Education Afternoon at the 53rd Annual Meeting of the Australian Mathematical Society

University of South Australia, Adelaide - Tuesday 29 September 2009

Sponsored by the International Centre of Excellence for Education in Mathematics

Inverse Symbolic Calculation:

symbols from numbers

Jonathan Borwein, FRSC www.carma.newcastle.edu.au/~jb616

Laureate Professor University of Newcastle, NSW

Director, Centre for **Computer Assisted Research Mathematics and Applications**

CARMA

Revised 26-09-09

ABSTRACT

Jonathan M. Borwein

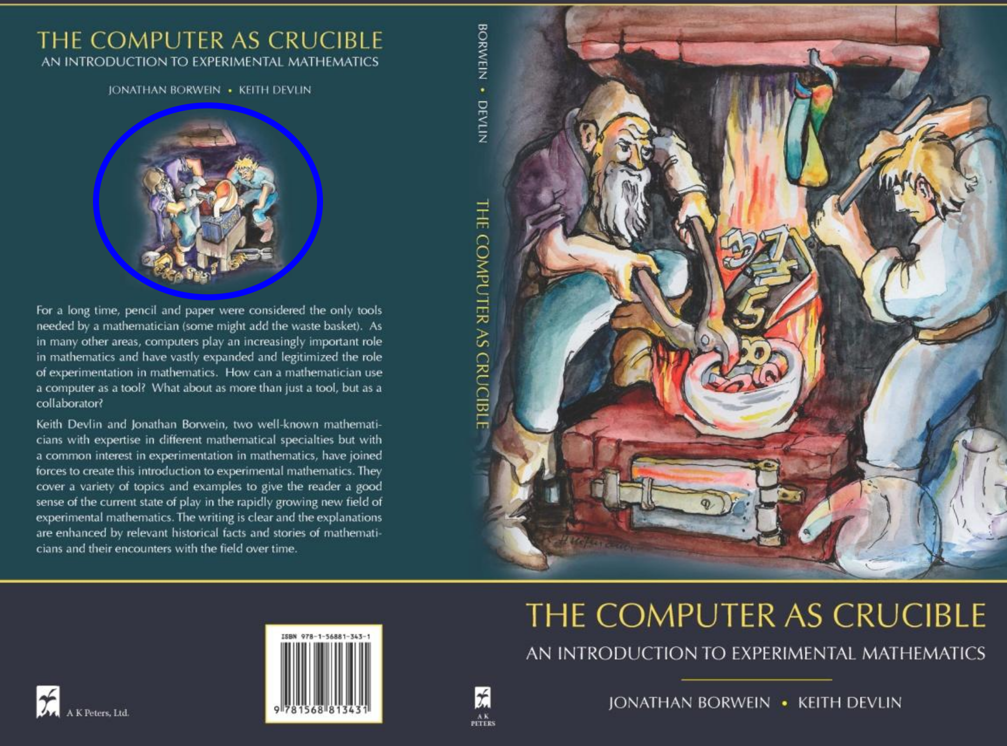
Director

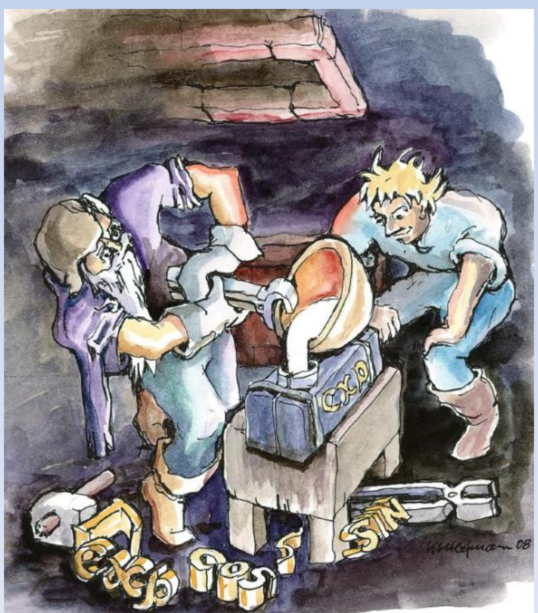
Newcastle Centre for

Computer Assisted Research Mathematics and its Applications (CARMA)

We are all familiar with the uses and misuses of calculators in the classroom and may take it for granted that they require mathematics as input and typically give numbers as output. I wish to show the power of calculators that invert this process: numbers go in and mathematics comes out. I shall demonstrate the ***Inverse Symbolic Calculator***, at http://ddrive.cs.dal.ca/~isc, and its implementation inside ***Maple*** as the **identify** function and will illustrate their use in teaching and research as tools of discovery.

"*The object of mathematical rigor is to sanction and legitimize the conquests of intuition, and there was never any other object for it.*" – Jacques Hadamard





Jonathan Borwein
Keith Devlin

with illustrations by Karl H. Hofmann

Contents

Francois Vieta (1540-1603)

Arithmetic is absolutely as much science as geometry [is]. Rational magnitudes are conveniently designated by numbers and irrational [magnitudes by irrational [numbers]. If someone measures magnitudes with numbers and by his calculation get them different from what they really are, it is not the reckoning's fault but the reckoner's.

Rather, says Proclus, ARITHMETIC IS MORE EXACT THAN GEOMETRY. To an accurate calculator, if the diameter is set to one unit, the circumference of the inscribed dodecagon will be the side of the binomial [i.e. square root of the difference $72 - \sqrt{3888}$. *Whoever declares any other result, will be mistaken, either the geometer in his measurements or the calculator in his numbers.*

- The inventor of 'x' and 'y'

OUTLINE

- Background and History

Part I. ISC1.0 and **Colour Calculator** in Action

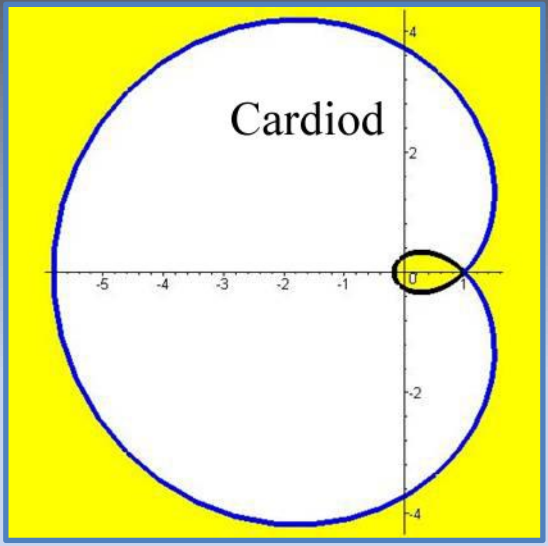
– Examples of **Identify** in action

Part II. Integer Relations

– What they are

– What they do

- Elementary examples
- Advanced examples



A Scatterplot Discovery

"The new availability of huge amounts of data, along with the statistical tools to crunch these numbers, offers a whole new way of understanding the world. Correlation supersedes causation, and science can advance even without coherent models, unified theories, or really any mechanistic explanation at all. There's no reason to cling to our old ways. It's time to ask: What can science learn from Google?" - Wired, 2008

BACKGROUND

- Knowing the answer is more-than-half the battle
  - Archimedes, Gauss, Hadamard, Russell, etc... all agree

What *is* that number?

“Who *was* that masked man?”

–*The Lone Ranger*
1950s tv western series

"*And yet since truth will sooner come out of error than from confusion.*"
- Francis Bacon, 1561-1626

CHRONLOGY of ISC and FRIENDS

1973 Sloane's **Handbook of Integer Sequences**

1978 Ferguson finds **PSLQ Integer Relation Algorithm**

1985 Sloane's **Encyclopaedia of Integer Sequences**

- with Plouffe (5,000 entries)

1990 Handbook of Real Numbers (100,000:16Mb)

1995 The Inverse Symbolic Calculator (ISC)

- binscripts/JAVA (10Gb: wanted by GNU)

1995 The Colour Calculator

1996 Sloane's Online Encyclopaedia (OEIS) (150,000)

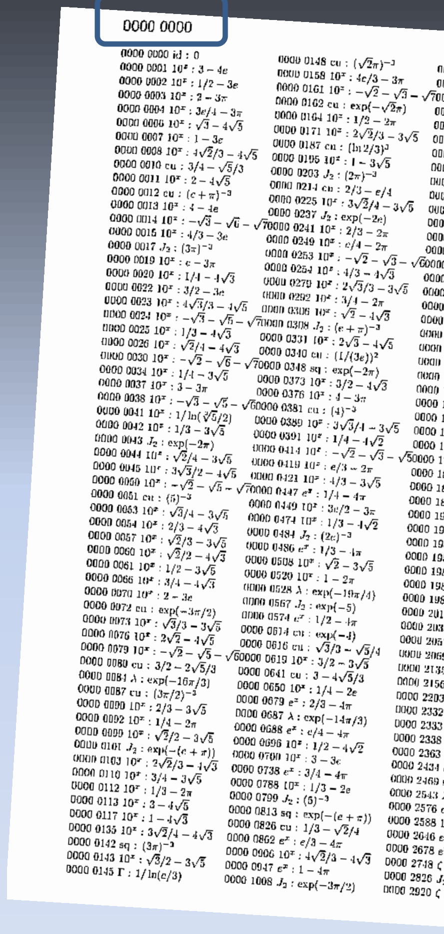
1999 "Identify" added to **Maple**

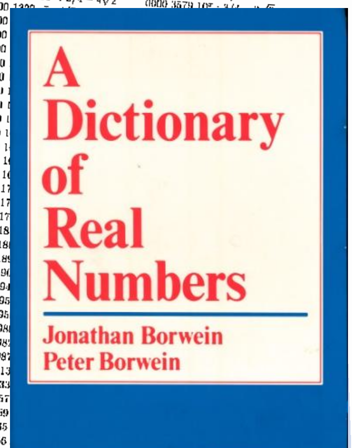
2007 ISC2.0 (Python + Cherry Pie) multi-threaded

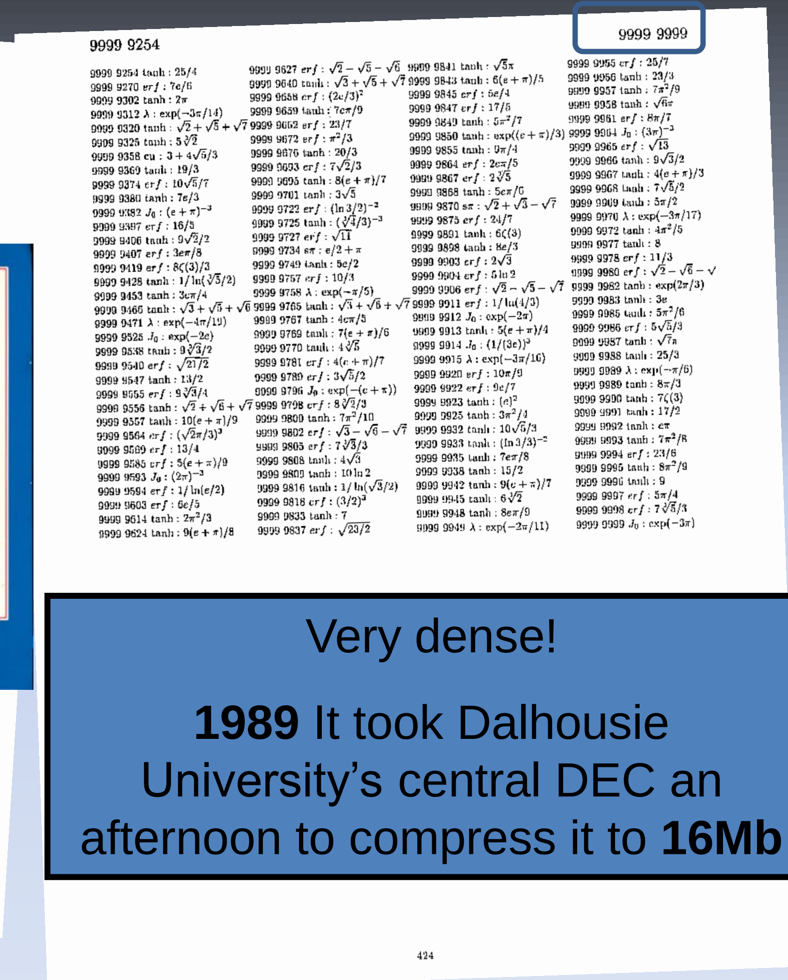
- less lookup, more preprocessing and computing

1988-90 A DICTIONARY of REAL NUMBERS

8 pages of preface and 424 of numbers in [0,1]





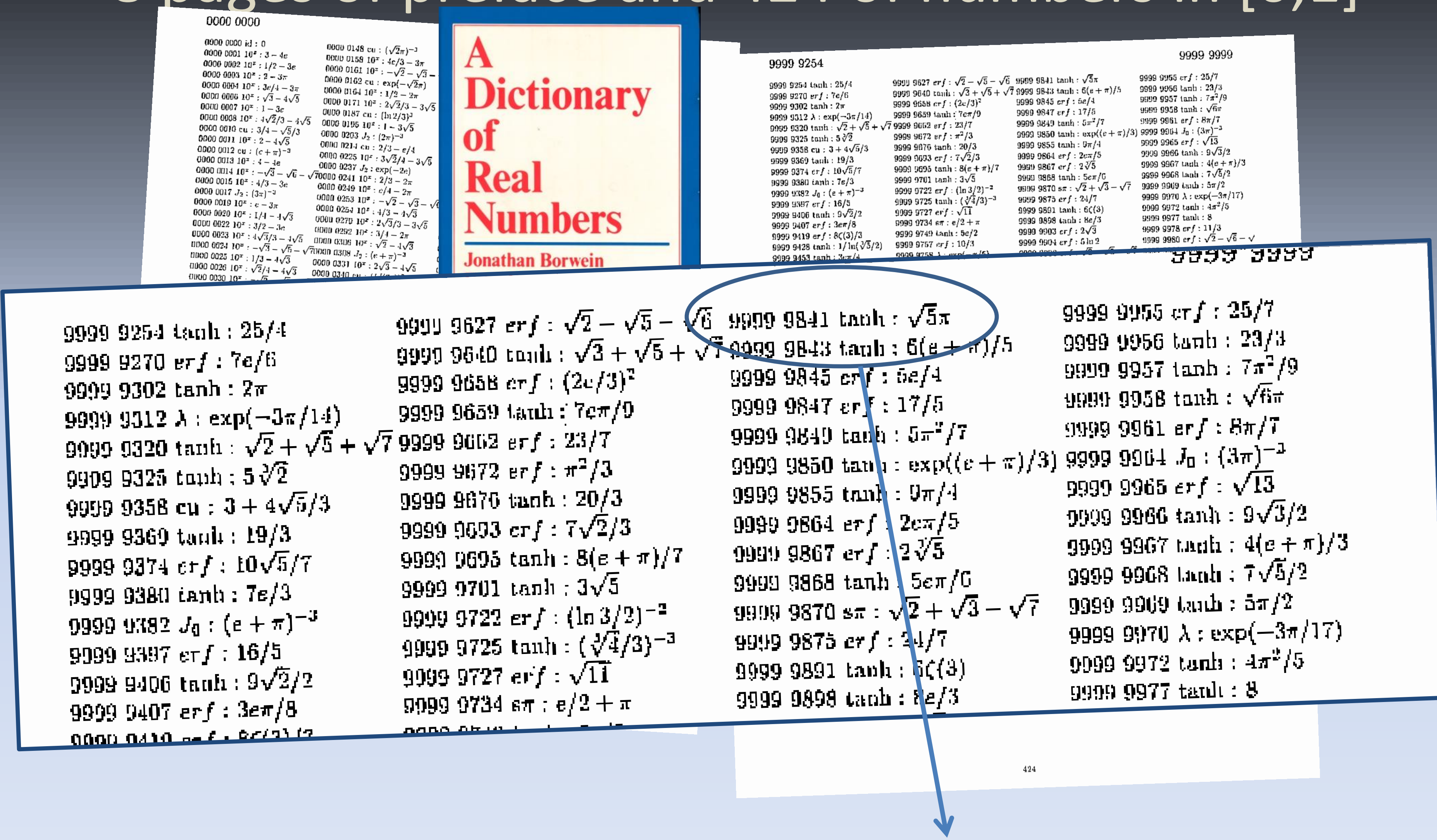


Very dense!

1989 It took Dalhousie University's central DEC an afternoon to compress it to **16Mb**

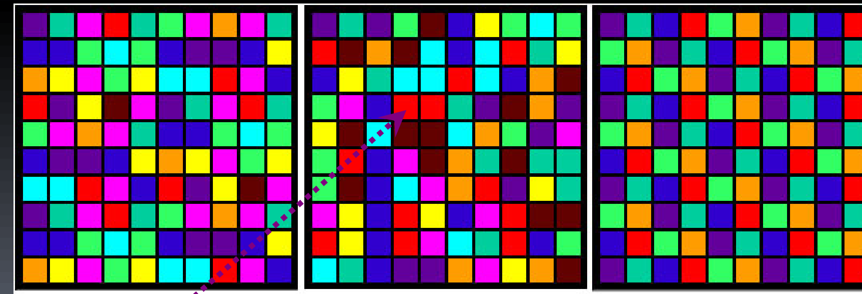
1988-90 A DICTIONARY of REAL NUMBERS

8 pages of preface and 424 of numbers in [0,1]



8 digits after the decimal point: $1+\tanh\left(\sqrt{5}\pi\right) = 1.9999984175$

COLOR and INVERSE CALCULATORS (1995)

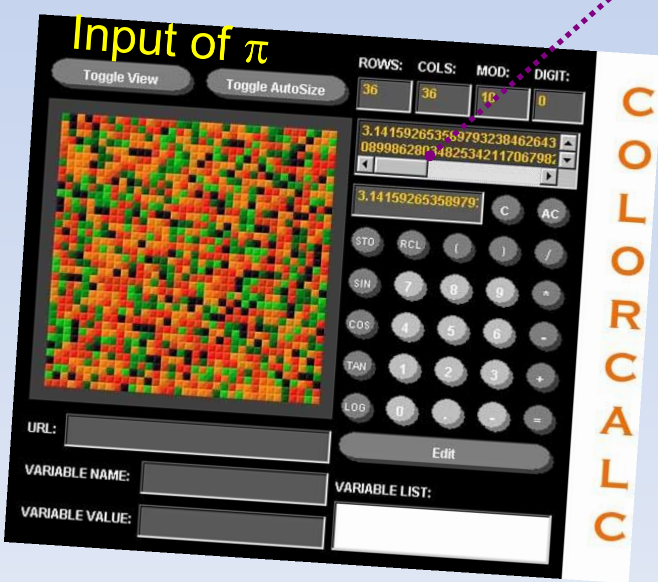


Archimedes: $223/71 < \pi < 22/7$

Inverse Symbolic Computation

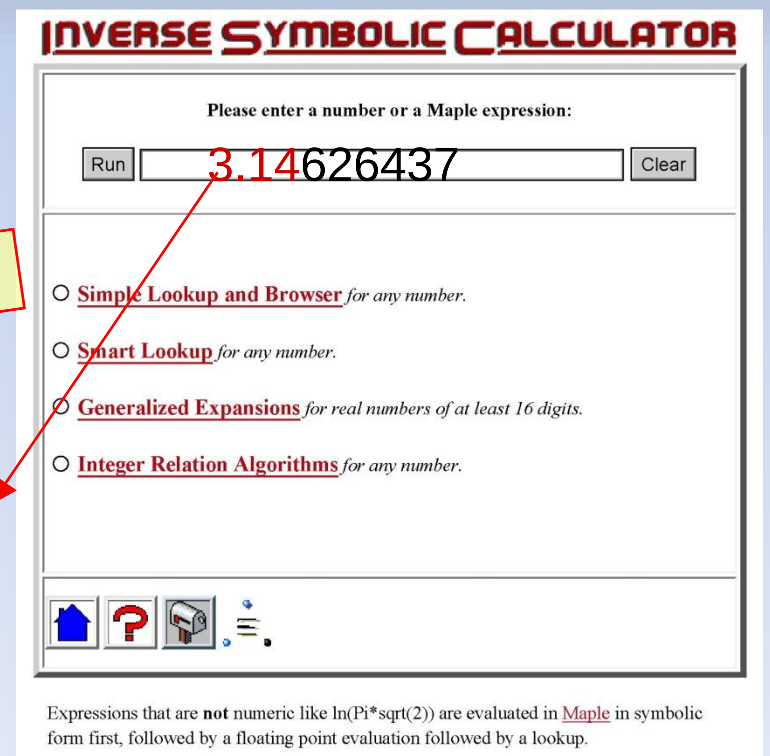
Inferring mathematical structure from numerical data

- Mixes *large table lookup*, integer relation methods and intelligent preprocessing – needs *micro-parallelism*
- It faces the "curse of exponentiality"
- Implemented as **identify** in Maple

