# Experimental Computation and Visual Theorems: The Computer as Collaborator

**Jonathan Borwein** FRSC FAAS FAA FBAS
(With Aragón, Bailey, P. Borwein, Skerritt, Straub, Tam, Wan, Zudilin, ...)

Centre for Computer Assisted Research Mathematics and its Applications
The University of Newcastle, Australia

http://carma.newcastle.edu.au/meetings/evims/

http://www.carma.newcastle.edu.au/jon/visuals-ext-abst.pdf

For 2014 Presentations

Revised 28-07-14

# Welcome to the Final Day

**ICMS 2014**

**Satellite Conference of ICM2014**

**The 4th International Congress on Mathematical Software**

August 5(Tue) - 9(Sat), 2014, Hanyang University, Seoul, Korea

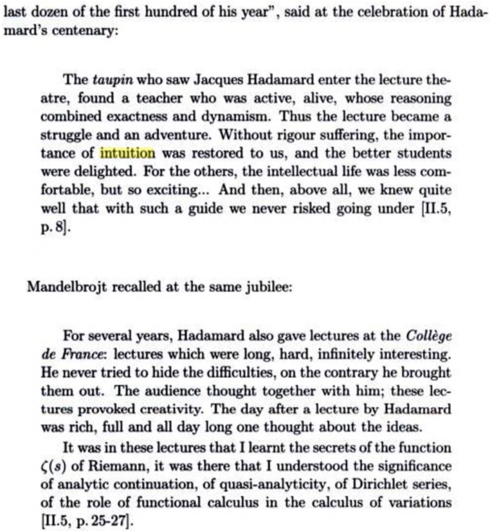
- Home
- Vision
- Invited Speakers
- Call for Demos/Tutorial/Posters
- Call For Session Proposal
- Sessions
- Submission
- Important Dates
- Committees
- Program
- Registration
- Conference Venue
- Accommodations
- Travel
- Bylaw
- Previous ICMS Links

**- News -**

- May 23: The proceedings (extended abstracts) of ICMS 2014 will be published by Springer as Volume 8592 of Lecture Notes in Computer Science.
- Jun 05: For requesting an official invitation letter, please refer to this page.
- Jun 24: Call for Demos/Tutorials/Posters (refer to this page.)
- July 01: For a partial ICM registration fee remission, refer to this page.
- July 22: A/V availability: beam projector, screen, microphone, speaker, and internet. NEW
- July 23: Information of limousine bus available in the Travel page. NEW
- July 24: Excursion to Namhansanseong planned on Aug 7 (after Tutorials/Demos/Posters sessions). NEW
- July 24: To see the registrants of ICMS2014, click here. NEW

## Dedicated to *Jacques Hadamard, A Universal Mathematician* (1998)

**"The object of mathematical rigor is to sanction and legitimize the conquests of intuition, and there was never any other object for it."**–JSH (1865-1963)



last dozen of the first hundred of his year", said at the celebration of Hadamard's centenary:

> The *taupin* who saw Jacques Hadamard enter the lecture theatre, found a teacher who was active, alive, whose reasoning combined exactness and dynamism. Thus the lecture became a struggle and an adventure. Without rigour suffering, the importance of intuition was restored to us, and the better students were delighted. For the others, the intellectual life was less comfortable, but so exciting... And then, above all, we knew quite well that with such a guide we never risked going under [II.5, p. 8].

Mandelbrojt recalled at the same jubilee:

> For several years, Hadamard also gave lectures at the *Collège de France*: lectures which were long, hard, infinitely interesting. He never tried to hide the difficulties, on the contrary he brought them out. The audience thought together with him; these lectures provoked creativity. The day after a lecture by Hadamard was rich, full and all day long one thought about the ideas.
>
> It was in these lectures that I learnt the secrets of the function $\zeta(s)$ of Riemann, it was there that I understood the significance of analytic continuation, of quasi-analyticity, of Dirichlet series, of the role of functional calculus in the calculus of variations [II.5, p. 25-27].

# Extended Abstract

Long before current graphic, visualisation and geometric tools were available, John E. Littlewood (1885-1977) wrote in his delightful *Miscellany*[1]:

> *A heavy warning used to be given [by lecturers] that pictures are not rigorous; this has never had its bluff called and has permanently frightened its victims into playing for safety. Some pictures, of course, are not rigorous, but I should say most are (and I use them whenever possible myself). [p. 53]*

[1] J.E. Littlewood, *A mathematician's miscellany*, London: Methuen (1953); Littlewood, J. E. and Bollobás, Béla, ed., *Littlewood's miscellany*, Cambridge University Press, 1986.

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Over the past decade, the role of visual computing in my own research has expanded dramatically.

In part this was made possible by the increasing speed and storage capabilities—and the growing ease of programming—of modern multi-core computing environments [BMC].

[1] J.E. Littlewood, *A mathematician's miscellany*, London: Methuen (1953); Littlewood, J. E. and Bollobás, Béla, ed., *Littlewood's miscellany*, Cambridge University Press, 1986.

But, at least as much, it has been driven by my group's paying more active attention to the possibilities for graphing, animating or simulating most mathematical research activities.

[2] See http://www.carma.newcastle.edu.au/jon/Completion.pdf and http://www.carma.newcastle.edu.au/jon/dr-fields11.pptx.

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- *I first briefly discuss both visual theorems and experimental computation.*
- *I then turn to dynamic geometry (iterative reflection methods [AB]) and matrix completion problems (applied to protein conformation [ABT]).[2] (Case studies I)*

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- *After an algorithmic interlude (Case studies II), I end with description of work from my group in probability (behaviour of short random walks [BS, BSWZ]) and transcendental number theory (normality of real numbers [AB3]). (Case studies III)*

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I will make a sample of the on-line presentation, based in part on:

- What we have seen and heard this week
- My inclinations on the day
- How I manage my time

# Key References and URLS


AB F. Aragon and J.M. Borwein,"Global convergence of a non-convex Douglas-Rachford iteration." J. Global Optim. **57**(3) (2013), 753–769.

AB3 F. Aragon, D. H. Bailey, J.M. Borwein and P.B. Borwein, "Walking on real numbers." *Mathematical Intelligencer*. **35**(1) (2013), 42–60.

