Effective Laguerre asymptotics

D. Borwein,[*] Jonathan M. Borwein[†] and Richard E. Crandall[‡]

January 18, 2007

Abstract

It is known that the generalized Laguerre polynomials can enjoy sub-exponential growth for large primary index. Specifically, for certain fixed parameter pairs (a, z) one has the large-n asymptotic

$$L_n^{(-a)}(-z) \sim C(a, z) n^{-a/2-1/4} e^{2\sqrt{nz}}.$$

We introduce a computationally motivated contour integral that allows highly efficient numerical evaluation of L_n, yet also leads to general asymptotic series over the full domain for sub-exponential behavior. We eventually lay out a fast algorithm for generation of the rather formidable expansion coefficients. Along the way we address the difficult problem of establishing effective (i.e. rigorous and explicit) error bounds on the general expansion. To this end, we avoid classical stationary-phase and steepest-descent techniques in favor of an "exp-arc" method that amounts to a natural bridge between converging series and effective asymptotics. Finally, we exhibit an absolutely convergent exp-arc series for Bessel-function evaluation as an alternative to conventional ascending-asymptotic switching.

[*]Department of Mathematics University of Western Ontario, London, ONT, N6A 5B7, Canada, dborwein@uwo.ca. Supported in part by NSERC.

[†]Faculty of Computer Science, Dalhousie University, Halifax NS, B3H2W5, CANADA. Email: jborwein@cs.dal.ca. J.M. Borwein's work is funded by NSERC and the Canada Research Chair Program

[‡]Center for Advanced Computation, Reed College, Portland, OR 97202 USA, Email: crandall@reed.edu

1 The challenge of "effectiveness"

The object of our interest will be the Laguerre polynomial we parameterize thus:

$$L_n^{(-a)}(-z) := \sum_{k=0}^{n} \binom{n-a}{n-k} \frac{z^k}{k!}. \tag{1}$$

Our use of negated parameters $-a, -z$ is intentional, for convenience in our analysis and in connection with related research, as we later explain.

We shall work on the difficult problem of establishing asymptotics *with* effective error bounds for the two-parameter domain

$$\mathcal{D} := \{(a, z) \in \mathcal{C} \times \mathcal{C} : z \notin (-\infty, 0]\}.$$

That is, parameter a is any complex number, while z is any complex number not on the negative-closed cut $(-\infty, 0]$. This $\mathcal{D}$ will turn out to be the precise domain of sub-exponential growth of $L_n^{(-a)}(-z)$. Herein, we say a function $f(n)$ *enjoys sub-exponential growth* if $\log\log|f(n)| \sim \delta \log n$, for some $0 < \delta < 1$, so for example $f(n) = an^b \exp(cn^\delta)$ for real constants a, b, c with $c > 0$ has this growth property.[1]

The reason for the negative-cut exclusion on z is simple: For z negative real, the Laguerre polynomial exhibits oscillatory behavior in large n, and is not of sub-exponential growth. Note also, from the definition (1), that

$$L_n^{(-a)}(0) = \binom{n-a}{n}, \tag{2}$$

which covers the case $z = 0$; again, not sub-exponential growth.

Now, for all $(a, z) \in \mathcal{D}$ we shall have (here and beyond we define $m := n + 1$, which tends to simplify notation throughout):

$$L_n^{(-a)}(-z) \sim S_n(a, z) \left(1 + O(m^{-1/2})\right), \tag{3}$$

where the sub-exponential term S is

$$\boxed{S_n(a, z) := \frac{e^{-z/2}}{2\sqrt{\pi}} \frac{e^{2\sqrt{mz}}}{z^{1/4-a/2}\, m^{1/4+a/2}}.} \tag{4}$$

In such expressions, $\Re(\sqrt{mz})$ is taken to be $\sqrt{m|z|}\cos(\theta/2)$ where $\theta := \arg(z) \in (-\pi, \pi]$ (we hereby adopt the convention $\arg(-1) := \pi$), and so for $(a, z) \in \mathcal{D}$ the expression (4) involves genuinely diverging growth in n, due to the sub-exponential $\exp(2\sqrt{mz})$ factor.

What we seek are effective bounds, for example to replace a logical error-bounding statement for an expression E in the following way:

$$\left\{E = O\left(\frac{1}{\sqrt{m}}\right)\right\} \quad \text{is replaced by} \quad \left\{E < \frac{C}{\sqrt{m}} \text{ for } m > m'\right\},$$

[1] We use "sub-exponential" in a sense distinct from that in number theory, in which field one typically refers to growth of order $\exp(d\log^\delta n)), \delta < 1$.

with both the constant C and the threshold m' being explicit. To do this, we eventually introduce a "Θ"-notation to simplify the nomenclature by replacing big-O when effectiveness is achieved.

Thus, our chief goal in the present paper is to establish effective bounds—starting with a suitable explicit big-O constant in (3)—for the entire domain $\mathcal{D}$ of (a, z) pairs, even to reach further, into arbitrary asymptotic orders. Moreover, we want to effect all of this is a systematic way, so that other researchers will have a symbolic scheme for generating effective terms. To these ends, we turn next to motivational matters, and the history of Laguerre asymptotics.

1.1 Research motives

A primary research motive for providing effective asymptotics lies in a beautiful Laguerre series for the incomplete gamma function (see [1]), namely [14]

$$\begin{aligned}\Gamma(a,z) &= z^a e^{-z} \cfrac{1}{z+\cfrac{1-a}{1+\cfrac{1}{z+\cfrac{2-a}{1+\cdots}}}} \\ &= \sum_{n=0}^{\infty} \frac{(1-a)_n}{(n+1)!} \frac{1}{L_n^{(-a)}(-z)\, L_{n+1}^{(-a)}(-z)}, \end{aligned} \tag{5}$$

where $(c)_n := c(c+1)\cdots(c+n-1)$ is the *Pochhammer symbol*. This series is valid whenever none of the Laguerre denominators has a zero. Thus an interesting sidelight is the research problem of establishing zero-free regions for Laguerre polynomials (see our Open Problems section).

Indeed, one may see in this formula one good reason to adopt, as we have, Laguerre superscript $(-a)$ and argument $(-z)$. The asymptotic in (3) is indeed an ingredient to a separate theory of R. Crandall [9] on the large-height behavior of the incomplete gamma function, which theory in turn applies to high-precision Riemann-zeta computations. This Riemann-zeta/incomplete-gamma connection relies in turn on both imaginary heights $|\Im(a)|, |\Im(z)|$ being very large. For example, to deal analytically-computationally with primes in the region of 10^{20}, one might need a rigorous evaluation of something quite stultifying, say

$$\Gamma\left(1 + 10^{10} i,\ 1 + 2 \cdot 10^{10} i\right).$$

In a word: To responsibly calculate results on prime numbers, using the analysis chain of *Laguerre-polynomial* $\to$ *incomplete-gamma* $\to$ *Riemann-zeta*, one must certainly know rigorous errors. It is this kind of generality of parameters that is incomplete in the literature. In this sense, a term $O(1/\sqrt{n+1})$ as in (3) will not suffice in matters of numerical rigor unless an implied big-O constant is known.

The Laguerre series for (5) suggests that the convergents of the continued fraction for the incomplete gamma function are closely related to Laguerre evaluations, and this is indeed so. In fact, another research motive we realize is that the continued-fraction theory for the $\Gamma(a,z)$ fractions is incomplete in the sense that the theory of S-fractions, which have been well-studied, does involve parameter restrictions. In fact, the present research began when we encountered great difficulty in estimating the convergence rate of the $\Gamma(a,z)$ fraction for arbitrary a, z; this is what led to our focus on the Laguerre asymptotics. So one new research avenue is opened by this Laguerre study; namely, new results on certain continued fractions lying outside the reach of S-fractions will surely accrue. Incidentally, we are aware that sub-exponential convergence results for the general incomplete-gamma-fraction *might* be attainable via the very complicated, seminal work of Jacobsen and Thron [15] on oval convergence regions, but at least our effective-Laguerre approach eventually yields the desired growth properties.

We should state an important caveat at this juncture: We are not intending to develop Laguerre asymptotics in order to *calculate* $\Gamma(a,z)$ via series (5); rather, as of this writing the present effective-asymptotic analysis is the only way we know to show subexponential convergence of the Γ continued fraction for the whole domain $\mathcal{D}$. If we may speak in the metamathematical vein, the intricacies of Laguerre asymptotics are not unexpected, given the corresponding intracies of complex continued fractions.

There are yet other research areas that can benefit from precise Laguerre asymptotics. The celebrated hydrogen wave-functions in quantum theory involve Laguerre polynomials L_n^m with n, m both positive integers [39, Ch. 4]. Beyond this, it turns out that Laguerre polynomials appear in some exactly solvable 3-body problems in quantum chemistry. In fact for the "helium-like" Hamiltonian operator for two mass-m electrons at r_1, r_2 and one mass-M nucleus at r_3:

$$H = -\frac{1}{2m}\left(\nabla_1^2 + \nabla_2^2\right) - \frac{1}{2M}\nabla_3^2 + \frac{1}{2}m\omega^2(r_{13}^2 + r_{23}^2) + \frac{\lambda}{r_{12}^2},$$

where $r_{jk} := r_j - r_k$ and the 2-electron repulsion is modeled here as an inverse-cube force with coupling λ, the exact wave-functions—indexed by integers n, l—are proportional to:

$$L_n^{(a)}\left(2m\omega r_{12}^2\right),$$

with

$$a := \sqrt{l(l+1) + 1/4 + \lambda m}.$$

Note that for vanishing $\lambda = 0$ the Laguerre super-index a is a half-odd integer $l + 1/2$, but that introducing a nonzero λ breaks this symmetry so that non half-integer a become involved. In either the standard hydrogen case or the 3-body models, the asymptotic form is that of the Fejér oscillatory variety (6); yet, analyticity studies in quantum theory can involve either space, time, or both continuations into the complex plane, in which cases rigorous asymptotics may be required. Thus there is a place for Laguerre asymptotics in quantum theory. As one last example: The so-called WKB theory for approximating the remote phase of a hydrogenic wave-function is problematic, yet the oscillatory Fejér–Perron theory we next outline yields precise phases easily. One can imagine, then, a modern computationalist's need for effective bounds in both the oscillatory and sub-exponential Laguerre cases.

Yet another application for growth theorems involves the Hermite polynomials, in turn expressible as

$$H_{2n}(x) = (-4)^n n!\, L_n^{(-1/2)}\left(x^2\right),$$

$$H_{2n+1}(x) = 2(-4)^n n!\, x\, L_n^{(+1/2)}\left(x^2\right),$$

and appearing at many important theoretical junctures in quantum theory. As with the aforementioned atomic wavefunctions, it is important to know the asymptotic behavior of the Hermite class. For one thing, space-time and space-energy propagators for the Schrödinger theory sometimes demand asymptotic analysis in regard to eigenvalue estimates. In any case, our present results do apply to the Hermite class, via the above even-odd identities (one simply asigns $x := \sqrt{-z}$ with our branch rule $\sqrt{-\rho} := i\sqrt{\rho}$ for positive real ρ). One interesting work—whose methods are distinct from our present ones—involves explicit bounds on Hermite oscillations [12]. And, for asymptotic analysis, there may be some hope in the interesting López–Temme expansion:

$$L_n^{(-a)}(-z) = (-x)^n \sum_{k=0}^{n} \frac{c_k H_{n-k}\left(\frac{a-z-1}{2x}\right)}{x^k (n-k)!},$$

where $x := \sqrt{-z-(1-a)/2}$ and the c_k are generated by a certain 4-th order recurrence relation [18]. Strikingly nonstandard as this representation may be, it is neverthless valid for *all* parameter pairs $(a, z) \in \mathcal{C} \times \mathcal{C}$.

1.2 Historical results

Laguerre asymptotics have long been established for certain restricted domains, and usually with noneffective asymptotics.[2] For example, in 1909 Fejér established that for z on the open cut $(-\infty, 0)$ and any real a, one has [30, Theorem 8.22.1]:

$$L_n^{(-a)}(-z) = \frac{e^{-z/2}}{\sqrt{\pi}(-z)^{1/4-a/2}\, m^{1/4+a/2}} \cos\left(2\sqrt{-mz} + a\pi/2 - \pi/4\right) + O\left(m^{-a/2-3/4}\right), \quad (6)$$

where we again use index $m := n+1$, which slightly alters the coefficients in such classical expansions, said coefficients being for powers $n^{-k/2}$ but we are now using powers $m^{-k/2}$. By 1921 Perron [28] had generalized the Fejér series to arbitrary orders, then for $z \notin (-\infty, 0]$ established a series consistent with (3) and (4), in essentially the form: [30, Theorem 8.22.3]

$$L_n^{(-a)}(-z) = S_n(a, z) \left(\sum_{k=0}^{N-1} \frac{C_k}{m^{k/2}} + O\left(m^{-N/2}\right) \right), \quad (7)$$

although this was for *real* a and so not for our general parameter domain $\mathcal{D}$. Note that $C_0 = 1$, consistent with our (3) and (4); however, one should take care that because we

[2]Some researchers use the term "realistic error bound" for big-O terms that have explicit structure. We prefer "effective bound," and when an expansion is bestowed with such a bound, we may say "effective expansion."

are using index $m := n + 1$, the coefficients C_k in the above formula differ slightly from the historical ones. Various of the Perron generalizations can be derived, for example, from the *Le Roy formula*, valid for $\Re(a) > 1$ and z negative real,

$$L_n^{(-a)}(-z) = (-z)^{a/2} e^{-z} \frac{1}{n!} \int_0^\infty t^{n-a/2} e^{-t} J_{-a}(2\sqrt{-zt})\, dt,$$

which formula is more than a century old (see [30, ch. 5.4, p. 99], where extension to more general parameters is discussed). That is, one may use the classical Bessel-expansion theory to obtain a Laguerre expansion. The complications attendant on this approach are a good reason to adopt, as we shall, a more universal contour integral representation.

It is not hard to see how the interplay between domain of z and the asymptotic character of (6) or (7) works. If we naively take $0 \neq z \notin (-\infty, 0)$ in the Fejér form (6), the cosine term has a sub-exponentially growing part $(1/2)\exp(2\sqrt{m|z|}\cos(\theta/2))$—so even the extra factor of $1/2$ contained in the S_n definition is explained in this way. Conversely, one can imagine moving from the sub-exponential form (7) to the oscillatory form (6) as z moves onto the cut $(-\infty, 0)$ from outside it, provided one remembers the mnemonic that there is in (3) or (7) a "buried" term like S_n but having $\exp(-2\sqrt{mz})$ in the numerator, such a term being sub-exponentially small. All of these heuristics and mnemonics can be remembered perhaps best by imagining a $\cosh(2\sqrt{mz})$ term in the definition of S_n, with the denominator's '2' removed. It will turn out that our rather involved effective-error analysis moves most cleanly, though, with $(a, z) \in \mathcal{D}$ and the $\exp(2\sqrt{mz})$ term intact.

Certain asymptotic properties may be gleaned from the formal generating function

$$u(t) := \sum_{n=0}^{\infty} L_n^{(-a)}(-z) t^n = (1-t)^{a-1} e^{\frac{zt}{1-t}}, \tag{8}$$

as can be derived from standard ordinary differential equation theory by applying the recurrence (10) in the next section to the generating function $\sum_n r_n t^n$. A more modern literature treatment that is again consistent with the heuristic (3)–(4), is given by Wipitski in [37], where (8) is invoked to yield a contour integral for $L_n^{(-a)}(-z)$. Then a stationary-phase approach yields precisely the correct asymptotic, at least for certain subregions of $\mathcal{D}$. It should be noted however that Wipitski's treatment is both elegant and non-rigorous, intended mostly as a computational guide. And once again, there is no estimate given on the $O(1/\sqrt{m})$ correction in (3). In fact, our present approach involves the *avoidance* of stationary-phase approach per se, in favor of an exponential-arcsin series development, as described in Section 5.

There is an interesting anecdote that reveals the difficulty inherent in Laguerre asymptotics. Namely, W. Van Assche in an interesting 1985 paper [34] used the expansion (7) for work on zero-distributions, only to find by 2001 that the C_1 term in that 1985 paper had been calculated incorrectly. The amended series is given in his correction note [35] as

$$L_n^{(-a)}(-z) \sim \frac{e^{-z/2}}{2\sqrt{\pi}} \frac{e^{2\sqrt{nz}}}{z^{1/4-a/2}\, n^{1/4+a/2}} \cdot \left(1 + \left(\frac{3 - 12a^2 + 24(1-a)z + 4z^2}{48\sqrt{z}}\right) \frac{1}{\sqrt{n}} + O\left(\frac{1}{n}\right)\right),$$

or in our own notation with $m := n + 1$,

$$\sim S_n(a,z)\left(1+\left(\frac{3-12a^2-24(1+a)z+4z^2}{48\sqrt{z}}\right)\frac{1}{\sqrt{m}}+O\left(\frac{1}{m}\right)\right). \tag{9}$$

Note the slight alteration used to obtain our $C_1/\sqrt{m}$ term. Van Assche credits T. Müller and F. Olver for aid in working out the correct $O(1/\sqrt{n})$ component.[3] This story suggests that even a low-order asymptotic development is nontrivial. Hence, we should anticipate the effective error-bounding project on which we embark to be at least as nontrivial. At any rate, along the way we shall streamline a symbolic-generation process for obtaining the C_k, and in this way shall give an algorithm for extending (9) to higher orders and provide effective bounds for the corresponding error terms E_k.

Absent the problem of incomplete domains (incomplete for our modern purposes), the classical authors certainly knew in principle how to establish effective error bounds. The excellent treatment of effectiveness for Laplace's method of steepest descent in [22] is a shining example. Even more illuminating is Olver's paper [21], which explains effective bounding and shows how unwieldy rigorous bounds can be. However, efficient algorithms for generating *explicit* effective big-O constants have only become practicable in recent times. We freely admit that here-and-now in the era of modern symbolic processing—and in spite of the complications into which we soon shall delve—the generation of explicit asymptotic terms and their associated errors is easier today than at any time in the history of the subject.

We note that many alternative Laguerre studies abound. One modern thrust—which we do not address in the present paper—involves asymptotic behavior when the subindex n is linked either to a or z, or both. For example, the paper [17] provides a large-n asymptotic expansion for $L_n^{(a)}(nx)$. Another treatment is [29], in which the author handles $L_n^{(nx)}(ny)$. In [8] the authors analyze cases of $L_n^{(a_n)}$ for which $a_n >> n$. One hopes that our present methods for effective bounding can be applied to such variants. Other important "linked" cases include the exponential monomial and the partial-exponential sum, respectively:

$$\boxed{L_n^{(-n)}(z) = \frac{(-1)^n}{n!}z^n; \qquad L_n^{(-n-1)}(z) = (-1)^n\sum_{k=0}^{n}\frac{z^k}{k!}.}$$

The paper [16] discusses such generalizations. An important point here is that any asymptotic theory with $n \approx a$ would have to take into account these useful special cases. On the subject of rigorous bounds, there is the interesting treatment [19] in which $L_\nu^{(\mu)}(x)$ is given upper (and lower!) bounds, for real x and $\Re(\mu) > -1$. One might call such results effective error bounds of zero-th order; in any case, they do not help the present treatment directly because of their parameter-domain restrictions.

[3]Accordingly, we hereby name the polynomial $C_1(a,z)$ the Perron–van Assche–Müller–Olver, or "PAMO" coefficient.

1.3 Further relations for Laguerre polynomials

Here we give further relations for Laguerre polynomials, including some relations we do not use in the present treatment, but which may nevertheless be helpful in future research on the problem of effective bounds.

We record what might be called initial values: $L_0^{(-a)}(-z) = 1$, and $L_{-1}^{(-a)}(-z) := 0$. We then note that the iteration ($n \geq 1$)

$$\boxed{r_n = \left(2 - \frac{A}{n}\right) r_{n-1} - \left(1 - \frac{B}{n}\right) r_{n-2},} \tag{10}$$

with initial values $r_{-1} = 0,\ r_0 := 1$, thus has an exact solution in the form

$$\boxed{r_n = L_n^{(-a)}(-z),} \tag{11}$$

where

$$z := B - A, \qquad a := B - 1,$$

is the parameter transformation that moves us between equivalent pairs (a, z) and (A, B). Thus, the complex domain in question for (A, B) pairs can be inferred from our caveat $(a, z) \in \mathcal{D}$.

The reason we have emphasized here the r_n-iteration (10) itself is that direct analysis of second-order recurrences have, on other problems, yielded strong results [5, 6, 7]. A "discrete" approach that attempts alternative asymptotic expressions for the r_n is therefore promising.

Perhaps just as promising for growth analysis is the Laguerre differential equation, which in our present parameterization reads

$$-z\frac{\partial^2 L}{\partial z^2} + (a - 1 - z)\frac{\partial L}{\partial z} + nL = 0, \tag{12}$$

where $L := L_n^{(-a)}(-z)$. In fact, various sharp results on Laguerre asymptotics have emerged from differential theory—there is the classical work of Erdélyi and Olver, plus modern work on combinations of differential, discrete, and saddle-point theory [11][13].

Next, for some hypergeometric connections, there is a hypergeometric form

$$\boxed{L_n^{(-a)}(-z) = \binom{n-a}{n} {}_1\mathrm{F}_1\left(-n, 1-a; -z\right),} \tag{13}$$

where ${}_1\mathrm{F}_1$ is also known as the *Kummer (confluent) hypergeometric function.* In references such as [1, §13.5.14] the Kummer function ${}_1\mathrm{F}_1$ asymptotics are consistent with aforementioned Laguerre forms, but again such formulae are established for restricted subregions of $\mathcal{D}$ and without effective bounds.

When n is small (in some sense not made rigorous here), and say $|z| \ll |a|$, an approximation follows from the representation (8), as

$$L_n^{(-a)}(-z) = \binom{n-a}{n} \sum_{j=0}^{n} \frac{(-n)_j}{(1-a)_j} \frac{(-z)^j}{j!}.$$

Because

$$\frac{(-z)^n}{(1-a)_n} = (-z)^n \frac{\Gamma(1-a)}{\Gamma(1-a+n)} \sim \frac{(-z)^n}{(1-a)^n}\left[1 + \frac{c_1(n)}{1-a} + \frac{c_2(n)}{(1-a)^2} + \ldots\right],$$

where the $c_j(n)$ are polynomials in n with $c_1(n) = -\frac{1}{2}n(n-1)$, this gives a first order approximation

$$\boxed{L_n^{(-a)}(-z) \sim \binom{n-a}{n}\left(1 + \frac{z}{1-a}\right)^n.}$$

More terms in such an expansion can be obtained via simple operations.[4]

We have not been rigorous about these small-n approximations, though we believe further research should certainly yield effective bounds. It is a fascinating phenomenon that the character of the above small-n expansion changes dramatically for large n—the behavior switches from a kind of modified power law to sub-exponential growth.

1.4 An outline of what follows

In Section 2 we obtain a tripartite contour representation for the generalized Laguerre polynomials and explore to which parts contribute predominantly. In Section 3 we provide effect bounds for each of three contour integrals we call c_1, d_1, e_1. We summarize the status of contour integration in Section 3.5. In Section 4 we attack a dominant term c_0 (of c_1), and reduce the effective-bounds problem to the study of a Bessel-class integral

$$\boxed{I(p,q) := \int_{-\pi/2}^{\pi/2} e^{-iq\omega} e^{p\cos\omega}\, d\omega.}$$

In Section 5 we conquer the asymptotics of I by exploiting the power series in x of $\exp(\tau \arcsin(x))$, calling this an "exp-arc" series. Section 6 then harvests our crop: Theorem 8 gives an effective expansion for the generalized Laguerre polynomials in the sub-exponential region. Section 6.2 then provides an algorithm for the purpose while Section 6.3 shows the method in numerical mode. In Section 7 we briefly consider the oscillatory domain before applying our results to the Bessel functions I_n and J_n of integral order. Finally, in Section 8 we visit or revisit some of the open problems thrown up by our study.

2 Contour representation

In this section we develop a highly efficient—both numerically and analytically—contour-integral representation for $L_n^{(-a)}(-z)$. First we indicate how experimental mathematics was employed to work out a good contour itself, then we proceed to provide effective bounds on segments of the contour, whence extracting a primary term that is sub-exponentially larger than the other terms.

[4]The authors are indebted to N. Temme for these small-n approximations.

2.1 Development of a "keyhole" contour

A well known contour integral has contour, Γ, encircling $s = 1 + 0i$ but avoiding the branch cut $(-\infty, 0]$:

$$L_n^{(-a)}(-z) = \frac{e^{-z}}{2\pi i} \int_\Gamma s^{-1-a} \left(1 - \frac{1}{s}\right)^{-n-1} e^{zs}\, ds. \tag{14}$$

This representation holds for all pairs $(a, z) \in \mathcal{C} \times \mathcal{C}$, n a non-negative integer, and may be proven to agree with the polynomial definition (1) via simple residue arithmetic, or via a Fourier transform of the generating function (8).[5]

However—and this is important—we found via *experimental mathematical* techniques that a certain kind of contour allows very accurate, efficient, and well behaved *numerical* Laguerre evaluations. Take $z \neq 0$, $m := n+1$ and assume $r := \sqrt{m/z}$ has $|r| > 1/2$. Then, use a circular contour centered at $s = 1/2$ with radius $|r|$. This contour will encompass $s = 1$, so the remaining requirement is to avoid the cut $s \in (-\infty, 0]$. But this can be done by cutting out a "wedge" from the negative-real arc of the circle, with the wedge's apex at $1/2 = 0i$. We actually tried such schemes with high-precision integration, to settle finally on the contour of Figure 1, where the aforementioned wedge has become a "keyhole" pattern consisting of cut-run D_1 and small, origin-centered circle E_1 of radius $1/2$.

So, adopting constraints and nomenclature:

$$z \neq 0, \quad \theta := \arg(z), \quad \omega_\pm := \pm\pi + \theta/2,$$

$$m := n + 1, \quad r := \sqrt{m/z} := \sqrt{m/|z|}e^{-i\theta/2}, \quad R := |r| > 1/2,$$

but no other constraints, we have the following representation for L:

$$L_n^{(-a)}(-z) \quad = \quad c_1 + d_1 + e_1, \tag{15}$$

where c_1, d_1, e_1 are the respective contributions from contour C_1, cut-discontinuity D_1, and contour E_1 from Figure 1. Exact formulae for said contributions are

$$c_1 \quad = \quad \frac{1}{2\pi} r^{-a} e^{-z/2} \int_{\omega_-}^{\omega_+} \mathcal{H}_m(a, z, e^{-i\omega}) e^{2\sqrt{mz}\cos\omega}\, d\omega, \tag{16}$$

$$d_1 \quad = \quad \frac{e^{-z}}{\pi} \sin(\pi a) \int_{1/2}^{R-1/2} T^{-1-a} \left(1 + \frac{1}{T}\right)^{-m} e^{-zT}\, dT, \tag{17}$$

$$e_1 \quad = \quad -\frac{e^{-z}}{4\pi} \int_{-\pi}^{\pi} \left(2e^{-i\omega}\right)^{1+a} \left(1 - 2e^{-i\omega}\right)^{-m} e^{i\omega + \frac{z}{2}e^{i\omega}}\, d\omega, \tag{18}$$

[5]The contour representation (14) can easily be continued to non-integer n, with care taken on the $(-n-1)$-th power, but our present treatment will only use nonnegative integer n.

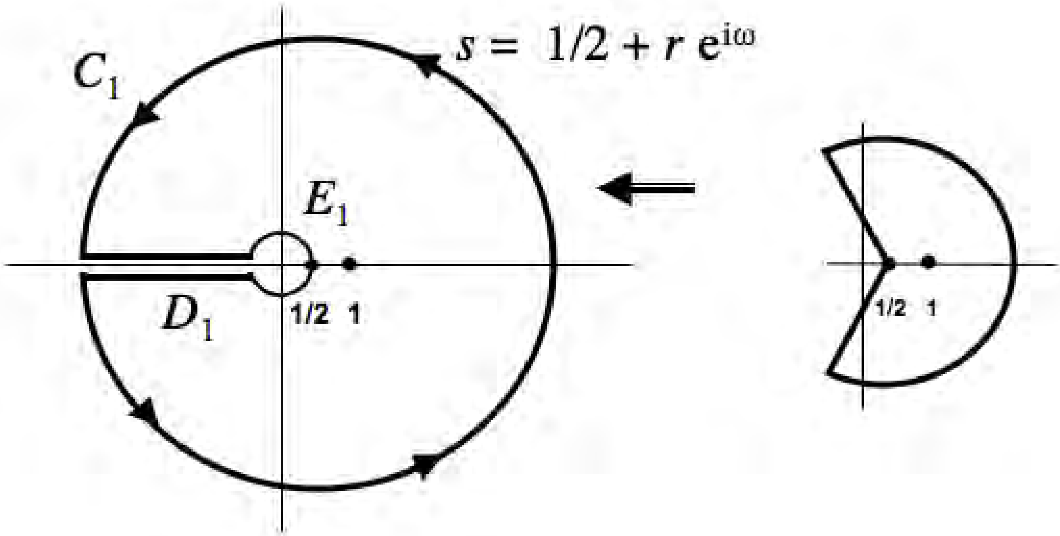

Figure 1: A numerically efficient "keyhole contour" for Laguerre evaluations $L_n^{(-a)}(-z)$, valid for all complex a, z with $z \neq 0$ and $n+1 > |z|/4$. Wedges with center-1/2—as pictured at right—are experimentally accurate, leading to a "keyhole" deformation avoiding the cut $s \in (-\infty, 0]$. It turns that for any pair $(a,z) \in \mathcal{D}$, the main arc C_1 gives the predominant contribution for large n, the D_1, E_1 components being sub-exponentially minuscule.

and when $m > \Re(a)$ we may write this last contribution by shrinking down the radius-1/2 contour segment to embrace the cut $(-1/2, 0]$, as

$$e_1 \quad = \quad \frac{e^{-z}}{\pi} \sin(\pi a) \int_0^{1/2} T^{-1-a} \left(1 + \frac{1}{T}\right)^{-m} e^{-zT}\, dT. \tag{19}$$

For the c_1 contribution above, we have used the function $\mathcal{H}$ defined by

$$\mathcal{H}_m(a,z,v) \quad := \quad v^a \left(1 + \frac{v}{2r}\right)^{-1-a} \left[F\left(\frac{v}{r}\right)\right]^m, \tag{20}$$

with

$$F(t) := \left(\frac{1+t/2}{1-t/2}\right) e^{-t}, \tag{21}$$

which for t small is: $1 + t^3/12 + t^5/80 + \cdots = 1 + O(t^3)$.

Note. The form e_1 given by (18) is more general than (19) as is exemplified by the case $a := 7/2, z := -1, m := 3$ for which $L_2^{(-7/2)}(1) = 31/8$ with the (18) form contributing correctly, while the integral (19) does not even exist. Another example is $a := 5, z :=$

$-1, m := 3$ for which (19) has a vanishing prefactor $\sin(\pi a)$, yet $e_1 \approx 9.513\ldots$ as provided by (18) is correct.

In any event, e_1, like d_1, is sub-exponentially small relative to c_1, for large n and $(a, z) \in \mathcal{D}$. It is to be stressed that decomposition (15) holds for all $(a, z) \in \mathcal{C} \times \mathcal{C}, z \neq 0$, as long as $m := n+1 > |z|/4$. We remind ourselves of (2) for $z = 0$. This means that such contour calculus applies to both oscillatory (Fejér) cases, where z is negative real, and sub-exponential (Perron) cases for the stated parameters. We shall soon be restricting (a, z) to the domain $\mathcal{D}$, which restriction will induce sub-exponential growth always, as in (3), with effective bounds attainable for various contour segments.

2.2 Numerical assessment of the contour

The triumvirate c_1, d_1, e_1 of integrals is suitable for accurate Laguerre computations, which computations do show that *integrals d_1, e_1 tend to be sub-exponentially small* relative to c_1. And this, of course, is our motive for identifying the contour terms in such a way. An example computation would be, from the defining series (1) with $(a, z) := (-i, -1-i)$:

$$\boxed{L_8^{(i)}(1+i) = -\frac{137}{288} + \frac{53}{45}i}$$

which to 20-place accuracy is

$$\approx -0.47569444444444444444 + 1.17777777777777777777i,$$

whereas the respective contributions from the C_1, D_1, E_1 segments of the keyhole contour of Figure 1 turn out to be

$$\boxed{\begin{aligned} c_1 &\approx -0.44406762576110996056 + 0.81722282272705891241i, \\ d_1 &\approx -0.03169598827425878852 + 0.36065591639721657587i, \\ e_1 &\approx 0.00006916959092430464 - 0.00010096134649771051i, \end{aligned}}$$

with the sum $c_1 + d_1 + e_1$ giving $L_8^{(i)}(1+i)$ to the implied precision.

Remarkably, this "keyhole-contour" approach has the additional, unexpected feature that for some parameter regions the contour evaluation of $L_n^{(a)}(-z)$ is actually *faster* then direct summation of the defining series (1). A typical example is: For the 13-digit evaluation

$$L_{50000}^{(25i/2)}(-30+3i) \approx (0.9275136583293 + 1.7406691595239i) \times 10^{1056}$$

the defining series (1) was, in our trials, three-times slower than the integral c_1 (the other two contour segments are well below the 13-digit significance).[6]

[6] Of course, series acceleration as in [3] would give the direct series a "leg up." Still, it is remarkable that contour integration would even be competitive. For the record, we compared *Mathematica*'s numerical integration to its own `LaguerreL[ ]` function.

3 Effective bounds on the contour components

3.1 Theta-calculus

We now introduce a notation useful in effective-error analysis. When two functions enjoy

$$|f(z)| \le |g(z)|$$

over some domain $z \in \mathcal{X}$, we shall say that

$$f(z) = \Theta(g(z))$$

on said domain. Thus, the Θ-notation is an effective replacement for big-O notation. For example, for z on the complex unit circle, we have

$$\frac{z}{z+1/2} = \Theta(2),$$

even though this fails for some z off the circle. This "theta-algebra" obeys certain relations, such as this for any finite function evaluations f_k:

$$|f_1 + f_2 + \dots| \le \Theta(f_1) + \Theta(f_2) + \dots,$$

from the classical triangle inequality. (Note however that one may *not* always write the right-hand side as $\Theta(f_1 + f_2 + \dots)$ because of possible cancelation.) Also easy is the multiplicative relation for arbitrary complex multipliers α:

$$\alpha\Theta(f(z)) = \Theta(\alpha f(z)).$$

When an expression g is unwieldy, we allow ourselves the luxury of denoting $\Theta(g)$ by $\Theta : g$, so that a long formula g can run arbitrarily to the right of the colon.

3.2 Effective bound for e_1

First we address the integral e_1 as given in relation (18). We need some preliminary lemmas, starting with a collection of polynomial estimates to transcendental functions.

Lemma 1 *The following inequalities hold:*

1. *On $\omega \in [-\pi, \pi]$ we have*
$$\log(5 - 4\cos\omega) \ge \frac{\log 9}{\pi^2}\omega^2.$$
2. *On $\omega \in [0, 1/2]$ we have*
$$\log(1+\omega) \ge \frac{4}{5}\omega.$$
3. *On $\omega \in [0, 1/\sqrt{2}]$ we have*
$$\arcsin\omega \le \frac{\pi}{\sqrt{8}}\omega.$$

4. On $\omega \in [-\pi/2, \pi/2]$ we have

$$\cos\omega \le 1 - \frac{4}{\pi^2}\omega^2.$$

Proof. All four are straight-forward calculus exercises. We illustrate with a proof the first which is slightly more subtle. Observe that 1. is equivalent to

$$\log(1 + 8\sin^2(x/2)) \ge \frac{\log 9}{\pi^2}x^2.$$

Since $\sin(t) \ge 2t/\pi$ when $0 \le t \le \pi/2$, we have

$$\log(1 + 8\sin^2(x/2)) \ge \log(1 + 8x^2/\pi^2).$$

Putting $u := 2x^2/\pi^2$, we see that it suffices to prove

$$\log(1 + 4u) \ge u \log 3 \text{ for } 0 \le u \le 2,$$

which follows by simple calculus. **QED**

Lemma 2 *Consider integrals of error-function class, specifically, for nonnegative real parameter μ,*

$$V_\mu(\alpha, \beta, \gamma) := \int_\gamma^\infty x^{2\mu} e^{2\alpha x - \beta x^2}\, dx,$$

where each of α, β, γ is real, with $\alpha, \beta > 0$. Then we always have the bound

$$V_0(\alpha, \beta, \gamma) = \Theta : \sqrt{\frac{\pi}{\beta}}\, e^{\frac{\alpha^2}{\beta}}.$$

If in addition $\gamma > 2\alpha/\beta$, we also have

$$V_\mu(\alpha, \beta, \gamma) = \Theta : \frac{1}{2}\, \frac{\Gamma(\mu + 1/2)}{(\beta - 2\alpha/\gamma)^{\mu+1/2}};$$

finally, if $\gamma > 4\alpha/\beta$ we have

$$V_\mu(\alpha, \beta, \gamma) = \Theta :\ 2^{\mu-1/2}\, \frac{\Gamma(\mu + 1/2)}{\beta^{\mu+1/2}},$$

Remark: The last two bounds we shall not use until a later section, but it is convenient to dispense with all these error-function results now.

Proof. Completing the exponent's square gives the first bound easily, since

$$V_0 = e^{\alpha^2/\beta} \int_\gamma^\infty e^{-\beta(x - \alpha/\beta)^2}\, dx$$

with the integral here being $\Theta\left(\sqrt{\pi/\beta}\right)$. For the second bound, it is elementary that for $x \geq \gamma$ one has $2\alpha x - \beta x^2 \leq x^2(2\alpha/\gamma - \beta)$, so that

$$V_\mu \leq \int_0^\infty x^{2\mu} e^{-(\beta-2\alpha/\gamma)x^2}\, dx$$

and the resulting bound follows. The final bound follows immediately from the previous bound, because $\gamma > 4\alpha/\beta$ implies $\beta - 2\alpha/\gamma > \beta/2$. **QED**

Theorem 1 *The contour contribution e_1 defined by (18), under conditions $(a, z) \in \mathcal{D}$ and m sufficiently large in the explicit sense*

$$\boxed{\begin{gathered} m := n + 1 > m_0 := |z|/4, \\ m > m_1 := 5\left(|\Re(z)| + (|\Im(a)| + |\Im(z)|/2)^2\right), \end{gathered}}$$

is bounded as

$$e_1 = \Theta : e^{-z/2} 2^{\Re(a)+3} \frac{1}{\sqrt{m}}.$$

Moreover, under alternative conditions $(a, z) \in \mathcal{D}$ and

$$m > m_0, \quad m > m_2 := \Re(a),$$

$$m > m_3 := -\frac{5}{4}|z| \cos\theta,$$

we have a bound

$$e_1 = \Theta : \frac{e^{-z}}{\pi} \sin(\pi a) \frac{2^{\Re(a)-m}}{m - \Re(a)}$$

Proof. First, in (18) the term $(1 - 2e^{-i\omega})^{-m}$ has absolute value $(5 - 4\cos\omega)^{-m/2}$. So we can write, recalling $\theta := \arg(z) \in (-\pi, \pi]$,

$$e_1 = \Theta : \frac{e^{-z/2}}{2\pi} 2^{\Re(a)} \int_{-\pi}^{\pi} e^{\omega \Im(a)} e^{-(m/2)\log(5-4\cos\omega)} e^{|z|(\cos(\omega+\theta)-\cos\theta)/2}\, d\omega.$$

Now the overall exponent here can be written

$$\omega \Im(a) - \frac{m}{2}\log(5 - 4\cos\omega) - \Re(z)\sin^2(\omega/2) - \frac{1}{2}\Im(z)\sin\omega.$$

Then, via Lemma 1 we can give an upper bound for this real expression when $\omega \geq 0$, namely

$$(|\Im(a)| + |\Im(z)|/2)\omega - \frac{\log 9}{2\pi^2} m\omega^2 - \Re(z)\sin^2(\omega/2).$$

Now define parameter

$$\alpha := |\Im(a)|/2 + |\Im(z)|/4,$$

and if $\Re(z) < 0$ define

$$\beta := \frac{m \log 9}{2\pi^2} + \Re(z)/4,$$

but if $\Re(z) \geq 0$ we drop the term $\Re(z)/4$ in the definition. With this nomenclature, we can write

$$\begin{aligned} e_1 := \ & \Theta : \frac{e^{-z/2}}{2\pi} 2^{\Re(a)} 2 \int_0^{\pi} e^{2\alpha\omega - \beta\omega^2} \, d\omega \\ = & \ \Theta : \frac{e^{-z/2}}{\pi} 2^{\Re(a)} V_0(\alpha, \beta, 0). \end{aligned}$$

Under the stated condition $m > m_1$, we have $|\Re(z)| < m/5$ and so

$$\beta \ > \ \left(\frac{\log 9}{2\pi^2} - \frac{1}{20} \right) m \ > \ \frac{m}{20}.$$

Also from the same fact $m > m_1$ we have

$$\alpha^2 < \frac{m}{20} < \beta.$$

whence Lemma 2 gives

$$e_1 = \Theta : \frac{e^{-z/2}}{\sqrt{\pi}} 2^{\Re(a)} \frac{e}{\sqrt{m/20}},$$

and the desired result follows.

For the second set of conditions on m, the fact of $m > m_2$ means that (18) and (19) are equivalent, and Lemma 1 allows:

$$e_1 = \Theta : \frac{e^{-z}}{\pi} \sin(\pi a) \int_0^{1/2} T^{m-1-\Re(a)} e^{-zT - 4mT/5} \, dT.$$

Since the integral is bounded for $m > m_3$ by $2^{\Re(a)-m}/(m - \Re(a))$, the desired bound follows. **QED**

3.3 Effective bound for d_1

Again we need an opening lemma:

Lemma 3 *Consider integrals of the incomplete–Bessel class, specifically*

$$W(\alpha, \beta) := \int_0^1 e^{-\alpha x - \frac{\beta}{x}} \frac{dx}{x},$$

where each of α, β is real, with $\beta > 0$. If $\beta > \alpha$ we have a bound

$$W = \Theta : \frac{1}{2} e^{-\beta - \alpha} \sqrt{\frac{\pi}{\beta}}.$$

Proof. We write

$$\begin{aligned} W(\alpha,\beta) &= \int_0^1 e^{\beta x-\alpha x}\, e^{-\beta x-\frac{\beta}{x}}\,\frac{dx}{x} \le e^{\beta-\alpha}\int_0^1 e^{-\beta x-\frac{\beta}{x}}\,\frac{dx}{x} \\ &= e^{\beta-\alpha}\int_0^\infty e^{-2\beta\cosh t}\,dt \le e^{-\beta-\alpha}\int_0^\infty e^{-\beta t^2}\,dt, \end{aligned}$$

which proves the theorem. (Incidentally, the integral with cosh in the exponent is the modified-bessel evaluation $K_0(2\beta)$ which is known to have the necessary bound [2, Lemma 1].) **QED**

Theorem 2 *The contour contribution* d_1 *defined by (17) can be bounded under conditions* $(a,z)\in\mathcal{D}$ *and*

$$\boxed{m > m_4 := 4|z|,}$$

as

$$d_1 = \Theta : \frac{e^{-z/2}}{\sqrt{\pi}} m^{|\Re(a)|/2-1/4}|z|^{-|\Re(a)|/2-1/4}\sin(\pi a)e^{-2\sqrt{m|z|}\cos^2\frac{\theta}{2}}.$$

Proof. From (17) we have (noting that $m > m_4$ means $R := \sqrt{m/|z|} > 2$,

$$d_1 = \frac{e^{-z/2}}{\pi}\sin(\pi a)\int_1^R (t-1/2)^{-1-a}e^{-zt+m(\log(1-1/(2t))-\log(1+1/(2t)))}\,dt.$$

On the interval $t\in(1,R)$ we have

$$(t-1/2)^{-1-a} = \Theta : (t-1/2)^{-1}(t-1/2)^{-\Re(a)}$$

which, since $R>2$, is $\Theta : \frac{2}{t}R^{|\Re(a)|}$. Thus

$$d_1 = \Theta : \frac{e^{-z/2}}{\pi}\sin(\pi a)2R^{|\Re(a)|}\int_{1/R}^1 \frac{d\tau}{\tau}e^{-\tau\sqrt{m|z|}\cos\theta-\sqrt{m|z|}/\tau}.$$

The desired bound then follows immediately from Lemma 3.

QED

3.4 Rigorous estimates on c_1

Having dispensed with d_1, e_1, we next show that the integration limits ω_-,ω_+ on the c_1 contribution can be changed—with only a sub-exponentially small error penalty—to $-\pi/2,\pi/2$ respectively, as we now establish. We start with

Lemma 4 *For any* v *on the unit circle* $\{e^{i\phi}:\phi\in(-\pi,\pi]\}$*, any complex* a*, and any real* $R > (1+|a|)/2$ *we have*

$$\left|\left(1+\frac{v}{2R}\right)^{-1-a}\right| \le \frac{1}{1-\frac{1+|a|}{2R}}.$$

Proof. From the binomial theorem,

$$\left(1+\frac{v}{2R}\right)^{-1-a} = 1+\frac{-1-a}{1!}\frac{v}{2R}+\frac{(-1-a)(-2-a)}{2!}\left(\frac{v}{2R}\right)^2+\dots.$$

The right-hand side is bounded in absolute value by

$$1+(1+|a|)\frac{1}{2R}+(1+|a|)^2\left(\frac{1}{2R}\right)^2+\dots$$

$$=\frac{1}{1-\frac{1+|a|}{2R}}.$$

QED

Lemma 5 *Let v be on the unit circle as in Lemma 4, let $R>1$ be real, and let m be a positive integer. For the function F appearing in (21), we have the bound*

$$\left|F\left(\frac{v}{R}\right)^m\right| \le e^{\frac{1}{6}\frac{m}{R^3}}.$$

Proof. From the definition (21) we have

$$F(v/R)^m = e^{\frac{1}{12}\frac{m}{R^3}(1+(3/5)/(2R)^2+(3/7)/(2R)^4+\dots)}.$$

For $R:=1$ the parenthetical infinite sum is no larger than 1.18... and is monotonic decreasing in R. **QED**

Lemma 6 *Let v be on the unit circle as in Lemma 4 and define for nonnegative integer m*

$$K := \left(1+\frac{v}{2R}\right)^{-1-a} F\left(\frac{v}{R}\right)^m.$$

For the assignments $(a,z)\in\mathcal{C}\times\mathcal{C}, z\neq 0, R:=\sqrt{m/|z|}>1, m>m_5:=|z|(1+|a|+|z|/2)^2$. Then

$$|K|\le 2.$$

Proof. From Lemmas 4, 5 we have

$$|K|\le \frac{1}{1-\frac{1+|a|}{2(1+|a|+|z|/2)}}e^{\frac{1}{6}\frac{|z|}{1+|a|+|z|/2}}.$$

The fact that the right-hand side is $\Theta(2)$ follows from the observation that the function

$$\frac{1}{1-\frac{Q}{2Q+x}}e^{\frac{1}{3}\frac{x}{2Q+x}}$$

for $Q\ge 1, x\in[0,\infty)$ is itself $\Theta(2)$, and this in turn follows easily from the substitution $y:=x/(2Q+x)$, for which the function to be bounded is $g(y):=(2/(1+y))e^{y/3}$ on $y\in[0,1/2]$— use of the derivative dg/dy settles this. **QED**

These lemmas in turn allow us to contract the range on the c_1 contour integral:

Theorem 3 *Decompose c_1 as defined by (16) into two terms,*

$$\boxed{c_1 := c_0 + c_2,}$$

with c_0 involving the integral's range contracted to $[-\pi/2, \pi/2]$, namely

$$\boxed{c_0 := \frac{1}{2\pi} r^{-a} e^{-z/2} \int_{-\pi/2}^{\pi/2} \mathcal{H}_m(a, z, e^{-i\omega}) e^{2\sqrt{mz}\cos\omega} \, d\omega,}$$

Then under conditions $(a, z) \in \mathcal{D}$ and

$$\boxed{m > m_5 := |z|(1 + |a| + |z|/2)^2}$$

we have a bound on the remaining integral

$$c_2 = \Theta : r^{-a} e^{-z/2} e^{\frac{3}{2}\pi|\Im(a)|}.$$

Proof. It is evident that c_2 is obtained from the definition (16) but with the integral replaced according to

$$\int_{\omega_-}^{\omega_+} \to \left\{ \int_{\omega_-}^{-\pi/2} + \int_{\pi/2}^{\omega_+} \right\}.$$

Over these domains of integration, however, we have $e^{2\sqrt{mz}\cos\omega} = \Theta(1)$, and from Lemmas 4, 5, 6 we have

$$c_2 = \Theta : \frac{1}{2\pi} r^{-a} e^{-z/2} \left\{ \int_{\omega_-}^{-\pi/2} + \int_{\pi/2}^{\omega_+} \right\} 2 \left| e^{-i\omega a} \right| \, d\omega,$$

and as the total support of the integrals cannot exceed π, the desired c_2 bound follows easily. **QED**

3.5 Summary of the contour decomposition for $L_n^{(-a)}(-z)$

The above machinations lead to the main result of the present section, namely a formula that decomposes the Laguerre evaluation, as in

Theorem 4 (Contour decomposition) *Let $(a, z) \in \mathcal{C} \times \mathcal{C}$ be an arbitrary parameter pair with $z \neq 0$ (with $z = 0$ cases resolved exactly by (2)). If*

$$\boxed{m := n + 1 = m_0 > |z|/4,}$$

then the contour decomposition

$$L_n^{(-a)}(-z) = c_0 + \mathcal{E}$$

holds, with c_0, c_2 as defined in Theorem 3 and $\boxed{\mathcal{E} := c_2 + d_1 + e_1.}$ *If an appropriate union of conditions on m, a, z across Theorems 1, 2, 3 holds, then we can write*

$$\boxed{L_n^{(-a)}(-z) = c_0 + S_n(a,z)\mathcal{E}_1,}$$

where $\mathcal{E}_1$ is sub-exponentially small, in the sense that for any positive ϵ the large-n behavior is

$$\mathcal{E}_1 = O\left(e^{-(2-\epsilon)\sqrt{m|z|}\cos\frac{\theta}{2}}\right).$$

Moreover, an effective big-O constant is available (the proof exhibits explicit forms).

Proof. The contour calculus is valid for all complex pairs (a, z) when $z \neq 0, R := \sqrt{m/|z|} > 1/2$, which conditions assure that the point $1 + 0i$ is contained in the contour. It remains to analyze $L_n^{(-a)}(-z) = c_0 + S_n(a,z)\,\mathcal{E}_1$. Consider, then, an appropriate union of conditions from the cited theorems, say

$$(a, z) \in \mathcal{D},$$

$$m > \max(m_0, m_1, m_2, m_3, m_4, m_5)$$

in which case we have, from Theorems 1, 2, 3 (in that respective order of Θ terms):

$$\mathcal{E}_1 = \Theta : \frac{e^{-z/2}}{\sqrt{\pi}} \sin(\pi a) \frac{2^{1+\Re(a)-m}}{m - \Re(a)} z^{1/4-a/2} m^{1/4+a/2} e^{-2\sqrt{m|z|}\cos\frac{\theta}{2}}$$

$$+\Theta : 2\left(\frac{m}{|z|}\right)^{(\Re(a)+|\Re(a)|)/2} e^{\frac{3}{2}\pi|\Im(a)|} e^{-2\sqrt{m|z|}\left(\cos^2\frac{\theta}{2}+\cos\frac{\theta}{2}\right)}$$

$$+\Theta : 2\sqrt{\pi}\;|mz|^{1/4} e^{\frac{3}{2}\pi|\Im(a)|} e^{-2\sqrt{m|z|}\cos\frac{\theta}{2}}.$$

This explicit bounding of the error term $\mathcal{E}_1$ proves the $O\left(e^{-(2-\epsilon)\sqrt{m|z|}\cos(\theta/2)}\right)$ statement of the theorem, while for any choice of ε an effective big-O constant can be read off at will. **QED**

Theorem 4 suggests—and we shall prove—that the c_0 term defined in Theorem 3 can be given a Perron-like series, essentially as in (7). This will establish that the *precise* (a, z)-parameter domain of large-n sub-exponential growth will be $\mathcal{D}$. Thus our remaining tasks are to provide:

1. A symbolic algorithm for the asymptotic coefficients C_k in (7), and
2. An effective bound for the E_N term.

Moreover, we wish to achieve these things over the whole domain of sub-exponential Laguerre growth, namely for all $(a, z) \in \mathcal{D}$.

4 Effective expansion for the $\mathcal{H}$-kernel

In the sense of Theorem 4, the dominant contribution to $L_n^{(a)}(-z)$ for $(a, z) \in \mathcal{D}$, with $r := \sqrt{m/z}, |r| > 1/2$, is

$$\boxed{c_0 \quad := \quad \frac{1}{2\pi} r^{-a} e^{-z/2} \int_{-\pi/2}^{\pi/2} \mathcal{H}_m \left(a, z, e^{-i\omega}\right) e^{2\sqrt{mz}\cos\omega} \; d\omega,} \tag{22}$$

with the integration kernel $\mathcal{H}$ defined, see (20) and (21), as

$$\boxed{\mathcal{H}_m(a, z, v) \quad := \quad v^a \left(1 + \frac{v}{2r}\right)^{-1-a} \left(\frac{1 + \frac{v}{2r}}{1 - \frac{v}{2r}}\right)^m e^{-mv/r},} \tag{23}$$

where in the integral we assign $v := e^{-i\omega}$. We need to obtain the growth properties of $\mathcal{H}$. This we do in the next three subsections. We suspect there is a more standard combinatorial route to Theorem 5 of Section 4.3, but it has so far eluded us.

4.1 Exponential form for $\mathcal{H}$

Lemma 7 *For $|v| = 1$ and $m > |z|/4$, the $\mathcal{H}$-kernel can be cast in the exponential form*

$$\mathcal{H}_m \quad := \quad v^a \exp\left\{\sum_{k\geq 1} \frac{a_k}{k} \frac{1}{m^{k/2}}\right\}, \tag{24}$$

where

$$\boxed{a_k \quad := \quad (1 + a)(-1)^k \left(\frac{v\sqrt{z}}{2}\right)^k + \left(1 - (-1)^k\right) \frac{k}{k+2} \left(\frac{v\sqrt{z}}{2}\right)^{k+2}.} \tag{25}$$

Moreover, we have the general coefficient bound

$$|a_k| \leq \left(\frac{\sqrt{z}}{2}\right)^k (1 + |a| + |z|/2). \tag{26}$$

Proof. (23) can be recast, with $\rho := v/(2r)$, as

$$\mathcal{H}_m := v^a \exp\left\{-(1 + a)\log(1 + \rho) - 2m\rho + m\left(\log(1 + \rho) - \log(1 - \rho)\right)\right\}. \tag{27}$$

Being as $|r| > 1/2$ and $|v| = 1$, the logarithmic series converge absolutely and we have

$$\mathcal{H}_m := v^a \exp\left\{\sum_{k\geq 1} \left[\frac{(1 + a)\,(-1)^k}{k} \left(\frac{v\sqrt{z}}{2}\right)^k g^k + \frac{2}{2k+1} \left(\frac{v\sqrt{z}}{2}\right)^{2k+1} g^{2k-2}\right]\right\}, \tag{28}$$

where $g := 1/\sqrt{m}$, and the precise form (25) for the a_k follows immediately. The given bound on $|a_k|$ is also immediate from (25). **QED**

4.2 Exponentiation of series

Though Lemma 7 is progress, we still need to exponentiate a series, in the sense that we want to know, for the following expansion, given the sequence (a_k),

$$\boxed{\exp\left\{\sum_{k\geq 1}\frac{a_k}{k}x^k\right\} =: \sum_{h\geq 0} A_h x^h,}$$

how the A_h depend on the a_k. The combinatorial answer is

$$A_h = \sum_{j=0}^{h}\frac{1}{j!}\, G_h(j;\vec{a}),$$

where

$$G_h(j;\vec{a}) := \sum_{h_1+\ldots+h_j=h}\frac{a_{h_1}\cdots a_{h_j}}{h_1\cdots h_j},$$

with the understanding $G_h(0,\vec{a}) := \delta_{0h}$ and that such combinatorial sums involve positive integer indices h_i. One result that can prove useful for such combinatorics is a bound on a simple instance of the G sum:

Lemma 8 *If all coefficients a_k are equal 1 then, for $j > 0$,*

$$\begin{aligned}
G_h(j;\vec{1}) &= \sum_{h_1+\ldots+h_j=h}\frac{1}{h_1\cdots h_j} \\
&= \Theta : \frac{1}{h}\left(2H_{h-j+1}\right)^{j-1}, \\
&= \Theta : \frac{1}{h}\left(2\gamma + 2\log h\right)^{j-1}.
\end{aligned}$$

Here, $H_q := 1 + 1/2 + \cdots + 1/q$ is the q-th Harmonic number, $H_0 := 0$, and γ is the Euler constant.

Remark. It turns out that G here enjoys a closed form of sorts, namely

$$G_h(j;\vec{1}) = \frac{j!}{h!}(-1)^{h-j}\mathcal{S}_h^{(j)},$$

where $\mathcal{S}$ denotes the Stirling number of the first kind, normalized via $x(x-1)\cdots(x-h+1) =: \sum_{j=0}^{h}\mathcal{S}_h^{(j)}x^j$. So the effective of our lemma is a rigorous bound on the growth of Stirling numbers; see [1, 24.1.3,III] and [31] for research on Stirling asymptotics.

Proof. The first Θ-estimate arises by induction. For notational convenience we omit the vector $\vec{1}$ and just use the symbol $G_h(j)$. Note $G_N(1) = 1/N$ and

$$G_N(2) = \sum_{j=1}^{N-1}\frac{1}{j(N-j)} = \frac{2}{N}H_{N-1}.$$

Generally we have

$$G_N(J) = \sum_{j=1}^{N-J+1} \frac{1}{j} G_{N-j}(J-1).$$

Now, assume by induction that $G_h(j) = \Theta : \frac{1}{h}\left(2H_{h-j+1}\right)^{j-1}$, holds for all $j < J$. Then

$$G(N,J) \leq \sum_{j=1}^{N-j+1} \frac{2^{J-2} H_{N-J-j+2}^{J-2}}{j(N-J)}$$

$$\leq \frac{2^{J-2}}{N} H_{N-J+1}^{J-2} \sum_{j=1}^{N-J+1} \left(\frac{1}{j} + \frac{1}{N-j}\right).$$

Now the parenthetical term is $H_{N-J+1} + H_{N-1} - H_J$ which, because $H_a - H_b \leq H_{a-b}$ for any positive integer indices $a > b$, is bounded above by $2H_{N-J+1}$, which proves the first Θ-bound of the theorem.

For the second θ-bound it suffices to show that $H_{n-1} > \gamma + \log(n)$, since H_j is increasing. We set $s_n := H_{n-1} - \log(n)$. Then, $s_1 = 0$ and for $n > 0$

$$s_{n+1} - s_n = \frac{1}{n} - \log\left(1 + \frac{1}{n}\right) > 0,$$

and so s_n increases and, by definition, tends to γ as $n \to \infty$. **QED**

Though we do not use Lemma 8 directly in what follows, it is useful in proving convergence for various sums $\sum A_h x^h$, and may matter in future research along our lines.

Lemma 9 *Let $y \geq 1$ and $x \in (-1,1)$ be real. Then in the expansion*

$$\exp\left\{ y \sum_{k \geq 1} \frac{x^k}{k} \right\} =: \sum_{h \geq 0} Y_h x^h$$

the coefficients Y_h enjoy the bound

$$Y_h = \Theta\left(y^h\right).$$

Proof. The left-hand side is $\exp(-y\log(1-x)) = (1-x)^{-y}$ whose binomial expansion has h-th coefficient ($h \geq 1$) equal to

$$\frac{y(y+1)\cdots(y+h-1)}{h!} \leq y^h \, \frac{1}{1} \frac{(1+1/y)}{2} \cdots \frac{1+(h-1)/y}{h} \leq y^h.$$

Finally, $Y_0 = 1 \leq 1$. **QED**

4.3 Effective expansions of exponentiated series

We are now in a position to contemplate an effective expansion for an exponentiated series, starting with

Lemma 10 *Assume complex vector $\vec{b}$ of defining coefficients b_k bounded as*

$$\boxed{|b_k| \le cd^k}$$

for positive real c, d with $c \ge 1$. Let real x satisfy $|x| < 1/(2cd)$. Then for any order $N \ge 0$ we have an effective expansion

$$\boxed{\exp\left\{\sum_{k\ge1} \frac{b_k}{k} x^k\right\} = \sum_{h=0}^{N-1} B_h x^h + \Theta\left(2c^N d^N x^N\right),}$$

where

$$\boxed{B_h = \sum_{j=0}^{h} \frac{G_h(j;\vec{b})}{j!}}$$

are the usual coefficients of the full formal exponentiation.

Proof. Denoting $f(x) := \exp\left\{\sum_{k\ge1} \frac{b_k}{k} x^k\right\}$ we have

$$f(x) = \sum_{h=0}^{N-1} B_h x^h + T_N.$$

where the remainder $T_N = \sum_{h\ge N} B_h x^h$, with all B_h coefficients given by a G-sum as in Lemma 10. Now,

$$|B_h| \le \sum_{j=0}^{h} \frac{|G_h(j;\vec{b})|}{j!},$$

but

$$|G_h(j, \vec{b}| \le |G_h(j, \vec{f})|$$

where $\vec{f} = (cd^k : k \ge 0)$. But by Lemma 9 we know that

$$|B_h| \le (cd)^h.$$

Therefore

$$|T_N| \le \sum_{h\ge N} (cd)^h x^h = \frac{c^N d^N x^N}{1 - cdx} = \Theta\left(2(cdx)^N\right).$$

QED

Finally we arrive at a general expansion—with effective remainder—for the $\mathcal{H}$-kernel:

Theorem 5 (Effective expansion for $\mathcal{H}$.) *For general complex $(a,z) \in \mathcal{C} \times \mathcal{C}$, assume $m > m_5 := |z|(1+|a|+|z|/2)^2$ and $|v| = 1$. Then for any expansion order $N \geq 0$ we have*

$$\mathcal{H}_m(a,z,v) = v^a \left(\sum_{h=0}^{N-1} \frac{A_h}{m^{h/2}} + \Theta : 2\left(\frac{m_5}{4m}\right)^{N/2} \right),$$

where

$$A_h := \sum_{j=0}^{h} \frac{G_h(j;\vec{a})}{j!},$$

$$G_h(j;\vec{a}) := \sum_{h_1+\ldots+h_j=h} \frac{a_{h_1}\cdots a_{h_j}}{h_1\cdots h_j},$$

with the defining coefficients a_k given in (25).

Proof. The result follows immediately from Lemma 10, on assigning $\vec{b} = \vec{a}$, with $x := 1/\sqrt{m}$, $c := 1+|a|+|z|/2$, $d := (1/2)\sqrt{|z|}$. **QED**

Note that Theorem 5 in the instance $N = 0$ implies our previous Lemma 6. It is interesting and suggestive that the threshold $m_5 := |z|(1+|a|+|z|/2)^2$ appears in both theorem and lemma rather naturally.

4.4 Effective integral form for c_0

To obtain a useful form for c_0, the dominant component of $L_n^{(-a)}(-z)$, we use Theorem 5:

Theorem 6 *For $(a,z) \in \mathcal{D}$ and $m > m_5$, the dominant component of Theorems 3 and 4, namely*

$$c_0 := \frac{1}{2\pi} r^{-a} e^{-z/2} \int_{-\pi/2}^{\pi/2} \mathcal{H}_m(a,z,e^{-i\omega}) e^{2\sqrt{mz}\cos\omega}\, d\omega,$$

can be given an effective form for any order $N \geq 0$, as

$$\boxed{\begin{aligned} c_0 = \frac{1}{2\pi} r^{-a} e^{-z/2} \sum_{h=0}^{N-1} \frac{1}{m^{h/2}} \int_{-\pi/2}^{\pi/2} e^{-i\omega a} A_h e^{2\sqrt{mz}\cos\omega}\, d\omega \\ + \boxed{S_n(a,z)}\, \mathcal{E}_{2,N}, \end{aligned}}$$

where the error term is bounded as

$$\boxed{\mathcal{E}_{2,N} = \Theta : \frac{\pi}{\sqrt{2}} \left(\frac{m_5}{4m}\right)^{N/2} \exp\left(\frac{\pi^2 \Im(a)^2 \sec\frac{\theta}{2}}{32\sqrt{m|z|}}\right) \sec^{1/2}\frac{\theta}{2}}$$

and the A_h are to be calculated as the first N coefficients of

$$\sum_{h=0}^{\infty} A_h x^h := \exp\left\{ \sum_{k\geq 1} \frac{a_k}{k} x^k \right\},$$

via (25) with $v := e^{-i\omega}$.

Proof. Inserting the effective $\mathcal{H}$-kernel expansion from Theorem 5 directly into the c_0 integral gives the indicated sum over $h \in [0, N-1]$ plus an error term

$$\Theta : \frac{1}{2\pi} r^{-a} e^{-z/2} 2 \left(\frac{m_5}{4m}\right)^{N/2} \int_{-\pi/2}^{\pi/2} e^{\omega \Im(a)} e^{-2\sqrt{m|z|}\cos(\theta/2)\cos\omega}\, d\omega.$$

Now using Lemma 1 on $\cos\omega$ and the V_0-part of Lemma 2 we obtain the $\mathcal{E}_{2,N}$ bound of the theorem. **QED**

With Theorem 6 we have come far enough to see that a Laguerre evaluation can be obtained—up to a sub-exponentially small relative error—via the A_h terms in said theorem. To this end, inspection of the defining relations reveals that in general we can decompose an A_h coefficient in terms of powers of $v := e^{-i\omega}$, namely we define $\alpha_{h,\mu}$ terms via

$$\boxed{A_h =: \sum_{u=0}^{h} \alpha_{h,u}(a,z) v^{h+2u}.} \tag{29}$$

For example,

$$\alpha_{00} = 1,$$
$$\alpha_{10} = -\frac{1+a}{2} z^{1/2},$$
$$\alpha_{11} = \frac{z^{3/2}}{12},$$
$$\alpha_{31} = \frac{1}{480}\left(5a^2 + 15a + 16\right) z^{5/2}.$$

The point being, we now have special formulae for the dominant contribution c_0, namely

$$\begin{aligned} c_0 \;=\; & \frac{1}{2\pi} r^{-a} e^{-z/2} \sum_{h=0}^{N-1} \frac{1}{m^{h/2}} \sum_{u=0}^{h} \alpha_{h,u}(a,z)\; \mathcal{I}(2\sqrt{mz}, a+h+2u) & (30)\\ & + S_n(a,z)\; \mathcal{E}_{2,N}, & (31) \end{aligned}$$

where the integral

$$\boxed{\mathcal{I}(p,q) := \int_{-\pi/2}^{\pi/2} e^{-iq\omega} e^{p\cos\omega}\, d\omega} \tag{32}$$

thus emerges as a fundamental entity for the research at hand.

5 $\mathcal{I}$-integrals and an "exp-arc" method

Having reduced the problem of sub-exponential Laguerre growth to a study of the $\mathcal{I}$-integrals (32), we next develop a method that is effective for both their numerical and theoretical estimation. This method amounts to the avoidance of stationary-phase techniques, employing instead various forms of exponential-arcsin series, as we see shortly.

Let us first define with a more general integral. For any complex pair (p, q), assume $\alpha, \beta \in (-\pi, \pi)$ and define

$$\boxed{I(p, q, \alpha, \beta) := \int_{\alpha}^{\beta} e^{-iq\omega} e^{p\cos\omega}\, d\omega,} \tag{33}$$

so that our special case (32) is simply

$$\mathcal{I}(p, q) := I(p, q, -\pi/2, \pi/2).$$

An aside is relevant here. The $\mathcal{I}$ integral can be written in terms of an *Anger function* $\mathbf{J}$ and *Weber function* $\mathbf{E}$—see [1]—as

$$\mathcal{I}(p, q) = \pi e^{iq\pi/2}(\mathbf{J}_{-q}(-ip) + i\mathbf{E}_{-q}(-ip)),$$

with an important special case

$$\mathcal{I}(p, 0) = \pi(J_0(-ip) + \mathbf{L}_0(p)),$$

where J here is the standard *Bessel function* and $\mathbf{L}$ denotes the *modified Struve function* [1]. Moreover, one may write the Bessel functions J_n of integer order n in terms of $\mathcal{I}$-integrals:

$$\boxed{J_n(z) = \frac{1}{2\pi}\left(e^{-i\pi n/2}\mathcal{I}(iz, n) + e^{i\pi n/2}\mathcal{I}(-iz, n)\right)}, \tag{34}$$

and the modified Bessel function, again of integer order n:

$$\boxed{I_n(z) = \frac{1}{2\pi}\left(\mathcal{I}(z, n) + (-1)^n\mathcal{I}(-z, n)\right)}, \tag{35}$$

about which representations we shall have more to say in a later section. These forms (34), (35) are easily derived directly from standard integral representations; see [1, 9.1.21], but note as with [1, 9.1.22], that non-integer n is more complicated because of a cut-term not unlike such terms we have encountered in the Laguerre problem for non-integer a parameter. (See relations (68, 69) for general indices ν on J_ν, I_ν.)

5.1 The exp-arc method explained

Now we investigate what we call *exponential-arcsine* ("exp-arc") series. First, for any complex τ and $x \in [-1, 1]$, one has a remarkable expansion (see [4]):

$$\boxed{e^{\tau \arcsin x} = 1 + \sum_{k=1}^{\infty} r_k(\tau)\,\frac{x^k}{k!},} \tag{36}$$

where the coefficients depend on the parity of the index, as

$$\boxed{r_{2m+1}(\tau) := \tau \prod_{j=1}^{m} \left(\tau^2 + (2j-1)^2\right), \qquad r_{2m}(\tau) := \prod_{j=1}^{m} \left(\tau^2 + (2j-2)^2\right).}$$

By differentiating with respect to x we obtain

$$\boxed{\frac{e^{\tau \arcsin x}}{\sqrt{1-x^2}} = \frac{1}{\tau} \sum_{k=0}^{\infty} r_{k+1}(\tau) \frac{x^k}{k!},}$$

valid for $x \in (-1, 1)$. In particular, we have the important expansion (here we define a function G, in passing)

$$G(\tau, x) := \frac{\cosh(\tau \arcsin x)}{\sqrt{1-x^2}} = \sum_{k=0}^{\infty} g_k(\tau) \frac{x^{2k}}{(2k)!}, \tag{37}$$

where

$$g_k(\tau) := \prod_{j=1}^{k} \left((2j-1)^2 + \tau^2\right).$$

The g coefficients are especially easy to remember, as the sequence $(g_0, g_1, \ldots)$ is simply $(1,\ (1^2+\tau^2),\ (1^2+\tau^2)(3^2+\tau^2),\ (1^2+\tau^2)(3^2+\tau^2)(5^2+\tau^2), \ldots)$.

These exp-arc expansions may be applied to the $I(p, q, \alpha, \beta)$ integrals as follows. From (33) and its subsequent manipulations we have

$$I(p, q, \alpha, \beta) = e^p \int_{\alpha}^{\beta} e^{-iq\omega} e^{-2p \sin^2(\omega/2)}\, d\omega$$

$$= 2e^p \int_{-\sin\frac{\alpha}{2}}^{\sin\frac{\beta}{2}} \frac{e^{-2iq \arcsin x}}{\sqrt{1-x^2}} e^{-2px^2}\, dx$$

$$\boxed{I(p, q, \alpha, \beta) = i e^p \sum_{k=0}^{\infty} \frac{r_{k+1}(-2iq)}{k!\ q} \int_{-\sin\frac{\alpha}{2}}^{\sin\frac{\beta}{2}} x^{2k} e^{-2px^2}\, dx.} \tag{38}$$

It is sometimes quite useful to define also $U(p, q, \psi) := I(p, q, 0, 2\psi)$, with the interesting partitioning $\boxed{U = U_1 + U_2}$ and respective series developments as follows:

$$\boxed{\begin{aligned} U_1(p, q, \psi) &= \frac{1}{2q} e^{p\cos(2\psi)} \sin(2q\psi) + 4p\, e^p \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)4^n} \prod_{k=1}^{n} \left(1 - \frac{4q^2}{(2k-1)^2}\right) \beta_{2n+1}(p, \psi), \\ U_2(p, q, \psi) &:= \frac{1}{q} e^{p\cos(2\psi)} \sin^2(q\psi) + 2pq\, e^p \sum_{n=1}^{\infty} \frac{4^n}{n^2 \binom{2n}{n}} \prod_{k=1}^{n-1} \left(1 - \frac{q^2}{k^2}\right) \beta_{2n}(p, \psi), \end{aligned}}$$

where β is an error-function-class integral

$$\beta_n(p,\psi) := \int_0^{\sin\psi} x^{n+1} e^{-2p\,x^2}\,dx = \frac{1}{2}\,(2p)^{-1-n/2}\left\{\Gamma(n/2+1) - \Gamma(n/2+1, 2p\,\sin^2\psi)\right\}. \tag{39}$$

With these prescriptions we can always write

$$I(p,q,\alpha,\beta) = U(p,q,\beta/2) - U(p,q,\alpha/2). \tag{40}$$

Even though we shall eventually be contemplating asymptotic expansions, convergence of such I-series is the rule:

Lemma 11 *For $\alpha, \beta \in (-\pi,\pi)$ and any complex pair (p,q), the series (38) converges absolutely.*

Proof. By relation (40) is enough to show that for $|\psi| < \pi/2$ each of U_1, U_2 converges absolutely. Since $\binom{2n}{n}4^{-n} = O(1/\sqrt{n})$, while $|\beta_n(p,\psi| \le |\sin\psi|^{n+2} e^{2|\Re(p)|}$, and the two product terms are bounded, the absolute convergence of both U_1, U_2 is assured. **QED**

Likewise, from (37), (38) we now have an absolutely convergent expansion for the special-case $\mathcal{I}$ integral:

$$\mathcal{I}(p,q) = 4e^p \int_0^{1/\sqrt{2}} G(-2iq,x) e^{-2px^2} \tag{41}$$

$$= 4e^p \sum_{k=0}^{\infty} \frac{g_k(-2iq)}{(2k)!} B_k(p), \tag{42}$$

where B_k is an error-function-class integral

$$B_k(p) := \beta_{2k-1}(p,\pi/4) = \int_0^{1/\sqrt{2}} x^{2k} e^{-2px^2}\,dx. \tag{43}$$

It is both computationally and theoretically important that B_k can be given a closed form (in terms of Γ- and incomplete Γ-functions) as well as a recursion relation. Namely, we have

$$B_k(p) = \frac{1}{2}\frac{1}{(2p)^{k+1/2}}\left\{\Gamma(k+1/2) - \Gamma(k+1/2,p)\right\}, \tag{44}$$

so that

$$B_0(p) = \frac{1}{\sqrt{8p}}\left(\sqrt{\pi} - \Gamma(1/2,p)\right) = \sqrt{\frac{\pi}{8p}}\,\mathrm{erf}\left(\sqrt{p}\right).$$

A recursion follows for $k > 0$:

$$B_k(p) := \frac{2k-1}{4p} B_{k-1} - \frac{2^{-k-3/2}}{p} e^{-p}, \tag{45}$$

We shall have more to say later about the computational efficacy of these relations. For the moment, we next focus upon theoretical applications relevant to the asymptotic nature of $\mathcal{I}$-integrals.

5.2 Effective expansion for the cosh-arc G-series

Our asymptotic analysis of representation (41) begins with a lemma that reveals how the G function (37) has an attractive self-similarity property. Namely, the function appears naturally in modified form within its own error terms.

Lemma 12 *For any complex τ the G function (37) can be given an effective expansion to any integer order $N \geq 0$, as*

$$\boxed{\begin{aligned} G(\tau, x) &:= \frac{\cosh(\tau \arcsin x)}{\sqrt{1-x^2}} \qquad (46) \\ &= \sum_{k=0}^{N-1} g_k(\tau)\frac{x^{2k}}{(2k)!} + g_N(\tau)\frac{x^{2N}}{(2N)!}\left(1+T_N(\tau,x)\right), \qquad (47) \end{aligned}}$$

with the error term T_N conditionally bounded over real $x \in [0, 1/\sqrt{2}]$ in the form

$$\begin{aligned} T_N &= \Theta : 1 \quad , \quad \text{if } N \geq 2x^2|\tau|^2 - 1; \\ &= \Theta : (\sqrt{2}\, x^2 + x|\tau|)e^{|\tau| \arcsin x} \quad , \quad \text{otherwise.} \end{aligned}$$

Proof. The error term is, by the definition of the $g_k(\tau)$, given by the absolutely convergent sum

$$T_N = h_N + h_N h_{N+1} + h_N h_{N+1} h_{N+2} + \ldots,$$

where

$$h_k := \frac{(2k+1)^2 + \tau^2}{(2k+1)(2k+2)} x^2.$$

When $|\tau|^2 x^2 \leq (N+1)/2$, and since $x^2 \leq 1/2$, it is immediate that for $k \geq N$ we have a bound:

$$|h_k| \leq \frac{(4k^2+4k+1)/2 + (N+1)/2}{4k^2+5k+2} \leq 1/2,$$

whence $T_N = \Theta : 1/2 + 1/4 + 1/8 + \ldots$, settling the first conditional bound of the theorem. In any case—i.e. any complex τ and any $x \in [0, 1/\sqrt{2}]$, we have for $j \geq 0$:

$$h_N h_{N+1} \cdots h_{N+j} = x^{2j} \prod_{k=0}^{j} \frac{(2N+2k+1)^2\left((2k+1)^2 + \tau^2 \frac{(2k+1)^2}{(2N+2k+1)^2}\right)}{(2N+2k+1)(2N+2k+2)(2k+1)^2}$$

$$= \Theta : \frac{g_{j+1}(|\tau|)}{(2j+1)!!^2}.$$

However, it is elementary that $(2j+1)!!^2 \geq (2j+1)!$ by simple factor-tallying, so

$$|T_N| \leq \frac{(1^2+|\tau|^2)\cdot 2}{2!}x^2 + \frac{(1^2+|\tau|^2)(3^2+|\tau|^2)\cdot 4}{4!}x^4 + \ldots$$

$$= x\frac{\partial}{\partial x}G(|\tau|, x),$$

where we have noticed that the right-hand series here is itself a differentiated "cosh-arc" series. Thus

$$T_N = \Theta : x\frac{\partial}{\partial x}\frac{\cosh(|\tau|\arcsin x)}{\sqrt{1-x^2}} = \Theta : \frac{x^2}{2\left(1-x^2\right)^{3/2}}\left(e^u + e^{-u}\right) + \frac{x|\tau|}{2\left(1-x^2\right)}\left(e^u - e^{-u}\right)$$

where $u := |\tau|\arcsin x$. Now by excluding $2x^2|\tau|^2 \le N+1$ for the second conditional bound of the lemma, we have $|\tau| \ge 1$, whence the e^{-u} terms can be ignored over $x \in [0, 1/\sqrt{2}]$, and the second conditional bound follows. **QED**

5.3 Effective expansion of the $\mathcal{I}$-integral

We need one more set of brief lemmas, all elementary but useful in regard to asymptotic analysis, including analysis that reaches beyond the present treatment. Consider the standard gamma function's common representation

$$\boxed{\Gamma(a,z) := \int_z^\infty t^{a-1}e^{-t}\,dt,}$$

for $z > 0$ and an alternative form, obtained by change of variables

$$\boxed{\Gamma(a,z) = z^a e^{-z}\int_0^\infty e^{-zs}(1+s)^{a-1}\,ds,}$$

the later integral representation being valid at least for conditions $((\Re(z) > 0)$ and $a \in \mathcal{C})$ or $((\Re(z) = 0)$ and $(\Re(a) < 0))$.

Lemma 13 *If real $\rho \ge 0$ and $\Re(z) > 0$,*

$$\int_0^\infty \frac{e^{-zs}}{(1+s)^\rho}\,ds = \Theta\left(\frac{2}{|z|}\right).$$

Accordingly, for real $a < 1$,

$$\Gamma(a,z) = \Theta\left(2|z|^{a-1}e^{-z}\right).$$

Proof. For $\rho = 0$ the result is trivial. For $\rho > 0$, integration by parts gives the integral as $1/z + (\rho/z)\Theta(1/\rho)$. The second result follows immediately from the alternative representation above. **QED**

The next lemma uses a standard gamma-recursion to bring gamma arguments of interest into the zone of applicability of Lemma 13:

Lemma 14 *For real a, integer $m \geq \max(0, a-1)$, $\Re(z) \geq 0, z \neq 0$, we have an effective expansion*

$$\Gamma(a,z) = z^{a-1}e^{-z}\left(1 + \frac{(a-1)}{z} + \cdots + \frac{(a-1)\cdots(a-m+1)}{z^{m-1}} + \Theta : 2 \cdot \frac{(a-1)\cdots(a-m)}{|z|^m}\right).$$

In particular, for integer $M \geq 0$, $\Re(z) \geq 0, z \neq 0$ and $|z| \geq 2M-1$, we have

$$\Gamma(M+1/2, z) = \Theta\left(2|z|^{M-1/2}e^{-z}\right).$$

Remark: Results such as these on incomplete-gamma errors appear in various texts on special functions, e.g., [22, §. 2.2, p. 110]. In the present treatment we have stated the bounds in a manner and style consistent with the rest of the present analysis.

Proof. The well known incomplete-gamma recursion, [1], for integer-depth $m \geq 0$ yields

$$\Gamma(a,z) = z^{a-1}e^{-z}\left(1 + \frac{(a-1)}{z} + \cdots + \frac{(a-1)\cdots(a-m+1)}{z^{m-1}}\right)$$

$$+(a-1)\cdots(a-m)\Gamma(a-m,z).$$

For real $a < 1$, take $m \geq 0$, whence Lemma 13 gives the error term as $\Theta : 2(a-1)\cdots(a-m)z^{a-m-1}e^{-z}$. Otherwise, take $m = \lfloor a+1 \rfloor$ and $|z| > 2(a-1)$ in which case

$$\Gamma(a,z) = z^{a-1}e^{-z} \cdot \Theta : 1 + 1/2 + 1/2^2 + \cdots + 1/2^{m-1} + 2/2^m$$

and the sum here is $\Theta(2)$. Finally, the assignment $a := M + 1/2$ yields the final theta bound as corollary. **QED**

The results of the present section may now be applied to a general, effective expansion of the $\mathcal{I}$-integral whenever $\Re(p)$ is sufficiently positive.

Theorem 7 **(Effective $\mathcal{I}$ expansion)** *For the integral*

$$\mathcal{I}(p,q) := I(p,q,-\pi/2,\pi/2) = \int_{-\pi/2}^{\pi/2} e^{-iq\omega}e^{p\cos\omega}\, d\omega,$$

assume an integer expansion order $N \geq 1$. Assume $\phi := \arg(p) \in (-\pi/2, \pi/2)$ and conditions

$$\Re(p) \geq 2N+1, \quad \Re(p) \geq 2\pi|q|^2.$$

Then we have an effective expansion

$$\mathcal{I}(p,q) = \sqrt{\frac{2\pi}{p}}\, e^p \left\{ \sum_{k=0}^{N-1} \frac{g_k(-2iq)}{k!\, 8^k}\frac{1}{p^k} + \Theta : \sqrt{\frac{8}{\pi p}}\, e^{-p}\cosh\left(\frac{\pi}{2}|q|\right) \right. \quad (48)$$

$$\left. + \frac{g_N(-2iq)}{N!\, 8^N}\frac{1}{p^N}\left(1 + \Theta : u_N \sec^{N+1/2}\phi\right)\right\}, \quad (49)$$

where the g_k *are as in the cosh-arc expansion (37), and we may take*

$$u_N := 1 + 2^N + \frac{2^{N+1}\Gamma(N+1)}{\sqrt{\pi}\Gamma(N+1/2)}; \tag{50}$$

however, with the extra condition $N \ge 4|q|^2 - 1$*, taking* $u_N := 1$ *suffices.*

Proof. Insertion of the series of Lemma 12 into representation (41) results in

$$\mathcal{I}(p,q) = 4e^p \left\{ \sum_{k=0}^{N} \frac{g_k(-2iq)}{(2k)!} B_k(p) + \frac{g_N(-2iq)}{(2N)!} \int_0^{1/\sqrt{2}} x^{2N} T_N(-2iq, x) e^{-2px^2}\, dx \right\},$$

where the $B_k(p)$ are given by (43). The sum over $k \in [0, N]$ here is thus

$$\frac{1}{2} \sum_{k=0}^{N} \frac{g_k(-2iq)}{(2k)!} \frac{\Gamma(k+1/2)}{(2p)^{k+1/2}}$$

$$+\ \Theta:\ e^{-p} \sum_{k=0}^{N} \frac{|g_k(-2iq)|}{(2k)!} \frac{1}{2^{k+1/2}p},$$

where the Θ-term here follows from Lemma 14 on our condition $\Re(p) \ge 2N+1$. But this very Θ-term is bounded above by

$$\frac{e^{-p}}{p\sqrt{2}} \sum_{k=0}^{N} \frac{g_k(2|q|)}{(2k)!} \frac{1}{2^k}$$

$$= \frac{e^{-p}}{p} \cosh\left(2|q| \arcsin\left(\frac{1}{\sqrt{2}}\right)\right),$$

so we have settled the summation and the $\cosh(\pi|q|/2)$ term in (48). Now consider the integral term

$$I_0 := \int_0^{1/\sqrt{2}} x^{2N} T_N(-2iq, x) e^{-2px^2}\, dx.$$

Define $\gamma := |q|^{-1}\sqrt{(N+1)/8}$. If $\gamma \ge 1/\sqrt{2}$ then by Lemma 12 we know $T_N = \Theta(1)$ and our theorem follows in the $u_N := 1$ case. Otherwise, $\gamma < 1/\sqrt{2}$ and we bound I_0 using two integrals

$$|I_0| \le \int_0^{1/\sqrt{2}} x^{2N} e^{-2\Re(p)x^2}\, dx + \int_\gamma^{1/\sqrt{2}} \left(\sqrt{2}\, x^{2N+2} + 2|q|\, x^{2N+1}\right) e^{2|q|\arcsin x - 2\Re(p)x^2}\, dx.$$

From Lemma 1 we know that the exponent here can be taken to be $\pi|q|x/\sqrt{2} - 2\Re(p)x^2$. For the assignments $\alpha := \pi|q|/\sqrt{8}$, $\beta := 2\Re(p)$ we have $\beta > 4\alpha/\gamma$ so that by Lemma 2 we have a bound

$$I_0 \ \le\ \frac{1}{2}\frac{\Gamma(N+1/2)}{(2\Re(p))^{N+1/2}} + \sqrt{2}\, V_{N+1}(\alpha,\beta,\gamma) + 2|q|\, V_{N+1/2}(\alpha,\beta,\gamma)$$

$$\leq \frac{1}{2}\frac{\Gamma(N+1/2)}{(2\Re(p))^{N+1/2}} + 2^N\frac{\Gamma(N+3/2)}{(2\Re(p))^{N+3/2}} + 2^{N+1}|q|\frac{\Gamma(N+1)}{(2\Re(p))^{N+1}}.$$

Using $|q|^2 \leq \Re(p)/(2\pi)$, $\Re(p) \geq 2N+1$, and writing $\Re(p) = p\cos\phi$ yields the theorem with the N-dependent u_N form.

QED

Note that the second (cosh) term of the expansion in Theorem 7 is relatively exponentially small, in that it decays like $e^{-\Re(p)}$, while the last term is a typical, effective "correction" to the asymptotic sum over $k \in [0, N]$.

6 Effective asymptotics for $L_n^{(-a)}(-z)$

At last we are in a position to provide explicit terms for Laguerre expansions in the sub-exponential-growth regime, which regime turns out to be precisely characterized by the parameter-pair requirement: $(a, z) \in \mathcal{D}$.

First, for convenience we recapitulate the thresholds for sufficiently large $m := n+1$ from our previous theorems, and add some more:

$$m_0 := |z|/4, \quad m_1 := 5\left(|\Re(z)| + (|\Im(a)| + |\Im(z)|/2)^2\right),$$

$$m_2 := \Re(a), \quad m_3 := -(5/4)|z|\cos\theta, \quad m_4 = 4|z|, \quad m_5 := |z|(1+|a|+|z|/2)^2,$$

$$m_6 := (2N+1)\sec^2(\theta/2)/|z|, \quad m_7 := 4\pi^2\sec^2(\theta/2)(|q|+3N-3)^4.$$

Here, $\theta := \arg(z)$ as before, while m_6, m_7 assume an asymptotic expansion order $N \geq 1$ in what follows. (We are aware that some of the m_i are masked by others; however it is best to assume all of the m_i are in force, because previous, partial results do depend on particular thresholds.)

Before delving into our main result, let us remind ourselves of previous nomenclature:

$$g_k(\tau) := \prod_{j=1}^{k}\left((2j-1)^2+\tau^2\right), \tag{51}$$

$$a_k := (1+a)(-1)^k\left(\frac{v\sqrt{z}}{2}\right)^k + \left(1-(-1)^k\right)\frac{k}{k+2}\left(\frac{v\sqrt{z}}{2}\right)^{k+2}, \tag{52}$$

$$\sum_{h=0}^{\infty} A_h x^h := \exp\left\{\sum_{k\geq 1}\frac{a_k}{k}x^k\right\}, \tag{53}$$

$$A_h =: \sum_{u=0}^{h}\alpha_{h,u}(a,z)v^{h+2u}. \tag{54}$$

Note that the $\alpha_{h,u}$ coefficients are thus implicitly defined, in terms of the original a_k functions. Notationally, we may write $\boxed{\alpha_{h,u} = [v^{h+2u}]A_h(v)}$, to indicate that we select the coefficient of v^{h+2u}.

6.1 The general sub-exponential expansion

Using Theorem 7 with $p := 2\sqrt{mz}$, and inserting this into formula (30), we arrive at our desired effective Laguerre expansion.

Theorem 8 (Effective Laguerre expansion) *Assume $(a, z) \in \mathcal{D}$. For asymptotic expansion order $N \geq 1$, and $m := n + 1$ sufficiently large in the sense $m > \max_{i\in[0,7]} m_i$, we have the expansion*

$$\boxed{L_n^{(-a)}(-z) = \frac{1}{2\sqrt{\pi}} \frac{e^{-z/2} e^{2\sqrt{mz}}}{z^{1/4-a/2}\ m^{1/4+a/2}} \left\{ \sum_{j=0}^{N-1} \frac{C_j}{m^{j/2}} + \frac{\overline{C}_N}{m^{N/2}} + \mathcal{E}_1 + \mathcal{E}_{3,N} \right\},} \tag{55}$$

where the expansion coefficients C_j are given finitely by

$$\boxed{C_j := \sum_{k=0}^{j} \frac{1}{16^k\ k!} \frac{1}{z^{k/2}} \sum_{u=0}^{j-k} \alpha_{j-k,u}(a, z)\, g_k(-2i(a + j - k + 2u)),} \tag{56}$$

in terms of (51) and (53), while the error term $\overline{C}_N$ is bounded as

$$\boxed{\begin{aligned}\overline{C}_N = \Theta : \sum_{v=1}^{N} \frac{1}{|z|^{v/2} 16^v v!} \left(1 + u_v \sec^{v+1/2} \frac{\theta}{2}\right) \sum_{u=0}^{N-v} |\alpha_{N-v,u}(a, z)\, g_v(-2i(a + N - v + 2u))| \\ + \Theta : 4(m_5/4)^{N/2} \sec^{1/2} \frac{\theta}{2},\end{aligned}}$$

with u_v taking the v-dependent form of (50) in Theorem 7.

The term $\mathcal{E}_1$ is sub-exponentially small (from Theorem 4), as is

$$\boxed{\mathcal{E}_{3,N} = \Theta : \sqrt{\frac{4}{\pi\sqrt{mz}}} e^{-2\sqrt{mz}} \sum_{h=0}^{N-1} \frac{1}{m^{h/2}} \sum_{u=0}^{h} |\alpha_{h,u}(a, z)| \cosh\left(\frac{\pi}{2}|a + h + 2u|\right).}$$

Proof. For brevity we leave out the details—all of which are straightforward, if tedious applications of the previous theorems and formulae. **QED**

We now have a resolution of the domain of sub-exponential growth, as:

Corollary 1 *For $(a, z) \in \mathcal{D}$, the Laguerre polynomial grows sub-exponentially, in the sense that for order $N \geq 1$, and any $\varepsilon > 0$,*

$$\boxed{L_n^{(-a)}(-z) = S_n(a, z) \left\{ \sum_{j=0}^{N-1} \frac{C_j}{m^{j/2}} + O\left(\frac{1}{m^{N/2}}\right) + O\left(e^{-(2-\varepsilon)\sqrt{m|z|}\cos(\theta/2)}\right) \right\},}$$

with all coefficients and the implied big-O constant effectively bounded via our previous theorems. Moreover, for $(a, z) \notin \mathcal{D}$, the large-$n$ growth is not sub-exponential. Thus, the precise domain of sub-exponential growth is characterized by $(a, z) \in \mathcal{D}$.

Proof. The given sub-exponential formula is a paraphrase of Theorem 8. Now assume $z \in (-\infty, 0]$. We already know $z = 0$ does not yield such growth (see (2). Now for z negative real, note that the integrals in (17, 19) are both decaying in large m. Finally, the integral in (16) has phase factor $|\exp(2\sqrt{mz}\cos\omega)| \leq 1$, and Lemma 6 show that c_1 also cannot grow sub-exponentially in m.

QED

This corollary echoes, of course, the classical Perron result (7), and we again admit that historical efforts derived the C_j coefficients in principle. What we have done up to this point is

a) established such Laguerre asymptotics over the full sub-exponential-growth domain of $(a, z) \in \mathcal{D}$;

b) established rigorous, explicit (i.e. effective) errors over said domain;

c) provided in passing an algorithm for generating the C_j and, perforce, effective bounding constants.

We now turn to the algorithmic and computational aspects of the general asymptotics.

6.2 Algorithm for explicit asymptotic coefficients

Theorem 8 indicates that, to obtain actual C_j coefficients, we need the cosh-arc numbers $g_k(\tau)$ and the $\alpha_{h,u}(a, z)$ coefficients. Observe that the chain of implications (51) amounts to an algorithm for generation of the C_k. All of this can proceed via symbolic processing, noting that v is simply a place-holder throughout.

Remarkably, there is another approach—a fast algorithm that bypasses much of the symbolic tedium. First, we have an explicit recursion for A_h, with $A_0 := 1$, as

$$\boxed{A_k = \frac{1}{k}\sum_{j=0}^{k-1} A_j\, a_{k-j}} \tag{57}$$

as follows from differentiating (53) logarithmically, and then comparing terms. Second, when we use (54) together with (57), we obtain a recursion *devoid* of the symbolic placeholder v, as

$$\boxed{\alpha_{k,u} = \frac{1}{k}\sum_{j=0}^{k-1}\left(\alpha_{j,u} b_{k-j} + \alpha_{j,u-1} d_{k-j}\right),} \tag{58}$$

where these new recursion coefficients are

$$\boxed{\begin{aligned} b_h &:= (-1)^h (1+a)\left(\frac{\sqrt{z}}{2}\right)^h, \\ d_h &:= \left(1-(-1)^h\right)\frac{h}{h+2}\left(\frac{\sqrt{z}}{2}\right)^{h+2}. \end{aligned}}$$

It is important that in practice we define $\alpha_{0,0} := 1$ and force any $\alpha_{j,u}$ with $u > j$ or $u < 0$ to vanish. In this sense, the collection of $\alpha_{k,u}$ make up a *lower-triangular matrix*, e.g. the entries for $k \le 3$ appear thus:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{(a+1)}{2} z^{1/2} & \frac{1}{12} z^{3/2} & 0 & 0 \\ \frac{a^2+3a+2}{8} z & -\frac{(a+1)}{24} z^2 & \frac{1}{288} z^3 & 0 \\ -\frac{a^3+6a^2+11a+6}{48} z^{3/2} & \frac{5a^2+15a+16}{480} z^{5/2} & -\frac{a+1}{576} z^{7/2} & \frac{1}{10368} z^{9/2} \end{pmatrix},$$

where $\alpha_{3,3}$ is the lower-right element here.

These observations lead to a fast algorithm for computation of the asymptotic coefficients:

– – – – –

Algorithm (Fast computation of Laguerre coefficients.) For given $(a, z) \in \mathcal{D}$ and desired expansion order N, this algorithm returns the asymptotic coefficients $(C_k : k \in [0, N])$ of relation (56), Theorem 8.

1) Set $\alpha_{0,0} := 1$ and for desired order N, calculate the lower-trianglar matrix elements $(\alpha_{k,u} : 0 \le u \le k \le N)$ via a recursion such as

$\alpha(k, u)\{$
 if$(k == 0)$ return $\delta_{0,u}$;
 return $\frac{1}{k} \sum_{j=0}^{k-1} (\alpha_{j,u} b_{k-j} + \alpha_{j,u-1} d_{k-j})$;
$\}$

or use an unrolled, equivalent loop (i.e., one may generate the left-hand column of the α-matrix, then fill in one row at a time, lexicographically).

2) Use the recursion $g_k(\tau) = \left((2k+1)^2 + \tau^2\right) g_{k-1}(\tau)$ and the lower-triangular matrix of α values to generate the C coefficients via (56).

– – – – –

We observe that the expensive sums in this algorithm are all *acyclic convolutions*. Thus, for numerical input (a, z) the algorithm complexity turns out to be $O\left(N^{2+\epsilon}\right)$ arithmetic operations, with the "2" part of the complexity power arising from the area of the lower-triangular sector.[7]

We employed the algorithm to generate exact asymptotic coefficients as follows:

$$C_0 = 1,$$

$$C_1 = \frac{-12a^2 - 24za + 4z^2 - 24z + 3}{48\sqrt{z}},$$

[7] For example, floating-point FFT-based convolutions of length L require $O(L \log L)$ operations, and we are calling that $O\left(L^{1+\epsilon}\right)$.

$$C_2 = \frac{1}{4608z}\left(144a^4 + 576za^3 + 480z^2a^2 + 1728za^2 - 360a^2 - 192z^3a \right.$$
$$\left. +\ 1152z^2a + 1584za + 16z^4 - 192z^3 + 312z^2 + 432z + 81\right),$$
$$C_3 = \frac{1}{3317760z^{3/2}}\left(-8640a^6 - 51840za^5 - 95040z^2a^4 - 259200za^4 + 75600a^4 \right.$$
$$-34560z^3a^3 - 622080z^2a^3 - 388800za^3 + 31680z^4a^2 - 103680z^3a^2 - 1395360z^2a^2$$
$$-129600za^2 - 139860a^2 - 5760z^5a + 69120z^4a + 60480z^3a - 1192320z^2a + 100440za$$
$$\left. +320z^6 - 5760z^5 + 7632z^4 + 129600z^3 - 267300z^2 + 48600z + 30375\right),$$

and so on.

Note that the C_1 form here agrees with the PAMO coefficient in (9). We were able to generate the full, symbolic $C_{64}(a, z)$ in about one minute of CPU on a typical desktop computer. To aid future researchers, we report that numerator of C_{64} has degree 128 in both a, z, while the denominator is

$$\begin{gathered}51966771566251701246166021641279466239698651933612951268704090451225 9\\ 134698859029589268254378668537757499729050302014891552685362781283752\\ 172777457411452513716217756260619192712841993912320000000000000000000\\ 0000000\end{gathered}$$

It is not unexpected that every prime number not exceeding 64 divides this denominator.

Attempts to simplify the formidable, bivariate $C_k(a, z)$ coefficients led us also to the following:

Conjecture 1 *Given index $k \geq 1$, consider the specific parameter assignments a in $\{-1/2, -3/2, \ldots, -1/2 - (k-1)\}$. Then for any such value of a, the coefficient $C_k(a, z)$ factors into a positive power of z times a rational polynomial in z, such that the total degree in z is $3k/2$.*

For instance with $k = 4$ we obtain, with total degree 6 in each case:

$$C_4(-7/2, z) = \frac{z^2\left(5z^4 + 300z^3 + 4842z^2 + 14580z - 63747\right)}{2488320}.$$

$$C_4(-5/2, z) = \frac{z^2\left(5z^4 + 180z^3 + 882z^2 - 10692z - 39771\right)}{2488320},$$

$$C_4(-3/2, z) = \frac{z\left(5z^5 + 60z^4 - 1038z^3 - 6804z^2 + 17253z + 29160\right)}{2488320},$$

$$C_4(-1/2, z) = \frac{z\left(5z^5 - 60z^4 - 918z^3 + 6804z^2 + 16605z - 48600\right)}{2488320}.$$

It seems very likely that Conjecture 1 is a harbinger of much more subtle structure that we have not yet been able to abstract.

6.3 Generating and verifying effective bounds

For the first nontrivial effective bound, we can employ the rigorous bound $\overline{C}_1$ in Theorem 8, with $N = 1$, to get an effective version of the original asymptotic (3), as

$$L_n^{(-a)}(-z) = S_n(a,z)\left(1 + \frac{\overline{C}_1}{\sqrt{m}} + \ \mathcal{E}_1 + \mathcal{E}_{3,1}\right),$$

with

$$\overline{C}_1 = \Theta : \frac{\left|1-4a^2\right|}{16|z|^{1/2}}\left(1 + 6\sec^{3/2}\frac{\theta}{2}\right) + 2|z|^{1/2}(1+|a|+|z|/2)\sec^{1/2}\frac{\theta}{2},$$

and we remind ourselves that $\mathcal{E}_1, \mathcal{E}_{3,1}$ are both sub-exponentially small.

At last we have an effective numerator, then, for the $1/\sqrt{m}$ asymptotic term. Though this effective numerator is almost surely nonoptimal, we are evidently on the right track, because the exact C_1 asymptotic coefficient above has very much the same form as does our $\overline{C}_1$ here (i.e., same degrees of appearance for a, z, and similar coefficients). And, it is easy to see that $\overline{C}_1$ is an upper bound on $|C_1|$ itself, as must of course be true.

In spite of the unwieldy character of the exact C_k coefficients, it is possible to verify numerically the asymptotic expansion, at least to a few orders. One good worked example is to take order $N = 4$, giving the partial series

$$\boxed{\frac{L_n^{(-a)}(-z)}{S_n(a,z)} \sim 1 + \frac{C_1}{m^{1/2}} + \frac{C_2}{m} + \frac{C_3}{m^{3/2}}}$$

and compare direct summation of (1) with the right-hand asymptotic piece. The results are (note that the S_n denominator here is of order 10^{64}):

$$\boxed{\frac{L_{5000}^{(-1-i)}(-1+i)}{S_{5000}} \approx 0.98514574 - \ 0.0080754i,}$$

from the direct sum, while the righthand-side sum over the $m^{-k/2}$ gives

$$\boxed{\approx 0.98514577 - \ 0.0080755i.}$$

This amounts to an absolute error of $\approx 9.80 \cdot 10^{-8}$, or more usefully stated, $\approx 2.45187/m^2$. The experimental situation is good, since the first missing asymptotic coefficient is $C_4(1+i, 1-i) \approx 2.48$.

However, regarding rigor, more important than these good approximations is the effective coefficient $\overline{C}_4$, which for the current parameters we get from Theorem 8 as

$$\overline{C}_4(1+i, 1-i) = \Theta : 202.63.$$

This amounts to a little over 2 decimals of penalty, in trade for rigor. If we ignore the sub-exponentially small error terms $\mathcal{E}_1, \mathcal{E}_{3,4}$ (which are well below the 7-digit accuracy threshold) then we find that our 7-decimal-accurate asymptotic piece is rigorously correct to 4 places. In a word: One can *prove* in this way that

$$\boxed{L_{5000}^{(-1-i)}(-1+i) = ((0.8053 \pm 0.0001) + \ (1.1483 \pm 0.0001)\ i) \cdot 10^{64},}$$

and it should be no surprise that this partial-but-rigorous asymptotic approach is radically faster—for such large n—than naive direct summation.[8]

So this is our technique for obtaining provable results about the large-n behavior of L_n. We have mentioned research motives in Section 1.1. One additional application might be to use such effective bounds to rule out zeros of L_n in the crossed (a, z)-plane, that is, to locate sufficiently large m such that the first terms $1 + C_1/\sqrt{m}$ cannot be overwhelmed by the rest of the terms.

7 The exp-arc method and oscillatory behavior

Though the exp-arc method has succeeded in establishing rigorous asymptotics for sub-exponential growth, there remains the issue of the Fejér form (6) for $z \neq 0$ on the cut $(-\infty, 0]$. Such oscillatory behavior can indeed be dealt with, but separate techniques come into play. For one thing, contour integrals must be handled differently.

7.1 Remarks on the oscillatory Laguerre regime

For this next analysis we shall proceed non-rigorously, in that effective bounds are problematic for z negative real. Yet, we shall still gain insight, and find some ways to provide at least some low-order bounds.

We remind ourselves that even on $z \in (-\infty, 0)$ the contour prescription of Figure 1 is valid, and the Laguerre polynomial is exactly the sum $c_1+d_1+e_1$, with $R := \sqrt{m/|z|} > 1/2$ being the only requirement for contour validity. However—and this is important—the dominant contribution (22) has to change, to involve an expanded integration interval; in fact, now we must use the contour integral c_1 itself as the main contribution:

$$\boxed{c_1 \quad := \quad \frac{1}{2\pi} r^{-a} e^{-z/2} \int_{-\pi/2}^{3\pi/2} \mathcal{H}_m \left(a, z, e^{-i\omega}\right) e^{2\sqrt{mz}\cos\omega} \, d\omega,} \tag{59}$$

where we have used our convention $\theta := \arg(z) = \pi$ for negative real z. This wider integration range is necessary because the procedure of Theorem 3, which peeled off the sub-exponentially small c_2 integral, fails when $\exp(2\sqrt{mz}\cos\omega)$ has constant magnitude, as it does for negative real z. Another way to view this is that we need to include *two* stationary points for $\cos\omega$ in the integration interval, namely $\omega = 0, \pi$.

These observations lead to an analysis of the wider integral

$$\boxed{\int_{-\pi/2}^{3\pi/2} e^{-iq\omega} e^{p\cos\omega} \, d\omega = \mathcal{I}(p, q) + e^{-i\pi q} \, \mathcal{I}(-p, q),}$$

easily established by bisecting the range $[-\pi/2, 3\pi/2]$. In turn, the main contributor to

[8]We recall that the direct series (1) admits of various accelerations.

Laguerre evaluations should be, rather than (30), the form

$$\begin{aligned} c_1 &= S_n(a,z)\,\mathcal{E}_{4,N} + \frac{1}{2\pi} r^{-a} e^{-z/2} \sum_{h=0}^{N-1} \frac{1}{m^{h/2}} \sum_{u=0}^{h} \alpha_{h,u}(a,z) \\ &\times \left\{ \mathcal{I}(2\sqrt{mz}, a+h+2u) + e^{-i\pi(a+h+2u)} \mathcal{I}(-2\sqrt{mz}, a+h+2u) \right\} \end{aligned} \tag{60}$$

where $\mathcal{E}_{4,N}$ is an N-th order error term which we shall not calculate here. These machinations do correctly give the leading (cosine) term of the classical Fejér expansion (6), and presumably, in analogy with Theorem 8, yield closed forms for the general coefficients in the Perron form for negative real z [30, Theorem 8.22.2]. Because we are not claiming rigor in this subsection—and especially as we do not yet have a comprehensive exp-arc approach to effective bounds in this case—we simply claim without proof, on the basis of (60), that the correct generalization of the Fejér series (6) for z on the open negative cut $(-\infty, 0)$, and any complex a, is

$$\begin{aligned} L_n^{(-a)}(-z) &\sim \frac{e^{-z/2}}{\sqrt{\pi}(-z)^{1/4-a/2}\, m^{1/4+a/2}} \\ \times &\left\{ \left(\sum_{k=0}^{\infty} \frac{A_k}{m^k} \right) \cos\left(2\sqrt{-mz} + a\pi/2 - \pi/4\right) + \left(\sum_{k=0}^{\infty} \frac{B_k}{m^{k+1/2}} \right) \sin\left(2\sqrt{-mz} + a\pi/2 - \pi/4\right) \right\}, \end{aligned} \tag{61}$$

where these oscillatory-series coefficients are directly related to the coefficients in Theorem 8 by

$$\begin{aligned} A_k &:= C_{2k}(a,z), \\ B_k &:= iC_{2k+1}(a,z). \end{aligned}$$

It turns out that for a real, every A_k, B_k is then real, whence the asymptotic has all real terms. An important observation is relevant here: The aforementioned Szegö Theorem 8.22.2 for the oscillatory Laguerre mode is stated in a fashion structurally different from our asymptotic (61); for one thing we are conjecturing that Sezgö's own $A_{\mathrm{odd}}, B_{\mathrm{even}}$—which are not defined quite like ours here—vanish.[9]

We did perform numerical experiments on the expansion (61). A worked example is for $n = 2880, a = -1+i, z = -16$, for which we calculated the prefactor P, the obvious angle χ, and the coefficients A_0, A_1, A_2, B_0, B_1 to obtain

$$\begin{aligned} L_{2880}^{(1-i)}(16) &\approx P \cdot \left\{ \left(1 + \frac{C_2}{m} + \frac{C_4}{m^2}\right)\cos\chi + \left(\frac{iC_1}{m^{1/2}} + \frac{iC_3}{m^{3/2}}\right)\sin\chi \right\} \\ &= (2.30\ldots) \ + \ (0.67839\ldots)\, i, \end{aligned}$$

correct to the implied precision. Note that the imaginary part is considerably more accurate—one of the typical complicating effects of allowing complex a parameter—and signaling the considerable difficulty of effective oscillatory expansions for general parameters.

[9] Our oscillatory asymptotic (61) has been verified to several terms by N. Temme; also he verifies our claim that the classical Szegö coefficients do vanish for the parities indicated.

7.2 Application of the exp-arc method to Bessel functions

The history of Bessel-function asymptotics is one of the great success stories in the annals of analysis. As early as 1823, Poisson developed the beginnings of Bessel asymptotics [36] [32], and eventually Hankel developed the classic, complete asymptotic series [1, §9.2.5], and effective bounds on error terms (for certain parameter domains) are well known, and in many cases optimal [36].

It is instructive to explore, at least partially, the application of our exp-arc method to Bessel expansions. We recall (34), (35) which we repeat here:

$$J_n(z) = \frac{1}{2\pi}\left(e^{-i\pi n/2}\mathcal{I}(iz,n) + e^{i\pi n/2}\mathcal{I}(-iz,n)\right), \tag{62}$$

$$I_n(z) = \frac{1}{2\pi}\left(\mathcal{I}(z,n) + (-1)^n\mathcal{I}(-z,n)\right), \tag{63}$$

both valid for *integer* n.

7.3 Asymptotics for Bessel functions I_n, J_n

Because our present methods are geared toward large-n growth, the modified Bessel function I_n, which for real argument is non-oscillatory, is easier to analyze with the exp-arc method. Theorem 7 implies an asymptotic expansion, based on our absolutely convergent series for $Re(z) > 0$, according to

$$\begin{aligned} I_n(z) &= \frac{2}{\pi}\sum_{k\geq 0}\frac{g_k(-2in)}{(2k)!}\left\{e^z B_k(z) + (-1)^n e^{-z}B_k(-z)\right\} \\ &\sim \frac{e^z}{\sqrt{2\pi z}}\sum_{k=0}^{\infty}\frac{g_k(-2in)}{k!\;8^k}\frac{1}{z^k}, \end{aligned} \tag{64}$$

where the first sum is convergent, exact, and the second sum agrees with the classical Hankel asymptotic [36] [1]. Moreover, under the conditions $\theta := \arg(z) \in (-\pi/2, \pi/2)$, $\Re(z) \geq \max\left(2N+1, 2\pi n^2\right)$, and $N > 4n^2 - 1$, the error on truncating the asymptotic series at $N-1$ summands inclusive is, again by Theorem 7,

$$\Theta : \frac{g_N(-2in)}{N!\;8^N}\frac{1}{z^N}\left(1 + \sec^{N+1/2}\theta\right),$$

plus, of course, some sub-exponential terms. Thus, typical rigorous error bounds in the literature are recovered, or at least suggested (e.g., real positive z yields a classic result that the first missing term with z^{-N} carries only a factor-of-2 magnitude penalty).

Now to a brief analysis of frank oscillatory behavior. Asymptotics for $J_n(z), \Re(z) > 0$ are possible via the exp-arc approach, but the details are rather intricate. Let us give at least an example of how an exp-arc series might be used to establish bounds. First, there is a useful lemma that often applies in oscillatory cases:

Lemma 15 *Let $F(x)$ be real, twice differentiable, with $|F''(x)| \geq \rho > 0$ for x on real interval (a, b). If a real function $G(x)$ on said interval has G/F' is monotonic, $|G| \leq M$, then*

$$\int_a^b G(x)e^{iF(x)}\ dx = \Theta : \frac{8M}{\sqrt{\rho}}.$$

Proof. This is proved in [33]. **QED**

Then, we follow with an implication specific to the present work:

Lemma 16 *For any complex p with $\Re(p) \geq 0$, and positive real ν, we have*

$$B_\nu(p) := \int_0^{1/\sqrt{2}} x^{2\nu} e^{-2px^2}\ dx = \Theta : \frac{9}{2^\nu} \frac{1}{\sqrt{|p|}}.$$

Proof. We establish two different upper bounds. First, if $\Re(p) > 0$ the integral is bounded as

$$B_\nu(p) = \Theta : \frac{1}{2^\nu} \int_0^\infty e^{-2\Re(p)x^2}\ dx = \frac{1}{2^{\nu+1}} \sqrt{\frac{\pi}{2\Re(p)}}.$$

Second, if $\Im(p) > 0$ we may use Lemma 15, with $G := x^{2\nu} e^{-2\Re(p)x^2}$ and $F := -2\Im(p)x^2$. Now this G has at most two branches of monotonicity, so we conclude

$$B_\nu(p) = \Theta : \frac{1}{2^\nu} \frac{8}{\sqrt{|\Im(p)|}}.$$

Then we simply observe that $9 \cdot 2^{-\nu}|p|^{-1/2}$ is always an upper bound for one of these two bounds.

QED

With these lemmas in hand, we can peel off some desired number of summands from (41) and perform integration by parts on the tail of the sum, as in the following example where just one term is peeled off:

$$\mathcal{I}(p, q) = 4e^p \left(B_0(p) + \int_0^{1/\sqrt{2}} f(x) e^{-2px^2}\ dx \right), \tag{65}$$

$$f(x) := \frac{\cos(2q \arcsin x)}{\sqrt{1 - x^2}} - 1.$$

Then integration by parts yields

$$\mathcal{I}(p, q) = 4e^p \left(B_0(p) - \frac{1}{4p} \frac{f(x)}{x}\Big|_{x=0}^{1/\sqrt{2}} + \frac{1}{4p} \int_0^{1/\sqrt{2}} \frac{d}{dx} \frac{f(x)}{x}\ dx \right).$$

These machinations lead to

Lemma 17 *For $\Re(p) > 0$ and any complex q we have a first-order expansion with effective error bound:*

$$\boxed{\mathcal{I}(p,q) = \sqrt{\frac{2\pi}{p}}e^p - \frac{2}{p}\cos\left(\frac{\pi}{2}q\right) \;+\; \Theta : \frac{\sqrt{2}}{p^2} \;+\; \Theta : C(q)\frac{e^p}{p^{3/2}},}$$

where

$$\boxed{C(q) := 9\sum_{k\geq 1}\frac{|g_k(-2iq)|\,(2k-1)}{(2k)!\;2^k}.}$$

Proof. The proof is a straightforward application of relation (44), Lemma 14, and Lemma (16).

QED

With the above ideas we have, at least, a first-order effective bound for Bessel functions in the oscillatory regime:

Theorem 9 *For integer n and $\Re(z) > 0$, we have an effective Bessel asymptotic, in the form*

$$\boxed{J_n(z) = \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{\pi}{2}n - \frac{\pi}{4}\right)}$$
$$+\Theta : \frac{\sqrt{2}}{\pi}\frac{1}{|z|^2}$$
$$+\Theta : \frac{D(n)}{|z|^{3/2}},$$

where

$$\boxed{D(n) := \frac{9}{\pi}\sum_{k\geq 1}\frac{|g_k(-2in)|\,(2k-1)}{(2k)!\;2^k}.}$$

Moreover, $D(n)$ can be given a closed form. In particular, for $J_0(z)$ the k-sum evaluates to 2, while for $J_1(z)$ the sum is $4\sqrt{2}-2$.

Proof. The proof follows from the sum (62) and Lemma 17. The specific evaluations of the k-sum arise from analyzing $(d/dx)f/x$ in relation (65), noting that because of the structure of the g_k, one only need find the k where g_k changes sign, thereby evaluating at most two sums without absolute-valuing every summand.

QED

Is this Theorem 9 as strong as historical knoeledge, such as the known effective bounds on the celebrated Hankel expansion—as expounded by Watson [36, Ch. VII]? No, but we have shown that effective bounds in the presence of oscillation can indeed arise from the exp-arc approach. Theorem 9 is thus displayed to convey the basic ideas; extension of our 1st-order effective term to higher orders, via the exp-arc method *per se*, remains an open research problem.

7.4 Geometrically convergent algorithms for J_n

Computationalists have know for decades that one way to evaluate Bessel functions uniformly in the argument z is to use the standard ascending series for small $|z|$, but an asymptotic series for large $|z|$. However, via the exp-arc method one can establish a converging series whose evaluation only involves *a single error-function evaluation*, followed by recursion and elementary algebra. In fact, the relations (44), (45) can be used to calculate $J_n(z)$ from relation (62) via the sum

$$J_n(z) = \frac{2}{\pi} \sum_{k=0}^{\infty} g_k(-2in)\,(b_k \cos\chi - c_k \sin\chi)\,, \tag{66}$$

with angle

$$\chi := z - \pi n/2 - \pi/4,$$

and the coefficients b_k, c_k determined by

$$b_k := B_k(iz)e^{i\pi/4} + B_k(-iz)e^{-i\pi/4},$$
$$ic_k := B_k(iz)e^{i\pi/4} - B_k(-iz)e^{-i\pi/4}.$$

Note that if z is real then each b_k, c_k is real, whence our series here has all real terms. Note that our recursion (45) likewise ignites a recursion amongst the b_k, c_k.

Note that (66) is actually the *classical Hankel asymptotic* if we replace B_k by its first term in (44), namely $(1/2)(2iz)^{-k-1/2}\Gamma(k+1/2)$; however, we already know that the sum (66) is always convergent. It is remakakble that we are using the same structure as the classical asymptotic, yet convergence for all complex z is guaranteed. Moreover, the $B_k(iz)$ are independent of the order n and so can be re-used if multiple $J_n(z)$ are desired for fixed z.

As just one experiment, we found that $J_{12}(8008+45i)$ is evaluated to 25 good decimals. using 60 terms (i.e. $k \in [0, 60]$) of the convergent sum. The primary points are a) the scheme is unconditionally convergent, and b) the b_k, c_k can be rapidly evaluated via recursion, after a single evaluation of $\operatorname{erf}\left(\sqrt{iz}\right)$. Incidentally, erf can be calculated via continued fraction, if one wants to avoid recourse to the standard ascending-asymptotic change of gears.[10] In this way, one has a universal Bessel-computation scheme happily devoid of asymptotic accuracy and stability issues.

7.5 Hadamard expansions and the work of R. Paris

Those acquainted with the intricacies of Bessel theory may observe that convergent expansion (64) is at least reminiscent of the convergent *Hadamard expansion* found in [36, p.204] for the modified Bessel function I_ν, as

$$I_\nu(z) \;=\; \frac{e^z}{\sqrt{2\pi z}\,\Gamma\left(\nu+\frac{1}{2}\right)} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}-\nu\right)_k}{(2z)^k k!} \left\{\Gamma\left(k+\nu+1/2\right) - \Gamma\left(k+\nu+1/2, 2z\right)\right\} \tag{67}$$

[10] One might argue that due to possible asymptotic schemes for incomplete-gamma (i.e. erf()), one is not truly avoiding the pitfalls of asymptotics. But this is wrong: It is known that incomplete-gamma (erf()) can be evaluated with always converging continued fractions, for example.

No. of summands K	$\|I_n - (\text{Hadamard})\|$	$\|I_n - (\text{exparc})\|$
32	10^{-7}	10^{-13}
64	10^{-9}	10^{-23}
128	10^{-10}	10^{-42}
256	10^{-11}	10^{-81}

Table 1: Comparison of Hadamard series (67) and exp-arc series (64) errors for Bessel evaluation $I_n(10); n = 4$. The Hadamard series error behaves as $K^{-n-1/2}$ while the exp-arc error behaves as $K^{-n-1/2}2^{-K}$ (which is geometrical (linear) convergence) [23].

Though both series are absolutely convergent, here are some important differences between this Hadamard expansion and our exp-arc forms (64, 66). For example we have given our convergent sum only for integer ν. Moreover, the exp-arc expansion is geometrically convergent, while the Hadamard expansion is genuinely slower. Also note the geometrically convergent analogue (66) for J_ν, integer ν. To be more concrete for the Bessel evaluation of $I_4(10)$, we have displayed various respective errors in Table 1.

The whole research area of convergent expansions related to classical, asymptotic ones has been pioneered in large part by R. Paris, whose works cover real and complex domains, saddle points, and the like [24] [25] [26] [27]. We should point out that Paris was able to develop within the last decade some similar, linearly convergent Bessel series by modifying the "tails" of Hadamard-class series. One open research problem is: What transformations move between Paris' convergent Bessel series and the exp-arc forms? One confounding aspect of this question is that the exp-arc method is only developed, so far, for integer indices. Also interesting: It may well be possible to develop effective bounds—of the type in the present treatment—along the lines of the Paris theory.

For general indices ν, it may also be possible to provide an exp-arc series using a representation valid for all cases ($\Re(z) > 0$) or ($\Re(z) = 0$ and $\Re(\nu) > 0$), namely [36, p. 176]

$$J_\nu(z) = \frac{1}{\pi}\int_0^\pi \cos(\nu t - z\sin t)\,dt - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-\nu t - z\sinh t}\,dt, \tag{68}$$

with a corresponding representation

$$I_\nu(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos t}\cos(\nu t)\,dt - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-\nu t - z\cosh t}\,dt, \tag{69}$$

itself valid for the same cases of z, ν. One wonders whether an exp-arc approach can be used to resolve the integrals here—which contribute when ν is not an integer—as exp-arc series.

8 Open problems

- How might one proceed with the exp-arc theory to obtain effective error bounds for oscillatory Laguerre modes, and-or oscillatory Bessel modes? We know that previous researchers have described how to give effective bounds in these cases (e.g., to our asymptotic (61), as in [30, §8.72]) but once again we stress: How can this be done explicitly, and for full parameter ranges?
- Though we did provide an analytic/symbolic algorithm for such, is there any hope for a fully closed form for the asymptotic-series coefficients $C_k(a, z)$ in $L_n/S_n \sim 1 + C_1/m^{1/2} + C_2/m + C_3/m^{3/2} + \ldots$? And what about our Conjecture 1?
- Where are the zeros in the complex z-plane—for fixed a—of $L_n^{(-a)}(-z)$? Are "most of" the zeros along some a-dependent ray, in some sense? Note that effective error bounds conceivably could help in addressing this problem—by ruling out vast zero-free regions. There is a considerable literature on this zero-free-region topic, especially for polynomials in real variables. For example, with $a := 0$ the Laguerre zeros are all real and negative; see [20, Ch. X] and references therein. There is also an interesting connection between Laguerre zeros and eigenvalues of certain (large) matrices [10].
- How can the discrete iteration (10) be used *directly* to glean information about sub-exponential growth? One would think that insertion of a formal asymptotic form into the recursion would force certain relations between coefficients—but this is easier said than done, at least for the current authors. One may ask the same question for the Laguerre differential equation (12) as starting point. A promising research avenue for a discrete-iterative approach to asymptotics is [38].
- It would be useful to establish the very most efficient way to calculate $J_n(z)$ with our converging series (66) and to know, for given arguments n, z how many terms of the exp-arc sum yield b good bits in the answer for $J_n(z)$. It should also be possible to extract the classical ascending series for J_n directly from our converging series.
- Can the integral pieces of (68, 69) be resolved as exp-arc series, to provide even more general, universally convergent I, J series (i.e. for noninteger ν)?
- Our highly efficient "keyhole" contour of Figure 1 was discovered experimentally. What other analytical problems might be approached in this (rather unexpected) fashion? For that matter, how might one properly use the celebrated Watson loop-integral lemma with error term [22, Theorem 5.1] on our keyhole contour to obtain similar effecive asymptotics?
- How can one go to arbitrary asymptotic-expansion orders with the exp-arc method in the presence of oscillatory behavior; beyond, say, Theorem 9? If this is possible for Bessel functions, it may well apply also to the more formidable Laguerre polynomials.
- What is the fastest was to evaluate the converging Bessel sum (66) to, say, extreme precision? If the argument z is real, what is a good way to avoid complex arithmetic per se for the evaluation of $\Gamma(1/2, iz)$—being as we know that all coefficients b_k, c_k will end up real-valued?

9 Acknowledgements

The authors are indebted to J. Buhler, R. Mayer, O'Yeat Chan, R. Paris, and N. Temme for the generous lending of expertise, ideas and suggestions.

References

[1] Ambramowitz and Stegun, *Handbook of Mathematical Functions*, NBS (now NIST), 1965.

[2] David H. Bailey, Jonathan M. Borwein and Richard E. Crandall, "Integrals of the Ising class," *J. Phys. A.: Math. Gen.*, to appear (2007). Also, D-drive Preprint #324, 2006, and `http://crd.lbl.gov/~dhbailey/dhbpapers/ising.pdf`.

[3] J.M. Borwein and P. B. Borwein, *Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity*, John Wiley, New York, 1987. Paperback edition 1998.

[4] Jonathan Borwein and Marc Chamberland, "Integer powers of Arcsin," Preprint 2005, [D-drive Preprint 288].

[5] Jonathan Borwein, Richard Crandall and Greg Fee, "On the Ramanujan AGM fraction. Part I: the Real-parameter Case," *Experimental Mathematics,* **13** (2004), 275–286. [CECM Preprint 2003:208/213]

[6] Jonathan Borwein and Richard Crandall, "On the Ramanujan AGM fraction. Part II: the Complex-parameter Case," *Experimental Mathematics,* **13** (2004), 287–296 . [CECM Preprint 2003:214]

[7] D. Borwein, J. Borwein, R. Crandall and R. Mayer, "On the dynamics of certain recurrence relations," *Ramanujan Journal* (Special issue for Richard Askey's 70th birthday), **13**, No. 1-3 (2007), 65–104. [CoLab Preprint 253]

[8] C. Bosbach and W. Gawronski, "Strong asymptotics for Laguerre polynomials with varying weights," *J. Comp. and Appl. Math.*, **99**, 77-89, 1998.

[9] R. Crandall, Unpublished Manuscript.

[10] H. Dette, "Strong approximation of eigenvalues of large dimensional Wishart matrices by roots of generalized Laguerre polynomials," preprint, 9 Oct 2001.

[11] A. Erdélyi, "Asymptotic forms for Laguerre polynomials," *J. Indian Math. Soc.*, Golden Jubilee Com- memoration Volume, 24 (1960), 235-250.

[12] W. Foster and I. Krasikov, "Explicit bounds for Hermite polynomials in the oscillatory region," *LMS J. Comput. Math.*, **3**, (2000), 307-314.

[13] C. Frenzen and R. Wong, "Uniform asymptotic expansions of Laguerre polynomials," *SIAM J. Math. Anal.*, **19**, (1988), 1232-1248

[14] P. Henrici, *Applied and Computational Complex Analysis: Special Functions, Integral Transforms, Asymptotics, Continued Fractions Vol 2 (Classics Library)*, John Wiley & Sons, 1988 (reprint from 1974).

[15] L. Jacobsen and W. Thron, “Oval convergence regions for continued fractions $K(a_n/1)$,” preprint, (1986).

[16] A. Kuijlaars and K. McLaughlin, “Riemann–Hilbert analysis for Laguerre polynomials with large negative parameter,” *Computational Methods and Function Theory*, **1** (2001), 205–233.

[17] J. López and N. Temme, “Convergent asymptotic expansions of Charlier, Laguerre and Jacobi polynomials,” Report MAS-E0320, Centrum voor Wiskunde en Informatica, 2003.

[18] J. López and N. Temme, “Approximations of orthogonal polynomials in terms of Hermite polynomials,” *Meth. and App. of Anal.*, **6**, (1999), 131-146.

[19] E. Love, “Inequalities for Laguerre functions,” *J. of Inequal. and Appl.*, **1** (1997), 293–299.

[20] L. Lorentzen and H. Waadeland, *Continued Fractions with Applications*, North-Holland, Amsterdam, 1992.

[21] F. Olver, “Why steepest descents?,” *SIAM Review*, **12** (1970), 228–247.

[22] F. Olver, *Introduction to Asymptotics and Special Functions*, Academic Press, New York, 1974.

[23] R. B. Paris, private communication (2007).

[24] R. B. Paris, “On the use of Hadamard expansions in hyperasymptotic evaluation of Laplace-type integrals. I: real variable,” JCAM 167 (2004) 293–319.

[25] R. B. Paris, “On the use of Hadamard expansions in hyperasymptotic evaluation of Laplace-type integrals. II: complex variable,” JCAM 167 (2004) 321–343.

[26] R. B. Paris, “On the use of Hadamard expansions in hyperasymptotic evaluation of Laplace-type integrals. III: clusters of saddles,” JCAM (2007) [to appear].

[27] R. B. Paris, “On the use of Hadamard expansions in hyperasymptotic evaluation of Laplace-type integrals. IV: poles,” JCAM (2007) [to appear].

[28] O. Perron, “Über das Verhalten einer ausgearteten hypergeometrischen Reihe bei unbegrenztem Wachstum eienes Parameters,” *Journal für die reine und angewandte Mathemaqtik*, **151** (1921), 63-78.

[29] A. Smith, “A new asymptotic form for the Laguerre polynomials,” *J. Math. Phys.* **33** (5), 1666-1671, May 1992.

[30] G. Szegö, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ. 23, Amer. Math. Soc., Providence, RI, fourth edition, 1975.

[31] N. Temme, “Asymptotic estimates of Stirling numbers,” *Stud. Appl. Math.*, **89** (1993), 223-243.

[32] S. Thangavelu, *Lectures on Hermite and Laguerre Expansions*, Math. Notes., Princeton Univ. Press, Princeton, N.J., 1993.

[33] E. Titchmarsh, *The theory of the Riemann zeta function*, Clarendon press, Oxford (1951, 1967).

[34] W. Van Assche, “Weighted zero distribution for polynomials orthogonal on an infinite interval,” *SIAM J. MATH. ANAL.*, **16**, 1317-1334, 1985.

[35] W. Van Assche, “Erratum to ’weighted zero distribution for polynomials orthogonal on an infinite interval,” *SIAM J. MATH. ANAL.*, **32**, 1169-1170, 2001.

[36] G. Watson, *Treatise on the theory of Bessel functions, 2nd Ed.*, Cambridge Math. Lib. Ed., Cambridge Univ. Press, 1966.

[37] S. Winitski, “Computing the incomplete gamma function to arbitrary precision,” LNCS 2667, 290, Springer–Verlag (2003).

[38] Z. Wang and R. Wong, “Liner difference equations with transition points,” Liu Bie Ju Centre for Math. Sci., Preprint Series, **11**, (2004).

[39] S. Wong, *Computational methods in physics and engineering*, Prentice–Hall, New Jersey, 1992.