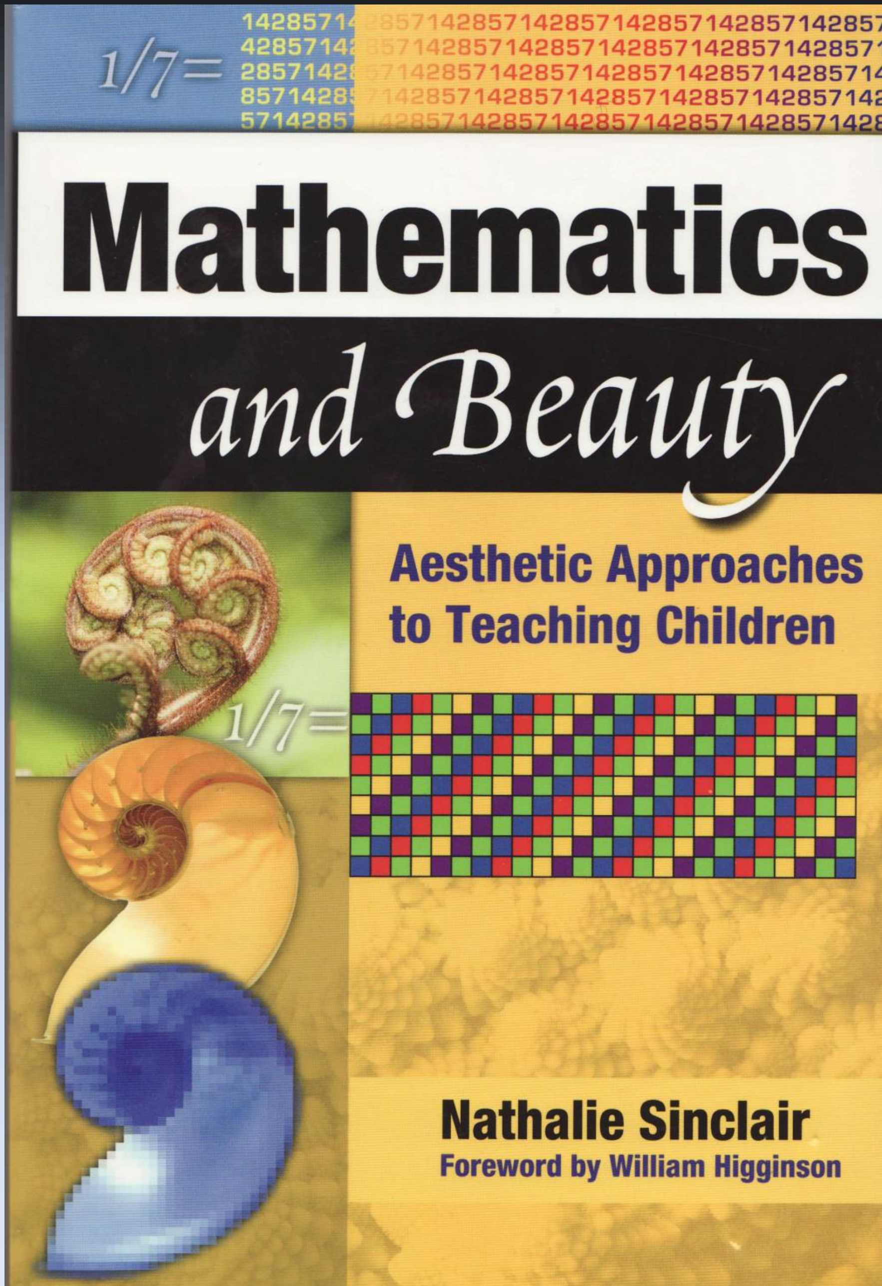


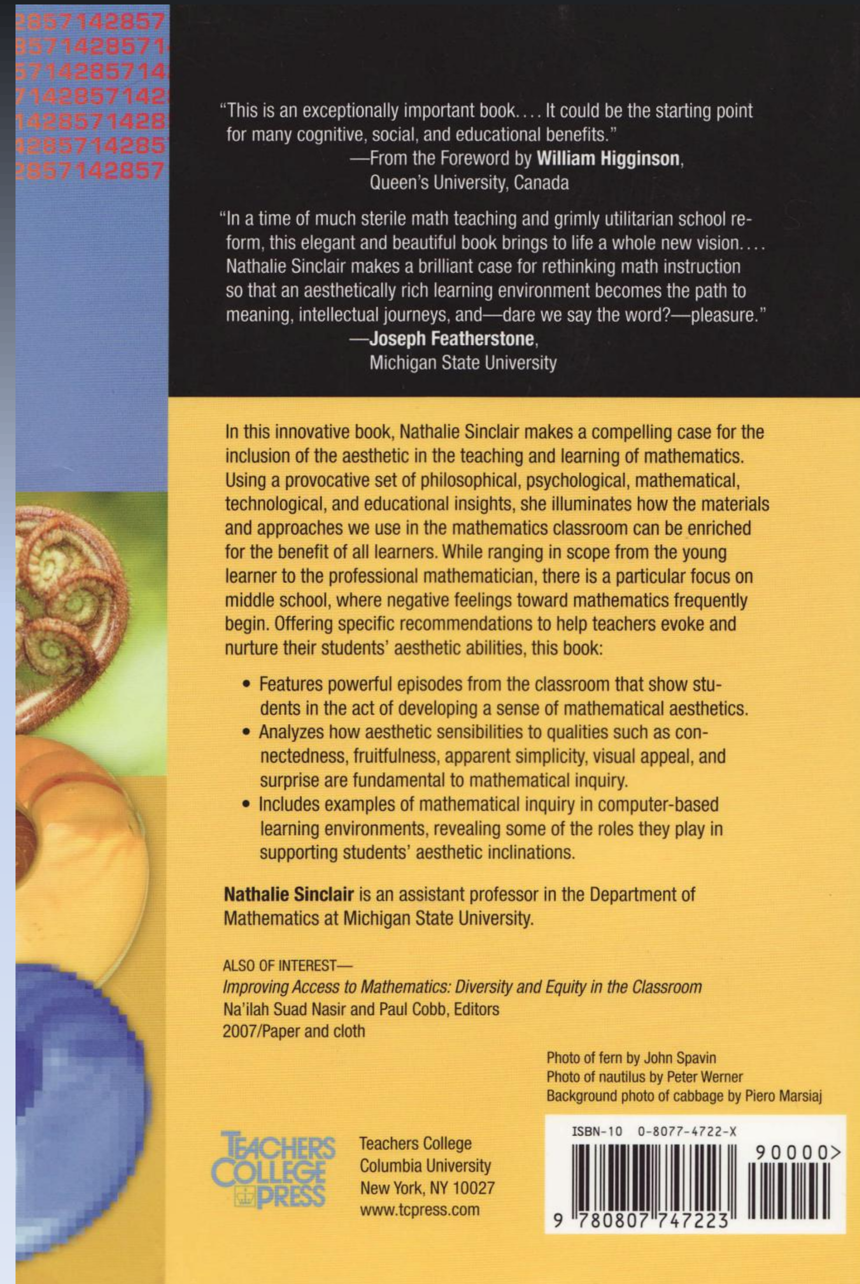
identify(sqrt(2.)+sqrt(3.))

$\sqrt{2} + \sqrt{3}$



MATHEMATICS and BEAUTY 2006





ENTERING

Knuth asked : $\sum_{k=1}^{\infty} \left\{ \frac{k^k}{k!\,e^k} - \frac{1}{\sqrt{2\pi\,k}} \right\} = ?$

evalf(Sum(k^k/k!/exp(k)-1/sqrt(2*Pi*k),k=1..infinity),16)

'**Simple Lookup**' fails.
'**Smart Lookup**' gives:

INVERSE SYMBOLIC CALCULATOR

The ISC is the **Inverse Symbolic Calculator**, a set of programs and specialized tables of mathematical constants dedicated to the identification of real numbers. It also serves as a way to produce identities with functions and real numbers. It is one of the main ongoing projects at the Centre for Experimental and Constructive Mathematics (CECM).

TOP 5% OF ALL WEB SITES POINT

Dr Matrix SCIENCE EXCELLENCE

INVERSE SYMBOL

Results of the search:

Maple output:

.08406950872765600

.8406950872765600e-1

Value to be looked up: .8406950872765600e-1 = K

Performing a smart lookup on .8406950872765600e-1:

Function	Result	Precision	Matches
K-2/3	.5825971579390106666666666	16	1

BOLIC CALCULATOR

579390106 was probably generated by one
s or found in one of the given tables.
Answers are given from shortest to longest description

Mixed constants with 5 operations
5825971579390106 = Zeta(1/2)/sr(2)/sr(Pi)

Browse around .5825971579390106.

$$\frac{\zeta(1/2)}{\sqrt{2\pi}}$$

Drive NSERC CRSNG Maplesoft

The ISC in Action

The Inverse Symbolic Calculator (ISC) uses a combination of lookup tables and integer relation algorithms in order to associate with a user-defined, truncated decimal expansion (represented as a floating point expression) a closed form representation for the real number.

ISC+ inverse symbolic calculator

Standard lookup results for **12.587886229548403854**

exp(1)+Pi^2

ISC The original ISC

The Dev Team: Nathan Singer, Andrew Shouldice, Lingyun Ye, Tomas Daske, Peter Dobcsanyi, Dante Manna, O-Yeat Chan, Jon Borwein

3.146264370 Try it!

19.99909998 Try it!

ISC The original ISC

The Dev Team: Nathan Singer, Andrew Shouldice, Lingyun Ye, Tomas Daske, Peter Dobcsanyi, Dante Manna, O-Yeat Chan, Jon Borwein

The ISC presently accepts either floating point expressions or correct Maple syntax as input. However, for Maple syntax requiring too long for evaluation, a timeout has been implemented.

Visit

Jon Borwein's Webpage

David Bailey's Webpage

Math Resources Portal

- ISC+ runs on Glooscap
- Less lookup & more algorithms than 1995

IDENTIFY and ISC IN ACTION

ISC does more for a naive user

identify does more for an experienced user

19.999099979

Advanced lookup results for **19.999099979**

exp(Pi)-Pi	19999099997918947
Pi-exp(Pi)	

ISC The original ISC

$$\prod_{n=2}^{\infty} \frac{n^2-1}{n^2+1} = 0.2720290549821332\ldots$$

Advanced lookup results for **0.2720290549821332**

Transform (K=0.2720290549821332)	Searched for	Description
K*1/2	.1360145274910666000000000	Pi/(exp(-Pi)-exp(Pi))

$$\prod_{n=2}^{\infty} \frac{n^3-1}{n^3+1} = 0.6666666666666667$$

$$> \text{identify}(3.140845070422535) = \frac{223}{71}$$

$$\int_0^1 \left|e^{i\pi x}+1\right| dx = 1.273239544735163$$

$$> \text{identify}(1.273239544735163) = \frac{4}{\pi}$$

4.599873743272336

```
> pslq(4.599873743272336,[1,Pi^2,Pi^4]);
```

[−360, 0, 0, −17], "Error is", $-4.730194857\ 10^{-13}$, "checking to", 26, *places*

$$4.599873743272336 = \frac{17}{360}\pi^4$$

A HOMEWORK CHALLENGE

What are

$$\lim_{n=\infty} \frac{1}{n} \sum_{k=1}^{n} \left\{\frac{n}{k}\right\}^2 = 0.2606614015078126229541 4...,$$

and

$$\sum_{k=1}^{\infty} \frac{1}{2^n n^2} = 0.5822405264650125059... ?$$

The answers are

$$\log(2\pi) - 1 - \gamma$$

and

$$\frac{\pi^2}{12} - \frac{1}{2} \log(2)^2.$$

Here

$$\gamma := \lim \sum_{k=1}^{n} \frac{1}{k} - \log n = 0.5772156649015328...$$

is Euler's mysterious (irrational?) constant.

How about $0.4380178794859424 12114 \ldots$?

Hint: first find 0.6309297535714574371 0... (Answer by email)

Greetings from The On-Line Encyclopedia of Integer Sequences!

1,1,4,9,25,64,169 | Search | Hints

Search: **1, 1, 4, 9, 25, 64, 169**

Displaying 1-1 of 1 results found. page 1

Format: long | short | internal | text   Sort: relevance | references | number   Highlight: on | off

A007598   F(n)^2, where F() = Fibonacci numbers A000045.   +20
(Formerly M3364)   41

0, **1, 1, 4, 9, 25, 64, 169**, 441, 1156, 3025, 7921, 20736, 54289, 142129, 372100, 974169, 2550409, 6677056, 17480761, 45765225, 119814916, 313679521, 821223649, 2149991424, 5628750625, 14736260449, 38580030724 (list; graph; listen)

OFFSET 0,4

COMMENT a(n)*(-1)^(n+1) = (2*(1-T(n,-3/2))/5), n>=0, with Chebyshev's polynomials T(n,x) of the first kind, is the r=-1 member of the r-family of sequences S_r(n) defined in A092184 where more information can be found. W. Lang (wolfdieter.lang_AT_physik_DOT_uni-karlsruhe_DOT_de), Oct 18 2004

Contribution from Giorgio Balzarotti (greenblue(AT)tiscali.it), Mar 11 2009: (Start)

Determinant of power series with alternate signs of gamma matrix with determinant 1!

a(n) = Determinant(A-A^2+ A^3-A^4+ A^5-... A^n)

where A is the submatrix A(1..2,1..2)= of the matrix with factorial determinant

A= [[1,1,1,1,1,1,...],[1,2,1,2,1,2,...],[1,2,3,1,2,3,...],[1,2,3,4,1,2,...],[1,2,3,4,5,1,...],[1,2,3,4,5,6,...],...]

note: Determinant A(1..n,1..n)= (n-1)!

a(n) is even with respect to signs of power of A.

See A158039...A158050 for sequence with matrix 2!, 3!... (End)

Contribution from Gary W. Adamson (qntmpkt(AT)yahoo.com), Apr 27 2009: (Start)

Equals the INVERT transform of (1, 3, 2, 2, 2,...). Example: a(7) = 169 = (1, 1, 4, 9, 25, 64) dot (2, 2, 2, 2, 3, 1) = (2 + 2 + 8 + 18 + 75 + 64) = 169. (End)

REFERENCES A. T. Benjamin and J. J. Quinn, Proofs that really count: the art of combinatorial proof, M.A.A. 2003, id. 8.

R. Honsberger, Mathematical Gems III, M.A.A., 1985, p. 130.

R. P. Stanley, Enumerative Combinatorics I, Example 4.7.14, p. 251.

LINKS T. D. Noe, Table of n, a(n) for n=0..200

Index entries for two-way infinite sequences

Index entries for sequences related to linear recurrences with constant coefficients

D. Foata and G.-N. Han, Nombres de Fibonacci et polynomes orthogonaux,

T. Mansour, A note on sum of k-th power of Horadam's sequence

T. Mansour, Squaring the terms of an ell-th order linear recurrence

P. Stanica, Generating functions, weighted and non-weighted sums of powers...

FORMULA a(0) = 0, a(1) = 1; a(n) = a(n-1) + Sum(a(n-i)) + k, 0 <= i < n where k = 1 when n is odd, or k = -1 when n is even. E.g. a(2) = 1 = 1 + (1 + 1 + 0) - 1, a(3) = 4 = 1 + (1 + 1 + 0) + 1, a(4) = 9 = 4 + (4 + 1 + 1 + 0) - 1, a(5) = 25 = 9 + (9 + 4 + 1 + 1 + 0) + 1. - Sadrul Habib Chowdhury

"Nature laughs at the difficulty of Integration" - Lagrange

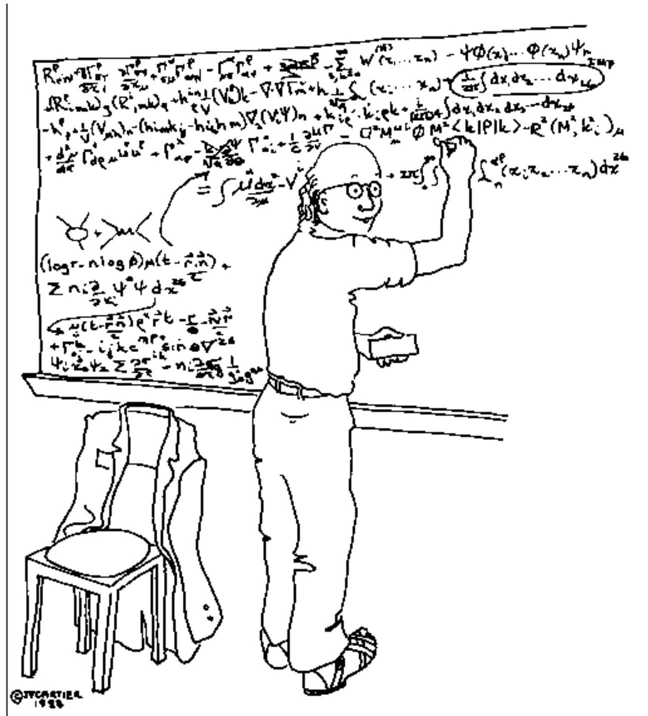
"A heavy warning used to be given [by lecturers] that pictures are not rigorous; this has never had its bluff called and has permanently frightened its victims into playing for safety. Some pictures, of course, are not rigorous, but I should say most are (and I use them whenever possible myself)." - J. E. Littlewood, 1885-1977

End of Part I

Part II on **Integer Relation Methods** is at

www.carma.newcastle.edu.au /~jb616/papers.html#TALKS

Some More Scenes from a Scientist's Life …



"At this point we notice that this equation is beautifully simplified if we assume that space–time has 92 dimensions."

PSLQ: INTEGER RELATION ALGORITHMS: WHAT THEY ARE

Let (x_n) be a vector of real numbers. An **integer relation algorithm** finds integers (a_n) such that

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0$$

or provides an **exclusion bound**

- i.e., testing linear independence over **Q**

- At present, the PSLQ algorithm of mathematician-sculptor *Helaman Ferguson* is the **best** known integer relation algorithm.
- High precision arithmetic software is required: at least $d \times n$ digits, where d is the size (in digits) of the largest of the integers a_k.

INTEGER RELATION ALGORITHMS: HOW THEY WORK

Let (x_n) be a vector of real numbers. An **integer relation algorithm** finds integers (a_n) such that

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0$$

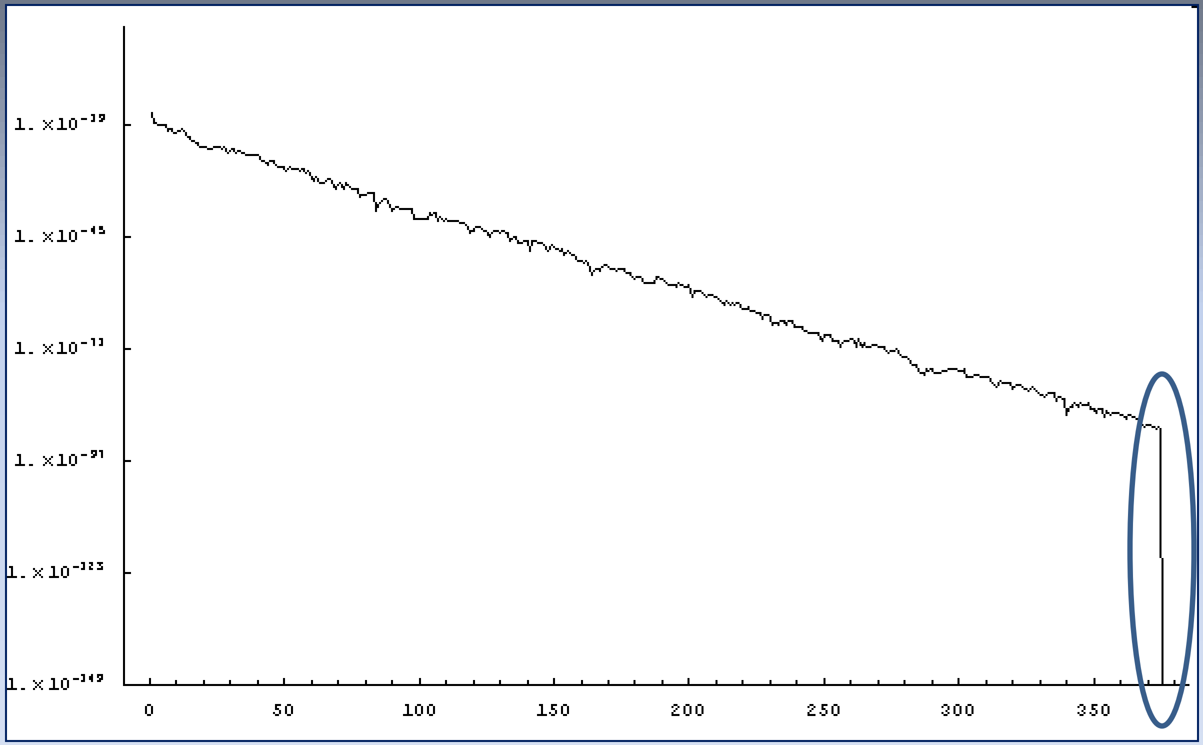
or provides an **exclusion bound.**

PSLQ operates by developing, iteratively, an integer matrix A that successively reduces the maximum absolute value of the entries of the vector $y = Ax$, until one of the entries of y is either zero or within roughly 10^{-p} of zero, where p is the numeric precision used.

Any integer relation detection scheme needs data to at least nd-digit precision: via a simple pigeonhole analysis. Assume the x vector does not satisfy an integer relation, with $|x_j| \leq 1$. Suppose all a_j satisfy $|a_j| \leq 10^d$. Then $\sum_{1\leq j\leq n} a_j x_j$ will assume one of $2^n 10^{nd}$ values in $[-n10^d, n10^d]$, depending on a. The average distance between these values is $2n2^{-n}10^{d-nd}$. Thus, an interval of size 10^{-p} around zero is likely to contain a spurious "relation" unless p is significantly larger than $nd - d$.

INTEGER RELATION ALGORITHMS: HOW THEY WORK

PSLQ is a combinatorial optimization algorithm designed for (pure) mathematics



The method is “self-diagnosing”---- the error drops precipitously when an identity is found. And basis coefficients are “small”.

TOP TEN ALGORITHMS

▶ Integer Relation Detection was recently ranked among "the 10 algorithms with the greatest influence on the development and practice of science and engineering in the 20th century." J. Dongarra, F. Sullivan, *Computing in Science & Engineering* **2** (2000), 22–23.

Also: Monte Carlo, Simplex, Krylov Subspace, QR Decomposition, Quicksort, ..., FFT, Fast Multipole Method.

- integer relation detection (PSLQ, 1997) was the most recent of the top ten

HELAMAN FERGUSON
SCULPTOR and MATHEMATICIAN

PROFILE: HELAMAN FERGUSON

Carving His Own Unique Niche, In Symbols and Stone

By refusing to choose between mathematics and art, a self-described "misfit" has found the place where parallel careers meet

BALTIMORE, MARYLAND—Helaman Ferguson's sculpture studio is set back from the road, hidden behind a construction site. Inside, pieces of art line shelves and cover tabletops. Ferguson, clad in a yellow plastic apron and a black T-shirt, serenely makes his way through the room. The 66-year-old is tall and white-haired, his bare arms revealing a strength requisite for his avocation.

The most striking work in the studio is a more than 2-meter-tall, 5-ton chunk of granite. When it is finished, it will stand in the entry to the science building at Macalester College in St. Paul, Minnesota. Right now, it is a mass of curving surfaces sloping in different directions, its surface still jagged with the rough grains left by the diamond-toothed chainsaw Ferguson uses to carve through the stone.

"I'm in my negative-Gaussian-curvature phase," Ferguson says. "Say we're going to shake hands, but we don't quite touch. OK, see the space between the two hands?" That saddle-shaped void, he explains, is a perfect example of negative Gaussian curvature. Our bodies contain many others, he adds: the line between the first finger's knuckle and the wrist, for instance, and where the neck meets the shoulders.

The topological jargon is no surprise: Ferguson spent 17 years as a mathematics professor at Brigham Young University (BYU) in Provo, Utah. What is unusual is how successfully he has pursued a dual career as mathematician and artist and the ease with which he blurs the categories. Math inspires and figures in almost all of Ferguson's artistic works. Through them, he has helped some mathematicians appreciate the artist's craft and aesthetic. And he's persuaded perhaps even more artists that math may not be as frighteningly elusive as they believe, or even if it is out of their reach, it's as beautiful as any work of art they might imagine. "The way he has brought together the worlds of science and the arts—this is an admirable thing," says Harvey Bricker, Ferguson's former college roommate.

Function-al form. The Fibonacci Fountain at the Maryland Science and Technology Center was inspired by the "golden ratio."

Twin callings

Ferguson himself finds it hard to say which calling came first. As a teenager in upstate New York, he learned stone carving as an informal apprentice to his adopted father, a stonemason. Artistically, however, he was more drawn to painting. After finishing high school in 1958, he wanted to study art as well as math. He chose Hamilton College, a liberal arts school in upstate New York near where he had spent most of his childhood, where he could do both.

After getting his math degree, he enrolled in a doctoral program in math at the University of Wisconsin, Madison. He paid for some of his living expenses by selling paintings. He also met and began dating an undergraduate art student, Claire. The couple married in 1963 and had their first child (of an eventual seven) in 1964. Ferguson dropped out of school for a couple of years to work as a computer programmer, then resumed his math studies. He obtained his master's degree in mathematics at BYU and a doctorate in group representations—a broad area of math that involves algebra, geometry, topology, and analysis—at the University of Washington, Seattle. In 1971, he accepted an appointment as assistant professor at BYU.

As a mathematician, Ferguson is perhaps best known for the algorithm he developed with BYU colleague Rodney Forcade. The algorithm, called PSLQ, finds mathematical relations among seemingly unrelated real numbers. Among many other applications, PSLQ provided an efficient way of computing isolated digits within pi and blazed a path for modeling hard-to-calculate particle interactions in quantum physics. In 2000, the journal *Computing in Science and Engineering* named it one of the top 10 algorithms of the 20th century.

Meanwhile, Ferguson's artistic career also developed apace. When he married Claire, a painter, the two struck a deal: "I get the floors, she gets the walls," he says. He began focusing more on sculpture. The art department at BYU allotted him some studio space, and he turned out a regular stream of work. He's done commissions for the Maryland Science and Technology Center, the University of California, Berkeley, the University of St. Thomas in St. Paul, and many other institutions. He has also designed small sculptures for awards presented by the Clay Mathematics Institute in Cambridge, Massachusetts, the Canadian Mathematical Society in Ontario, and the Association for Computing Machinery in New York City.

CREDITS: J. MOGLIA/SCIENCE

He has worked to keep a foot in each of the "two cultures." While at BYU, he taught a course each year for honors students called Qualitative Mathematics and Its Aesthetics. Both art students and math students enrolled: the artists looking for a palatable way to take in a math requirement, and the math students lured by the promise of higher level mathematics. Ferguson delivered on both ends. He taught concepts mathematicians don't normally encounter until graduate school, such as braid theory. Artists could relate to braids as physical objects, rope or hair that can be woven into a specific form. But students were also asked to write down an algebra to go along with how the braid was formed—a noncommutative algebra.

"Some of these folks were in there because they were either afraid of or hated math," says Ferguson. At the end of the semester, however, "quite a few art students wanted a follow-on semester—more math, more art."

Bridging

Ferguson, who left BYU in 1988, now devotes most of his time to his art. For his large-scale or complicated pieces, he uses computer programs such as Mathematica to form and refine the shape he wants the finished piece to take. "With sculpture, you want a piece to be a unit so it has direct impact as a form," he says. "Sculptures are complicated enough already." With computer programs, he says, before even putting hand to stone "you can walk around [the piece] and see a different view; you can touch it and reshape it to make it simpler and more direct."

Once the design is in place, Ferguson turns to the task of carving the stone. He works alone, without assistants, using both chisels and assorted power tools. Finally comes a lengthy smoothing process, going from 20-grit sandpaper to as fine as 8500-grit. Ferguson has to work "wet" much of the time, using water to wash down the fine particles of stone that could otherwise become deposited in his lungs. For some of the work, he dons gloves made of woven stainless steel and a positive-pressure facemask. A large sculpture can take several months to complete, working flat-out.

Granite is Ferguson's favorite medium. "Mathematics is kind of timeless," he says, "so incorporating mathematical themes and ideas into geologically old stone—that's something that has great aesthetic appeal to me." He also likes the idea that his sculptures will be around for millions or even billions of years.

The finished sculptures vary widely in appearance. Some are delicate, with looped projections or intricate imprints, and are small enough to hold in one's hand. Others are massive, meant to be touched, even climbed on (as many children have discovered). As a rule, they also contain much more detail than meets the eye. "My work generally involves a circle of ideas," says Ferguson. People he interacts with, new information he obtains, mathematics he has had on his mind—all of these become "part of the design consideration."

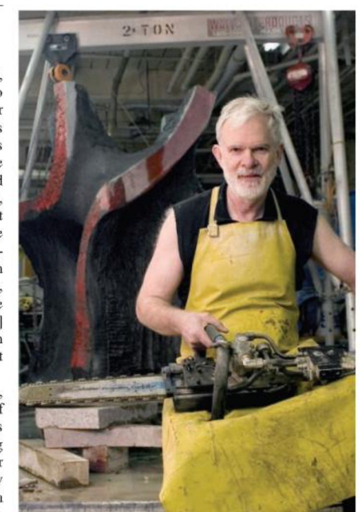
Twisted. Braids and knots turn up in many of Ferguson's works, including these small metal sculptures

As an example, he cites an architectural-scale sculpture recently installed outside his alma mater Hamilton College's new science building. The work, made of 10-centimeter-thick granite, centers on a pair of massive disks representing the planets Mars and Venus. "Venus" is exactly 161 centimeters in diameter—the height of the average female Hamilton student, taken from the records of one of the college's psychology professors. "Mars" is 174 centimeters in diameter—the average male student's height. The disks are inlaid with tiles in a pattern defined by the Poincaré and Beltrami-Klein models of plane hyperbolic geometry.

Ferguson's admirers say his artwork goes far beyond academic exercises. David Broadhurst, a physicist at the Open University in Milton Keynes, U.K., learned about Ferguson's sculpture after using the PSLQ algorithm in his research in quantum mechanics. He compares Ferguson's artistic renderings of math to Fournier playing the Bach cello suites, "giving expression to abstract forms, whose beauty is preexistent to the interpretation, yet recreated in a widely accessible medium."

For his part, Ferguson says his lifelong project to embody mathematics in mass and form is very much in the spirit of the times—and he credits technology with making it all possible. "We're living in the golden age of art, we really are. But it's also the golden age of science," he says. "Today, young people have seen more art and science in, say, their first 25 years of life than anyone in the years before that." With the collaborations between computer scientists and artists, and tools for art being used as tools for scientific exploration and invention, Ferguson suggests we may be in the midst of a second Renaissance. "It's a great time to be alive," says Ferguson, "because there are more places for misfits like myself to survive."

–KATHERINE UNGER



Tough medium. A diamond-toothed chainsaw helps Ferguson carve through granite rocks that are up to a billion years old.

CREDITS: J. MOGLIA/SCIENCE

Katherine Unger is a writer in Washington, D.C.

Peter Borwein
in front of
Helaman Ferguson's
work

CMS Meeting
December 2003
SFU Harbour Centre

Ferguson uses high tech tools and micro engineering at NIST to build monumental math sculptures

MADELUNG's CONSTANT

David Borwein CMS Career Award

$$= \sum_{n,m,p}{}' \frac{(-1)^{n+m+p}}{\sqrt{n^2+m^2+p^2}}$$

This polished solid silicon bronze sculpture is inspired by the work of David Borwein, his sons and colleagues, on the conditional series above for salt, Madelung's constant. This series can be summed to uncountably many constants; one is Madelung's constant for **electro-chemical stability of sodium chloride**.

This constant is a period of an elliptic curve, a real surface in four dimensions. There are uncountably many ways to imagine that surface in three dimensions; one has negative gaussian curvature and is the tangible form of this sculpture. (As described by the artist.)


THEORIES
PROOFS
S. Harris

INTEGER RELATION ALGORITHMS: WHAT THEY DO: ELEMENTARY EXAMPLES

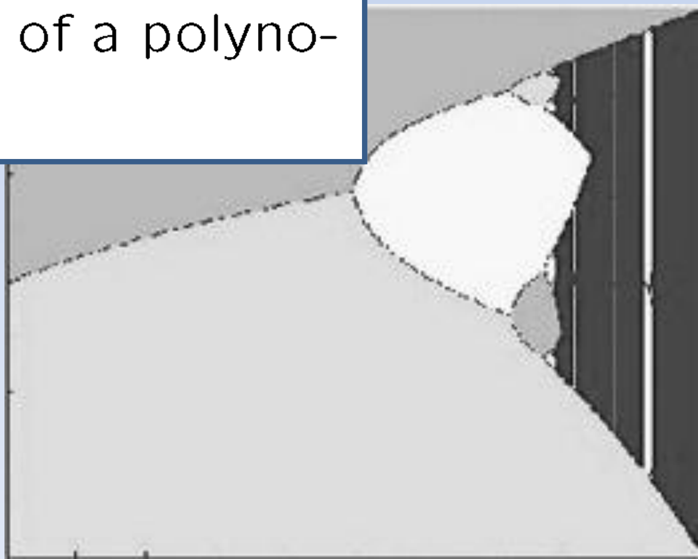
ALGEBRAIC NUMBERS

Compute α to sufficiently high precision ($O(n^2)$) and apply LLL to the vector

$$(1, \alpha, \alpha^2, \cdots, \alpha^{n-1}).$$

- Solution integers a_i are coefficients of a polynomial likely satisfied by α.

An application was to determine explicitly the 4th and 5th bifurcation points of the logistics curve have degrees 256.



FINALIZING FORMULAE

- If we suspect an identity PSLQ is powerful.

  - (*Machin's Formula*) We try PSLQ on

$$[\arctan(1), \arctan(\frac{1}{5}), \arctan(\frac{1}{239})]$$

and recover [1, -4, 1]. That is,

$$\frac{\pi}{4} = 4\arctan(\frac{1}{5}) - \arctan(\frac{1}{239}).$$

[Used on all serious computations of π from 1706 (100 digits) to 1973 (1 million).]

If we try with arctan(1/238) we obtain huge integers

  - (*Dase's 'mental' Formula*) We try PSLQ on

$$[\arctan(1), \arctan(\frac{1}{2}), \arctan(\frac{1}{5}), \arctan(\frac{1}{8})]$$

and recover [-1, 1, 1, 1]. That is,

$$\frac{\pi}{4} = \arctan(\frac{1}{2}) + \arctan(\frac{1}{5}) + \arctan(\frac{1}{8}).$$

[Used by Dase for 200 digits in 1844.]

In his head

INTEGER RELATIONS in MAPLE

```
> with(IntegerRelations);Digits:=25;
```

$$[LLL, LinearDependency, PSLQ]$$

$$Digits := 25 \qquad (2)$$

```
> PSLQ([Pi,arctan(1/2),arctan(1/5),arctan(1/8)]);
```

$$[1, -4, -4, -4] \qquad (3)$$

```
> PSLQ([Pi,arctan(1/2),arctan(1/5),arctan(1/9)]);
```

$$[10129, 2473744, -4734091, -2207521] \qquad (4)$$

```
> pslq(Pi,[arctan(1/2),arctan(1/5),arctan(1/8)]);
```

[1, 4, 4, 4], "Error is", $-2.\ 10^{-35}$, "checking to", 35, *places*

$$\pi = 4\arctan\left(\frac{1}{2}\right) + 4\arctan\left(\frac{1}{5}\right) + 4\arctan\left(\frac{1}{8}\right) \qquad (5)$$

```
> a:=evalf(sqrt(3)+sqrt(5));identify(a);
```

$$a := 3.968118785068666989936620$$

$$\sqrt{3} + \sqrt{5} \qquad (6)$$

```
> ?identify
```

- *Maple* also implements the Wilf-Zeilberger algorithm
- *Mathematica* can only recognize algebraic numbers

INTEGER RELATION ALGORITHMS: WHAT THEY DO: ADVANCED EXAMPLES

- THE BBP FORMULA FOR PI
- PHYSICAL INTEGRALS
  - ISING AND QUANTUM FIELD THEORY
- APERY SUMS
  - AND GENERATING FUNCTIONS
- RAMANUJAN SERIES FOR $1/\pi^N$

The BBP FORMULA for Pi

In 1996 Bailey, P. Borwein and Plouffe, using PSLQ for months, discovered this formula for π:

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k}\left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right)$$

Indeed, this formula permits one to directly calculate binary or hexadecimal (base-16) digits of π beginning at an arbitrary starting position n, without needing to calculate any of the first n-1 digits.

A finalist for the **Edge of Computation Prize**, it has been used in compilers, in a record web computation, and in a trillion-digit computation of Pi.

PHYSICAL INTEGRALS (2006-2008)

The following integrals arise independently in mathematical physics in Quantum Field Theory and in Ising Theory:

$$C_n = \frac{4}{n!}\int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n (u_j + 1/u_j)\right)^2} \frac{du_1}{u_1} \cdots \frac{du_n}{u_n}$$

We first showed that this can be transformed to a 1-D integral:

$$C_n = \frac{2^n}{n!}\int_0^\infty t K_0^n(t)\, dt$$

where K_0 is a **modified Bessel function**. We then (with care) computed 400-digit numerical values (over-kill but who knew), from which we found with **PSLQ** these (now proven) **arithmetic** results:

$$C_3 = \mathrm{L}_{-3}(2) := \sum_{n\geq 0} \left\{ \frac{1}{(3n+1)^2} - \frac{1}{(3n+2)^2} \right\}$$

$$C_4 = \frac{7}{12}\zeta(3)$$

$$\lim_{n\to\infty} C_n = 2e^{-2\gamma}$$

IDENTIFYING THE LIMIT WITH THE **ISC** (2.0)

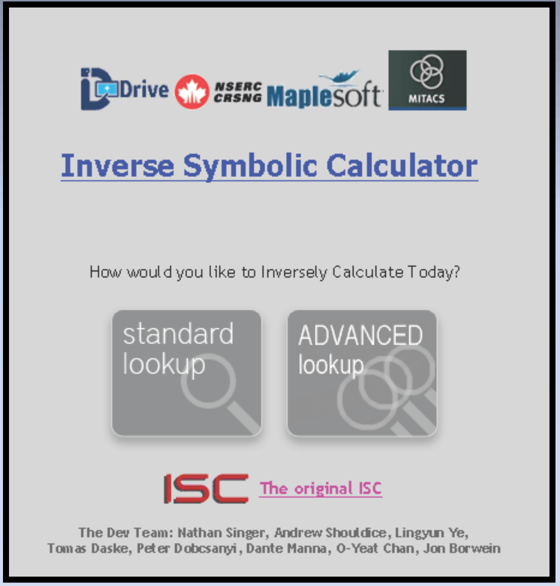
We discovered the limit result as follows: We first calculated:

$$C_{1024} \;=\; 0.63047350337438679612204019271087890435458\ldots$$

We then used the Inverse Symbolic Calculator, the online numerical constant recognition facility available at:
http://ddrive.cs.dal.ca/~isc/portal

Output: **Mixed constants, 2 with elementary transforms.**

.6304735033743867 = sr(2)^2/exp(gamma)^2



In other words,

$$C_{1024} \;\approx\; 2e^{-2\gamma}$$

References. Bailey, Borwein and Crandall, “Integrals of the Ising Class," *J. Phys. A.*, **39** (2006)
Bailey, Borwein, Broadhurst and Glasser, “Elliptic integral representation of Bessel moments," *J. Phys. A*, **41** (2008) [IoP Select]

APERY-LIKE SUMMATIONS

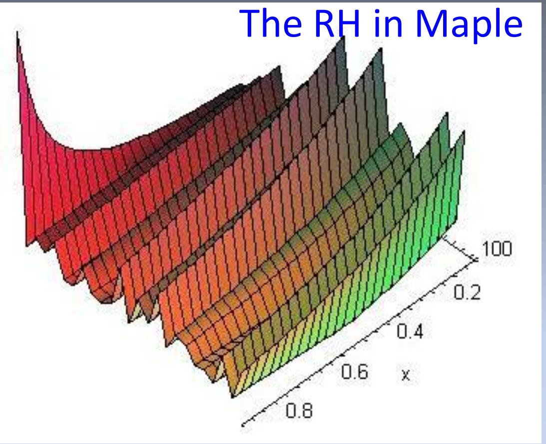
The following formulas for $\zeta(s)$ have been known for many decades:

$$\zeta(2) = 3 \sum_{k=1}^{\infty} \frac{1}{k^2 \binom{2k}{k}},$$

$$\zeta(3) = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3 \binom{2k}{k}},$$

$$\zeta(4) = \frac{36}{17} \sum_{k=1}^{\infty} \frac{1}{k^4 \binom{2k}{k}}.$$

for $\mathrm{Re}(s) > 1$
$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$



These results have led many to speculate that

$$Q_5 := \zeta(5) / \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^5 \binom{2k}{k}}$$

might be some nice rational or algebraic value.

Sadly (?), PSLQ calculations have established that if Q_5 satisfies a polynomial with **degree** at most **25**, then at least **one coefficient** has **380** digits. But positive results exist.

APERY OGF'S

1. via PSLQ to 5,000 digits (120 terms)

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$$\zeta(2) = \frac{\pi^2}{6}, \zeta(4) = \frac{\pi^4}{90}, \zeta(6) = \frac{\pi^6}{945}, \ldots$$

Euler (1707-73)

$$\mathcal{Z}(x) = 3\sum_{k=1}^{\infty} \frac{1}{\binom{2k}{k}(k^2 - x^2)} \prod_{n=1}^{k-1} \frac{4x^2 - n^2}{x^2 - n^2}$$

$$= \sum_{k=0}^{\infty} \zeta(2k+2) x^{2k} = \sum_{n=1}^{\infty} \frac{1}{n^2 - x^2}$$

$$= \frac{1 - \pi x \cot(\pi x)}{2x^2}$$

1

2005 Bailey, Bradley & JMB *discovered and proved* - in 3Ms - three *equivalent* binomial identities

2. reduced as hoped

2

$$3n^2 \sum_{k=n+1}^{2n} \frac{\prod_{m=n+1}^{k-1} \frac{4n^2 - m^2}{n^2 - m^2}}{\binom{2k}{k}\left(k^2 - n^2\right)} = \frac{1}{\binom{2n}{n}} - \frac{1}{\binom{3n}{n}}$$

3

$${}_3F_2\left(\begin{matrix} 3n, n+1, -n \\ 2n+1, n+1/2 \end{matrix}; \frac{1}{4}\right) = \frac{\binom{2n}{n}}{\binom{3n}{n}}$$

3. was easily computer proven (Wilf-Zeilberger) **(now 2 human proofs)**

NEW RAMANUJAN-LIKE IDENTITIES

Guillera (around 2003) found Ramanujan-like identities, including:

$$\frac{128}{\pi^2} = \sum_{n=0}^{\infty} (-1)^n r(n)^5 (13 + 180n + 820n^2) \left(\frac{1}{32}\right)^{2n}$$

$$\frac{8}{\pi^2} = \sum_{n=0}^{\infty} (-1)^n r(n)^5 (1 + 8n + 20n^2) \left(\frac{1}{2}\right)^{2n}$$

$$\frac{32}{\pi^3} \stackrel{?}{=} \sum_{n=0}^{\infty} r(n)^7 (1 + 14n + 76n^2 + 168n^3) \left(\frac{1}{8}\right)^{2n}.$$

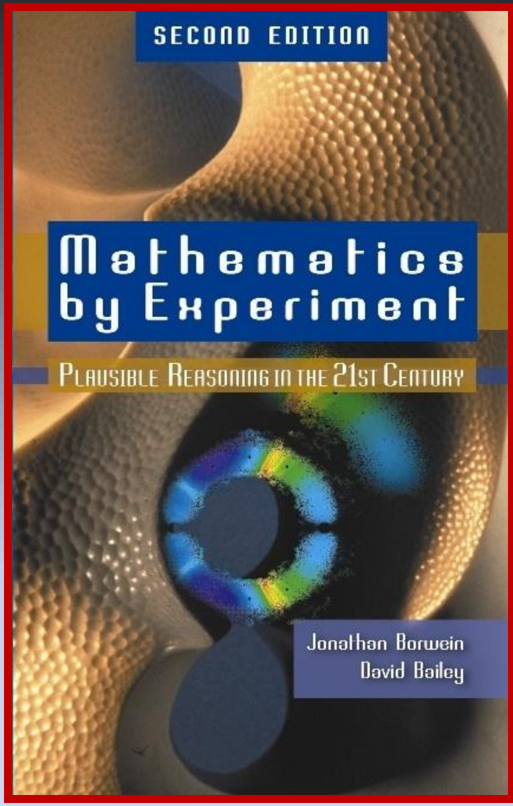
where

$$r(n) = \frac{(1/2)_n}{n!} = \frac{1/2 \cdot 3/2 \cdot \cdots \cdot (2n-1)/2}{n!} = \frac{\Gamma(n+1/2)}{\sqrt{\pi}\,\Gamma(n+1)}$$

Guillera proved the first two using the Wilf-Zeilberger algorithm. He ascribed the third to Gourevich, who found it using integer relation methods. **It is true but has no hint of a proof…**

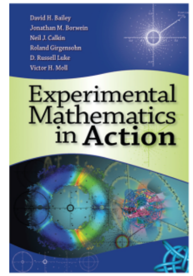
As far as we can tell there are no higher-order analogues!

REFERENCES



Experiencing Experimental Mathematics

Experimental Mathematics in Action
David H. Bailey, Jonathan M. Borwein, Neil J. Calkin, Roland Girgensohn, D. Russell Luke, Victor H. Moll

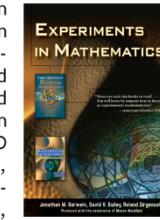


"David H. Bailey et al. have done a fantastic job to provide very comprehensive and fruitful examples and demonstrations on how experimental mathematics acts in a very broad area of both pure and applied mathematical research, in both academic and industry. Anyone who is interested in experimental mathematics should, without any doubt, read this book!"
—*Gazette* of the Australian Mathematical Society

978-1-56881-271-7; Hardcover; $49.00

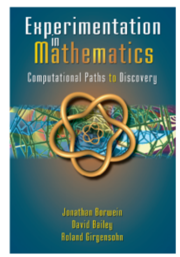
Experiments in Mathematics (CD)
Jonathan M. Borwein, David H. Bailey, Roland Girgensohn

In the short time since the first edition of *Mathematics by Experiment: Plausible Reasoning in the 21st Century* and *Experimentation in Mathematics: Computational Paths to Discovery,* there has been a noticeable upsurge in interest in using computers to do real mathematics. The authors have updated and enhanced the book files and are now making them available in PDF format on a CD-ROM. This CD provides several "smart" features, including hyperlinks for all numbered equations, all Internet URLs, bibliographic references, and an augmented search facility assists one with locating a particular mathematical formula or expression.



978-1-56881-283-0; CD; $49.00

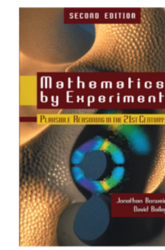
Experimentation in Mathematics
Computational Paths to Discovery
Jonathan M. Borwein, David H. Bailey, Roland Girgensohn



978-1-56881-136-9; Hardcover; $59.00

Mathematics by Experiment Second Edition
Plausible Reasoning in the 21st Century
Jonathan M. Borwein, David H. Bailey

"These are such fun books to read! Actually, calling them books does not do them justice. They have the liveliness and feel of great Web sites, with their bite-size fascinating factoids and their many human- and math-interest stories and other gems. But do not be fooled by the lighthearted, immensely entertaining style. You are going to learn more math (experimental or otherwise) than you ever did from any two single volumes. Not only that, you will learn by osmosis how to become an experimental mathematician."
—*American Scientist Online*



978-1-56881-442-1; Hardcover; $69.00

D.H. Bailey and JMB, "PSLQ: an Algorithm to Discover Integer Relations," *Computeralgebra Rundbrief*, October 2009.

JMB and P. Lisoněk, "Applications of integer relation algorithms," *Discrete Mathematics*, **217** (2000), 65–82.

- www.experimentalmath.info is our website