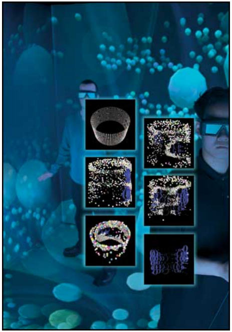
ABT F. Aragon, J. M.Borwein, and M. Tam, '"Douglas-Rachford feasibility methods for matrix completion problems. *ANZIAM Journal*. Accepted March 2014. Available at `http://arxiv.org/abs/1308.4243`.

BS J.M. Borwein and A. Straub, "Mahler measures, short walks and logsine integrals." *Theoretical Computer Science*. Special issue on *Symbolic and Numeric Computation*. **479** (1) (2013), 4-21. DOI: `http://link.springer.com/article/10.1016/j.tcs.2012.10.025`.

BSC J.M. Borwein, M. Skerritt and C. Maitland, "Computation of a lower bound to Giuga's primality conjecture." *Integers* **13** (2013). Online Sept 2013 at #A67, `http://www.westga.edu/~integers/cgi-bin/get.cgi`.

BSWZ J.M. Borwein, A. Straub, J. Wan and W. Zudilin (with an Appendix by Don Zagier), "Densities of short uniform random walks." *Can. J. Math.* **64**(5), (2012), 961-990. `http://dx.doi.org/10.4153/CJM-2011-079-2`.

# ...and 3D?



**NAMS 2005**. KnotPlot in a Cave

*Considerable obstacles generally present themselves to the beginner, in studying the elements of Solid Geometry, from the practice which has hitherto uniformly prevailed in this country, of never submitting to the eye of the student, the figures on whose properties he is reasoning, but of drawing perspective representations of them upon a plane.*

...

*I hope that I shall never be obliged to have recourse to a perspective drawing of any figure whose parts are not in the same plane.—Augustus De Morgan*

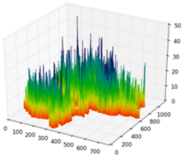
*In Adrian Rice,"What Makes a Great Mathematics Teacher?" MAA Monthly, 1999.*

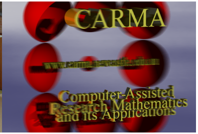
# Contents

# Visual Theorems: Animation, Simulation and Stereo . . .

See `http://vis.carma.newcastle.edu.au/`: Stoneham movie








Cinderella, 3.14 min of Pi, Catalan's constant and Passive 3D

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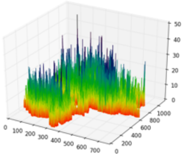
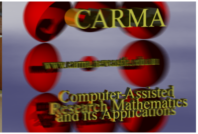
See `http://vis.carma.newcastle.edu.au/`: Stoneham movie

> *The latest developments in computer and video technology have provided a multiplicity of computational and symbolic tools that have rejuvenated mathematics and mathematics education. Two important examples of this revitalization are experimental mathematics and visual theorems*
> — *ICMI Study* ***19*** (2012)





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# Visualising large matrices

Large matrices often have structure that pictures will reveal but which numeric data may obscure.

- The picture shows a $25 \times 25$ *Hilbert* matrix on the left and on the right a matrix required to have 50% sparsity and non-zero entries random in $[0, 1]$.

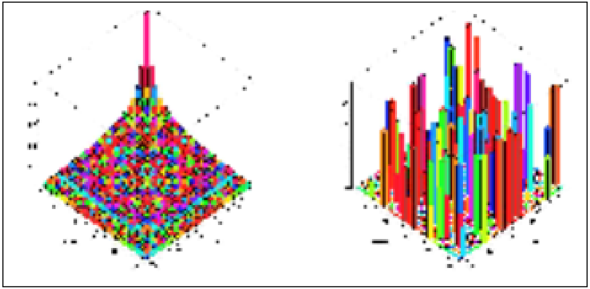

Figure : The Hilbert matrix (L) and a sparse random matrix (R)

# Contents

# Visualising large matrices

The $4 \times 4$ Hilbert matrix is

$$\begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/3 & 1/4 & 1/5 \\ 1/3 & 1/4 & 1/5 & 1/6 \\ 1/4 & 1/5 & 1/6 & 1/7 \end{bmatrix}$$

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Hilbert matrices are notoriously unstable numerically. The left of the Figure shows the inverse of the $20 \times 20$ Hilbert matrix computed *symbolically exactly*. The middle shows enormous *numerical errors* if one uses 10 digit precision, and the right even if one uses 20 digits.

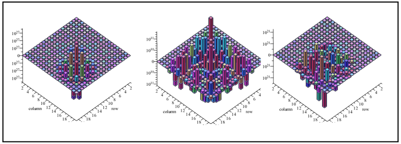
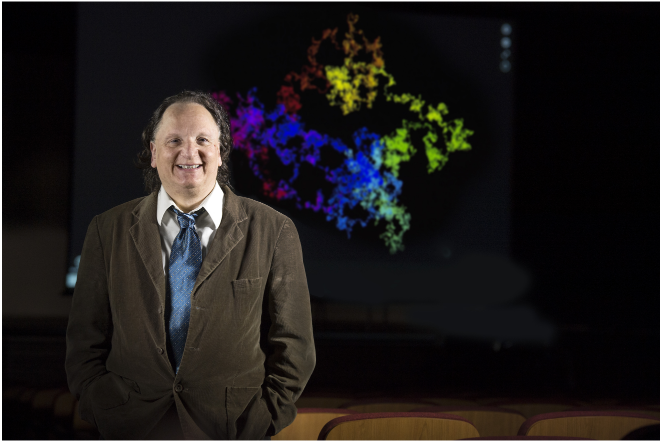



Figure : Inverse $20 \times 20$ Hilbert matrix (L) and 2 numerical inverses (R)

# Me and my collaborators



**MAA 3.14**

`http://www.carma.newcastle.edu.au/jon/pi-monthly.pdf`

# 2012 walk on $\pi$ (went *viral*)

Biggest mathematics picture ever?

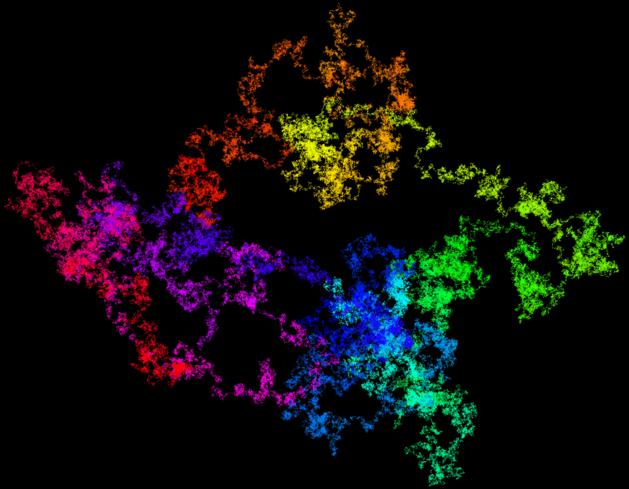


Figure : Walk on first 100 billion base-4 digits of $\pi$ (normal?).

# 2012 walk on $\pi$ (went *viral*)

Biggest mathematics picture ever?

Resolution: $372{,}224 \times 290{,}218$ pixels (108 gigapixels)

Computation: took roughly a month where several parts of the algorithm were run in parallel with 20 threads on CARMA's MacPro cluster.

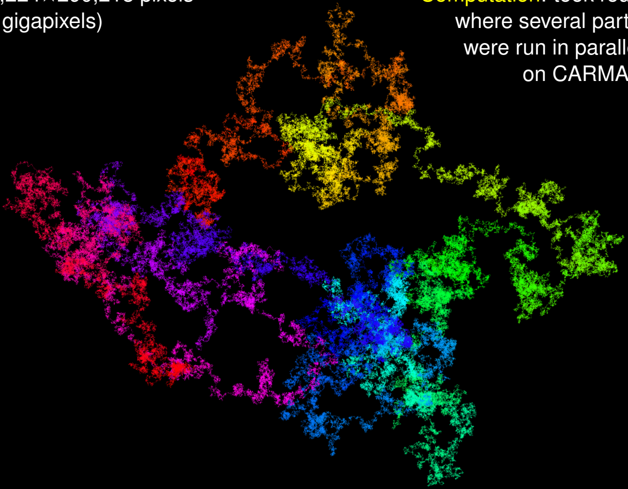
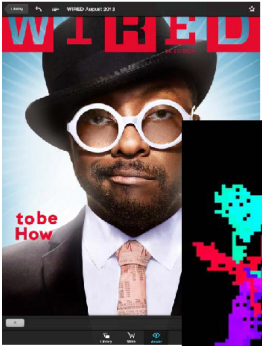


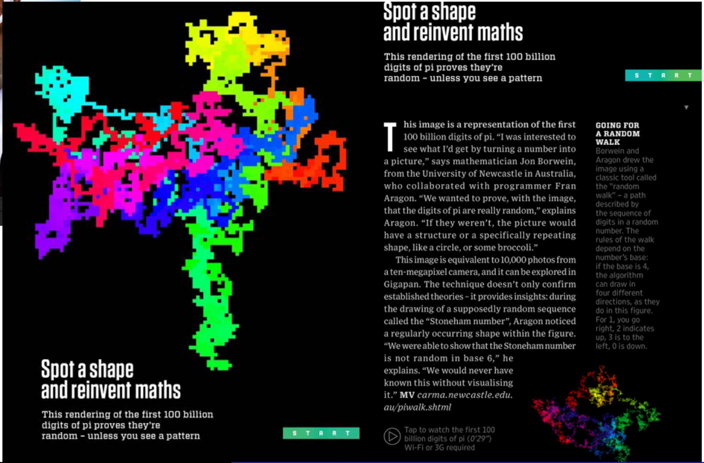
Figure : Walk on first 100 billion base-4 digits of $\pi$ (normal?).

http://gigapan.org/gigapans/106803

# Outreach: images and animations led to high-level research which went viral




## Wired UK August 2013



Spot a shape and reinvent maths

This rendering of the first 100 billion digits of pi proves they're random – unless you see a pattern

START

This image is a representation of the first 100 billion digits of pi. "I was interested to see what I'd get by turning a number into a picture," says mathematician Jon Borwein, from the University of Newcastle in Australia, who collaborated with programmer Fran Aragon. "We wanted to prove, with the image, that the digits of pi are really random," explains Aragon. "If they weren't, the picture would have a structure or a specifically repeating shape, like a circle, or some broccoli."

This image is equivalent to 10,000 photos from a ten-megapixel camera, and it can be explored in Gigapan. The technique doesn't only confirm established theories – it provides insights: during the drawing of a supposedly random sequence called the "Stoneham number", Aragon noticed a regularly occurring shape within the figure. "We were able to show that the Stoneham number is not random in base 6," he explains. "We would never have known this without visualising it." **MV** *carma.newcastle.edu.au/piwalk.shtml*

**GOING FOR A RANDOM WALK**
Borwein and Aragon drew the image using a classic tool called the "random walk" – a path described by the sequence of digits in a random number. The rules of the walk depend on the number's base: if the base is 4, the algorithm can draw in four different directions, as they do in this figure. For 1, you go right, 2 indicates up, 3 is to the left, 0 is down.

Tap to watch the first 100 billion digits of pi (0'29")
Wi-Fi or 3G required

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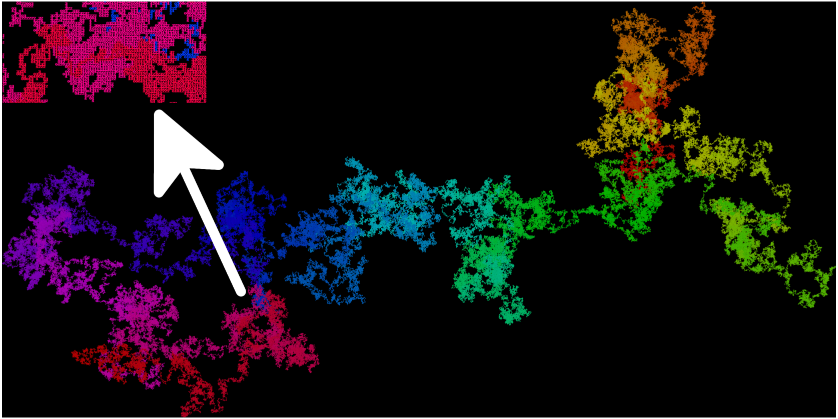


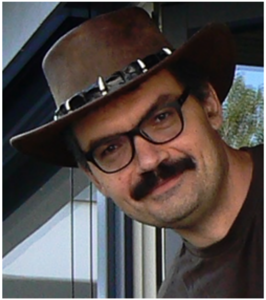
- 100 billion base four digits of $\pi$ on Gigapan
- Really big pictures are often better than movies (NASA and AMS)

# My number-walk collaborators

# My short-walk collaborators

James Wan

Armin Straub



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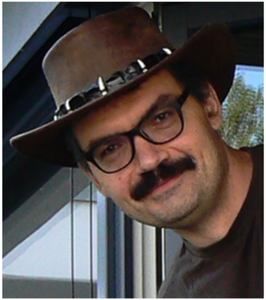
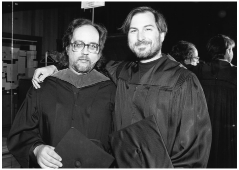


Wadim Zudilin

- Plus Dirk Nuyens  and Don Zagier, ...

# Dedication: To my friend Richard E. Crandall (1947-2012)

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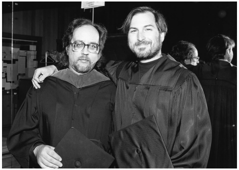


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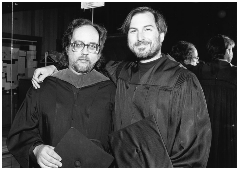   

- A remarkable man and a brilliant (physical and computational) scientist and inventor, from Reed College
  - Chief scientist for *NeXT*
  - *Apple* distinguished scientist
  - and *High Performance Computing* head
- Developer of the *Pixar* compression format
  - and the *iPod shuffle*

http://en.wikipedia.org/wiki/Richard_Crandall

# Some early conclusions:

So I am sure they get made

**Key ideas**: randomness, normality of numbers, planar walks, and fractals


**How not to experiment**

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Maths can be done *experimentally* (it is fun)

- using computer algebra, numerical computation and graphics: SNaG
- computations, tables and pictures are experimental data
- but you can not stop thinking

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DHB and JMB, **Exploratory Experimentation in Mathematics** (2011), www.ams.org/notices/201110/rtx111001410p.pdf

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*I imagine the world conqueror must feel thus, who, after one kingdom is scarcely conquered, stretches out his arms for others.*

**Carl Friedrich Gauss**
(1777-1855)

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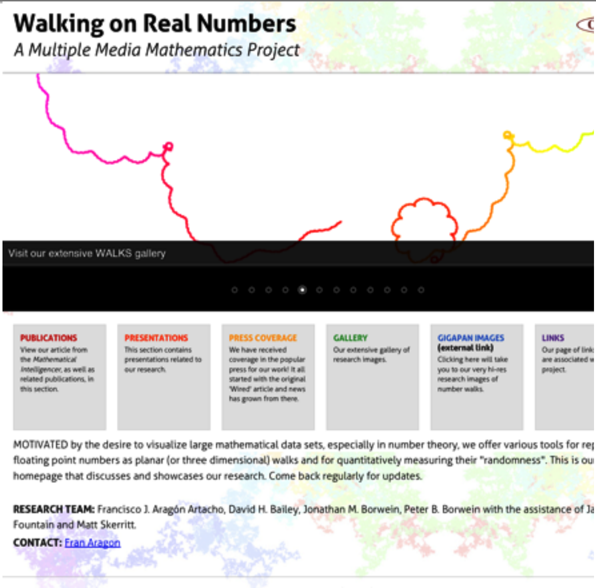
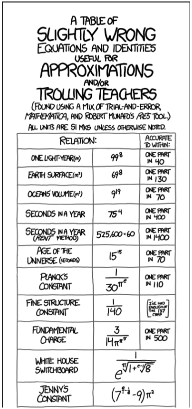
(1777-1855)

- In an **1808** letter to his friend Farkas (father of Janos Bolyai)
- Archimedes, Euler, Gauss are the big three



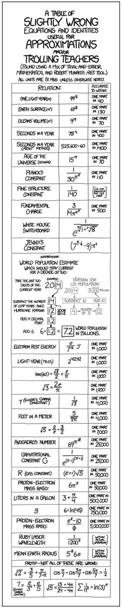



A TABLE OF SLIGHTLY WRONG EQUATIONS AND IDENTITIES USEFUL FOR APPROXIMATIONS AND/OR TROLLING TEACHERS

(FOUND USING A MIX OF TRIAL-AND-ERROR, MATHEMATICA, AND ROBERT MUNAFO'S RIES TOOL.)

ALL UNITS ARE SI MKS UNLESS OTHERWISE NOTED.

| RELATION: | | ACCURATE TO WITHIN: |
|---|---|---|
| ONE LIGHT-YEAR(m) | $99^8$ | ONE PART IN 40 |
| EARTH SURFACE($m^2$) | $69^8$ | ONE PART IN 130 |
| OCEANS' VOLUME($m^3$) | $9^{19}$ | ONE PART IN 70 |
| SECONDS IN A YEAR | $75^4$ | ONE PART IN 400 |
| SECONDS IN A YEAR (RENT METHOD) | $525{,}600\cdot 60$ | ONE PART IN 1400 |
| AGE OF THE UNIVERSE (SECONDS) | $15^{15}$ | ONE PART IN 70 |
| PLANCK'S CONSTANT | $\frac{1}{30^{\pi^e}}$ | ONE PART IN 110 |
| FINE STRUCTURE CONSTANT | $\frac{1}{140}$ | [I'VE HAD ENOUGH OF THIS 137 CRAP] |
| FUNDAMENTAL CHARGE | $\frac{3}{14\pi^{\pi^\pi}}$ | ONE PART IN 500 |
| WHITE HOUSE SWITCHBOARD | $\frac{1}{e^{\sqrt[\pi]{1+\sqrt[e-1]{8}}}}$ | |
| JENNY'S CONSTANT | $(7^{\frac{e}{\pi}-\frac{1}{e}}-9)\pi^2$ | |



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INTERMISSION:
WORLD POPULATION ESTIMATE WHICH SHOULD STAY CURRENT FOR A DECADE OR TWO:

TAKE THE LAST TWO DIGITS OF THE CURRENT YEAR — EXAMPLE: $20\boxed{14}$ → $\boxed{14}$
SUBTRACT THE NUMBER OF LEAP YEARS SINCE HURRICANE KATRINA (2008, 2012) → $\boxed{12}$
ADD A DECIMAL POINT → $\boxed{1.2}$
ADD 6: $6+\boxed{1.2}=\boxed{7.2}$ WORLD POPULATION IN BILLIONS.

VERSION FOR US POPULATION: $20\boxed{14}$ → SUBTRACT 10 → $\boxed{4}$ → MULTIPLY BY 3 → $\boxed{12}$ → ADD 10 → $3\boxed{22}$ MILLION

| | | |
|---|---|---|
| ELECTRON REST ENERGY | $\frac{e}{7^{16}}J$ | ONE PART IN 1,000 |
| LIGHT-YEAR (MILES) | $2^{42.42}$ | ONE PART IN 1,000 |
| $\sin(60^\circ)=\frac{\sqrt{3}}{2}=\frac{e}{\pi}$ | | ONE PART IN 1,000 |
| $\sqrt{3}=\frac{2e}{\pi}$ | | ONE PART IN 1,000 |
| $\gamma$ (EULER'S GAMMA CONSTANT) | $\frac{1}{\sqrt{3}}$ | ONE PART IN 4,000 |
| FEET IN A METER | $\frac{5}{\sqrt{9\pi}}$ | ONE PART IN 4,000 |
| $\sqrt{5}=\frac{2}{e}+\frac{3}{2}$ | | ONE PART IN 7,000 |
| AVOGADRO'S NUMBER | $69^{\pi^{\sqrt{5}}}$ | ONE PART IN 25,000 |
| GRAVITATIONAL CONSTANT G | $\frac{1}{e^{(\pi-1)^{(\pi+1)}}}$ | ONE PART IN 25,000 |
| R (GAS CONSTANT) | $(e+1)\sqrt{5}$ | ONE PART IN 50,000 |
| PROTON-ELECTRON MASS RATIO: | $6\pi^5$ | ONE PART IN 50,000 |
| LITERS IN A GALLON | $3+\frac{\pi}{4}$ | ONE PART IN 500,000 |
| $g$ | $6+\ln(45)$ | ONE PART IN 750,000 |
| PROTON-ELECTRON MASS RATIO: | $\frac{e^8-10}{\phi}$ | ONE PART IN 5,000,000 |
| RUBY LASER WAVELENGTH | $\frac{1}{1200^2}$ | [WITHIN ACTUAL VARIATION] |
| MEAN EARTH RADIUS | $5^8 6e$ | [WITHIN ACTUAL VARIATION] |

PROTIP—NOT ALL OF THESE ARE WRONG:

$\sqrt{2}=\frac{3}{5}+\frac{\pi}{7-\pi}$ $\quad$ $\cos\frac{\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{5\pi}{7}=\frac{1}{2}$

$\gamma=\frac{e}{3^4}+\frac{e}{5}$ (EULER'S GAMMA CONSTANT) $\quad$ $\sqrt{5}=\frac{13+4\pi}{24-4\pi}$ $\quad$ $\sum\frac{1}{n^n}=\ln(3)^e$

Almost all I mention is accessible at `http://carma.newcastle.edu.au/walks/`

# Contents

# Computer Assisted Research Maths: what it is?

*Experimental mathematics is the use of a computer to run computations—sometimes no more than trial-and- error tests—to look for patterns, to identify particular numbers and sequences, to gather evidence in support of specific mathematical assertions that may themselves arise by computational means, including search.*

# Computer Assisted Research Maths: what it is?

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*Like contemporary chemists — and before them the alchemists of old—who mix various substances together in a crucible and heat them to a high temperature to see what happens, today's experimental mathematicians put a hopefully potent mix of numbers, formulas, and algorithms into a computer in the hope that something of interest emerges. (JMB-Devlin, Crucible 2008, p. 1)*

- Quoted in International Council on Mathematical Instruction *Study 19: On Proof and Proving*, 2012

# Experimental Mathematics: Integer Relation Methods

**Secure Knowledge without Proof.** Given real numbers $\beta, \alpha_1, \alpha_2, \ldots, \alpha_n$, Helaman Ferguson's integer relation method (PSLQ), finds a nontrivial linear relation of the form

$$a_0\beta + a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n = 0, \tag{1}$$

where $a_i$ are integers—if one exists and provides an exclusion bound otherwise.

PROFILE: HELAMAN FERGUSON

**Carving His Own Unique Niche, In Symbols and Stone**

By refusing to choose between mathematics and art, a self-described "misfit" has found the place where parallel careers meet

**CMS D. Borwein Prize:** Madelung

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- $\beta = 1, \alpha_i = \alpha^i$ means $\alpha$ is algebraic of degree $n$
- **2000** *Computing in Science & Engineering*: PSLQ one of top 10 algorithms of 20th century

  (**2001** CISE article on *Grand Challenges* (JB-PB))

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# PSLQ in action

In all serious computations of $\pi$ from 1700 (by John Machin) until 1980 some version of a *Machin formula* was used. These write

$$\arctan(1) = a_1 \cdot \arctan\left(\frac{1}{p_1}\right) + a_2 \cdot \arctan\left(\frac{1}{p_2}\right) + \cdots + a_n \cdot \arctan\left(\frac{1}{p_n}\right) \tag{2}$$

for rationals $a_1, a_2, \ldots, a_n$ and integers $p_1, p_2, \ldots, p_n > 1$.

Recall the Taylor series $\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$. Combined with (2) this computes $\pi = 4\arctan(1)$ efficiently, especially if the $p_n$ are not too small.

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For instance, Machin found

$$\pi = 16 \arctan\left(\frac{1}{5}\right) - 4 \arctan\left(\frac{1}{239}\right)$$

while Euler discovered

$$\arctan(1) = \arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{5}\right) + \arctan\left(\frac{1}{8}\right). \tag{3}$$

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- I have a function '`pslq`' in *Maple*. When input data for PSLQ it *predicts* an answer to the precision requested. And checks it to ten digits more (or some other precision).
- This makes the code a real *experimental tool* as it predicts and confirms.

# PSLQ in action

prepping for class

```
> pslq(arctan(1),[arctan(1/2),arctan(1/5), arctan(1/8)],20);;
```

[1, 1, 1, 1], "Error is", 0., "checking to", 30, *places*

$$\frac{1}{4}\pi = \arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{5}\right) + \arctan\left(\frac{1}{8}\right)$$

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> pslq(arctan(1),[arctan(1/2),arctan(1/3), arctan(1/8)],20);
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[1, 1, 1, 0], "Error is", $-1.\ 10^{-30}$, "checking to", 30, *places*

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```
> pslq(arctan(1),[arctan(1/2),arctan(1/5), arctan(1/9)],20);
```

[42613, 72375, 22013, −40066], "Error is", $2.31604649037\ 10^{-15}$, "checking to", 30, *places*

$$\frac{1}{4}\pi = \frac{72375}{42613}\arctan\left(\frac{1}{2}\right) + \frac{22013}{42613}\arctan\left(\frac{1}{5}\right) - \frac{40066}{42613}\arctan\left(\frac{1}{9}\right)$$

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- The third shows that when no relation exists the code may find a good approximation but using very large rationals.

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- The third shows that when no relation exists the code may find a good approximation but using very large rationals.
- So it diagnoses failure because it uses large coefficients and because it is not true to the requested 30 places.

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# Digital Assistance

By digital assistance I mean use of *artefacts* as:

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  - Primarily **numeric** packages start with the proprietary MATLAB and public counterpart *Octave* or the statistical package *R*.
  - The dynamic **geometry** offerings include *Cinderella*, *Geometer's SketchPad*, *Cabri* and the freeware *Geogebra*.

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- Specialized Packages or General Purpose Languages such as Fortran, C++, Python, CPLEX, PARI, SnapPea, and MAGMA.

# Digital Assistance

- Web Applications such as: Sloane's Encyclopedia of Integer Sequences, the Inverse Symbolic Calculator, Fractal Explorer, Jeff Weeks' Topological Games, or Euclid in Java.[3]
  - Most of the functionality of the ISC is built into the "identify" function *Maple* starting with version 9.5. For example, `identify(4.45033263602792)` returns $\sqrt{3}+e$. As always, the experienced will extract more than the novice.

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- Web Databases including Google, MathSciNet, ArXiv, GitHub, Wikipedia, MathWorld, MacTutor, Amazon, Wolfram Alpha, the DLMF (all formulas of which are accessible in MathML, as bitmaps, and in TEX) and many more that are not always so viewed.

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# Digital Assistance

All entail *data-mining* . Franklin argues *"exploratory experimentation"* facilitated by *"widening technology"*, as in finance, pharmacology, astrophysics, medicine, and biotechnology, is leading to a reassessment of what legitimates experiment; in that a "*local model*" is not now prerequisite. Sørenson says *experimental mathematics* is following similar tracks.

> *These aspects of exploratory experimentation and wide instrumentation originate from the philosophy of (natural) science and have not been much developed in the context of experimental mathematics. However, I claim that e.g. the importance of wide instrumentation for an exploratory approach to experiments that includes concept formation also pertain to mathematics.*

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In consequence, boundaries between mathematics and natural sciences and between inductive and deductive reasoning are blurred and getting more so.

I leave the philosophically-vexing if mathematically-minor question as to if genuine *mathematical experiments* exist even if one embraces a fully idealist notion of mathematical existence. They sure feel like they do.

# Top Ten Algorithms (20C): all but one well used in CARMA

## Algorithms for the Ages

**"Great algorithms are the poetry of computation," says Francis Sullivan of the Institute for Defense Analyses' Center for Computing Sciences in Bowie, Maryland. He and Jack Dongarra of the University of Tennessee and Oak Ridge National Laboratory have put together a sampling that might have made Robert Frost beam with pride--had the poet been a computer jock. Their list of 10 algorithms having "the greatest influence on the development and practice of science and engineering in the 20th century" appears in the January/February issue of Computing in Science & Engineering. If you use a computer, some of these algorithms are no doubt crunching your data as you read this. The drum roll, please:**

1. **1946: The Metropolis Algorithm for Monte Carlo.** Through the use of random processes, this algorithm offers an efficient way to stumble toward answers to problems that are too complicated to solve exactly.
2. **1947: Simplex Method for Linear Programming.** An elegant solution to a common problem in planning and decision-making.
3. **1950: Krylov Subspace Iteration Method.** A technique for rapidly solving the linear equations that abound in scientific computation.
4. **1951: The Decompositional Approach to Matrix Computations.** A suite of techniques for numerical linear algebra.
5. **1957: The Fortran Optimizing Compiler.** Turns high-level code into efficient computer-readable code.
6. **1959: QR Algorithm for Computing Eigenvalues.** Another crucial matrix operation made swift and practical.
7. **1962: Quicksort Algorithms for Sorting.** For the efficient handling of large databases.
8. **1965: Fast Fourier Transform.** Perhaps the most ubiquitous algorithm in use today, it breaks down waveforms (like sound) into periodic components.
9. **1977: Integer Relation Detection.** A fast method for spotting simple equations satisfied by collections of seemingly unrelated numbers.
10. **1987: Fast Multipole Method.** A breakthrough in dealing with the complexity of n-body calculations, applied in problems ranging from celestial mechanics to protein folding.

From *Random Samples*, Science page 799, February 4, 2000.

# Experimental Mathematics: PSLQ is core to CARMA

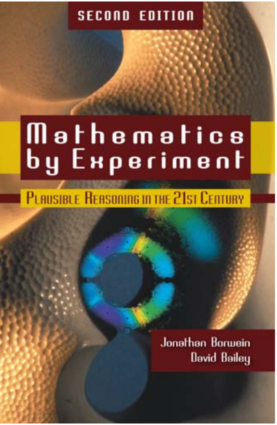
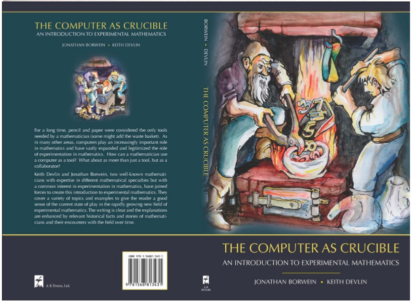
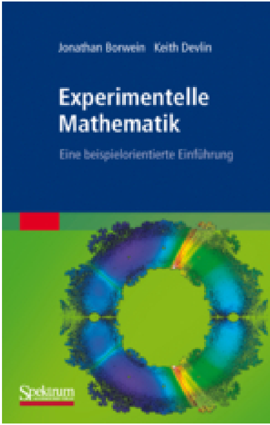



Figure 6.3. Three images quantized at quality 50 (L), 48 (C) and 75 (R). Courtesy of Mason Macklem.




**Experimental Mathematics** (2004-08, 2009, 2010)

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# Simulation in *pure* mathematics

Pure mathematicians have not often though of simulation as a relevant tool.
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It is given for complex numbers $a$ and $b$ by

$$\mathscr{R}(a,b) = \cfrac{a}{1+\cfrac{b^2}{1+\cfrac{4a^2}{1+\cfrac{9b^2}{1+\ddots}}}}. \tag{4}$$

As often I first tried to compute—$R(1,1)$—and had little luck.

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We eventually determined from highly sophisticated arguments that:

# Simulation in pure mathematics

**Theorem (Six formulae for $\mathscr{R}(a,a), a > 0$)**

$$\begin{aligned}
\mathscr{R}(a,a) &= \int_0^\infty \frac{\operatorname{sech}\left(\frac{\pi x}{2a}\right)}{1+x^2}\,\mathrm{d}x \\
&= 2a\sum_{k=1}^\infty \frac{(-1)^{k+1}}{1+(2k-1)a} \\
&= \frac{1}{2}\left(\psi\left(\frac{3}{4}+\frac{1}{4a}\right)-\psi\left(\frac{1}{4}+\frac{1}{4a}\right)\right) \\
&= \frac{2a}{1+a}\,{}_2F_1\left(\left.\begin{matrix}\frac{1}{2a}+\frac{1}{2},1\\ \frac{1}{2a}+\frac{3}{2}\end{matrix}\right|-1\right) \\
&= 2\int_0^1 \frac{t^{1/a}}{1+t^2}\,\mathrm{d}t \\
&= \int_0^\infty e^{-x/a}\operatorname{sech}(x)\,\mathrm{d}x.
\end{aligned}$$

# Simulation in pure mathematics

Here ${}_2F_1$ is the hypergeometric function. If you do not know $\psi$ ('psi') , you can easily look it up once you can say 'psi'.
Notice that

$$\mathscr{R}(a,a) = 2\int_0^1 \frac{t^{1/a}}{1+t^2}\,\mathrm{d}t$$

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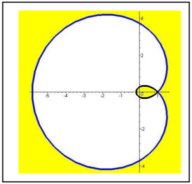
- After making no progress analytically, Crandall and I decided in 2003, taking a somewhat arbitrary criterion for convergence, to colour yellow points for which the fraction seemed to converge.
- We sampled one million points and reasoned a few thousand mis-categorisations would not damage the experiment.



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Since for positive $a, b$ the fraction satisfies

$$\mathscr{R}(\frac{a+b}{2}, \sqrt{ab}) = \frac{\mathscr{R}(a,b) + \mathscr{R}(b,a)}{2}$$

this gave us enormous impetus to continue our eventually successful hunt for a proof.

# Contents

# Reflection methods

Let $S \subseteq \mathbb{R}^m$. The (nearest point or metric) projection onto $S$ is the (set-valued) mapping,

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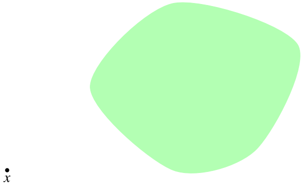
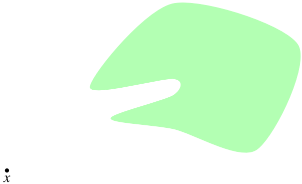
The reflection w.r.t. $S$ is the (set-valued) mapping,

$$R_S := 2P_S - I.$$



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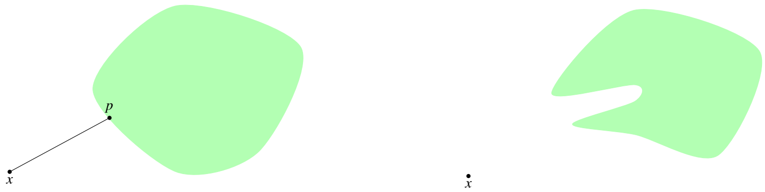

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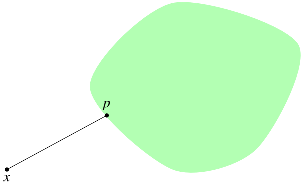


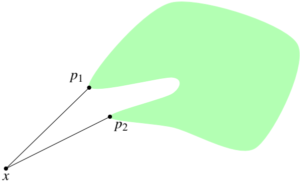

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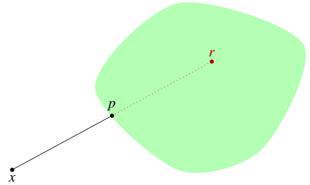


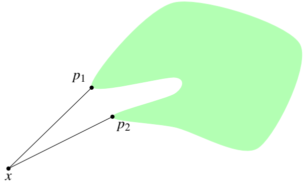

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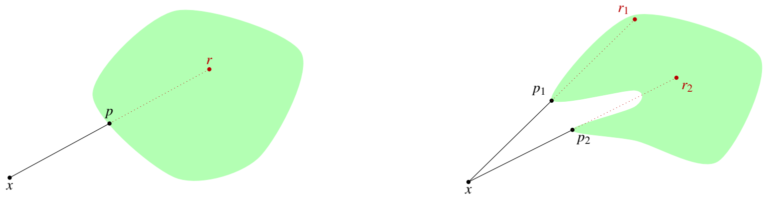


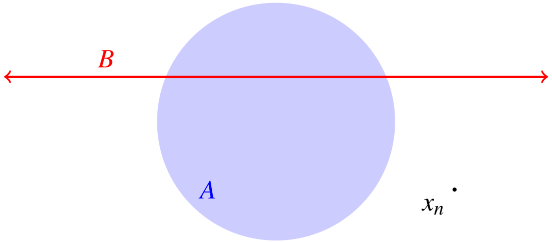
# The Douglas–Rachford Algorithm (1956–1979– )

**Theorem (Douglas–Rachford in finite dimensions)**

Suppose $A, B \subseteq \mathbb{R}^m$ are closed and convex. For any $x_0 \in \mathbb{R}^m$ define

$$x_{n+1} := T_{A,B} x_n \text{ where } T_{A,B} := \frac{I + R_B R_A}{2}.$$

If $A \cap B \neq \emptyset$, then $x_n \to x$ such that $P_A x \in A \cap B$. Else $\|x_n\| \to +\infty$.

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((non)-convex Phase retrieval)

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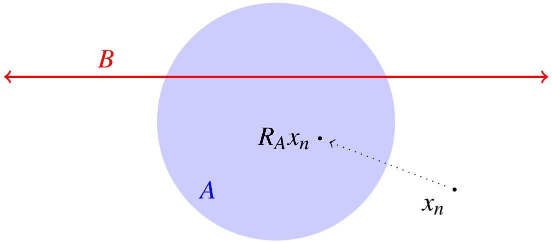


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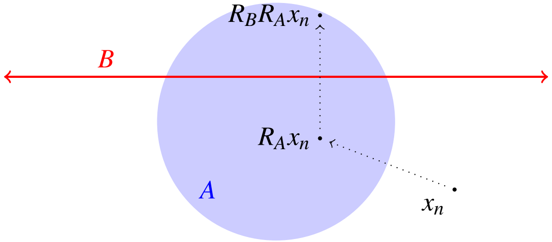


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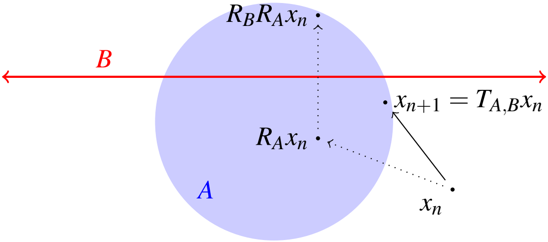


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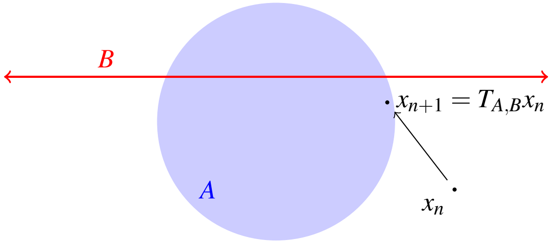


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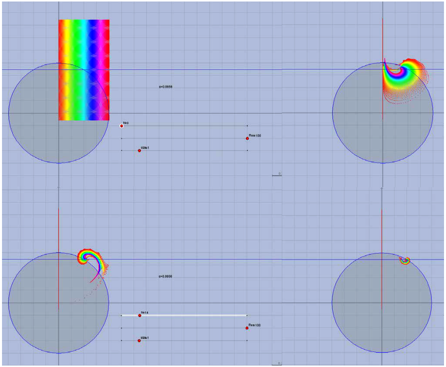
- Some local and fewer global convergence results.
- Much empirical evidence for this and other non-convex settings.
  - both numeric and geometric (Cinderella/SAGE)
  - http://carma.newcastle.edu.au/jon/expansion.html



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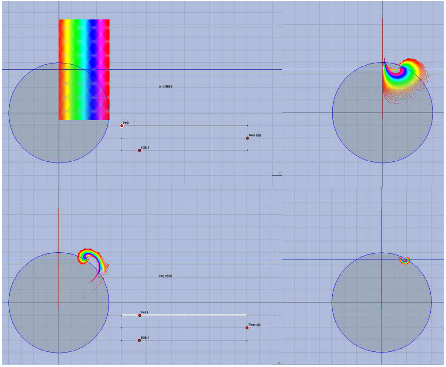

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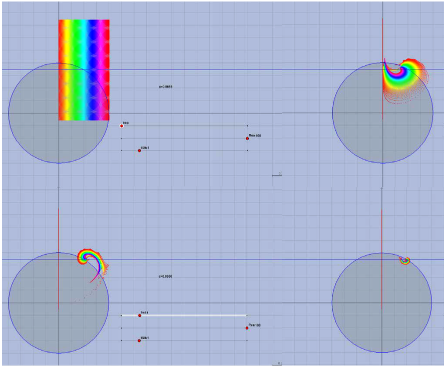

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- $20000$ starting points coloured by distance from $y$-axis
- after $0, 7, 14, 21$ steps
- a “*generic* visual theorem”?

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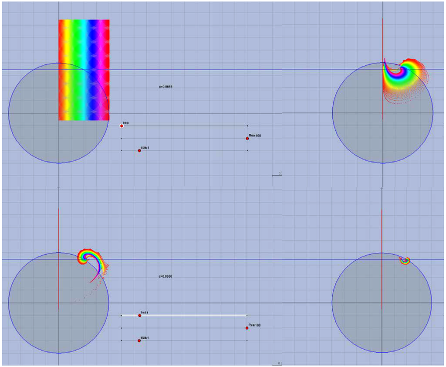

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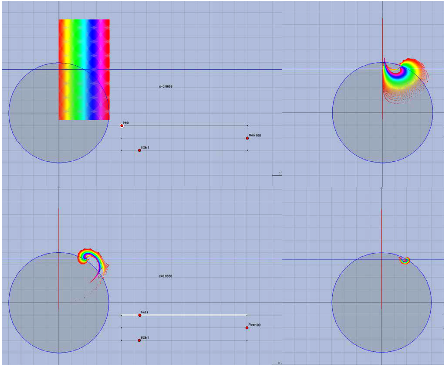

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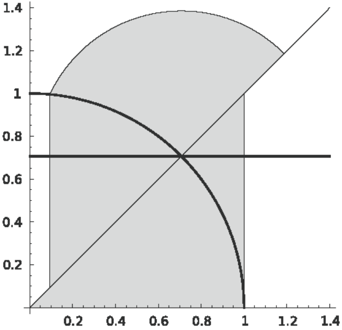
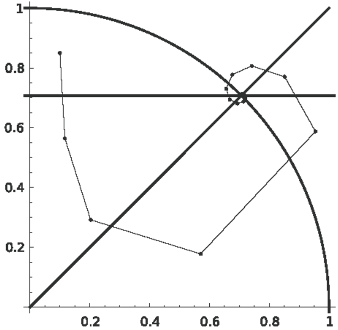
- note the *error* from using only $14$ digit computation.

# Works for $B$ affine and $A$ a 'sphere'

What we can *prove* (L) and what we can *see* (R)



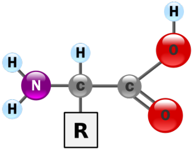




**Proven region of convergence in grey**

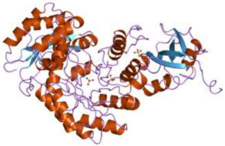
# Contents

# