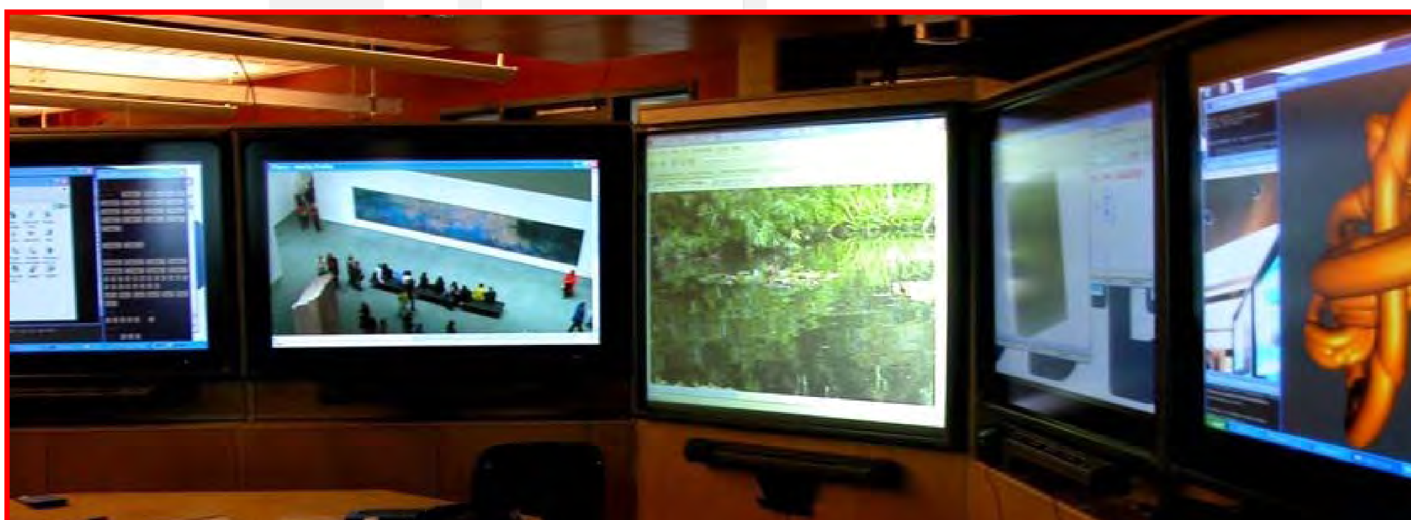
—*Bill Casselman, Graphics Editor*
(notices-cover@ams.org)



East meets West: Collaboration goes National

Welcome to D-DRIVE whose mandate is to study and develop resources specific to distributed research in the sciences with first client groups being the following communities

- High Performance Computing
- Mathematical and Computational Science Research
- Science Outreach
    - ✓ Educational
    - ✓ Research



Centre seen as 'serious nirvana'

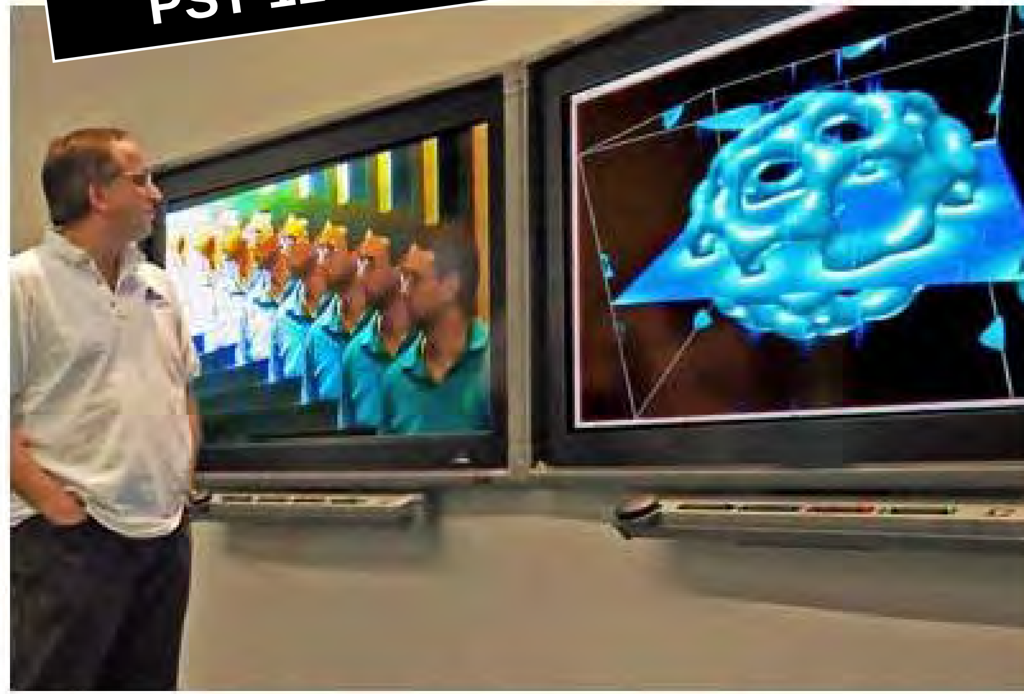
April 07, 2005 , vol. 32, no. 7

By Carol Thorbes

The 2,500 square metre IRMACS research centre

Trans-Canada Seminar Thursdays PST 11.30 MST 12.30 AST 3.30

Move over creators of Max Head-room, Matrix and Metropolis. What researchers can accomplish at Simon Fraser University's IRMACS centre rivals the high tech feats of the most memorable futuristic films.



SFU mathematician and IRMACS executive director Peter Borwein (left) communicates with IRMACS collaboration and visualization coordinator Brian Corrie. To the right of them another plasma display portrays a 3D image of a molecular structure.

- ✓The building is a also a 190cpu G5 Grid
- ✓At the official April opening, I gave one of the four presentations from D-DRIVE

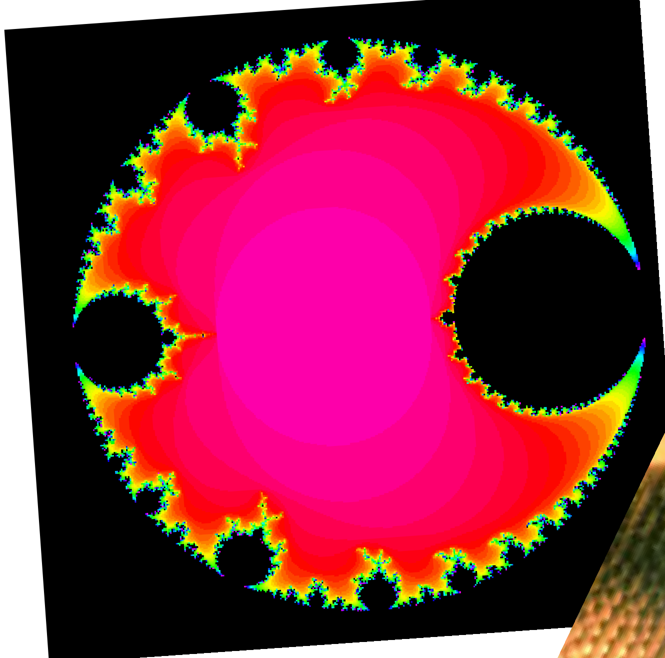
The $14 million centre's acronym stands for interdisciplinary research in the mathematical and computational sciences. The centre's expansive view of the

from atop
in echoes its
al as a facility
tering
research
s whose
is the computer.

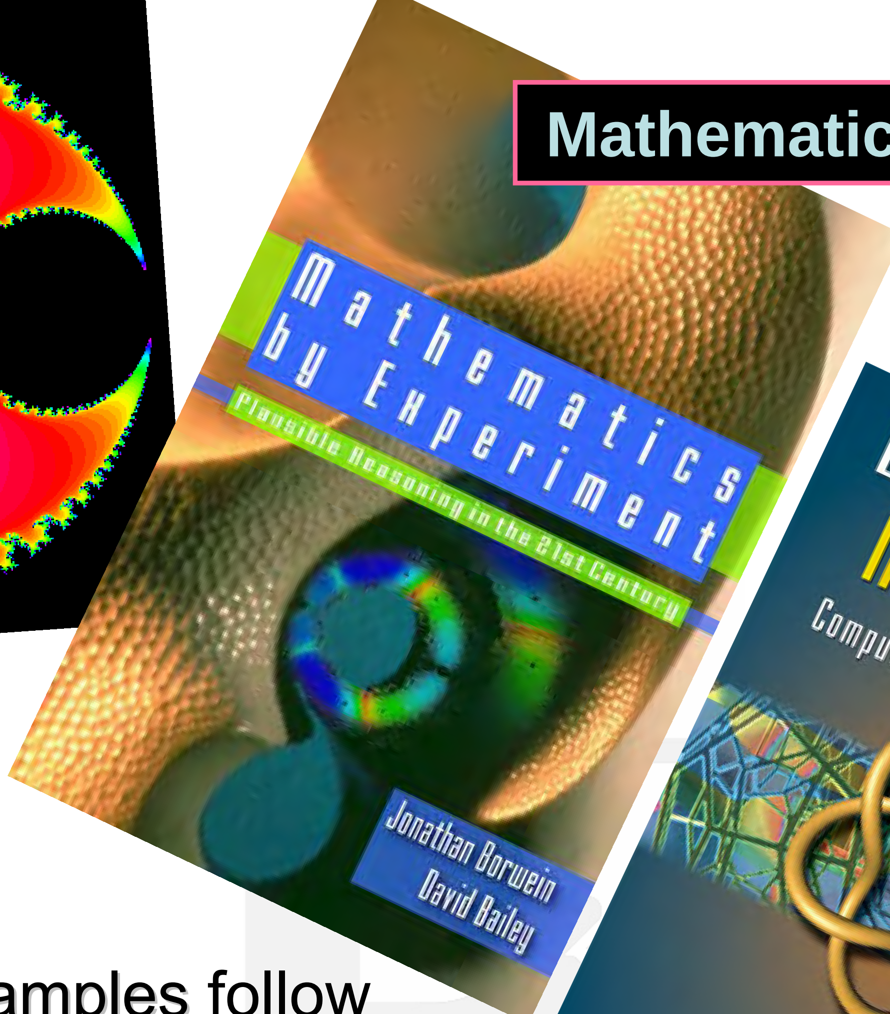
cted 2,500 square metre space atop the applied sciences building, the centre has eight
ng rooms and a presentation theatre, seating up to 100 people. They are equipped with
ble computational, multimedia, internet and remote conferencing (including satellite)
technology. High performance distributed computing and clustering technology, designed at SFU, and
access to WestGrid, an ultra high speed, interprovincial network, with shared computing and multimedia

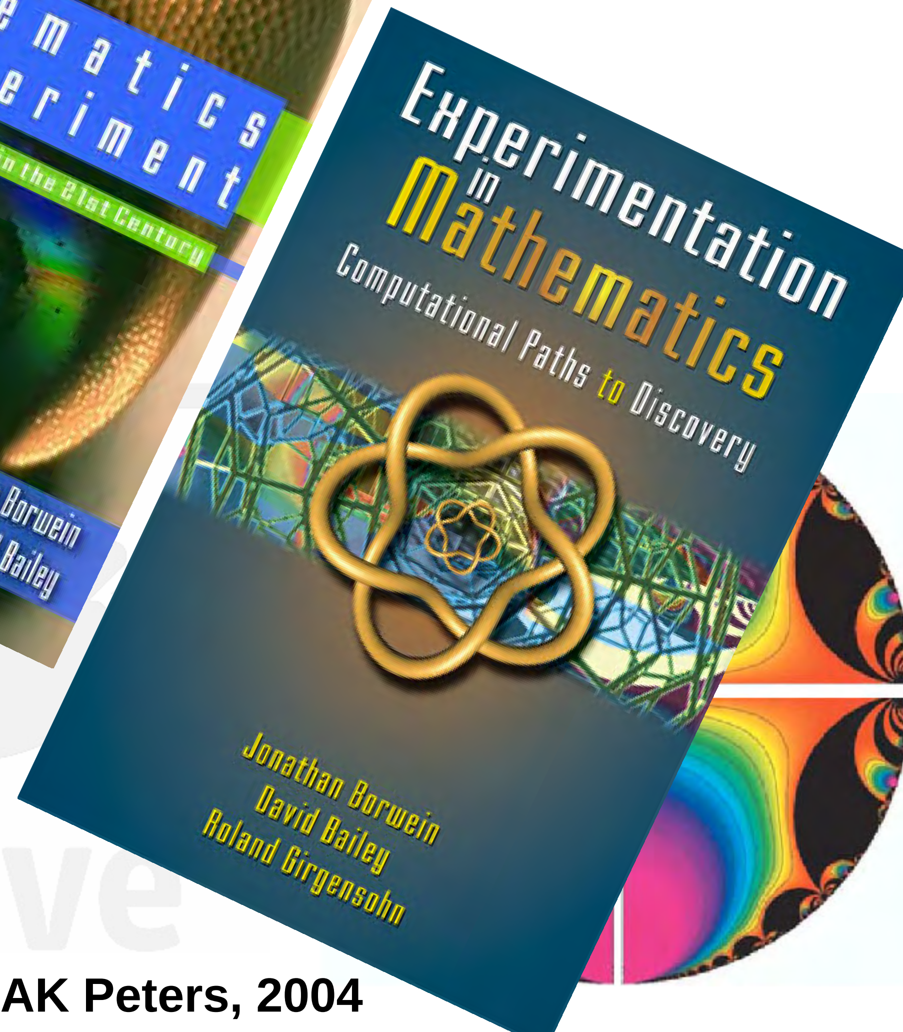
The present
SFU - Colab
Pierre Boulanger
SFU - Colab
University of Alberta
The AG in Action
in CoLab
westgrid...
UBC - MAGIC
Netera AG

Mathematical Data Mining



An unusual Mandelbrot parameterization





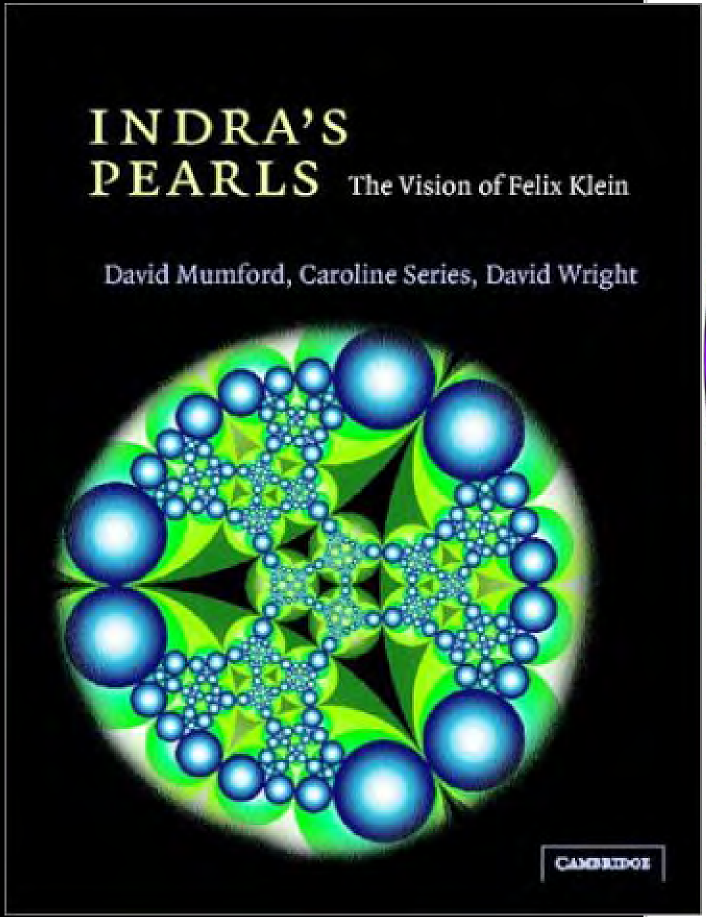
Various visual examples follow

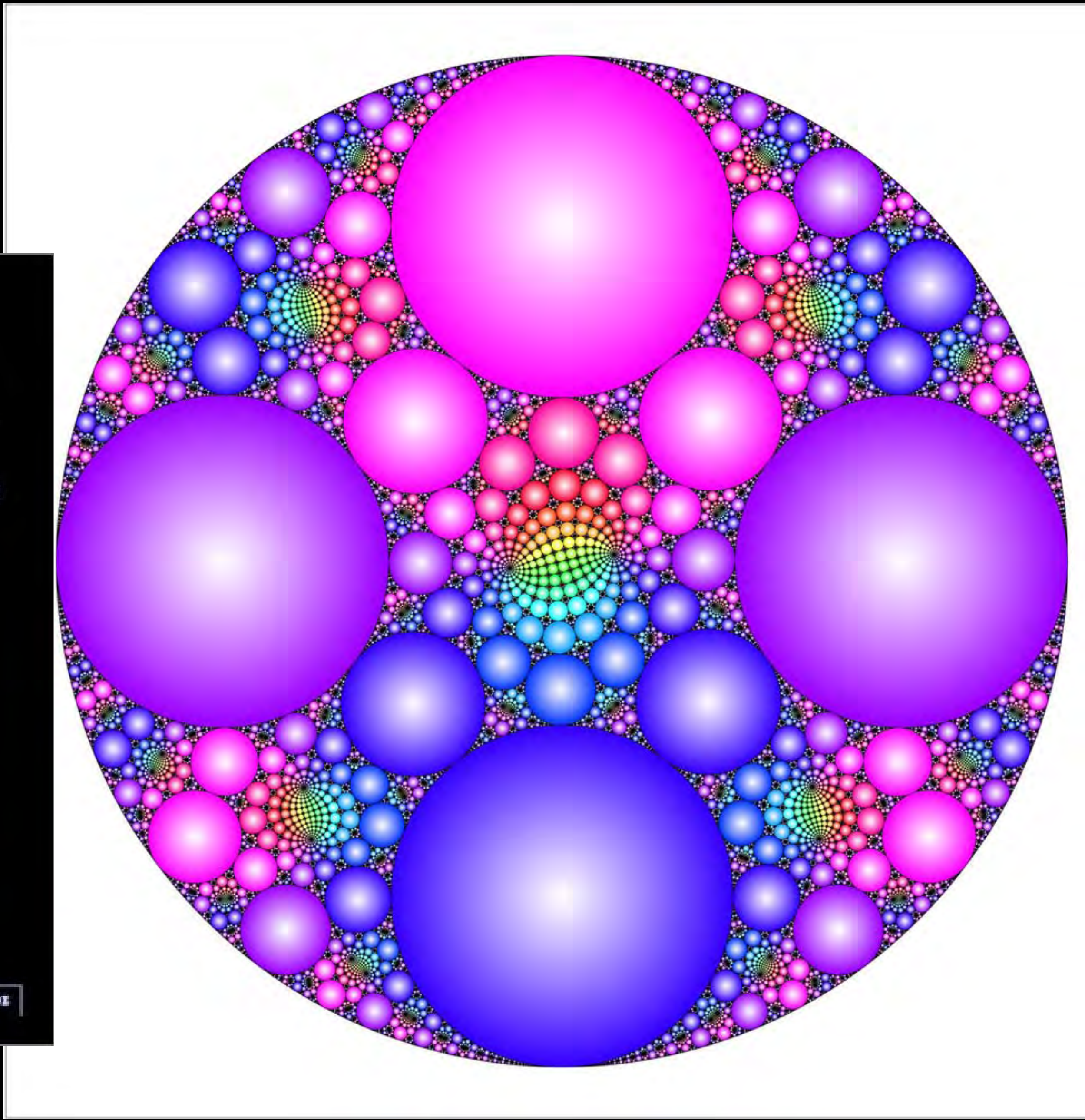
- ✓ Roots of `1/-1' polynomials
- ✓ Ramanujan's fraction
- ✓ Sparsity and Pseudospectra

AK Peters, 2004

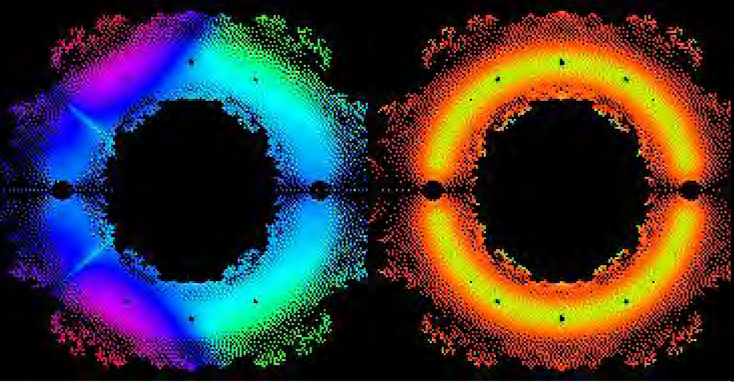
(CD in press)

Indra's Pearls
A merging of 19th and 21st Centuries



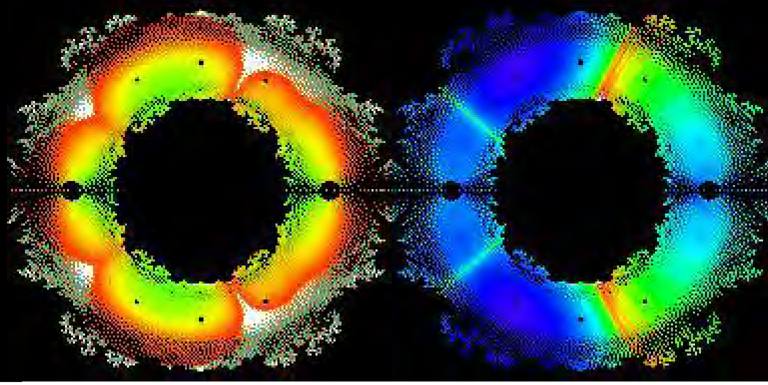


http://klein.math.okstate.edu/IndrasPearls/



Roots of Zeros

What you draw is what you see (visible patterns in number theory)



Striking fractal patterns formed by plotting complex zeros for all polynomials in powers of x with coefficients 1 and -1 to degree 18

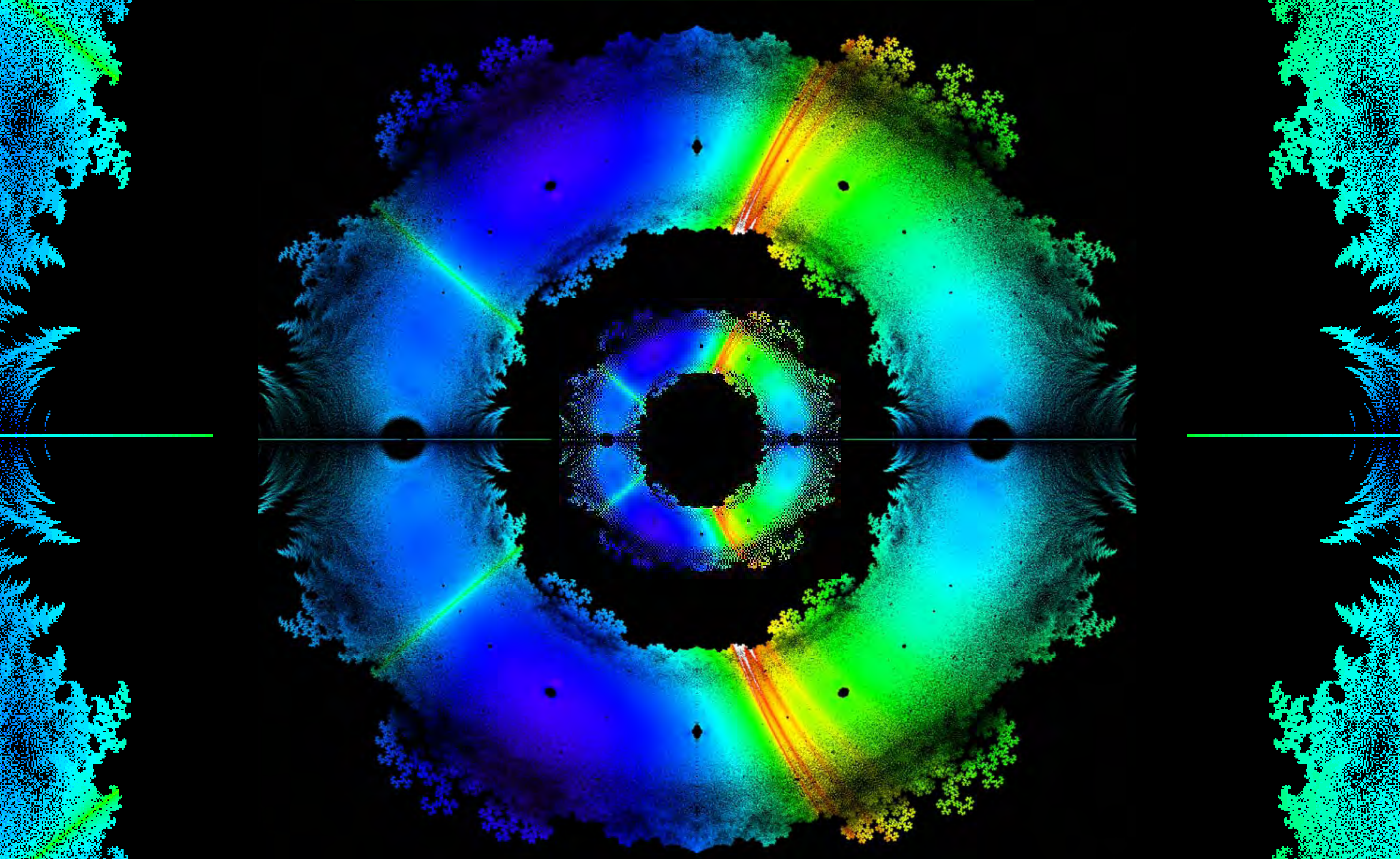
Coloration is by sensitivity of polynomials to slight variation around the values of the zeros. The color scale represents a normalized sensitivity to the range of values; red is insensitive to violet which is strongly sensitive.

- All zeros are pictured (at **3600 dpi**)
- Figure 1b is colored by their local density
- Figure 1d shows sensitivity relative to the $\mathbf{x^9}$ term
- **The white and orange striations are not understood**

A wide variety of patterns and features become visible, leading researchers to totally unexpected mathematical results

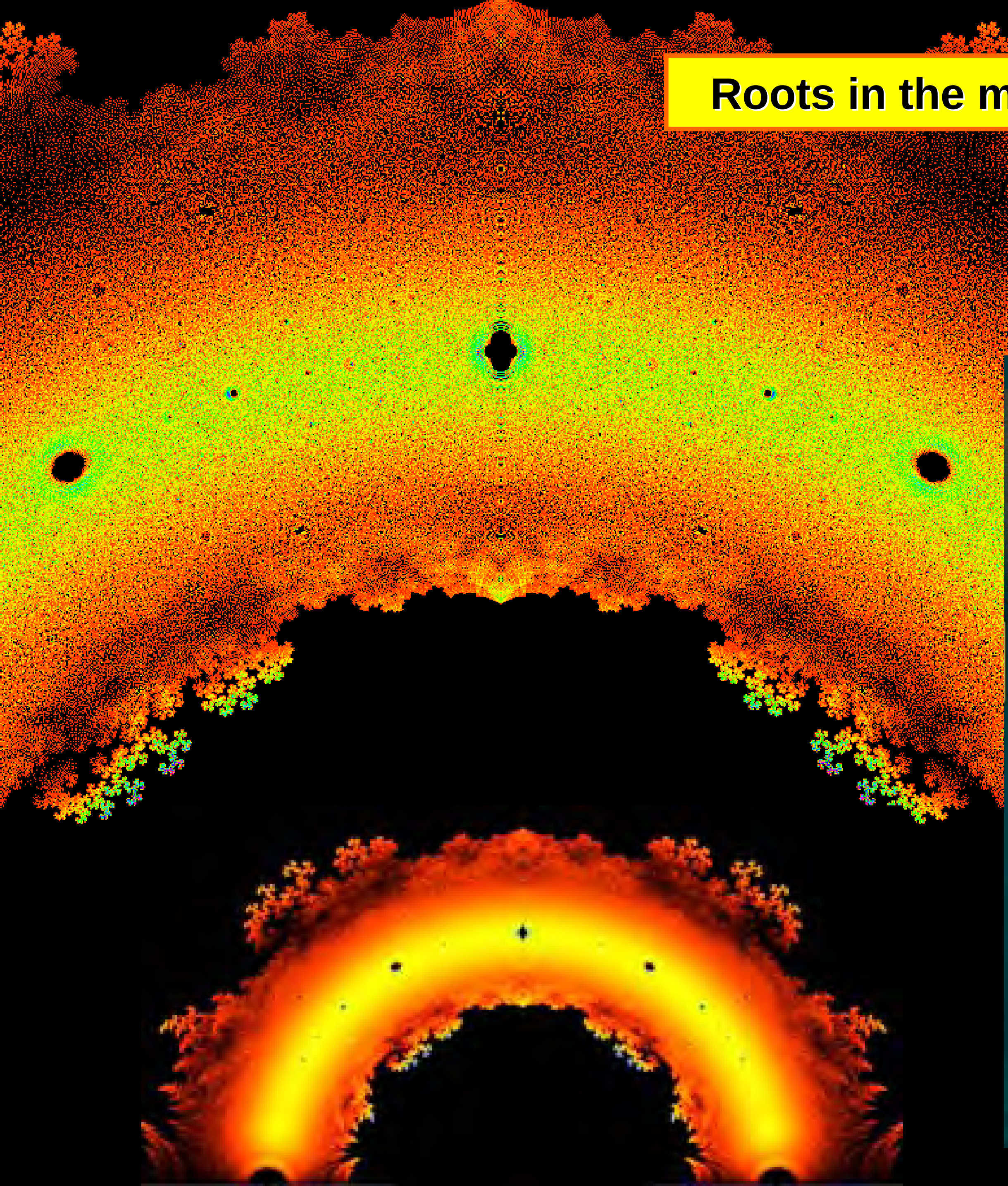
"The idea that we could make biology mathematical, I think, perhaps is not working, but what is happening, strangely enough, is that maybe mathematics will become biological!" Greg Chaitin, Interview, 2000.

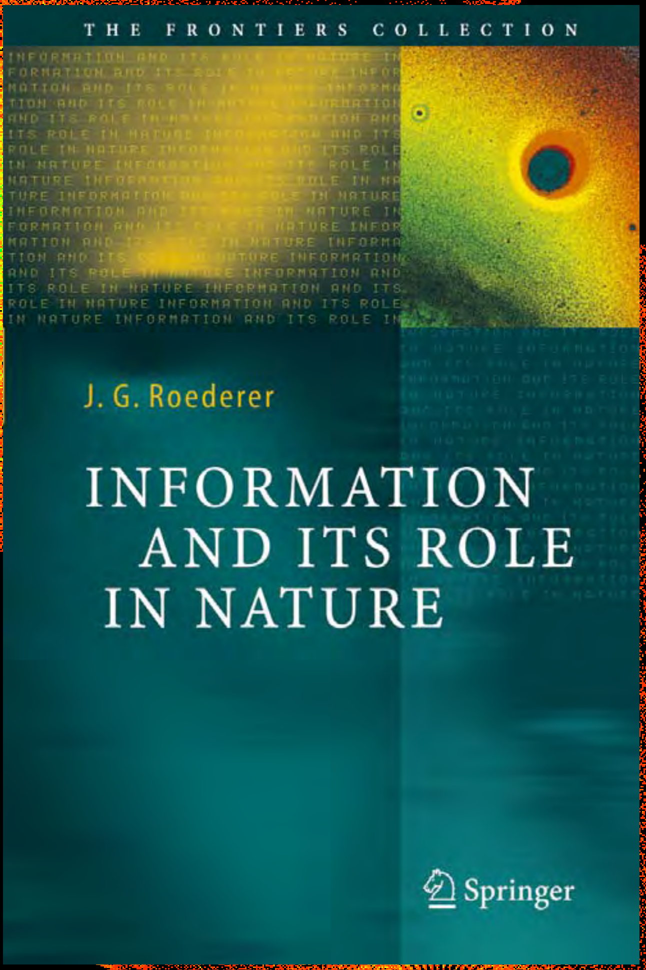
The TIFF on THREE SCALES

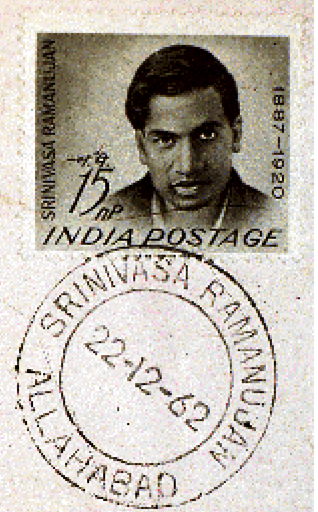


Pictures are more democratic
but they come from formulae

Roots in the most stable colouring

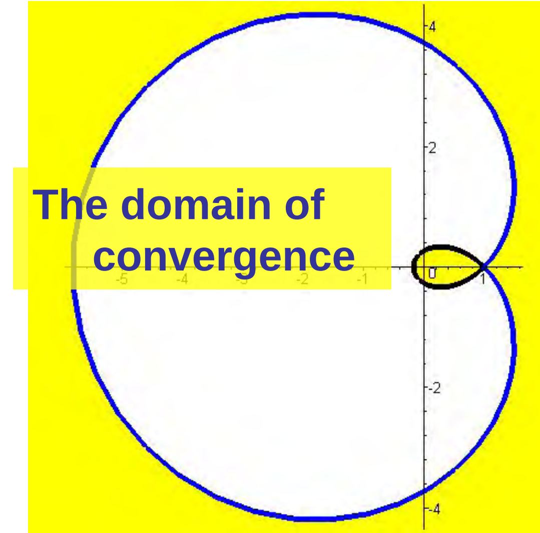






Ramanujan's Arithmetic-Geometric Continued fraction (CF)

$$R_\eta(a,b) = \cfrac{a}{\eta + \cfrac{b^2}{\eta + \cfrac{4a^2}{\eta + \cfrac{9b^2}{\eta + \ddots}}}}$$



A cardioid

- ❑ For a,b>0 the CF satisfies a lovely symmetrization

$$\mathcal{R}_\eta\left(\frac{a+b}{2}, \sqrt{ab}\right) = \frac{\mathcal{R}_\eta(a,b) + \mathcal{R}_\eta(b,a)}{2}$$

- ❑ Computing directly was too hard even just 4 places of $\mathcal{R}_1(1,1) = \log 2$

We wished to know for which a/b in **C** this all held

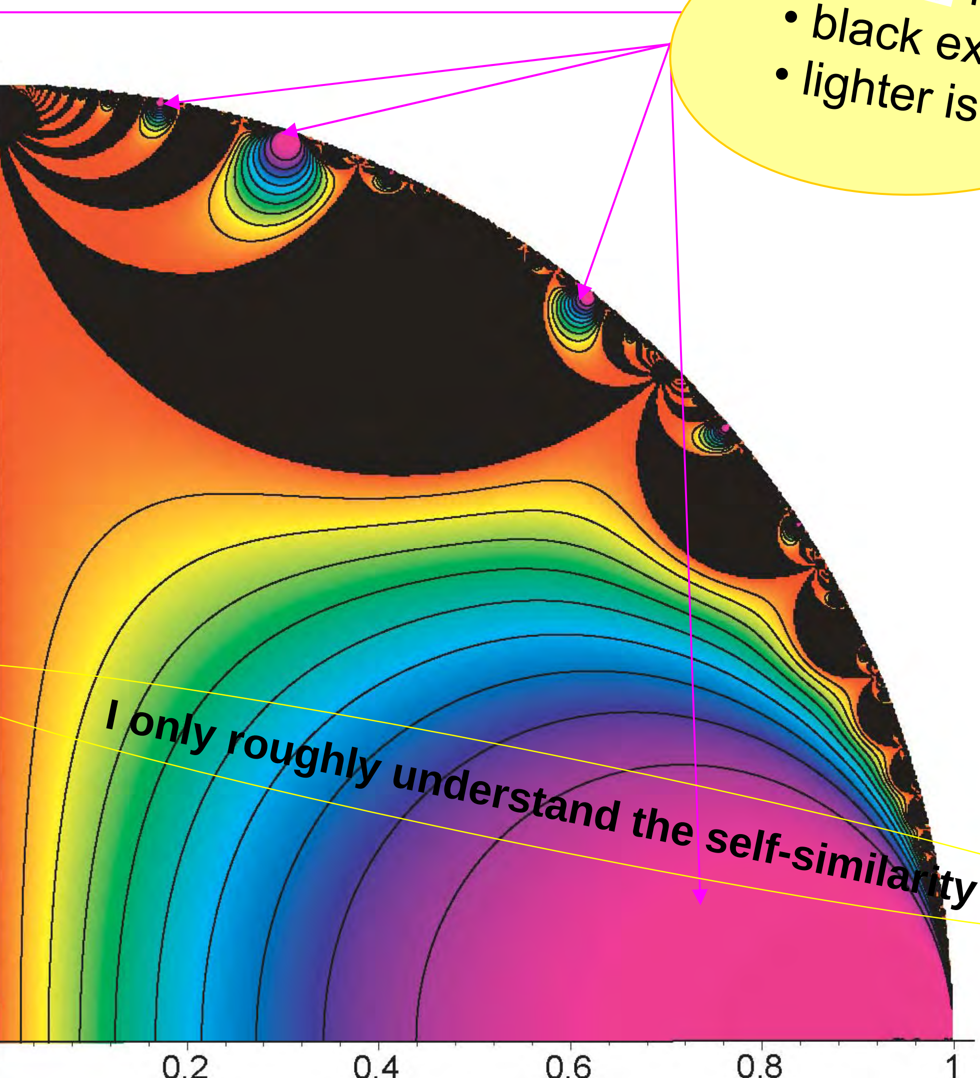
- ✓ The **scatterplot** revealed a precise **cardioid** where r=a/b.
  - ✓ which discovery it remained to prove?

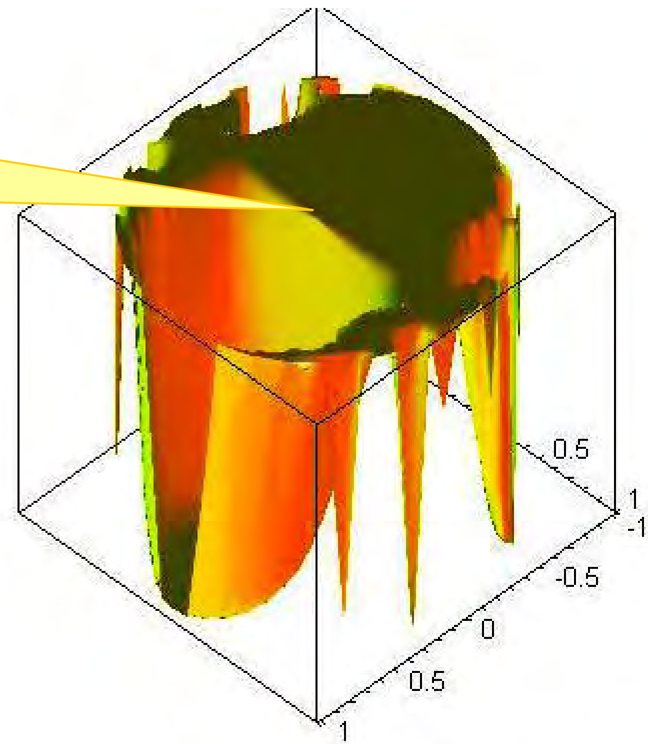
$r^2 - 2r\{2 - \cos(\theta)\} + 1 = 0$

$$\left|\frac{a+b}{2}\right| \geq \sqrt{|ab|}$$

FRACTAL of a Modular Inequality

$$\mathcal{R} = \frac{|\sum_{n \in \mathbf{Z}} (-1)^n q^{n^2}|}{|\sum_{n \in \mathbf{Z}} q^{n^2}|}$$



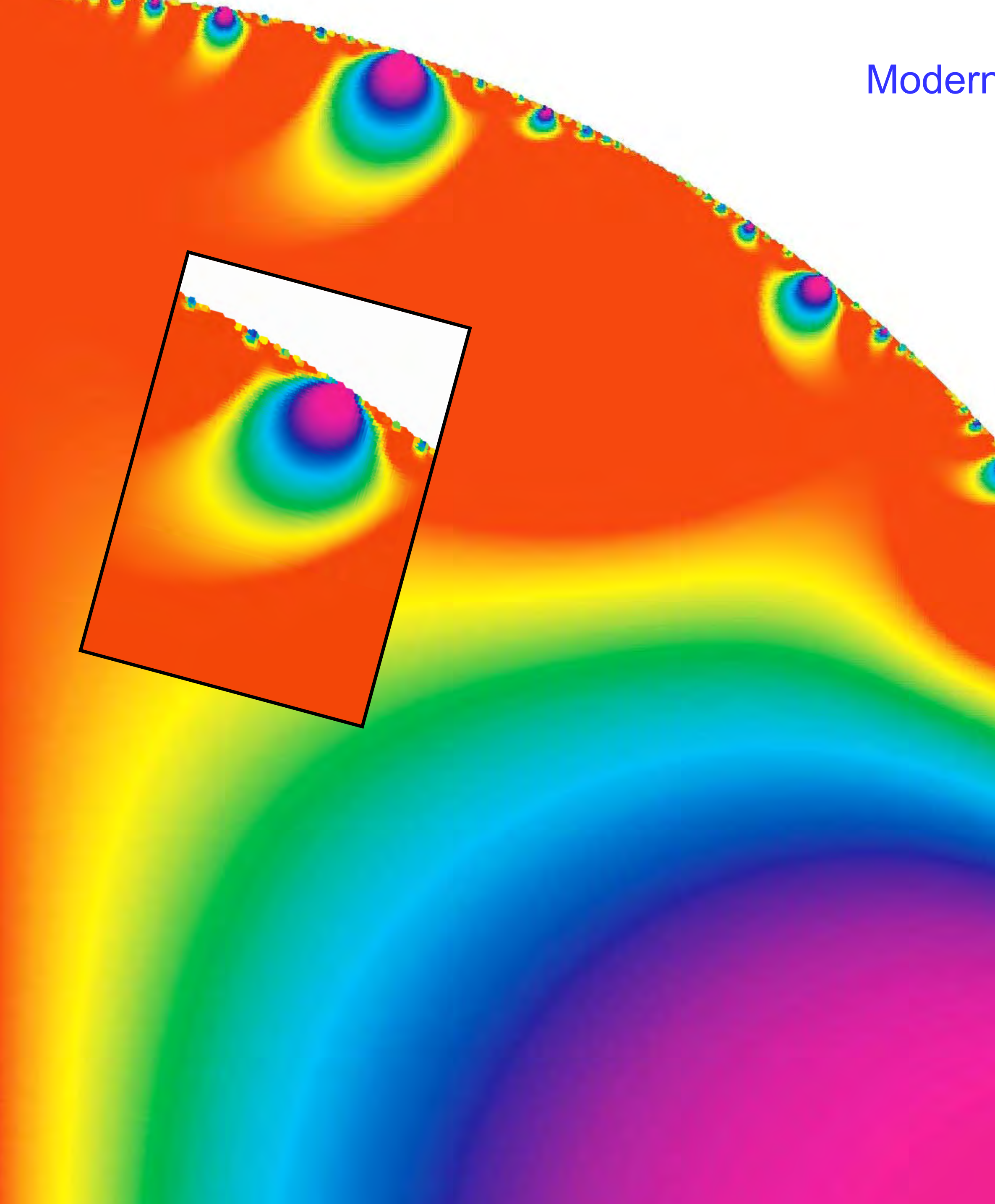


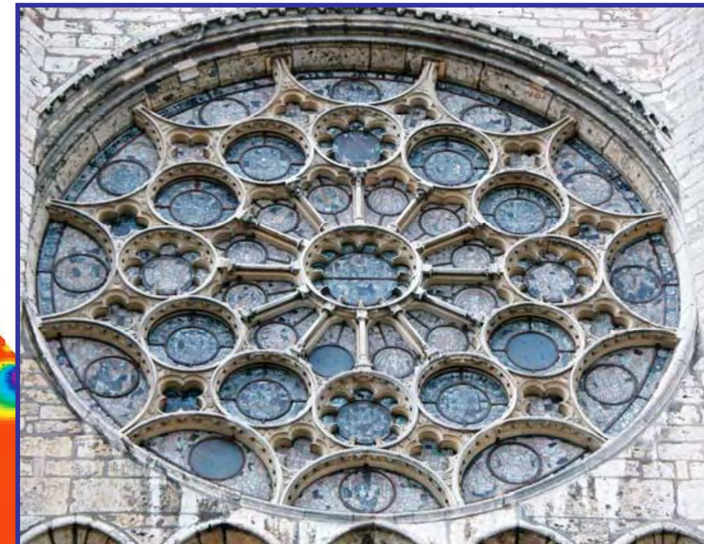
- ✓ related to Ramanujan's continued fraction
- ✓ took several hours to print
- ✓ Crandall/Apple has parallel print mode

Mathematics and the aesthetic

Modern approaches to an ancient affinity

(CMS-Springer, 2005)

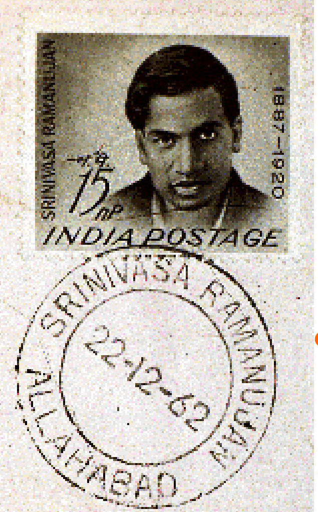




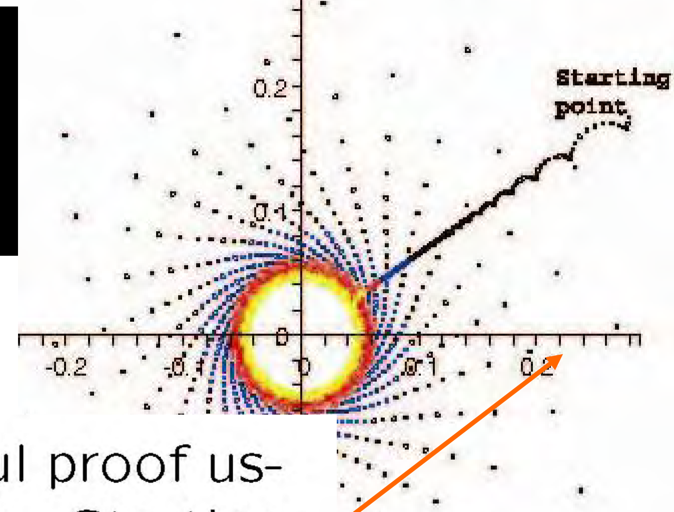
Why should I refuse a good dinner simply because I don't understand the digestive processes involved?

Oliver Heaviside (1850 - 1925)

✓ **when criticized for his daring use of operators before they could be justified formally**



Ramanujan's Arithmetic-Geometric Continued fraction



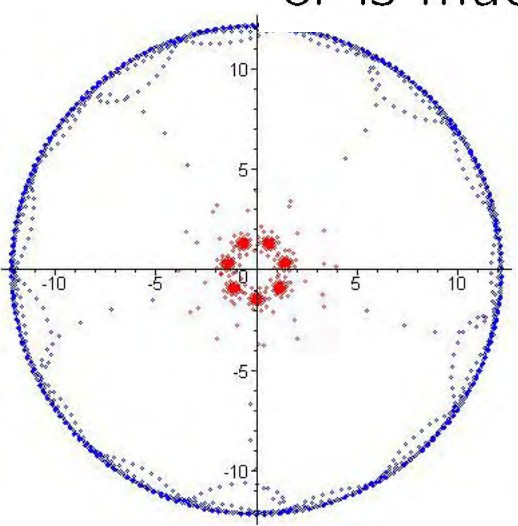
1. The Blackbox

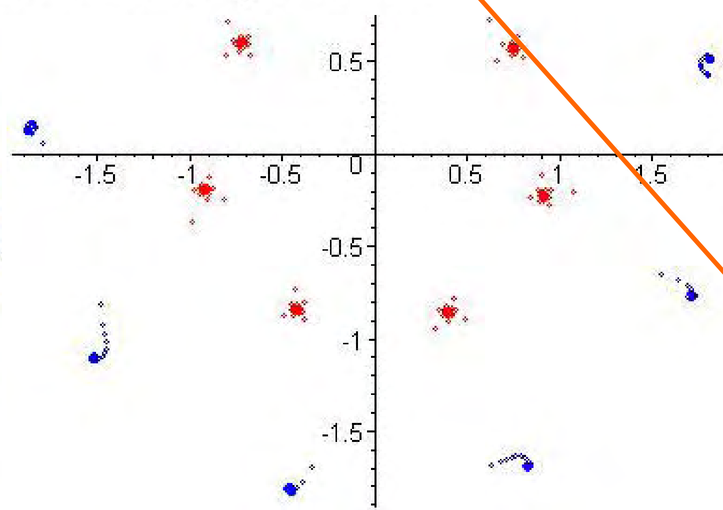
Six months later we had a beautiful proof using genuinely new dynamical results. Starting from the *dynamical system* $t_0 := t_1 := 1$:

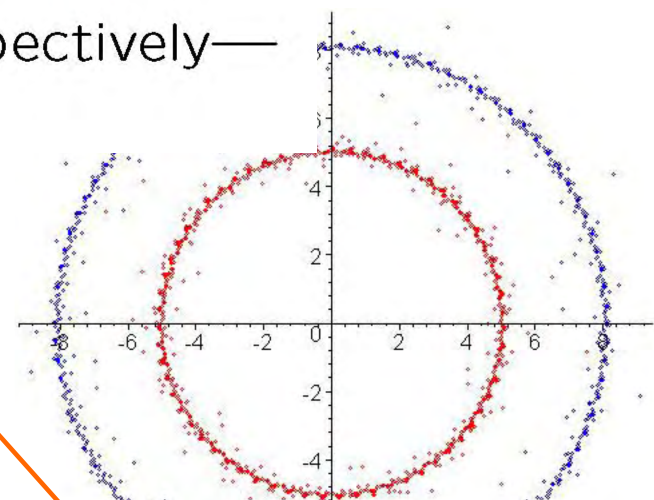
$$t_n \leftarrow \frac{1}{n} t_{n-1} + \omega_{n-1} \left(1 - \frac{1}{n}\right) t_{n-2},$$

2. Seeing convergence

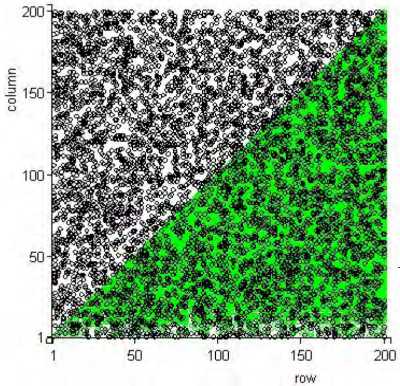
where $\omega_n = a^2, b^2$ for n even, odd respectively—or is much more general.*





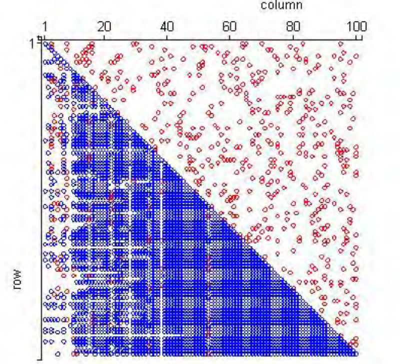


3. Attractors. *Normalizing* by $n^{1/2}$ three cases appear



Pseudospectra or Stabilizing Eigenvalues

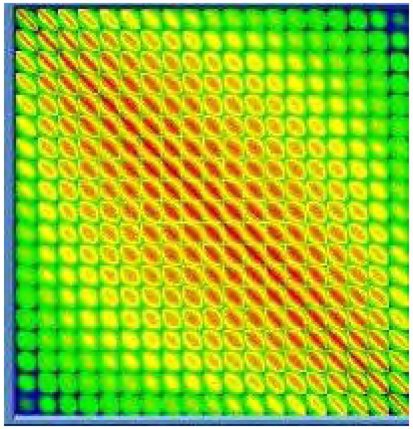
Gaussian elimination of random sparse (10%-15%) matrices



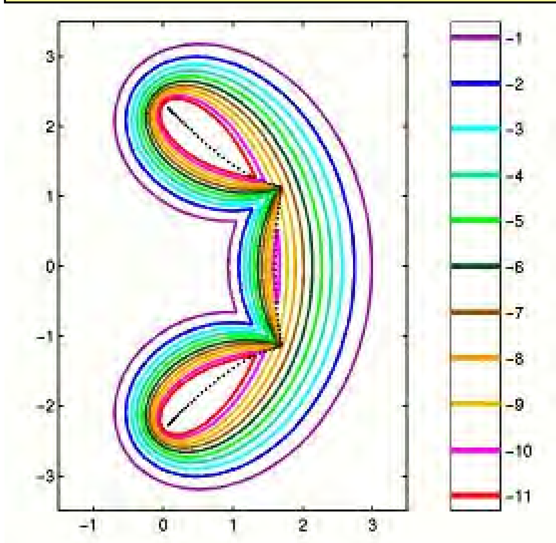
‘Large’ (10^5 to 10^8) Matrices must be seen

- ✓ sparsity and its preservation
- ✓ conditioning and ill-conditioning
- ✓ eigenvalues
- ✓ singular values (helping Google work)

A dense inverse



Pseudospectrum of a banded matrix



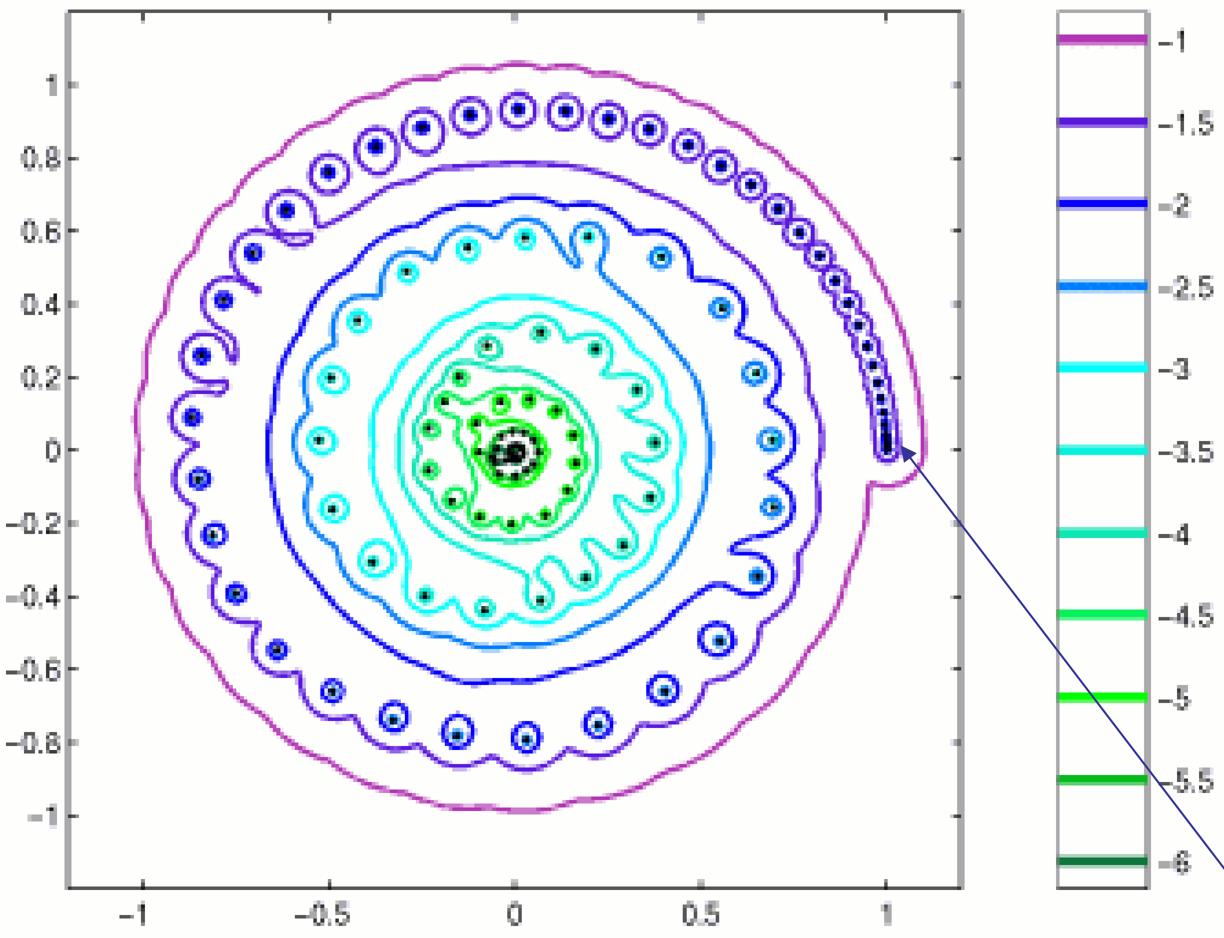
The ε-pseudospectrum of A is:

$$\sigma_\varepsilon(A) = \{x : \exists \lambda \text{ s.t. } \|Ax - \lambda x\| \le \varepsilon\}$$

- ✓ for ε = 0 we recover the eigenvalues
- ✓ full pseudospectrum carries much more information

http://web.comlab.ox.ac.uk/projects/pseudospectra

An Early Use of Pseudospectra (Landau, 1977)



$\varepsilon = 10^t$

An infinite dimensional integral equation in laser theory

- ✓discretized to a matrix of dimension **600**
- ✓projected onto a well chosen invariant subspace of dimension **109**

Rob Scharein's KnotPlot

Perko pair knots

These are the famous Perko pair knots, listed as distinct knots in many knot tables since the 19th century, until Kenneth Perko showed in 1974 that they were in fact the same knot. He proved the equivalence by showing a sequence of diagrams leading from one to the other. The following sequence is a different demonstration of the same fact, obtained by relaxing the two knots using KnotPlot.

A movie of the deformation is included as one of the standard KnotPlot demos. View it by first installing KnotPlot, then click on the "DemoA" panel and then "Perko pair".

Perko A (10_{161})

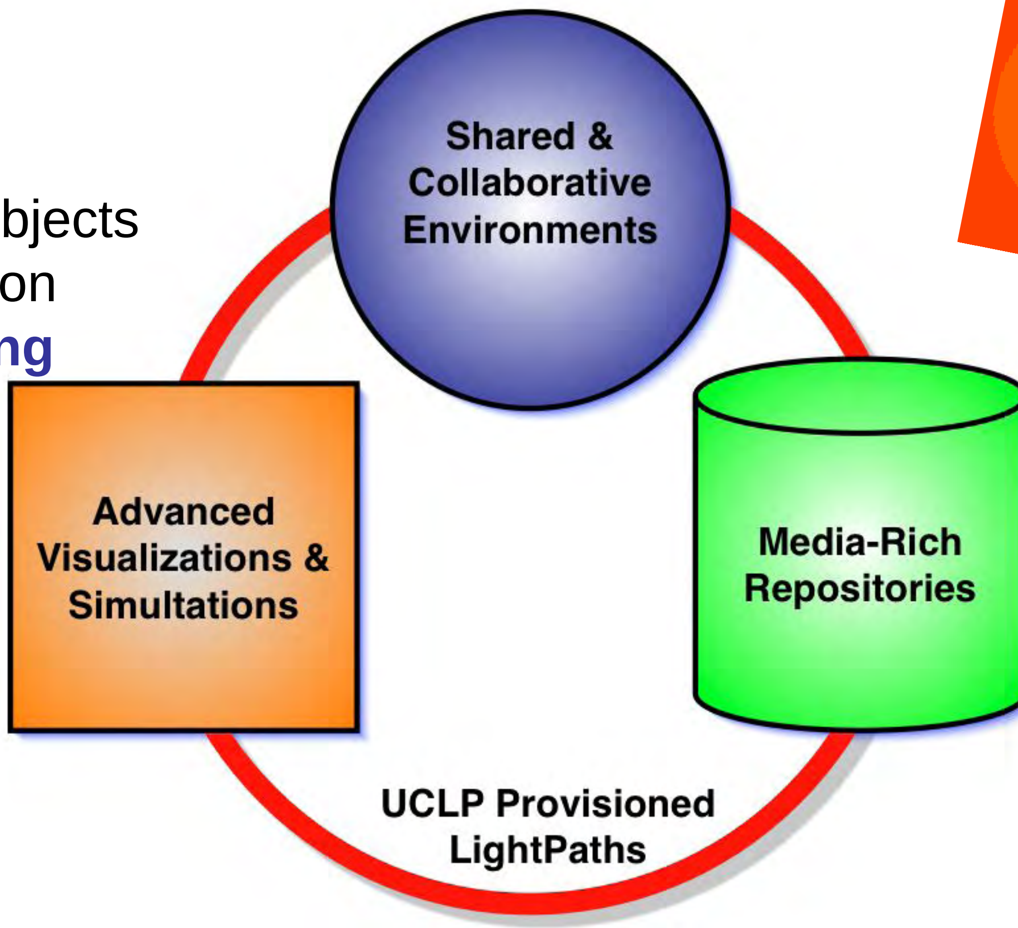
Perko B (10_{162})

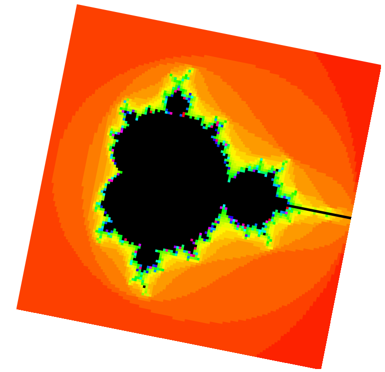
Visualization

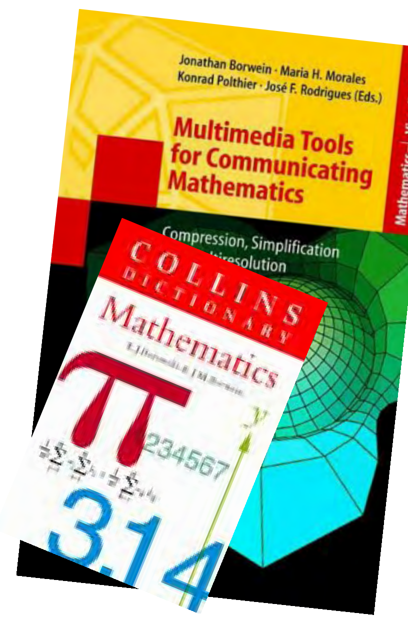
Advanced Networking …

Components include

- **AccessGrid**
- **UCLP** for
  - ✓ haptics
  - ✓ learning objects
  - ✓ visualization
- **Grid Computing**







Haptics in the MLP

Haptic Devices extend the world of I/O into the tangible and tactile

Drive

Dalhousie Distributed Research Institute and Virtual Environment

We aim to link multiple devices together such that two or more users may interact at a distance

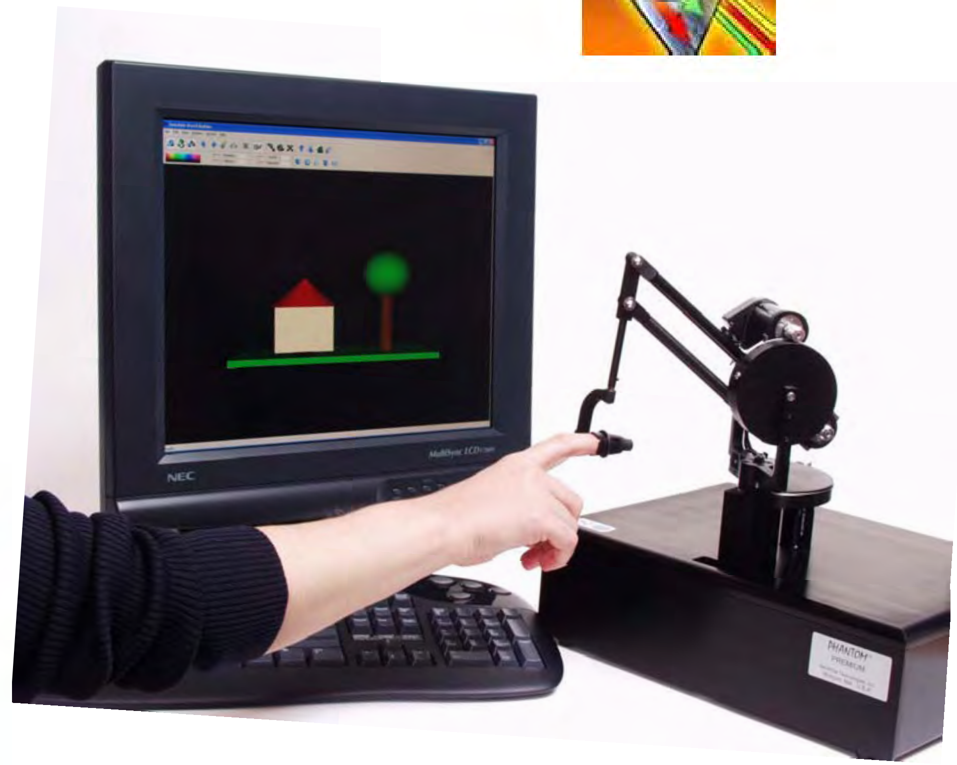
- in Museums and elsewhere
- Kinesiology, HCI ….

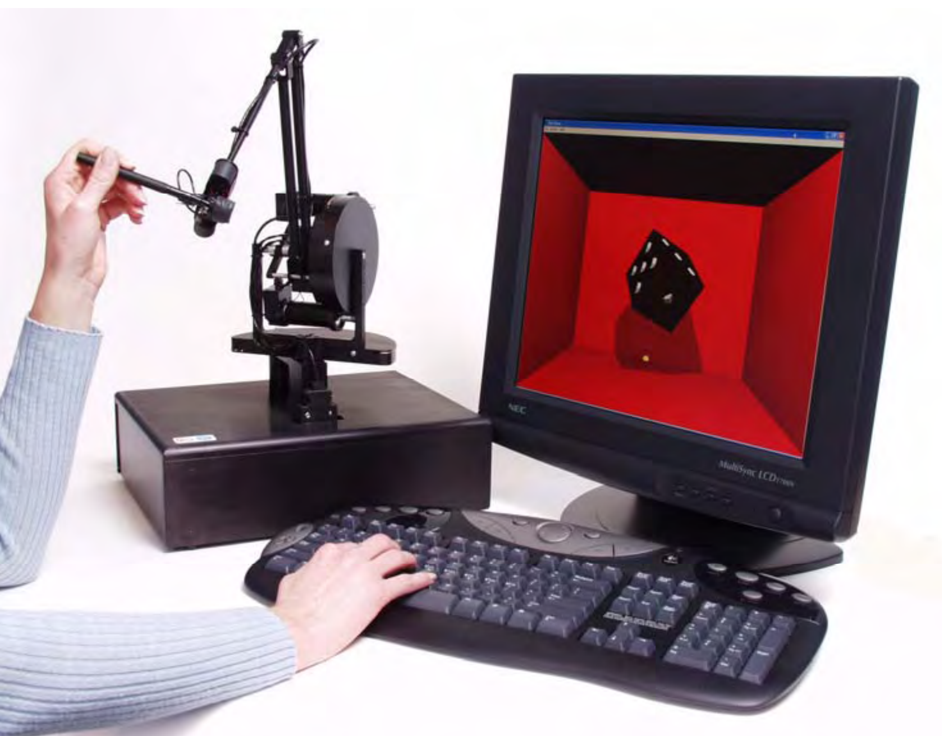
Sensable's **Phantom Omni**

And what they do

MEDIALIGHTPATHS

Force feedback informs the user of his virtual environment adding an increased depth to human computer interaction





The user feels the contours of the virtual die via resistance from the arm of the device

Generic Code Optimization

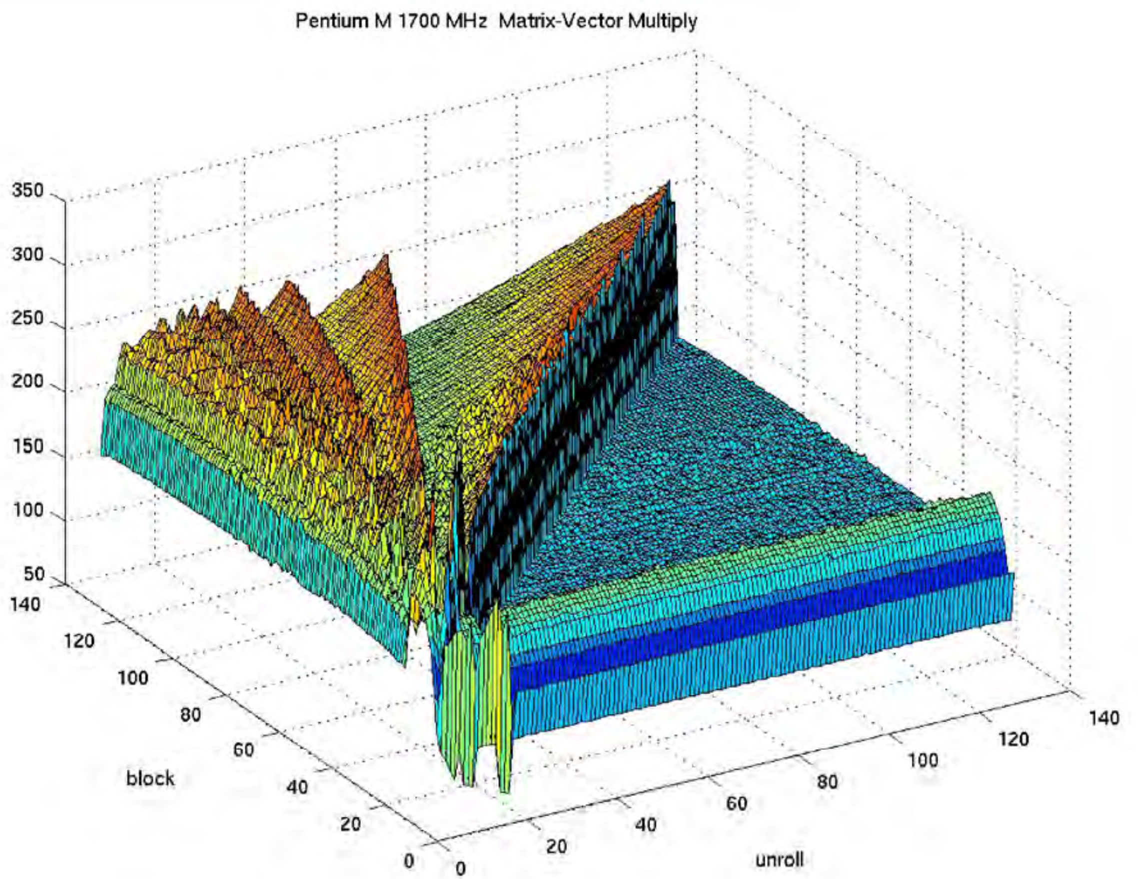
Experimentation with DGEMV (matrix-vector multiply):

128x128=16,384 cases.

Experiment took 30+ hours to run.

Best performance = 338 Mflop/s with blocking=11 unrolling=11

Original performance = 232 Mflop/s



Visual Representation of Automatic Code Parallelization

Outline. What is HIGH PERFORMANCE MATHEMATICS?

1. Visual Data Mining in Mathematics.

- ✓ Fractals, Polynomials, Continued Fractions, Pseudospectra

2. High Precision Mathematics.

3. Integer Relation Methods.

- ✓ Chaos, Zeta and the Riemann Hypothesis, HexPi and Normality

4. Inverse Symbolic Computation.

- ✓ A problem of Knuth, $\pi/8$, Extreme Quadrature

5. The Future is Here.

- ✓ Examples and Issues

6. Conclusion.

- ✓ Engines of Discovery. The 21st Century Revolution
  - ✓ Long Range Plan for HPC in Canada

hpcs 2005

Drive

A WARMUP Computational Proof

- Suppose we know that $1<\alpha<10$ and that α is an integer
  - **computing α to 1 significant place** **with a certificate** will prove the value of α. *Euclid's method* is basic to such ideas.
- Likewise, suppose we <u>know</u> α is *algebraic of degree d and length l*

  (coefficient sum in absolute value)

  If P is polynomial of degree D & length L **EITHER** $P(\alpha) = 0$ **OR**

$$|P(\alpha)| \geq \frac{1}{L^{d-1} l^D}$$

Example (MAA, April 2005). Prove that

$$\int_{-\infty}^{\infty} \frac{y^2}{1+4\,y+y^6-2\,y^4-4\,y^3+2\,y^5+3\,y^2}\,dy = \pi$$

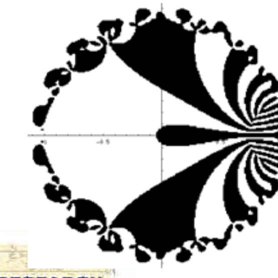
Proof. Purely **qualitative analysis** with partial fractions and arctans shows integral is $\pi\,\beta$ where β is algebraic of degree *much* less than **100 (actually 6)**, length *much* less than **100,000,000**.

✓ With **P(x)=x-1** (D=1,L=2, d=6, L=?), this means *checking* the identity to **100** places is plenty **PROOF:**

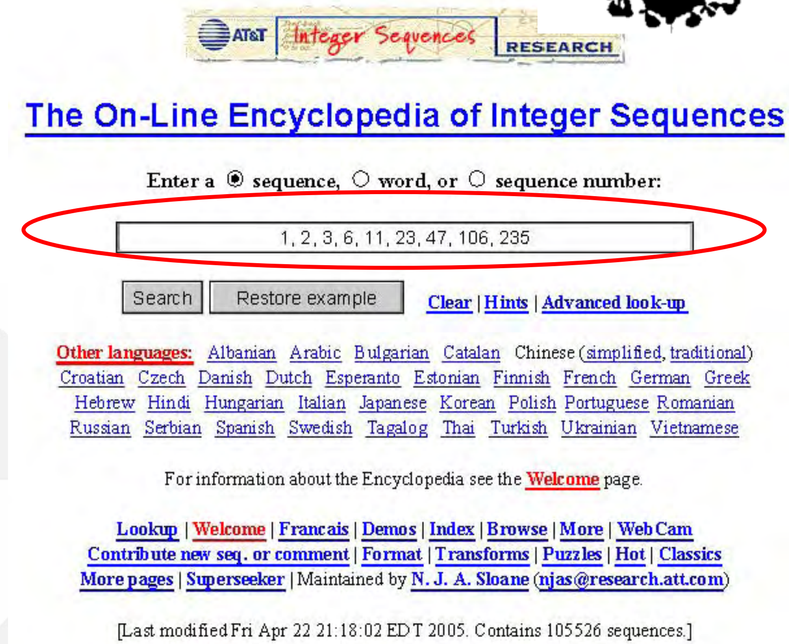
$$|\beta - 1| < 1/(32L) \mapsto \beta = 1$$

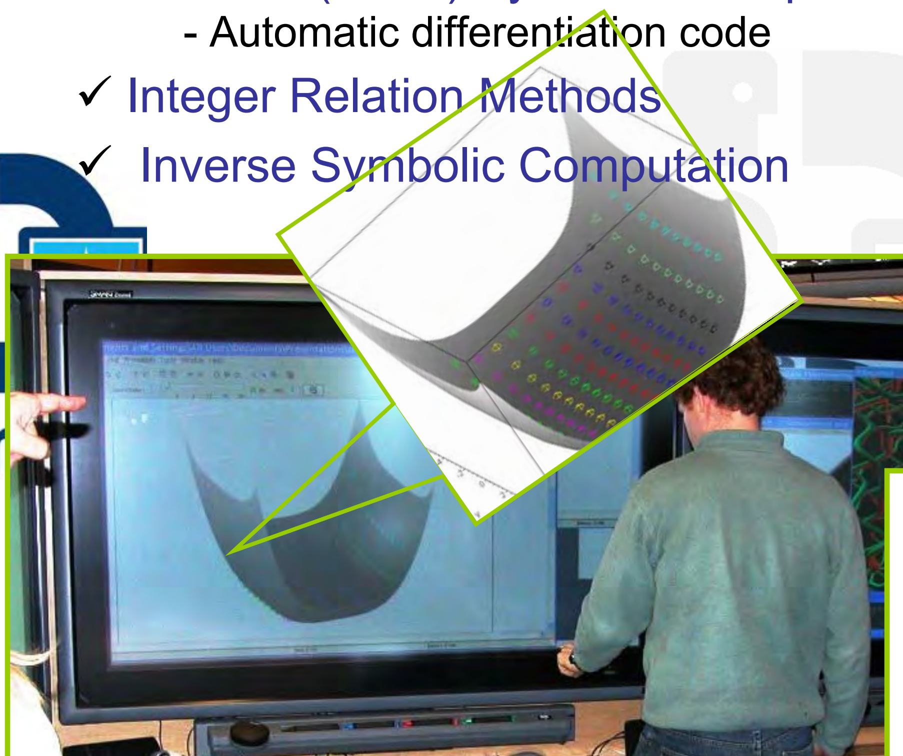
✓ A fully symbolic Maple proof followed. **QED**

Fast High Precision Numeric Computation (and Quadrature)



- Central to my work - with Dave Bailey - meshed with visualization, randomized checks, many web interfaces and
  - ✓ Massive (serial) Symbolic Computation
    - Automatic differentiation code
  - ✓ Integer Relation Methods
  - ✓ Inverse Symbolic Computation





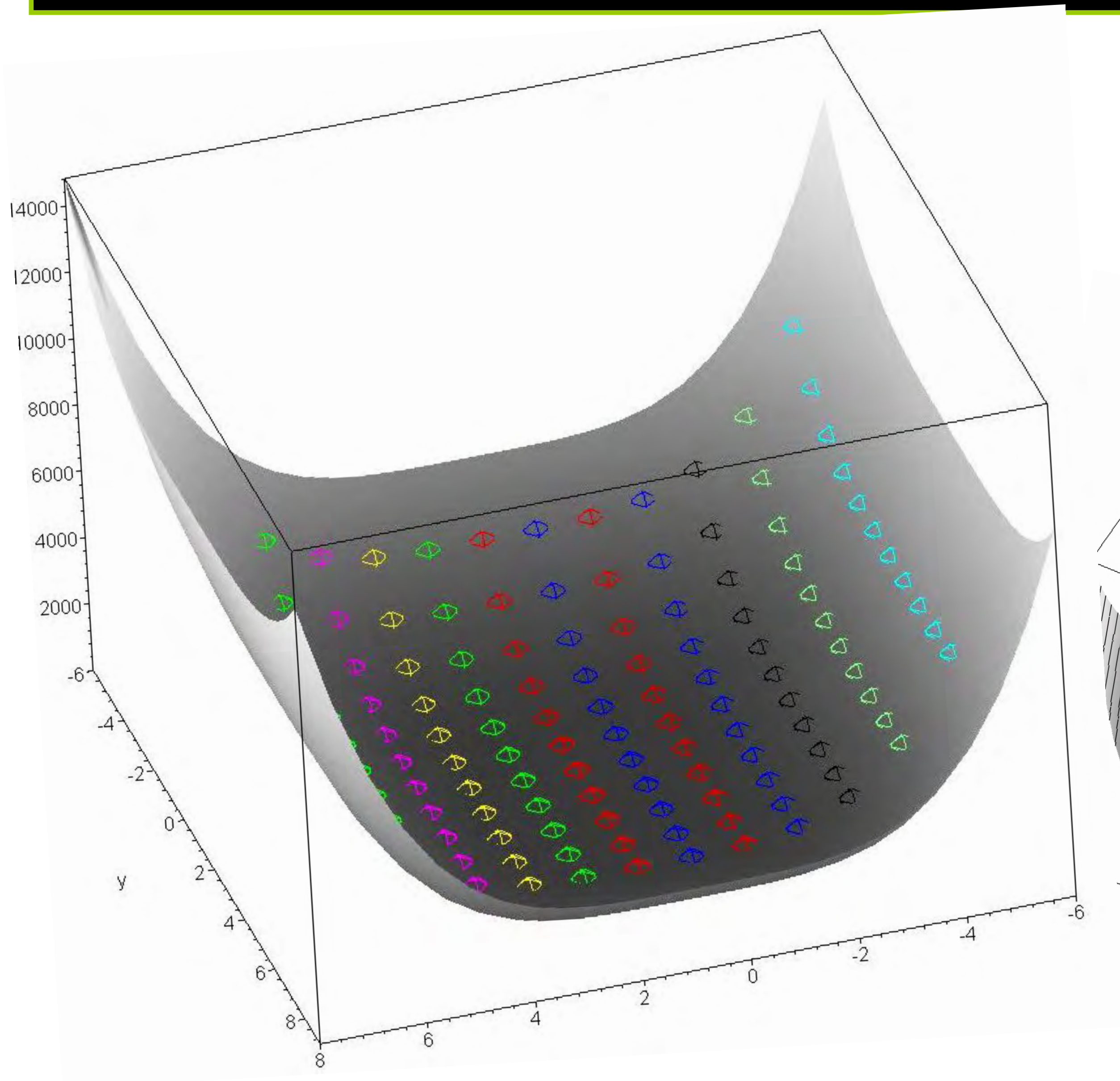
Parallel derivative free optimization in ***Maple***

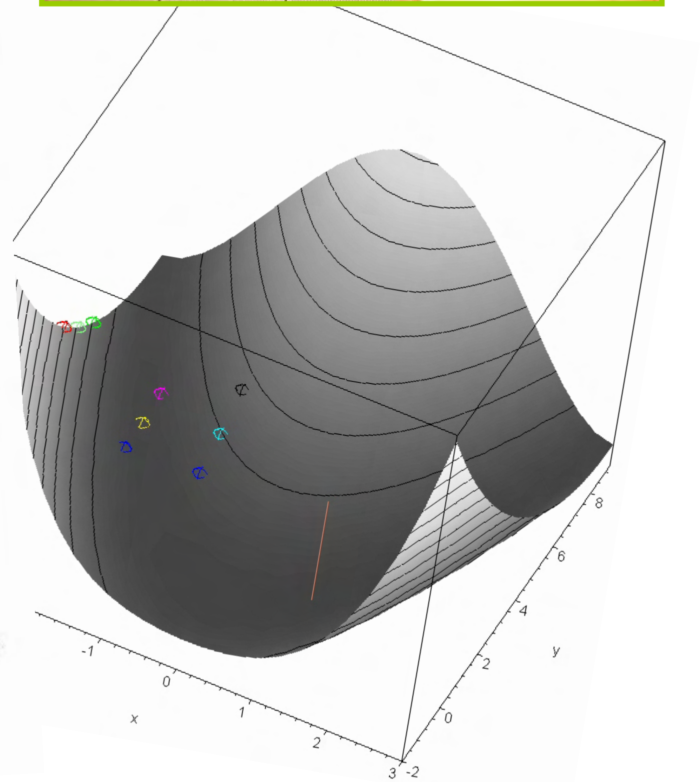
Other useful tools : Parallel Maple

- Sloane's online sequence database
- Salvy and Zimmerman's generating function package '*gfun*'
- Automatic identity proving: Wilf-Zeilberger method for hypergeometric functions

Maple on SFU 192 cpu `bugaboo' cluster

- different node sets are in different colors





Greetings from the On-Line Encyclopedia of Integer Sequences!

AT&T Integer Sequences RESEARCH

Matches (up to a limit of 30) found for 1 2 3 6 11 23 47 106 235 :

[It may take a few minutes to search the whole database, depending on how many matches are found (the second and later looku[ps] are faster)]

An Exemplary Database

ID Number: A000055 (Formerly M0791 and N0299)

URL: http://www.research.att.com/projects/OEIS?Anum=A000055

Sequence: 1,1,1,1,2,3,6,11,23,47,106,235,551,1301,3159,7741,19320, 48629,123867,317955,823065,2144505,5623756,14828074, 39299897,104636890,279793450,751065460,2023443032, 5469566585,14830871802,40330829030,109972410221

Name: Number of trees with n unlabeled nodes.

Comments: Also, number of unlabeled 2-gonal 2-trees with n 2-gons.

References F. Bergeron, G. Labelle and P. Leroux, Combinatorial Species and Tree-Like Structures, Camb. 1998, p. 279.

N. L. Biggs et al., Graph Theory 1736-1936, Oxford, 1976, p. 49.

S. R. Finch, Mathematical Constants, Cambridge, 2003, pp. 295-316.

D. D. Grant, The stability index of graphs, pp. 29-52 of Combinatorial Mathematics (Proceedings 2nd Australian Conf.), Lect. Notes Math. 403, 1974.

F. Harary, Graph Theory. Addison-Wesley, Reading, MA, 1969, p. 232.

F. Harary and E. M. Palmer, Graphical Enumeration, Academic Press, NY, 1973, p. 58 and 244.

D. E. Knuth, Fundamental Algorithms, 3d Ed. 1997, pp. 386-88.

R. C. Read and R. J. Wilson, An Atlas of Graphs, Oxford, 1998.

J. Riordan, An Introduction to Combinatorial Analysis, Wiley, 1958, p. 138.

Integrated real time use

- moderated
- 100,000 entries
- grows daily
- AP book had 5,000

Links: P. J. Cameron, Sequences realized by oligomorphic permutation groups J. Integ. Seqs. Vol

Steven Finch, Otter's Tree Enumeration Constants

E. M. Rains and N. J. A. Sloane, On Cayley's Enumeration of Alkanes (or 4-Valent Trees).

N. J. A. Sloane, Illustration of initial terms

E. W. Weisstein, Link to a section of The World of Mathematics.

Index entries for sequences related to trees

Index entries for "core" sequences

G. Labelle, C. Lamathe and P. Leroux, Labeled and unlabeled enumeration of k-gonal 2-tree

Formula: G.f.: A(x) = 1 + T(x)-T^2(x)/2+T(x^2)/2, where T(x) = x + x^2 + 2*x^3 + ...

Fast Arithmetic (Complexity Reduction in Action)

Multiplication

✓ Karatsuba multiplication (200 digits +) or Fast Fourier Transform (FFT)

✓ in ranges from 100 to 1,000,000,000,000 digits

- The other operations
  ✓ via Newton's method $\times, \div, \sqrt{\cdot}$
- Elementary and special functions
  ✓ via Elliptic integrals and Gauss AGM

$$O\left(n^{\log_2(3)}\right)$$

For example:

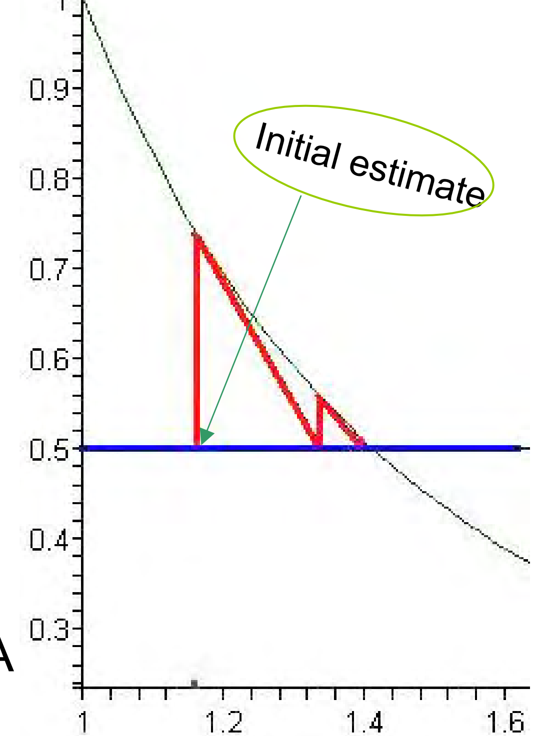
$$\begin{aligned}
&\left(a + c \cdot 10^N\right) \times \left(b + d \cdot 10^N\right) \\
&= ab + (ad + bc) \cdot 10^N + cd \cdot 10^{2N} \\
&= ab + \underbrace{\{(a+c)(b+d) - ab - cd\}}_{\text{three multiplications}} \cdot 10^N + cd \cdot 10^{2N}
\end{aligned}$$

Karatsuba replaces one 'times' by many 'plus'

FFT multiplication of multi-billion digit numbers reduces centuries to minutes. Trillions must be done with Karatsuba!

Newton's Method for Elementary Operations and Functions

$$x \leftarrow x - \frac{f(x)}{\frac{d}{dx} f(x)}$$

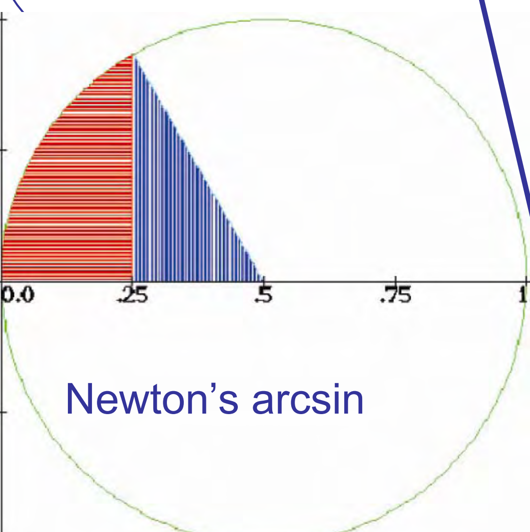


1. Doubles precision at each step
   - ✓ Newton is **self correcting** and **quadratically convergent**
2. Consequences for work needed:
   - ✓ division = **4 x mult**: 1/x = A
   - ✓ sqrt = **6 x mult**: $1/x^2 = A$

$$x \leftarrow x(2 - xA)$$

$$x \leftarrow 1/2\, x\left(3 - x^2 A\right)$$

Now multiply by A



Newton's arcsin

3. For the **logarithm** we approximate by **elliptic integrals (AGM)** which admit **quadratic transformations**: near zero

$$\frac{d}{dk} K(k) \sim \log\left(\frac{4}{k}\right)$$

4. We use **Newton** to obtain the **complex exponential**
   - ✓ hence ***all*** **elementary functions** are fast computable

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   - ✓ Chaos, Zeta & Riemann Hypothesis, HexPi &Normality
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   - ✓ A problem of Knuth, $\pi/8$, Extreme Quadrature
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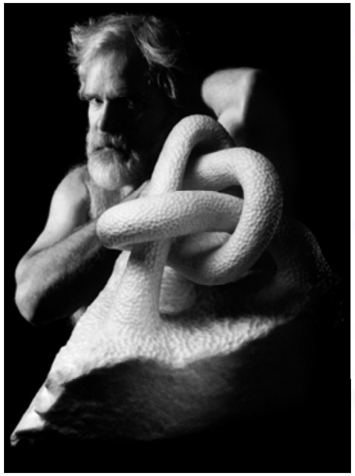
   - ✓ Engines of Discovery. The 21st Century Revolution
     - ✓ Long Range Plan for HPC in Canada

Integer Relation Methods

The PSLQ Integer Relation Algorithm

Let (x_n) be a vector of real numbers. An integer relation algorithm finds integers (a_n) such that

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0$$



- At the present time, the PSLQ algorithm of mathematician-sculptor Helaman Ferguson is the best-known integer relation algorithm.
- PSLQ was named one of ten "algorithms of the century" by *Computing in Science and Engineering*.
- High precision arithmetic software is required: at least d x n digits, where d is the size (in digits) of the largest of the integers a_k.

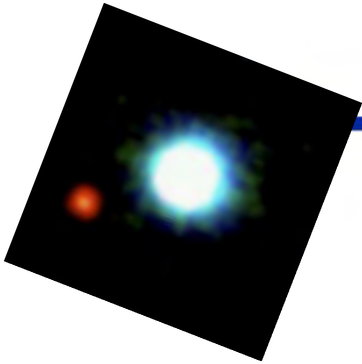
An Immediate Use

To see if α is algebraic of degree N, consider $(1,\alpha,\alpha^2,\ldots,\alpha^N)$

Peter Borwein in front of Helaman Ferguson's work

CMS Meeting
December 2003
SFU Harbour Centre

Ferguson uses high tech tools and micro engineering at NIST to build monumental math sculptures



Application of PSLQ: Bifurcation Points in Chaos Theory

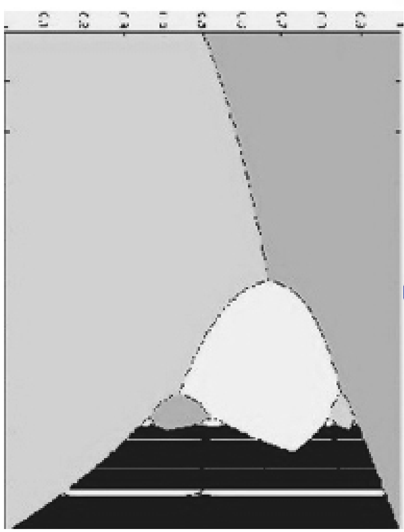
B_3 = 3.54409035955… is third bifurcation point of the logistic iteration of chaos theory:

$$x_{n+1} = r x_n (1 - x_n)$$

i.e., B_3 is the smallest r such that the iteration exhibits 8-way periodicity instead of 4-way periodicity.

In 1990, a predecessor to PSLQ found that B_3 is a root of the polynomial

$$0 = 4913 + 2108t^2 - 604t^3 - 977t^4 + 8t^5 + 44t^6 + 392t^7 - 193t^8 - 40t^9 + 48t^{10} - 12t^{11} + t^{12}$$



Recently B_4 was identified as the root of a 256-degree polynomial by a much more challenging computation.

These results have subsequently been proven formally.

- The proofs use **Groebner basis** techniques
- Another useful part of the HPM toolkit

PSLQ and Zeta

Riemann (1826-66)

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$$\zeta(2) = \frac{\pi^2}{6}, \zeta(4) = \frac{\pi^4}{90}, \zeta(6) = \frac{\pi^6}{945}, \cdots$$

Euler (1707-73)

2005. Bailey, Bradley & JMB *discovered and proved* - in Maple - three *equivalent* binomial identities

$$\mathcal{Z}(x) = 3 \sum_{k=1}^{\infty} \frac{1}{\binom{2k}{k}(k^2 - x^2)} \prod_{n=1}^{k-1} \frac{4x^2 - n^2}{x^2 - n^2}$$

$$= \sum_{k=0}^{\infty} \zeta(2k+2) x^{2k} = \sum_{n=1}^{\infty} \frac{1}{n^2 - x^2}$$

$$= \frac{1 - \pi x \cot(\pi x)}{2x^2}$$

1

1. via **PSLQ** to 50,000 digits (250 terms)

2

2. reduced as hoped

$$3n^2 \sum_{k=n+1}^{2n} \frac{\prod_{m=n+1}^{k-1} \frac{4n^2 - m^2}{n^2 - m^2}}{\binom{2k}{k}(k^2 - n^2)} = \frac{1}{\binom{2n}{n}} - \frac{1}{\binom{3n}{n}}$$

3

$${}_3F_2\left(\begin{matrix} 3n, n+1, -n \\ 2n+1, n+1/2 \end{matrix}; \frac{1}{4}\right) = \frac{\binom{2n}{n}}{\binom{3n}{n}}$$

3. was easily **computer proven** (Wilf-Zeilberger)

Wilf-Zeilberger Algorithm

is a form of <u>automated telescoping</u>: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left\{ \frac{1}{n} - \frac{1}{n+1} \right\} = 1$

✓ AMS Steele Research Prize winner. In Maple 9.5 set:

$$F := \frac{(3\,n+k-1)!\,(n+k)!\,(-n+k-1)!\,(2\,n)!\,(n-1/2)!\,(1/4)^k}{(3\,n-1)!\,n!\,(-n-1)!\,(2\,n+k)!\,(n-1/2+k)!\,k!}, \quad r := \frac{\binom{2\,n}{n}}{\binom{3\,n}{n}}$$

and execute:

```
> with(SumTools[Hypergeometric]):
> WZMethod(F,r,n,k,'certify'): certify;
```

which returns the certificate

```
           /     2                          \
           \11 n  + 1 + 6 n + k + 5 k n/ k
         - --------------------------------
            3 (n - k + 1) (2 n + k + 1) n
```

This proves that summing $F(n,k)$ over k produces $r(n)$, as asserted.

If this were a philosophy talk I should discuss the following two quotes and defend our philosophy of mathematics:

Abstract of the future *We show in a certain precise sense that the **Goldbach Conjecture** is true with probability larger than 0.99999 and that its complete truth could be determined with a budget of 10 billion.*

"Secure Mathematical Knowledge"

"It is a waste of money to get absolute certainty, unless the conjectured identity in question is known to imply the Riemann Hypothesis."

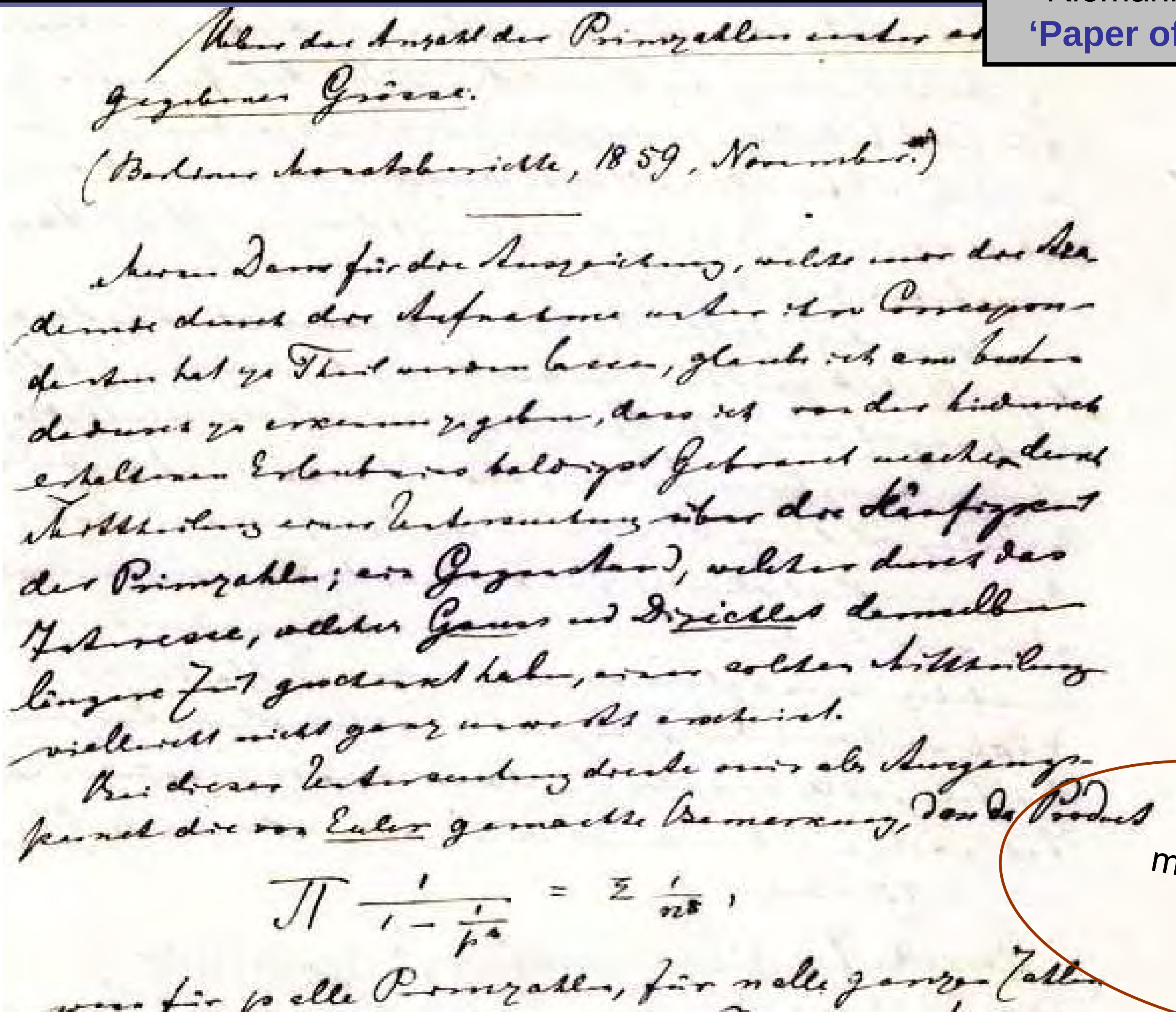
Doron Zeilberger, 1993

- ✓ **Goldbach**: every even number (>2) is a sum of two primes?
- ✓ So we will look at the **Riemann Hypothesis** …

Uber die Anzahl der Primzahlen unter einer Gegebenen Grosse

On the number of primes less than a given quantity

Riemann's six page 1859
'Paper of the Millennium'?


Über die Anzahl der Primzahlen unter einer gegebenen Grösse.
(Berliner Monatsberichte, 1859, November.)

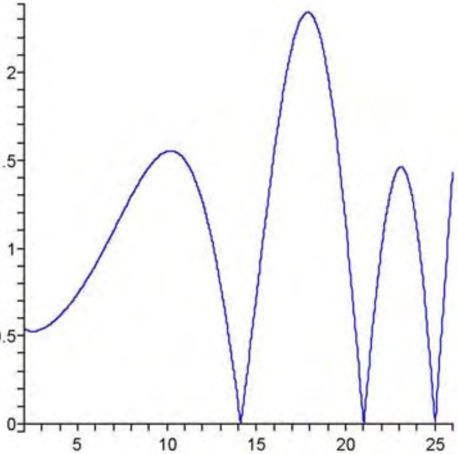
Meinen Dank für die Auszeichnung, welche mir die Akademie durch die Aufnahme unter ihre Correspondenten hat zu Theil werden lassen, glaube ich am besten dadurch zu erkennen zu geben, dass ich von der hierdurch erhaltenen Erlaubniss baldigst Gebrauch mache durch Mittheilung einer Untersuchung über die Häufigkeit der Primzahlen; ein Gegenstand, welcher durch das Interesse, welches Gauss und Dirichlet demselben längere Zeit geschenkt haben, einer solchen Mittheilung vielleicht nicht ganz unwerth erscheint.
Bei dieser Untersuchung diente mir als Ausgangspunkt die von Euler gemachte Bemerkung, dass das Product

$$\prod \frac{1}{1-\frac{1}{p^s}} = \sum \frac{1}{n^s},$$

wenn für p alle Primzahlen, für n alle ganzen Zahlen

RH is so important because it yields precise results on distribution and behaviour of primes

Euler's product makes the key link between **primes and ζ**



The Modulus of Zeta and the **Riemann Hypothesis** (A Millennium Problem)

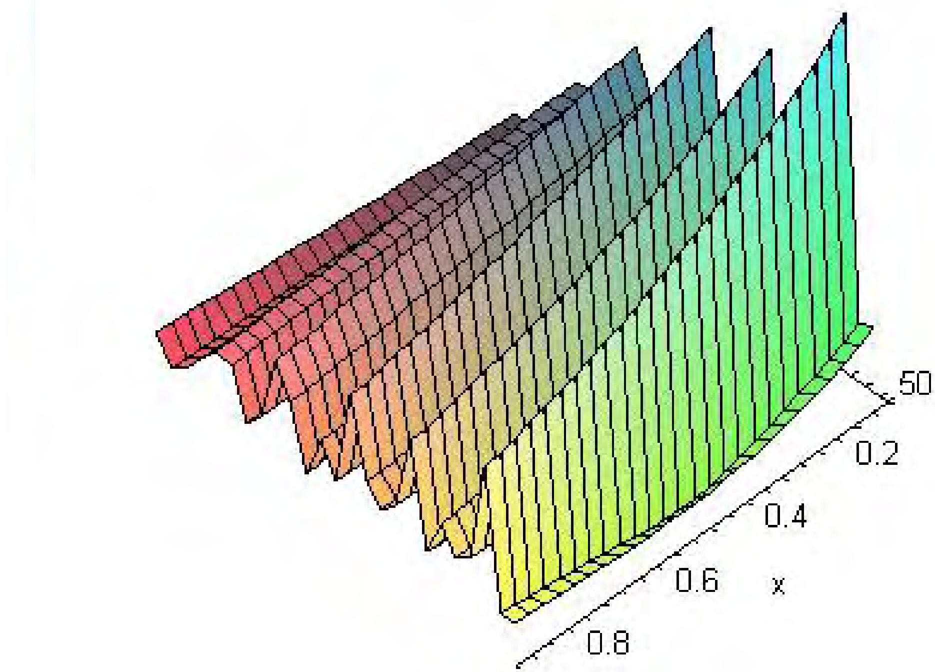
The imaginary parts of first 4 zeroes are:

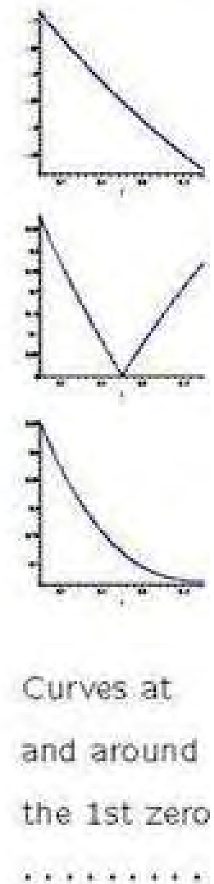
14.134725142
21.022039639

25.010857580
30.424876126

The first 1.5 billion are on the *critical line*

Yet at 10^{22} the ***"Law of small numbers"*** still rules (Odlyzko)

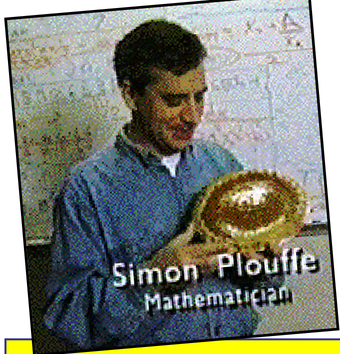




'All non-real zeros have real part one-half'

(The Riemann Hypothesis)

Note the **monotonicity** of **$x \to |\zeta(x+iy)|$** is **equivalent to RH** (discovered in a Calgary class in 2002 by Zvengrowski and Saidak)



PSLQ and Hex Digits of Pi

$$\log 2 = \sum_{n=1}^{\infty} \frac{1}{k\,2^k}$$

My brother made the observation that this log formula allows one to compute binary digits of log 2 *without* knowing the previous ones! (a **BBP formula**)

Bailey, Plouffe and he hunted for such a formula for Pi. Three months later **the computer** - doing **bootstrapped PSLQ** hunts - **returned**:

$$\pi = 4F(1/4, 5/4; 1; -1/4) + 2\arctan(1/2) - \log 5$$

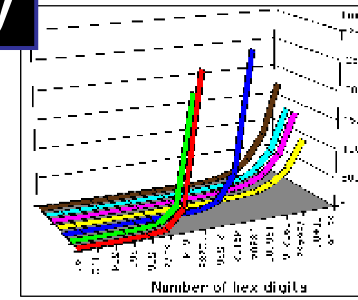
- this reduced to

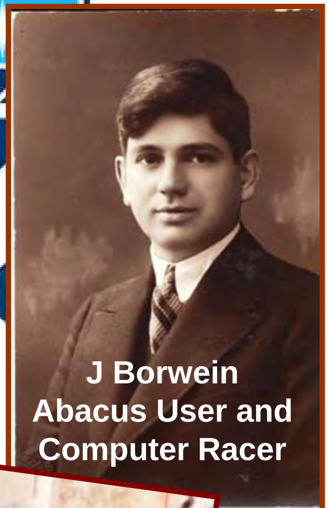
$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i}\left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6}\right)$$

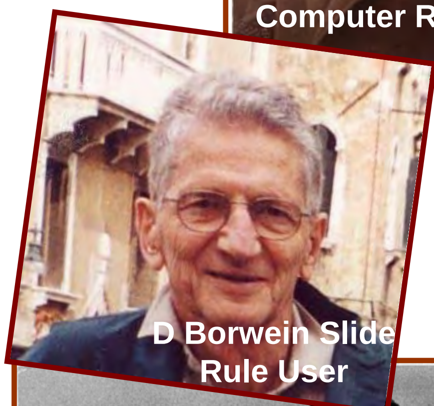
which *Maple*, *Mathematica* and humans can easily prove.

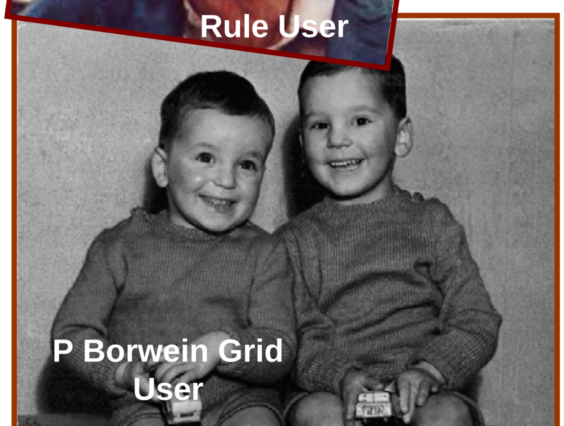
- ✓ A triumph for "**reverse engineered mathematics**" - algorithm design
- ✓ No such formula exists base-ten (provably)

The pre-designed Algorithm ran the next day




J Borwein Abacus User and Computer Racer


D Borwein Slide Rule User


P Borwein Grid User

ALGORITHMIC PROPERTIES

(1) produces a modest-length string hex or binary digits of π, beginning at an arbitrary position, using no prior bits;

(2) is implementable on any modern computer;

(3) requires no multiple precision software;

(4) requires very little memory; and


T. Borwein Game Player

(5) has a computational cost growing only slightly faster than the digit position.

PiHex– A distributed effort to calculate Pi

http://www.cecm.sfu.ca/projects/pihex/

Getting Started   Latest Headlines   www.icbc.ca

Join PiHex
- Download
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Who am I?

Email me!

174962

hits since the counter last reset.

PiHex

A distributed effort to calculate Pi.

Percival 2004

The Quadrillionth Bit of Pi is '0'!
The Forty Trillionth Bit of Pi is '0'!
The Five Trillionth Bit of Pi is '0'!

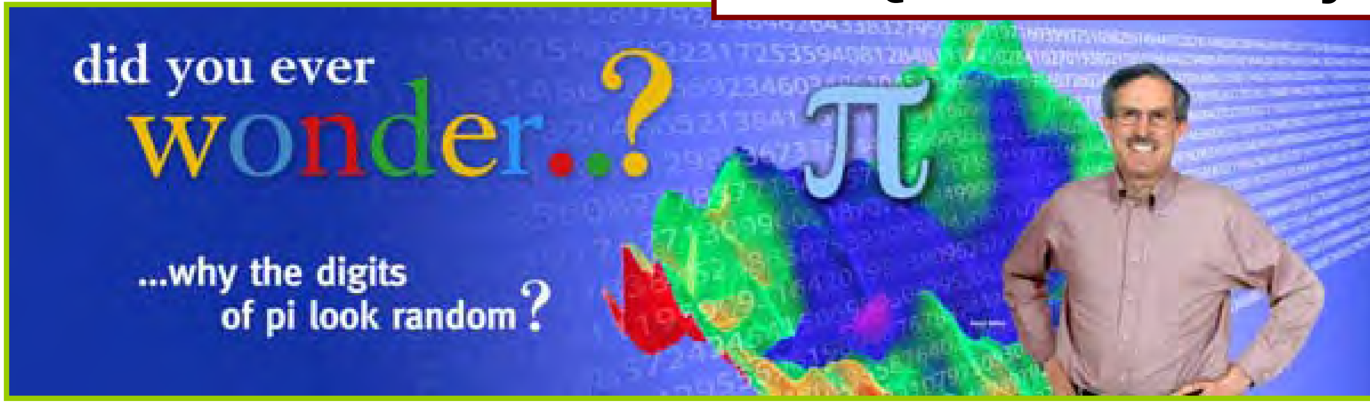
PiHex was a distributed computing project which used idle computing power to set three records for calculating specific bits of Pi. PiHex has now finished.

Position	Hex Digits Beginning At This Position
10^6	26C65E52CB4593
10^7	17AF5863EFED8D
10^8	ECB840E21926EC
10^9	85895585A0428B
10^{10}	921C73C6838FB2
10^{11}	9C381872D27596
1.25×10^{12}	07E45733CC790B
2.5×10^{14}	E6216B069CB6C1

Undergraduate **Colin Percival's** grid computation **PiHex** rivaled **Finding Nemo**

1999 on 1736 PCS in 56 countries using 1.2 million Pentium2 cpu-hours

PSLQ and Normality of Digits



Bailey and Crandall observed that BBP numbers most probably are normal and make it precise with a hypothesis on the behaviour of a dynamical system.

• For example Pi is normal in Hexadecimal if the iteration below, starting at zero, is uniformly distributed in [0,1]

$$x_n = \left\{16x_{n-1} + \frac{120n^2 - 89n + 16}{512n^4 - 1024n^3 + 712n^2 - 206n + 21}\right\}$$

Consider the hex digit stream:

$$d_n = \lfloor 16x_n \rfloor$$

✓ We have checked that this gives first million hex-digits of Pi.

✓ Is this always the case? The weak Law of Large Numbers implies this is **very probably true**!

Pi to 1.5 trillion places in 20 steps

This fourth order algorithm was used on all big-π computations from 1986 to 2001

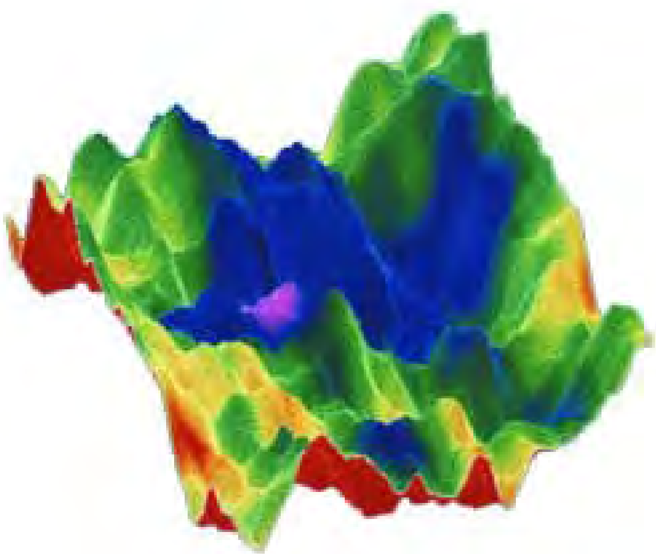
$$y_1 = \frac{1-\sqrt[4]{1-y_0{}^4}}{1+\sqrt[4]{1-y_0{}^4}}, a_1 = a_0\left(1+y_1\right)^4 - 2^3 y_1\left(1+y_1+y_1{}^2\right)$$
$$y_2 = \frac{1-\sqrt[4]{1-y_1{}^4}}{1+\sqrt[4]{1-y_1{}^4}}, a_2 = a_1\left(1+y_2\right)^4 - 2^5 y_2\left(1+y_2+y_2{}^2\right)$$
$$y_3 = \frac{1-\sqrt[4]{1-y_2{}^4}}{1+\sqrt[4]{1-y_2{}^4}}, a_3 = a_2\left(1+y_3\right)^4 - 2^7 y_3\left(1+y_3+y_3{}^2\right)$$
$$y_4 = \frac{1-\sqrt[4]{1-y_3{}^4}}{1+\sqrt[4]{1-y_3{}^4}}, a_4 = a_3\left(1+y_4\right)^4 - 2^9 y_4\left(1+y_4+y_4{}^2\right)$$
$$y_5 = \frac{1-\sqrt[4]{1-y_4{}^4}}{1+\sqrt[4]{1 \ldots}}$$
$$y_7 = \frac{1-\sqrt[4]{1-y_6{}^4}}{1+\sqrt[4]{1-y_6{}^4}}, a_7 = a_6\left(1+y_7\right)^4 - 2^{15} y_7\left(1+y_7+y_7{}^2\right)$$
$$y_8 = \frac{1-\sqrt[4]{1-y_7{}^4}}{1+\sqrt[4]{1-y_7{}^4}}, a_8 = a_7\left(1+y_8\right)^4 - 2^{17} y_8\left(1+y_8+y_8{}^2\right)$$
$$y_9 = \frac{1-\sqrt[4]{1-y_8{}^4}}{1+\sqrt[4]{1-y_8{}^4}}, a_9 = a_8\left(1+y_9\right)^4 - 2^{19} y_9\left(1+y_9+y_9{}^2\right)$$
$$y_{10} = \frac{1-\sqrt[4]{1-y_9{}^4}}{1+\sqrt[4]{1-y_9{}^4}}, a_{10} = a_9\left(1+y_{10}\right)^4 - 2^{21} y_{10}\left(1+y_{10}+y_{10}{}^2\right)$$
$$y_{11} = \frac{1-\sqrt[4]{1-y_{10}{}^4}}{1+\sqrt[4]{1-y_{10}{}^4}}, a_{11} = a_{10}\left(1+y_{11}\right)^4 - 2^{23} y_{11}\left(1+y_{11}+y_{11}{}^2\right)$$
$$y_{12} = \frac{1-\sqrt[4]{1-y_{11}{}^4}}{1+\sqrt[4]{1-y_{11}{}^4}}, a_{12} = a_{11}\left(1+y_{12}\right)^4 - 2^{25} y_{12}\left(1+y_{12}+y_{12}{}^2\right)$$
$$y_{13} = \frac{1-\sqrt[4]{1-y_{12}{}^4}}{1+\sqrt[4]{1-y_{12}{}^4}}, a_{13} = a_{12}\left(1+y_{13}\right)^4 - 2^{27} y_{13}\left(1+y_{13}+y_{13}{}^2\right)$$
$$y_{14} = \frac{1-\sqrt[4]{1-y_{13}{}^4}}{\ldots}$$
$$\ldots - 2^{33} y_{16}\left(1+y_{16}+y_{16}{}^2\right)$$
$$y_{17} = \frac{1-\sqrt[4]{1-y_{16}{}^4}}{1+\sqrt[4]{1-y_{16}{}^4}}, a_{17} = a_{16}\left(1+y_{17}\right)^4 - 2^{35} y_{17}\left(1+y_{17}+y_{17}{}^2\right)$$
$$y_{18} = \frac{1-\sqrt[4]{1-y_{17}{}^4}}{1+\sqrt[4]{1-y_{17}{}^4}}, a_{18} = a_{17}\left(1+y_{18}\right)^4 - 2^{37} y_{18}\left(1+y_{18}+y_{18}{}^2\right)$$
$$y_{19} = \frac{1-\sqrt[4]{1-y_{18}{}^4}}{1+\sqrt[4]{1-y_{18}{}^4}}, a_{19} = a_{18}\left(1+y_{19}\right)^4 - 2^{39} y_{19}\left(1+y_{19}+y_{19}{}^2\right)$$
$$y_{20} = \frac{1-\sqrt[4]{1-y_{19}{}^4}}{1+\sqrt[4]{1-y_{19}{}^4}}, \mathbf{a_{20}} = a_{19}\left(1+y_{20}\right)^4 - 2^{41} y_{20}\left(1+y_{20}+y_{20}{}^2\right).$$

These equations specify an algebraic number: $1/\pi \approx a_{20}$

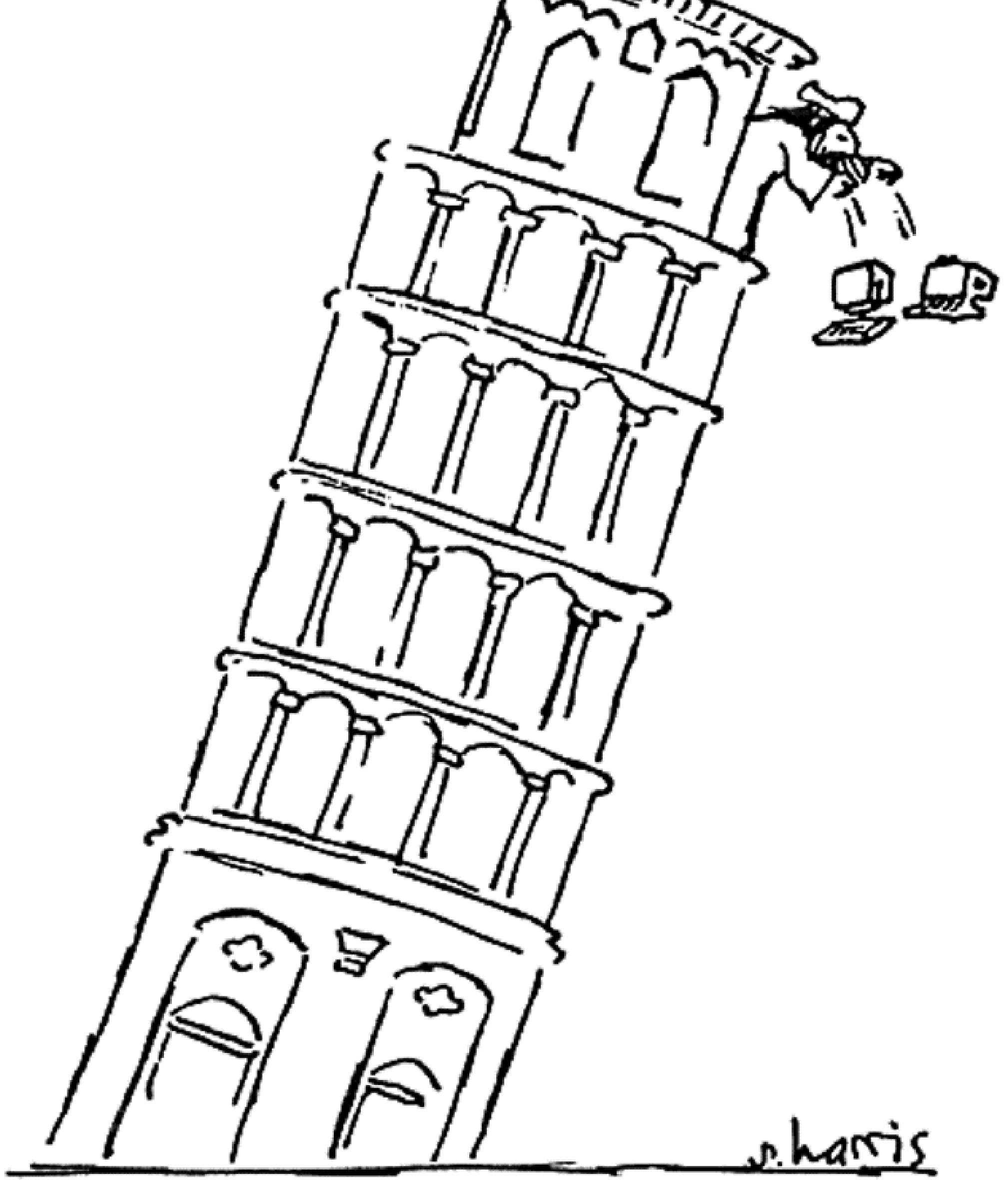
Set $a_0 = 6 - 4\sqrt{2}$ and $y_0 = \sqrt{2} - 1$. Iterate

$$y_{k+1} = \frac{1-(1-y_k^4)^{1/4}}{1+(1-y_k^4)^{1/4}} \quad \text{and}$$
$$a_{k+1} = a_k(1+y_{k+1})^4 - 2^{2k+3} y_{k+1}(1+y_{k+1}+y_{k+1}^2).$$

Then $1/a_k$ **converges quartically** to π



A random walk on a million digits of Pi



IF THERE WERE COMPUTERS IN GALILEO'S TIME

Outline. What is HIGH PERFORMANCE MATHEMATICS?

1. Visual Data Mining in Mathematics.

- ✓ Fractals, Polynomials, Continued Fractions, Pseudospectra

2. High Precision Mathematics.

3. Integer Relation Methods.

- ✓ Chaos, Zeta and the Riemann Hypothesis, HexPi and Normality

4. Inverse Symbolic Computation.

- ✓ A problem of Knuth, $\pi/8$, Extreme Quadrature

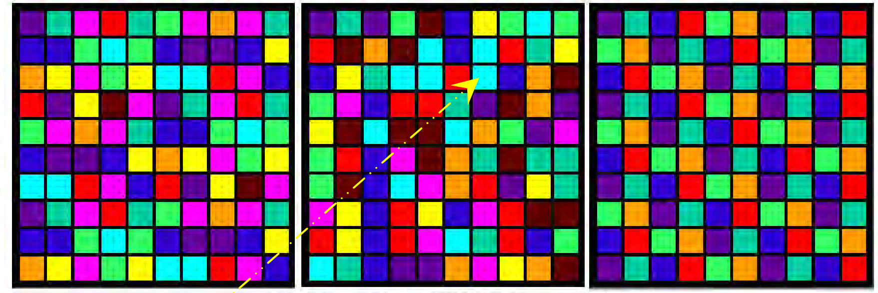
5. The Future is Here.

- ✓ Examples and Issues

6. Conclusion.

- ✓ Engines of Discovery. The 21st Century Revolution
  - ✓ Long Range Plan for HPC in Canada

An Inverse and a Color Calculator

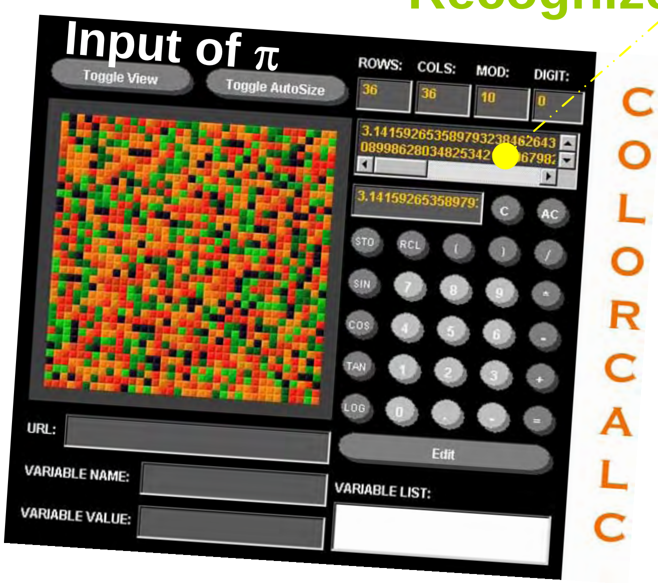


Archimedes: $223/71 < \pi < 22/7$

Inverse Symbolic Computation

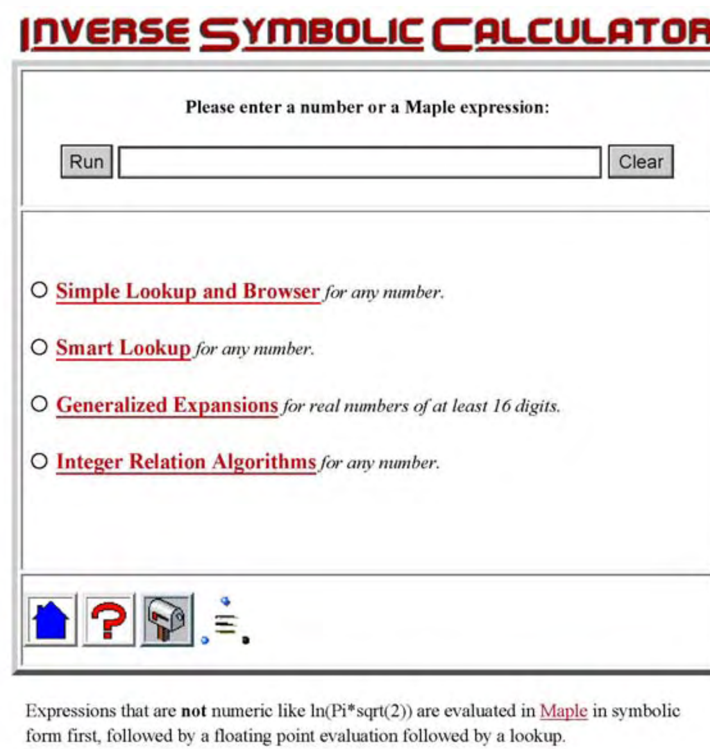
- “**Inferring symbolic structure from numerical data**”
- Mixes *large table lookup*, integer relation methods and intelligent preprocessing – needs ***micro-parallelism***
- It faces the “curse of exponentiality”

➢ Implemented as **identify** in Maple and **Recognize** in Mathematica



identify(sqrt(2.)+sqrt(3.))

$$\sqrt{2}+\sqrt{3}$$

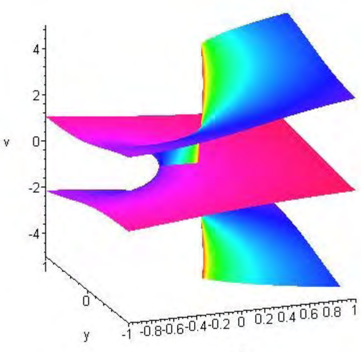


Knuth's Problem – we can know the answer first

A guided proof followed on **asking why** Maple could compute the answer so fast.

The answer is **Lambert's W** which solves

W exp(W) = x



W's **Riemann** surface

Donald Knuth* asked for a closed form evaluation of:

$$\sum_{k=1}^{\infty} \left\{ \frac{k^k}{k!\, e^k} - \frac{1}{\sqrt{2\,\pi\,k}} \right\} = -0.084069508727655\ldots.$$

- **2000 CE.** It is easy to compute 20 or 200 digits of this sum

† ISC is shown on next slide

∡ The 'smart lookup' facility in the *Inverse Symbolic Calculator*† rapidly returns

$$0.084069508727655 \approx \frac{2}{3} + \frac{\zeta\,(1/2)}{\sqrt{2\,\pi}}.$$

We thus have a prediction which *Maple* 9.5 on a laptop confirms to 100 places in under 6 seconds and to 500 in 40 seconds. *** ARGUABLY WE ARE DONE**

ENTERING

evalf(Sum(k^k/k!/exp(k)-1/sqrt(2*Pi*k),k=1..infinity),16)

'Simple Lookup' fails;
'Smart Look up' gives:

INVERSE SYMBOLIC CALCULATOR

The ISC is the **Inverse Symbolic Calculator**, a set of programs and specialized tables of mathematical constants dedicated to the identification of real numbers. It also serves as a way to produce identities with functions and real numbers. It is one of the main ongoing projects at the Centre for Experimental and Constructive Mathematics (CECM).

Results of the search:

Maple output:

.08406950872765600

.8406950872765600e-1

Value to be looked up: .8406950872765600e-1 = K

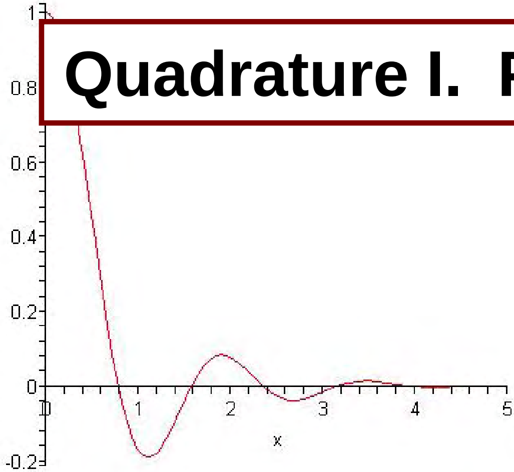
Performing a smart lookup on .8406950872765600e-1:

Function	Result	Precision	Matches
K-2/3	.5825971579390106666666666	16	1

579390106 was probably generated by one
s or found in one of the given tables.
Answers are given from shortest to longest description

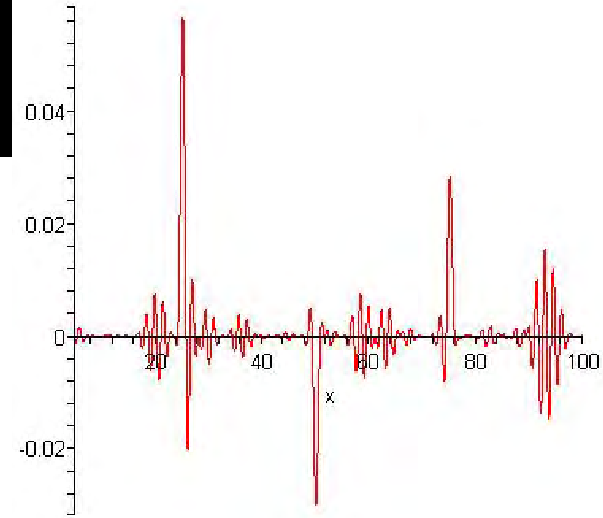
Mixed constants with 5 operations
5825971579390106 = Zeta(1/2)/sr(2)/sr(Pi)

Browse around .5825971579390106.



Quadrature I. Pi/8?

A numerically challenging integral



$$\int_0^\infty \cos(2x) \prod_{n=1}^{\infty} \cos\left(\frac{x}{n}\right)\, dx \stackrel{?}{=} \frac{\pi}{8}.$$

But π/8 is

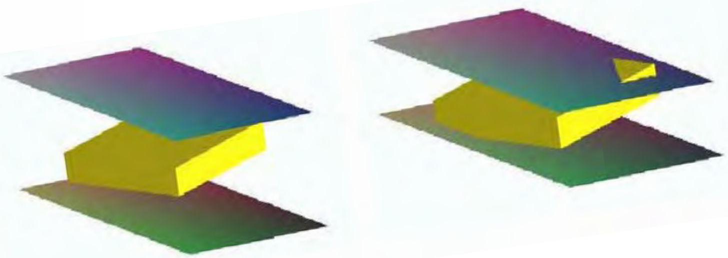
0.392699081698724154807830422909937860524645434

while the integral is

0.392699081698724154807830422909937860524646174

A careful *tanh-sinh quadrature* **proves** this difference after **43 correct digits**

✓ Fourier analysis explains this as happening when a hyperplane meets a hypercube



Before and After

Quadrature II. Hyperbolic Knots

$$\frac{24}{7\sqrt{7}} \int_{\pi/3}^{\pi/2} \log \left| \frac{\tan t + \sqrt{7}}{\tan t - \sqrt{7}} \right| dt \stackrel{?}{=} L_{-7}(2) \qquad (@)$$

where

$$L_{-7}(s) = \sum_{n=0}^{\infty} \left[\frac{1}{(7n+1)^s} + \frac{1}{(7n+2)^s} - \frac{1}{(7n+3)^s} + \frac{1}{(7n+4)^s} - \frac{1}{(7n+5)^s} - \frac{1}{(7n+6)^s} \right].$$

"Identity" (@) has been verified to 20,000 places. I have *no idea* of how to prove it.

✓ Easiest of 998 empirical results linking physics/topology (LHS) to number theory (RHS). [JMB-Broadhurst]

We have certain knowledge without proof

Extreme Quadrature …
20,000 Digits (50 Certified)
on 1024 CPUs

Ш. The integral was split at the nasty interior singularity
Ш. The sum was `easy'.
Ш. All fast arithmetic & function evaluation ideas used


Perko knots 10_{162} and 10_{163} agree: a dynamic proof

Run-times and speedup ratios on the Virginia Tech G5 Cluster

CPUs	Init	Integral #1	Integral #2	Total	Speedup
1	*190013	*1534652	*1026692	*2751357	1.00
16	12266	101647	64720	178633	15.40
64	3022	24771	16586	44379	62.00
256	770	6333	4194	11297	243.55
1024	199	1536	1034	2769	993.63

Parallel run times (in seconds) and speedup ratios for the 20,000-digit problem

Expected and unexpected scientific spinoffs

- **1986-1996**. Cray used quartic-Pi to check machines in factory
- **1986**. Complex FFT sped up by factor of two
- **2002**. Kanada used hex-pi (20hrs not 300hrs to check computation)
- **2005**. Virginia Tech (this integral pushed the limits)
- **1995-** Math Resources (next overhead)


THEORIES
PROOFS
S. Harris

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5. The Future is Here. (What is D-DRIVE?)

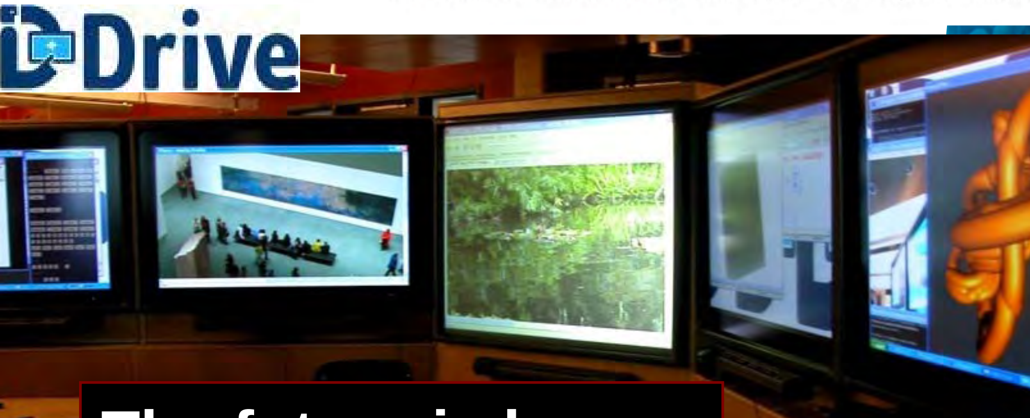
- ✓ **Examples and Issues**

6. Conclusion.

- ✓ Engines of Discovery. The 21st Century Revolution
  - ✓ Long Range Plan for HPC in Canada

WestGrid

www.westgrid.ca

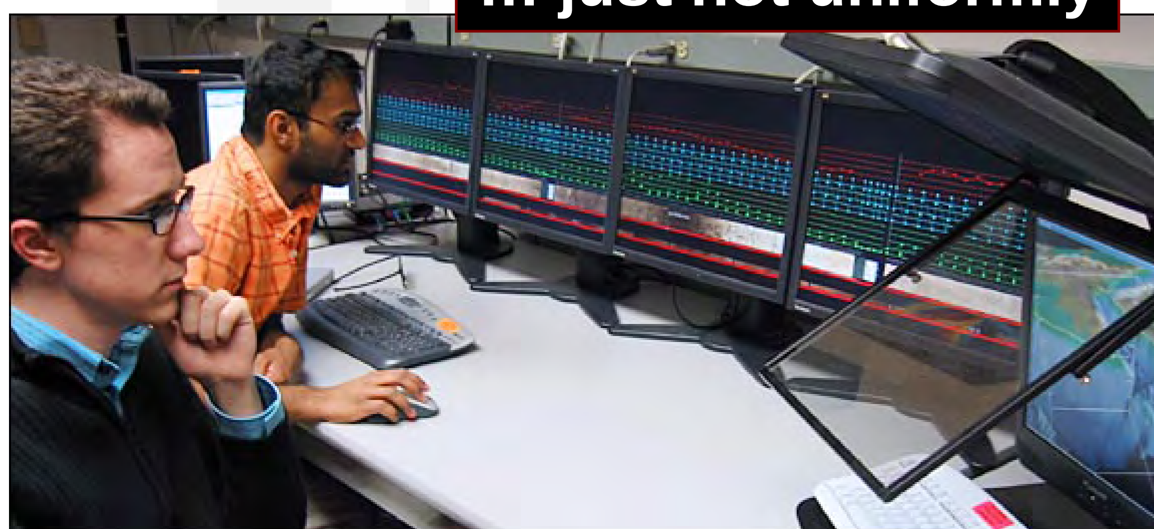


The future is here…

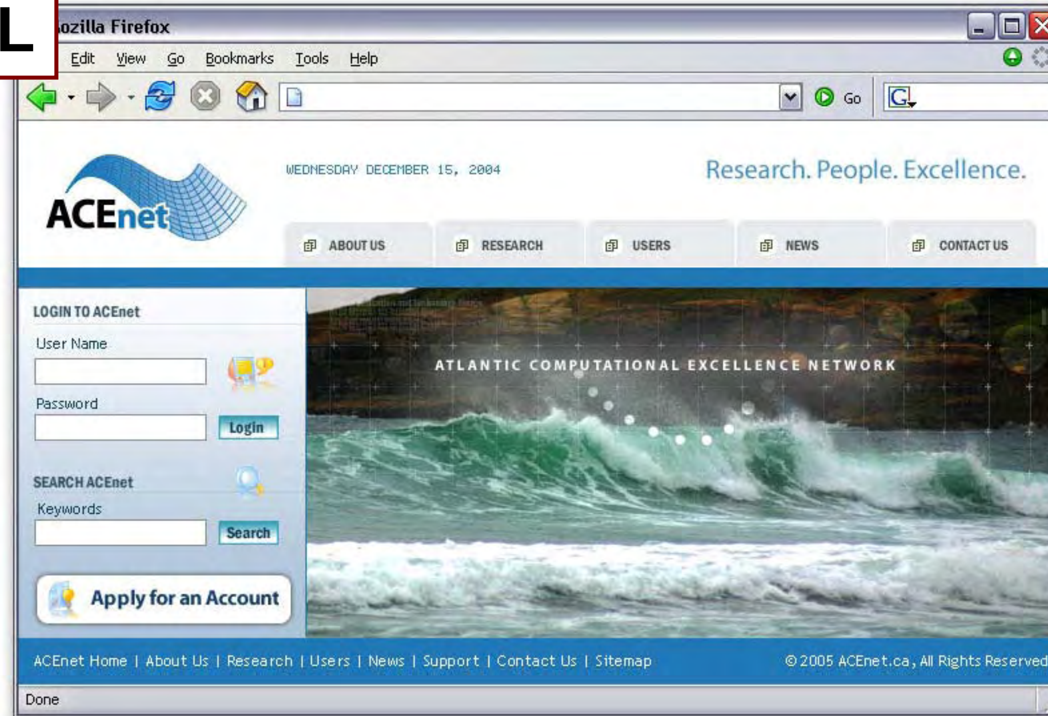
… just not uniformly

Remote Visualization via Access Grid

- The touch sensitive interactive D-DRIVE
- An Immersive 'Cave' Polyhedra
- and the 3D GeoWall



a. ACENet and HPC@DAL

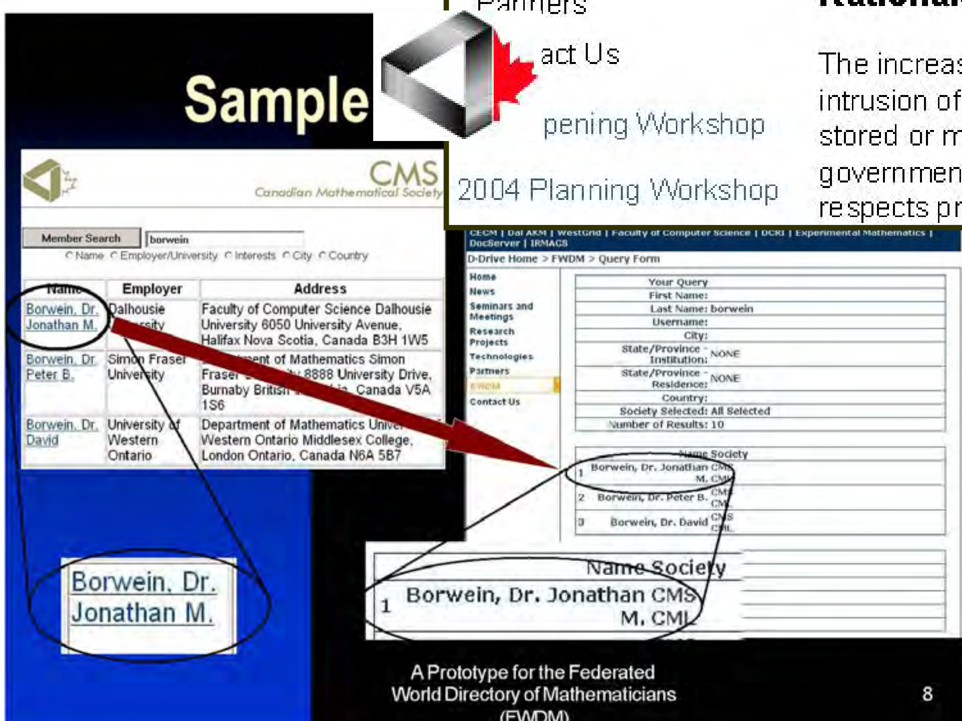


ACEnet completes the
Pan Canadian Consortia

Dalhousie's role will be in collaboration, visualization, and large data-set storage

b. Advanced Knowledge Management

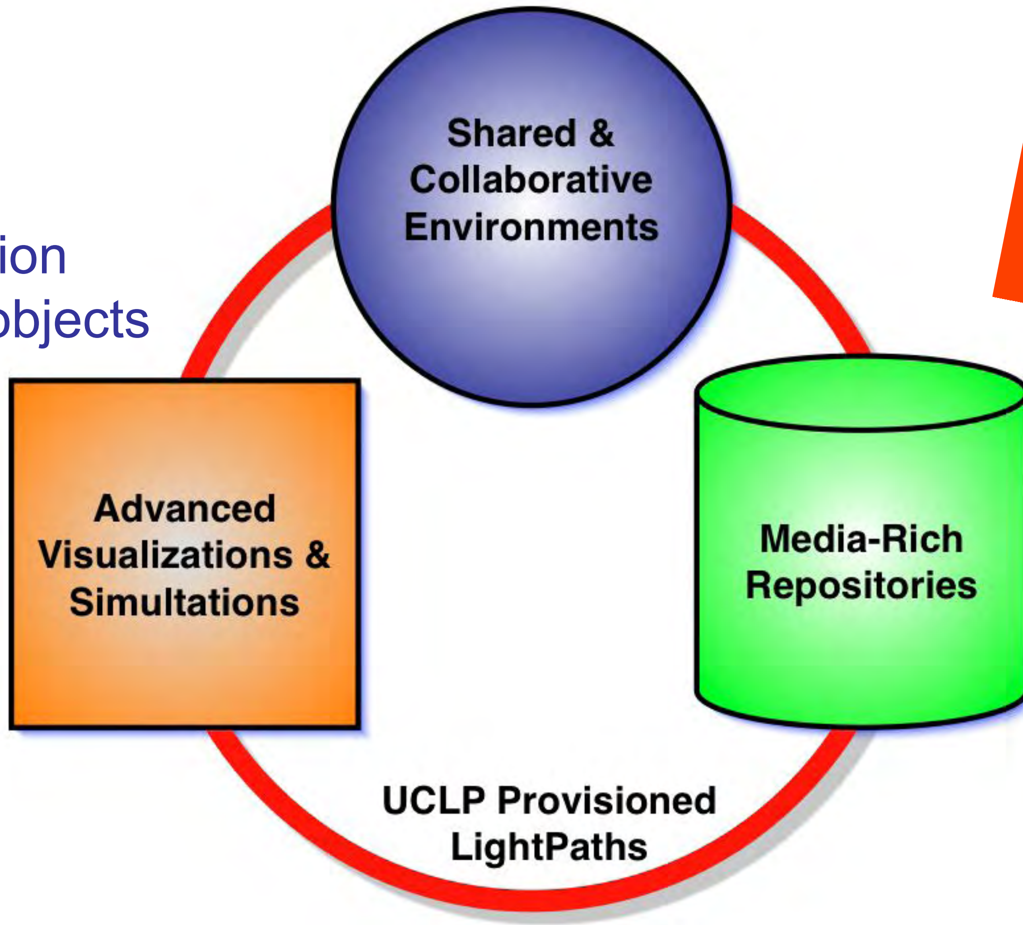
Projects include
- PSL
- FWDM (IMU)
- CiteSeer

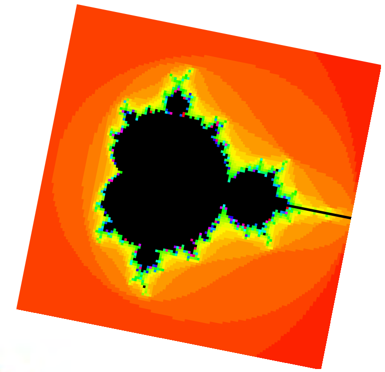


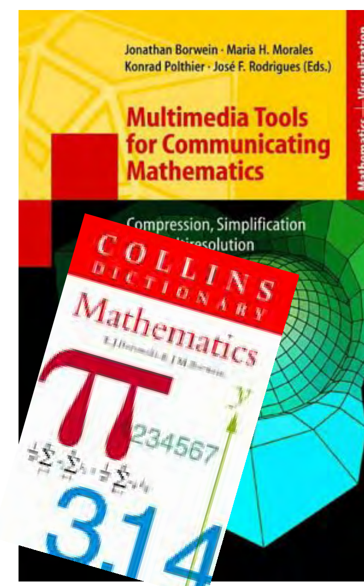
Diverse partners include
- ✓ International Mathematical Union
- ✓ CMS
- ✓ Symantec and IBM

c. Advanced Networking …

These include
- AccessGrid
- UCLP for
  - visualization
  - learning objects
  - haptics







Haptics in the MLP

Haptic Devices extend the world of I/O into the tangible and tactile

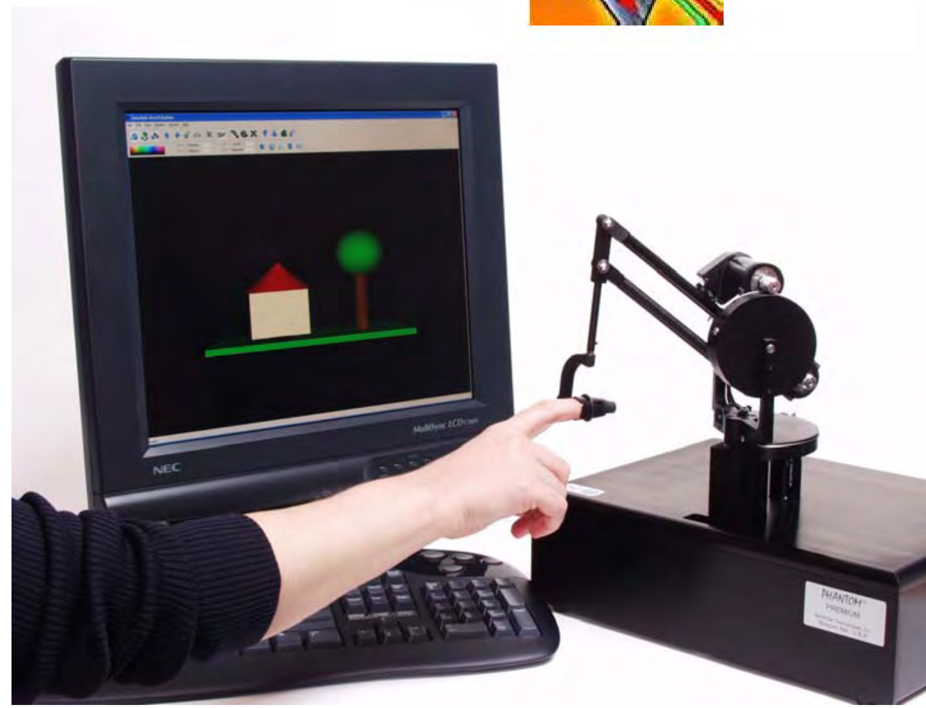
We aim to link multiple devices together such that two or more users may interact at a distance

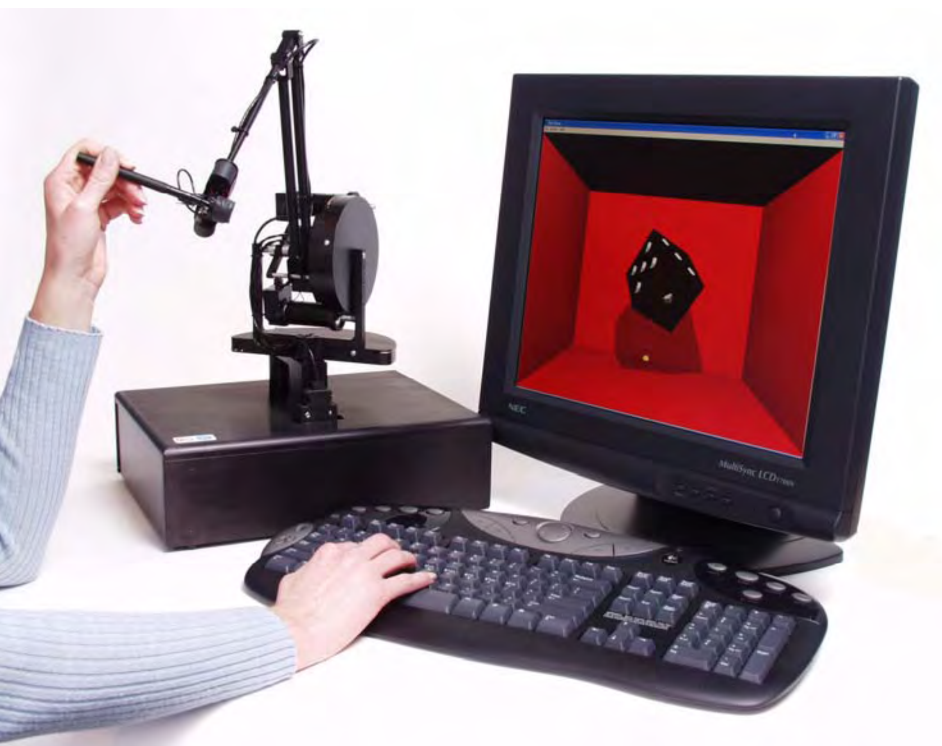
• in Museums and elsewhere

Sensable's Phantom Omni

What these Haptic devices do

- Force feedback informs the user of his virtual environment adding an increased depth to human computer interaction

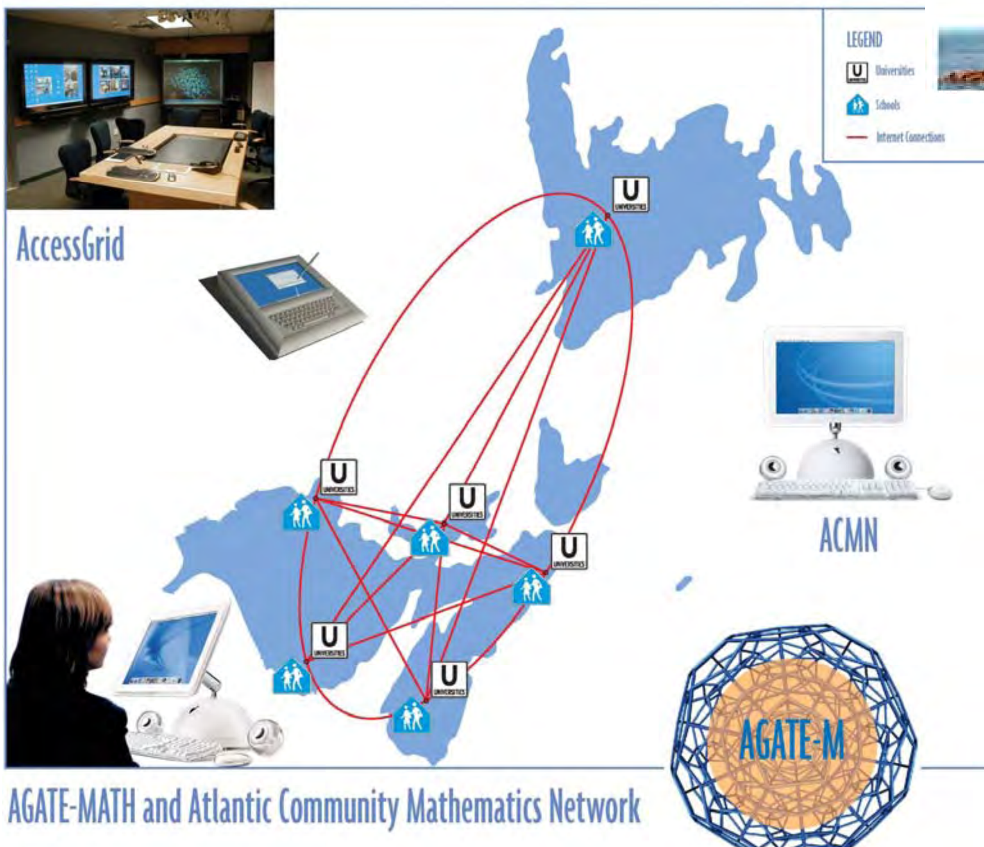




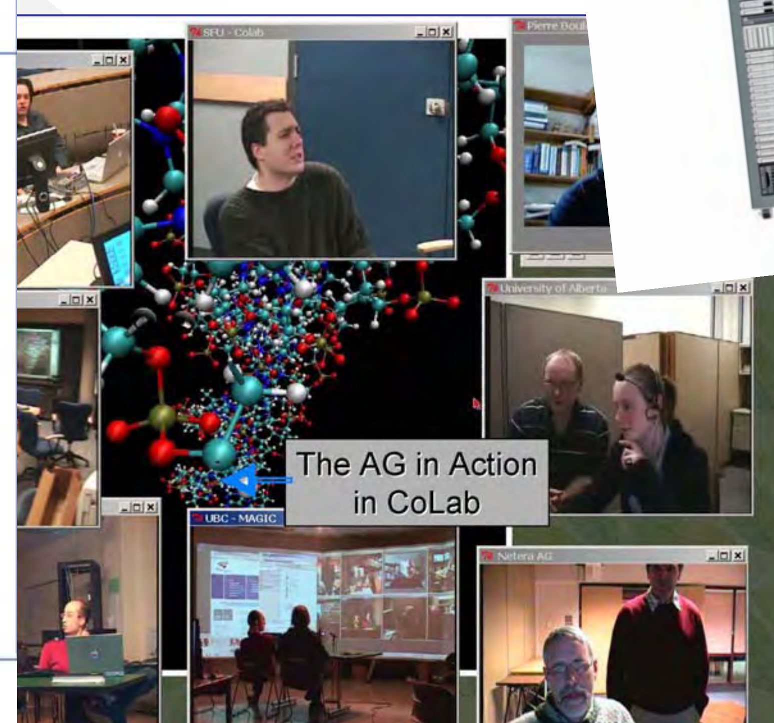
- The user feels the contours of the virtual die via resistance from the arm of the device

d. Access Grid, AGATE and Apple

- First 25 teachers identified



agate Math



magate h

Atlantic Gateway to Mathematics

AGATE-MATH was recently established for the purpose of improving, encouraging, and supporting the teaching of mathematical sciences, in Atlantic Canada and elsewhere.

Vision Statement

The discipline of Mathematics is beautiful and important in its own right. At the same time mathematics and mathematical competency are critical to most other scientific disciplines and are pervasive in modern society. Cell phones, Google, e-banking, internet security, "Finding Nemo," all use enormously sophisticated mathematics, as do countless more obvious examples from medical imaging to mutual funds.

Mathematics is a fundamental component of the language of science. Consequently, mastery of basic mathematics is critical for sustaining interest not only in the pursuit of science but also in understanding the sciences (physical, biological, artificial, social and human) that affect our lives. Successful scientists and engineers typically report a serious early engagement with mathematics as one of their formative experiences. Base competency and interest in mathematics and science are often achieved or lost before the end of high school and likely by the end of elementary grades.

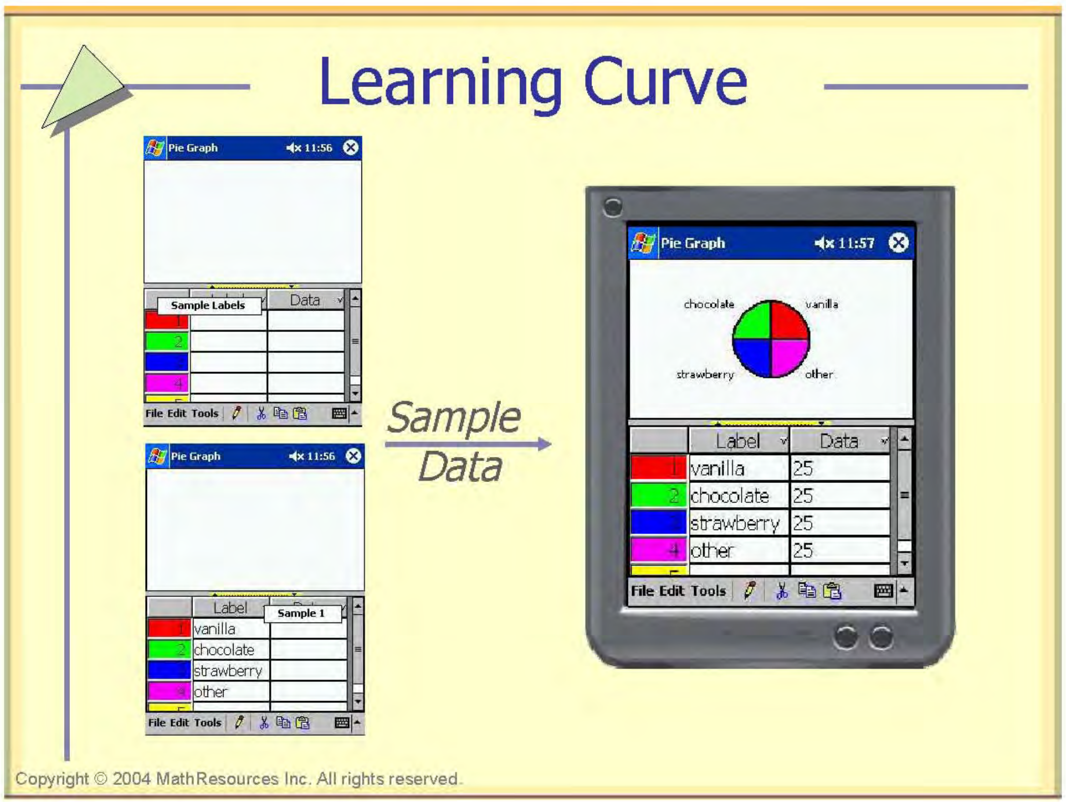
Goals of AGATE-M

- To create a network linking everyone with an interest in math education.
- To enable easy communication between teachers and researchers.
- To strengthen the sense of community amongst those who share the goal of improving math education.
- To provide a forum for the discussion of current issues.
- To offer enrichment resources through web based resources.
- To facilitate the dissemination of knowledge and experience.
- To stimulate enthusiasm and creative thinking in our community.

e. University – Industry links

MITACS – MRI

putting high end science
on a hand held



D4 Wednesday, December 15, 2004 BUSINESS

Try your hand at new math

Firm develops software to help guide kids through maze of numbers

By GREG MacVICAR

Ron Fitzgerald says math is a language — and most students are illiterate.

The president of Halifax software company **MathResources Inc.** wants to change that. That's why Mr. Fitzgerald and his wife quit their jobs as book editors in Toronto in 1994.

Ten years later, he says his company … graphing calcula- … ftware for hand-

… over the next … that we can build … have $40 million … nue," Mr. Fitzger- … d-storey suite on

… fessor friends — … nd Jonathan Bor- … athResources Inc. … ted to create new … n of an interactive

… months, they spent … Mr. Fitzgerald's … e development and

… f 1995 we had spent … Mr. Fitzgerald says. … ne — John Lindsay … with a line of credit

… another $300,000. … now the chairman of … Inc.'s nine-member … ors. There are 30

… software was re- … MathResource was … gh school, college and … nts.

… thousand copies of it … ice," Mr. Fitzgerald … asn't a coup in the

… lectronic dictionaries … nd we're going to be … laughing.

… y decided to "move … nd create software for … nts. Let's Do Math: … designed for grades 4 … sed in late 1998. … ing respectably good … e product," Mr. Fitzger-

… released next year under

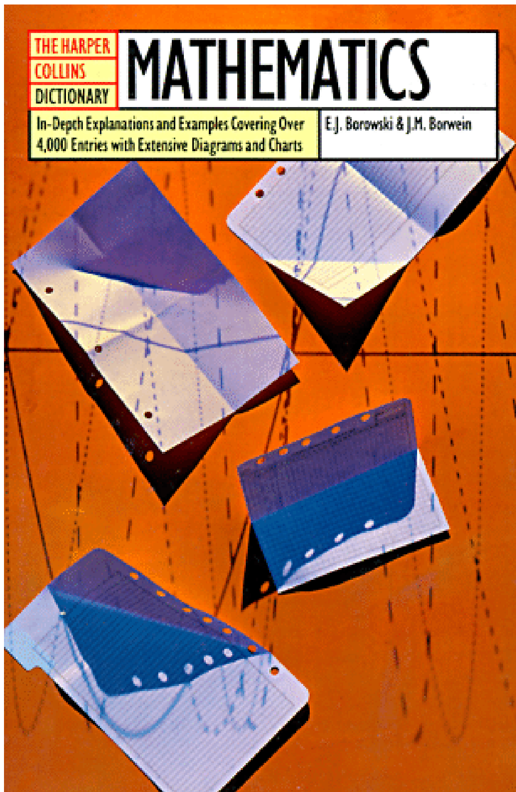
… r. Fitzgerald hopes will … pany really profitable in … ture is MRI-Graphing … graphing and calculating … and-held computers.

Ronald Fitzgerald, president of MathResources Inc., holds a hand-held computer capable of … conventional computers and running the company's mathematics programs.

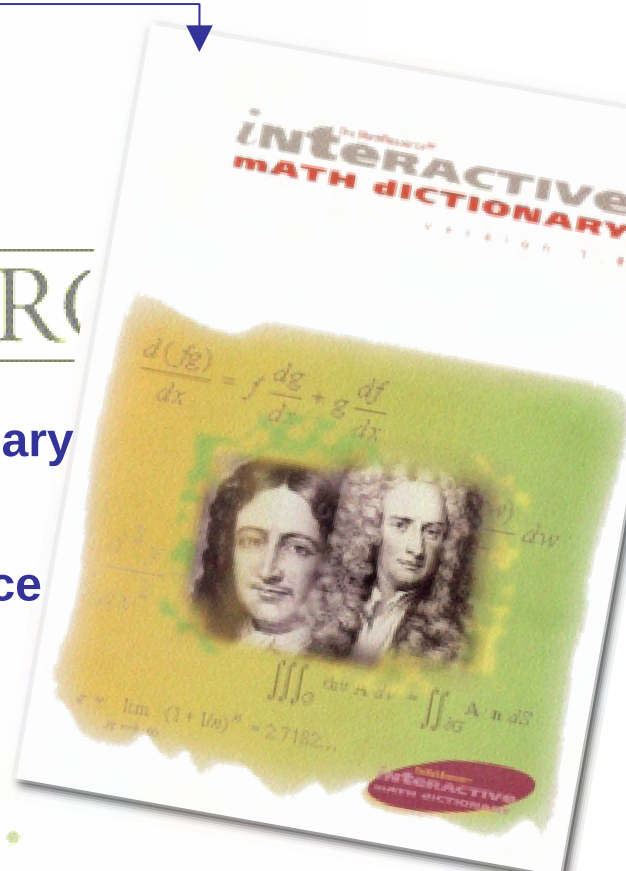
mitacs

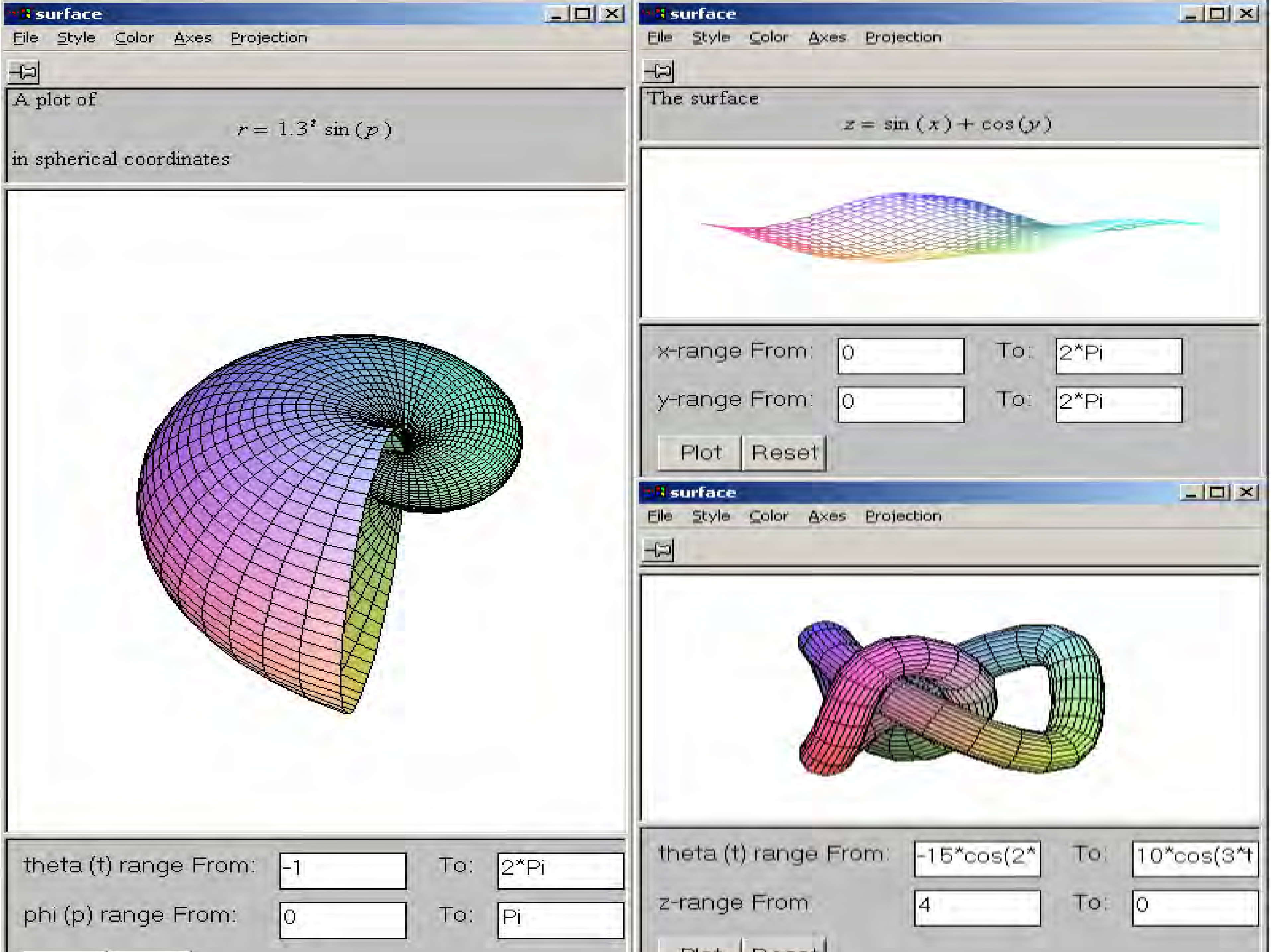
MathResources Inc.

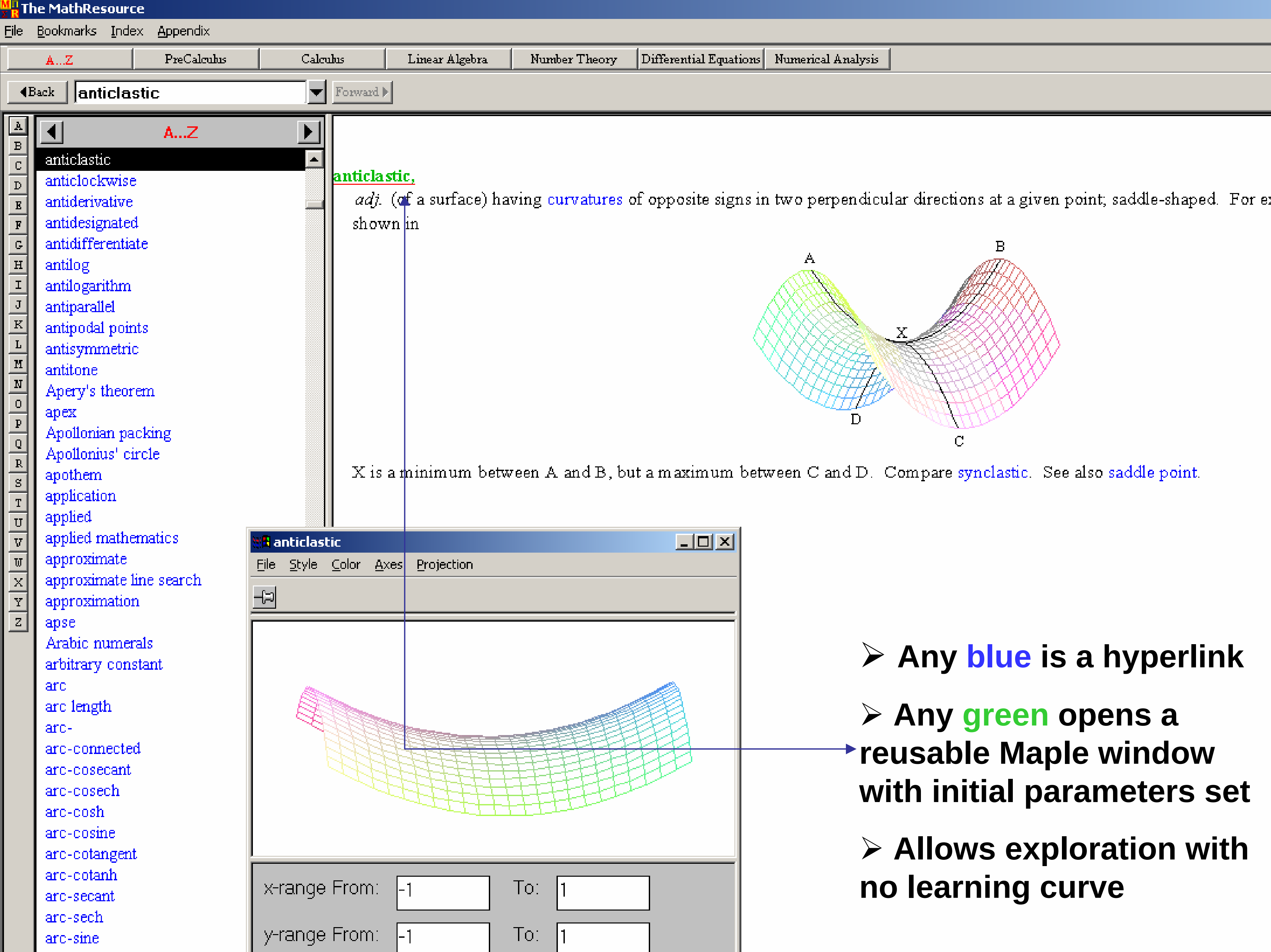
MRI’s First Product in Mid-nineties



- ►Built on **Harper Collins dictionary** - an IP adventure!
- ► **Maple** inside the **MathResource**
- ► **Data base** now in **Maple 9.5**
- ►**CONVERGENCE?**




surface
File Style Color Axes Projection
A plot of
r = 1.3^t sin(p)
in spherical coordinates
theta (t) range From: -1 To: 2*Pi
phi (p) range From: 0 To: Pi
surface
File Style Color Axes Projection
The surface
z = sin(x) + cos(y)
x-range From: 0 To: 2*Pi
y-range From: 0 To: 2*Pi
Plot Reset
surface
File Style Color Axes Projection
theta (t) range From: -15*cos(2* To: 10*cos(3*t
z-range From 4 To: 0



- Any blue is a hyperlink
- Any green opens a reusable Maple window with initial parameters set
- Allows exploration with no learning curve

Building on products such as:

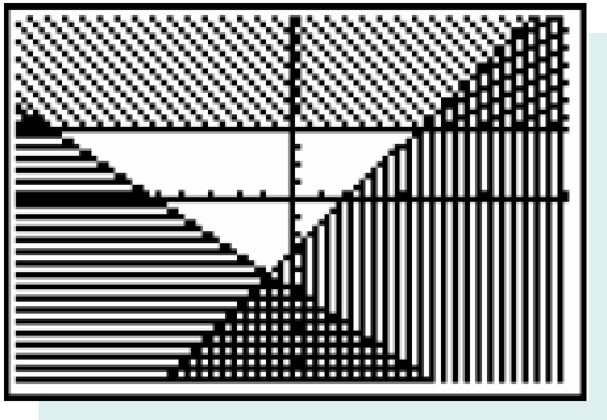
MRI Graphing Calculator & Robert Morris College

Ed Clark, an instructor at Robert Morris College, has been using the MRI Graphing Calculator with his students. Ed says:

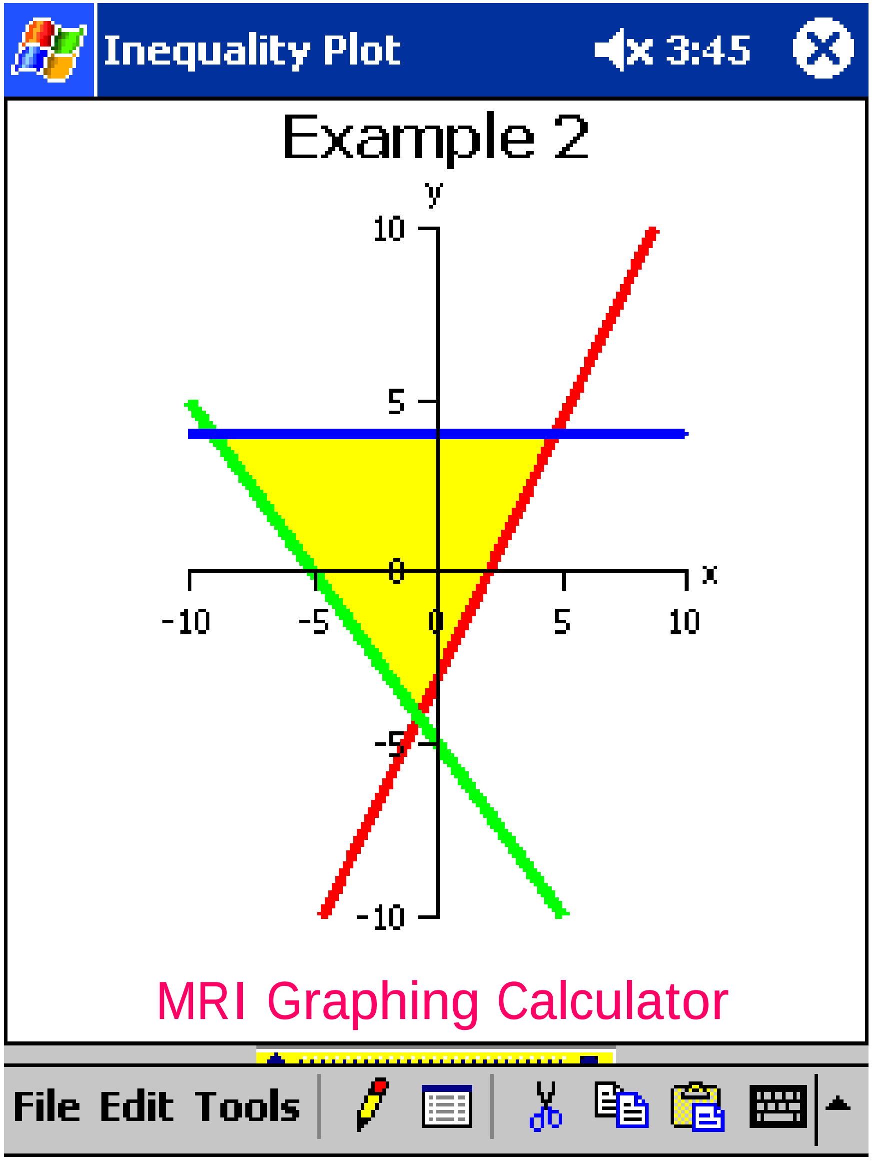
- "The learning curve for the MRI Graphing Calculator is very very short."
- "Just the fact that a handheld computer displays color is huge."

MathResources Inc.

Graphing in Color

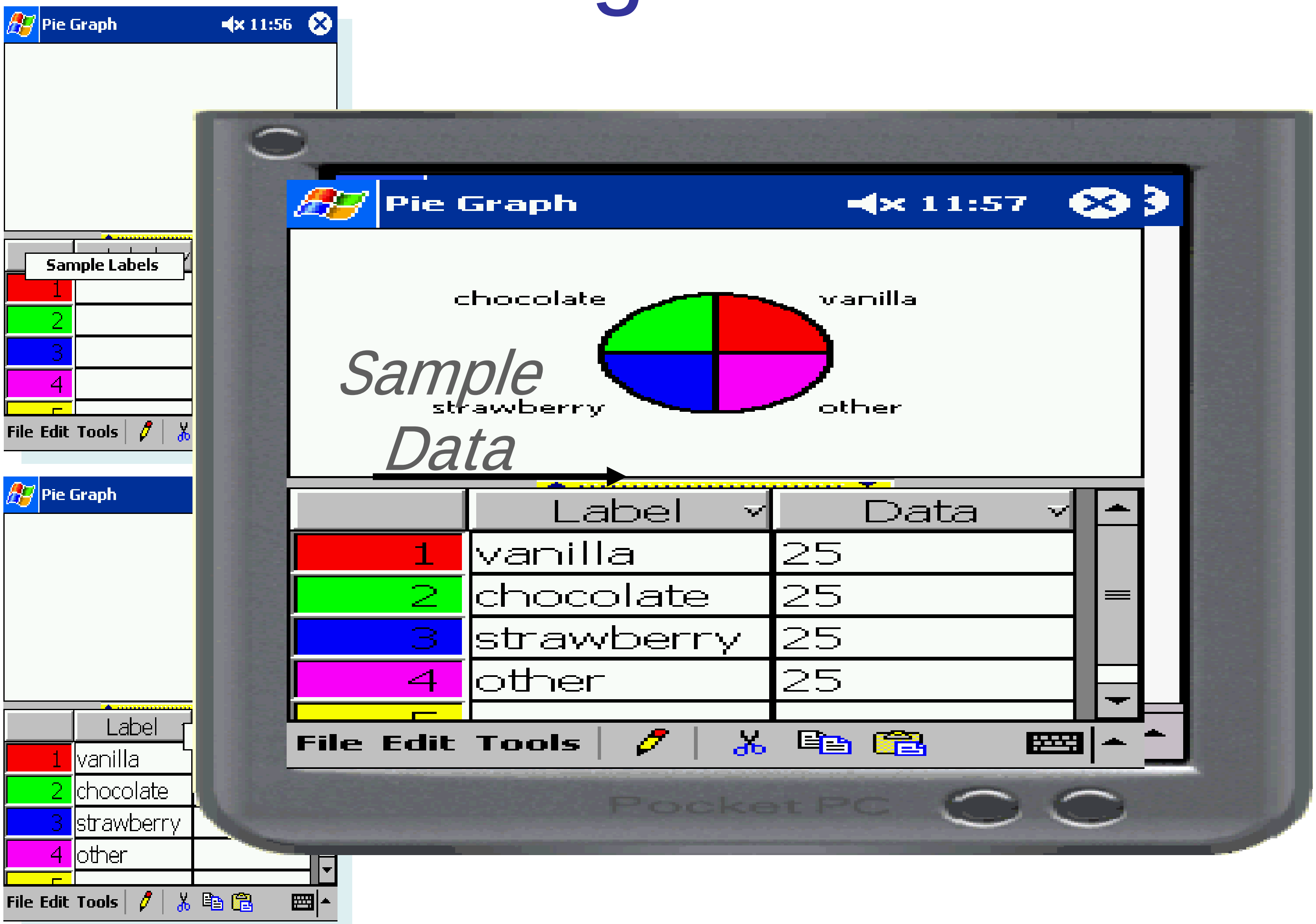


Traditional
Graphing Calculator



MRI Graphing Calculator

Learning Curve



A selection of appropriate virtual manipulables

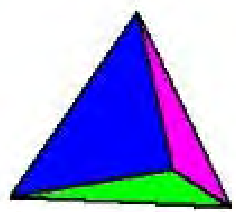
Parabola
Paradox
Parallel
Parallelogram
Parameter
Parametric equation
Parentheses
Partial product of an infinite product
Partial sum of an infinite series
Pascal's triangle
Pascal, Blaise
Peg game
Pentagon
Pentagonal number
Percent
Percentage change
Percentage decrease
Percentage increase
Percentile
Perfect number
Perfect square
Perfect square trinomial
Perimeter
Period of a function
Permutation
Perpendicular
Perpendicular bisector
Phase shift
Pi
Pick's formula
Pictograph
Pie graph
Pint
Place value
Plane
Plane figure
Plane of symmetry
Plane symmetry

Platonic solids

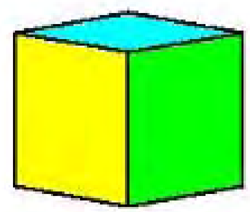
Also called regular polyhedra.

The five special polyhedra where all of the faces of each polyhedron are congruent regular polygons and the same number of polygons meet at each vertex. The ancient Greeks proved that there are only five platonic solids. They are: cube, tetrahedron, octahedron, dodecahedron, and icosahedron.

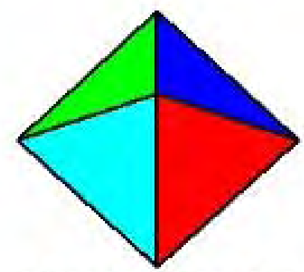
Click on one of the polyhedra below and drag the mouse to rotate it. By right clicking on one of the polyhedra you can change to a wire frame view.



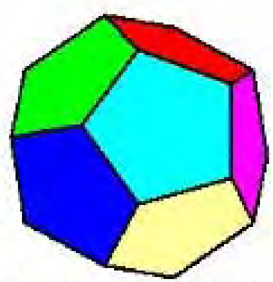
A Regular Tetrahedron



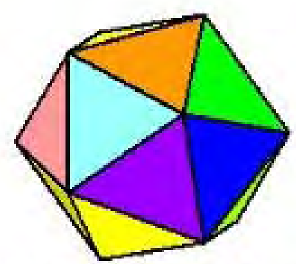
A Cube



A Regular Octahedron



A Regular Dodecahedron



A Regular Icosahedron

Let's Do Math: Tools & Things

File Bookmarks Index View Tools Help

A...Z Full Quick Algebra Arithmetic Geometry Measurement Number Theory Extensions

Back Probability Forward

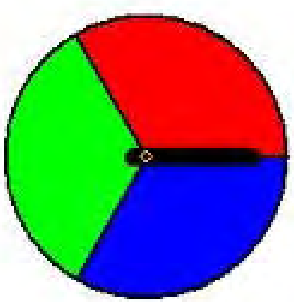
Index

Pint
Place value
Plane
Plane figure
Plane of symmetry
Plane symmetry
Platonic solids
Plotting
Plus sign
Point
Point symmetry
Point-slope form of equation of line
Polygon
Polygonal numbers
Polyhedron
Polynomial
Polynomial equation
Polynomial function
Population
Positional system of numeration
Positive integer
Positive number
Positive sign
Postulate
Pound
Power of a number
Power of ten
Power property of logarithms
Precision of measurement

Probability

Probability is used extensively in business and manufacturing. Manufacturers often base a product guarantee on the results of extensive research and the probability of an item being defective.

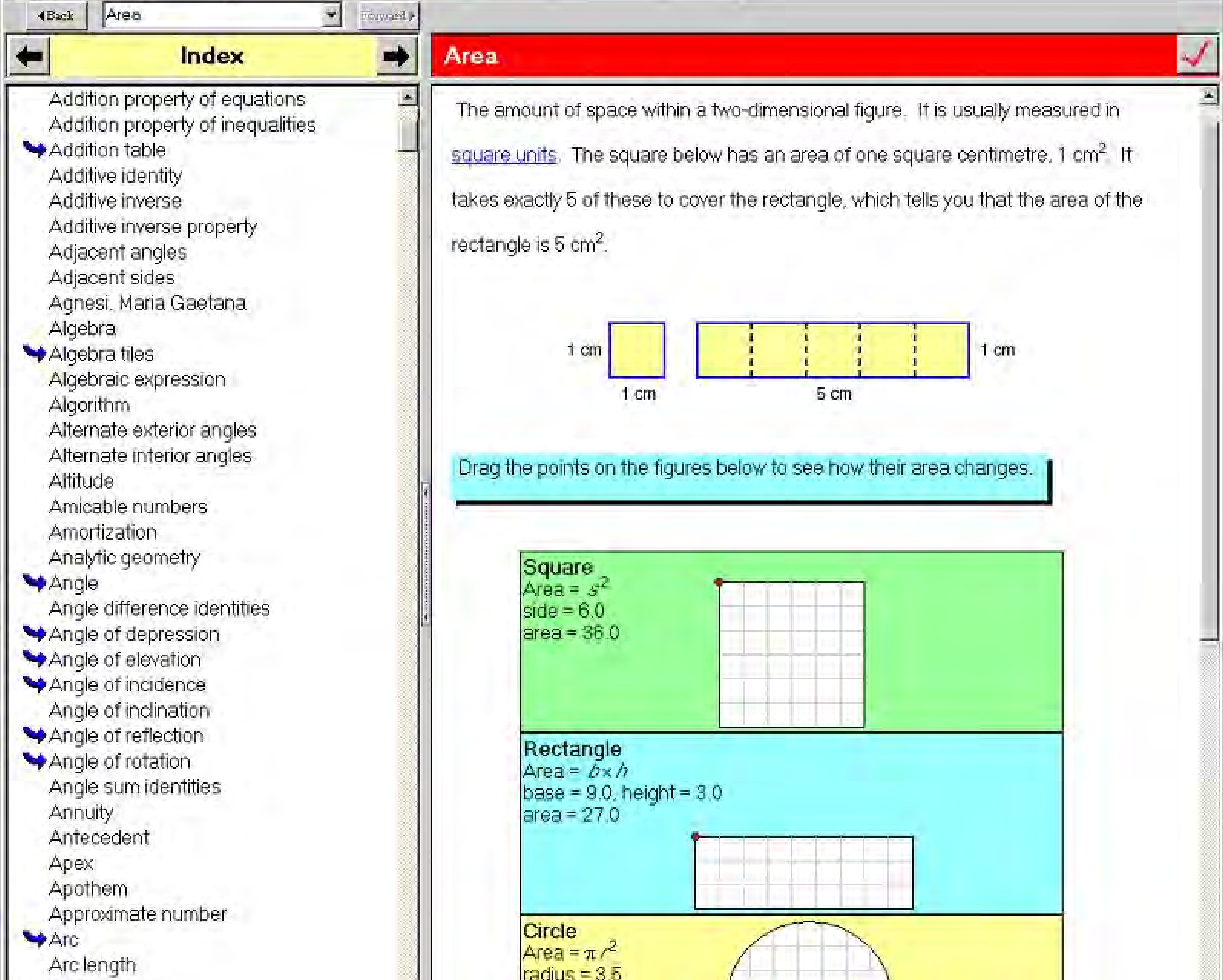
Choose the number of sectors, from 2 to 6. You can also click on an angle measure and change it. All angles must be positive whole numbers and add up to 360°. Enter the number of spins and click the 'Start' button to begin spinning the needle.

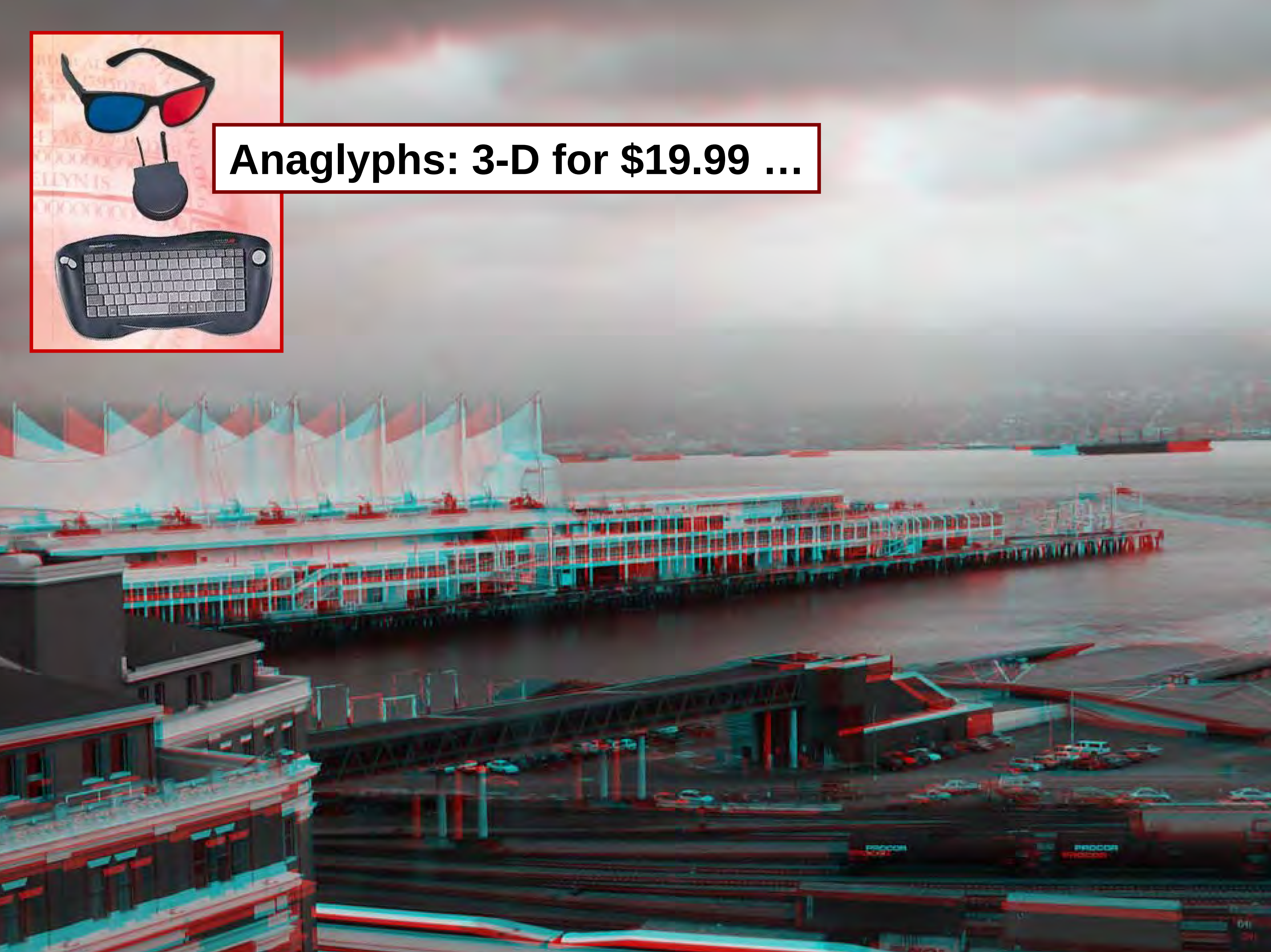


Sector	Angle (°)	Frequency	Theoretical Probability	Experimental Estimate
(red)	120	0	0.333	0.000
(green)	120	0	0.333	0.000
(blue)	120	0	0.333	0.000

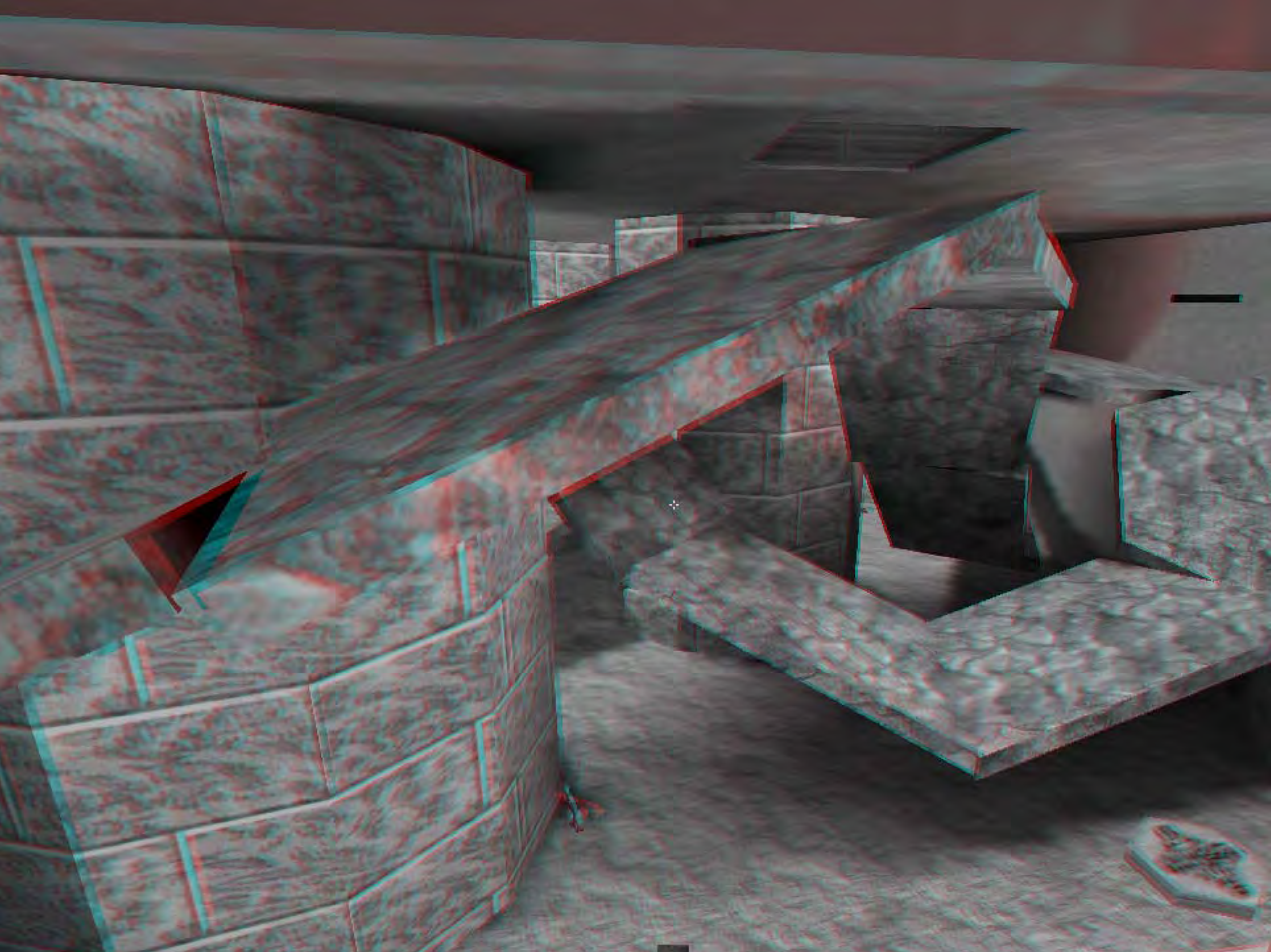
Total = 360°

Total Number of Spins = 0


Back
Area
Forward
Index
Area
Addition property of equations
Addition property of inequalities
Addition table
Additive identity
Additive inverse
Additive inverse property
Adjacent angles
Adjacent sides
Agnesi, Maria Gaetana
Algebra
Algebra tiles
Algebraic expression
Algorithm
Alternate exterior angles
Alternate interior angles
Altitude
Amicable numbers
Amortization
Analytic geometry
Angle
Angle difference identities
Angle of depression
Angle of elevation
Angle of incidence
Angle of inclination
Angle of reflection
Angle of rotation
Angle sum identities
Annuity
Antecedent
Apex
Apothem
Approximate number
Arc
Arc length
The amount of space within a two-dimensional figure. It is usually measured in square units. The square below has an area of one square centimetre, 1 cm². It takes exactly 5 of these to cover the rectangle, which tells you that the area of the rectangle is 5 cm².
1 cm
1 cm
5 cm
1 cm
Drag the points on the figures below to see how their area changes.
Square
Area = s^2
side = 6.0
area = 36.0
Rectangle
Area = $b \times h$
base = 9.0, height = 3.0
area = 27.0
Circle
Area = πr^2
radius = 3.5



Anaglyphs: 3-D for $19.99 …



Outline. What is HIGH PERFORMANCE MATHEMATICS?

1. Visual Data Mining in Mathematics.
   - ✓ Fractals, Polynomials, Continued Fractions, Pseudospectra
2. High Precision Mathematics.
3. Integer Relation Methods.
   - ✓ Chaos, Zeta and the Riemann Hypothesis, HexPi and Normality
4. Inverse Symbolic Computation.
   - ✓ A problem of Knuth, $\pi/8$, Extreme Quadrature
5. The Future is Here.
   - ✓ Examples and Issues
6. Conclusion.
   - ✓ Engines of Discovery. The 21st Century Revolution
   - ✓ Long Range Plan for HPC in Canada

hpcs 2005

Drive

CONCLUSION

ENGINES OF DISCOVERY:
The 21st Century Revolution

The Long Range Plan for High Performance Computing in Canada

The LRP tells a Story

- The Story
- Executive Summary
- Main Chapters
  - Technology
  - Operations
  - HQP
  - Budget
- 25 Case Studies
  - many sidebars

One Day ...

High-performance computing (HPC) affects the lives of Canadians every day. We can best explain this by telling you a story. It's about an ordinary family on an ordinary day, Russ, Susan, and Kerri Sheppard. They live on a farm 15 kilometres outside Wyoming, Ontario. The land first produced oil, and now it yields milk; and that's just fine locally.

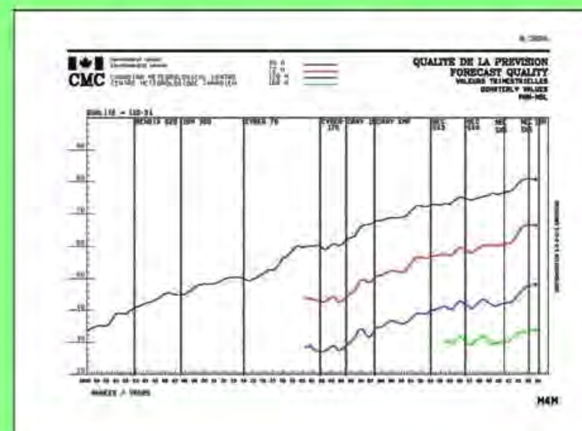
Their day, Thursday, May 29, 2003, begins at 4:30 am when the alarm goes off. A busy day, Susan Zhong-Sheppard will fly to Toronto to see her father, Wu Zhong, at Toronto General Hospital; he's very sick from a stroke. She takes a quick shower and packs a day bag for her 6 am flight from Sarnia's Chris Hadfield airport. Russ Sheppard will stay home at their dairy farm, but his day always starts early. Their young daughter Kerri can sleep three more hours until school.

Waiting, Russ looks outside and thinks, *It's been a dryish spring. Where's the rain?*

In their farmhouse kitchen on a family-sized table sits a PC with a high-speed Internet line. He logs on and finds the Farmer Daily site. He then chooses the Environment Canada link, clicks on Ontario, and then scans down for Sarnia-Lambton.

WEATHER PREDICTION

The "quality" of a five-day forecast in the year 2003 was equivalent to that of a 36-hour forecast in 1963 [REF 1]. The quality of daily forecasts has risen sharply by roughly one day per decade of research and HPC progress. Accurate forecasts transform into billions of dollars saved annually in agriculture and in natural disasters. Using a model developed at Dalhousie University (Prof. Keith Thompson), the Meteorological Service of Canada has recently been able to predict coastal flooding in Atlantic Canada early enough for the residents to take preventative action.



The backbone that makes so much of our individual science possible

REFERENCES


Enigma

J.M. Borwein and D.H. Bailey, *Mathematics by Experiment: Plausible Reasoning in the 21st Century* A.K. Peters, 2003.

J.M. Borwein, D.H. Bailey and R. Girgensohn, *Experimentation in Mathematics: Computational Paths to Discovery,* A.K. Peters, 2004.

D.H. Bailey and J.M Borwein, "Experimental Mathematics: Examples, Methods and Implications," *Notices Amer. Math. Soc.*, **52** No. 5 (2005), 502-514.

"The object of mathematical rigor is to sanction and legitimize the conquests of intuition, and there was never any other object for it."

- J. Hadamard quoted at length in E. Borel, *Lecons sur la theorie des fonctions*, 1928.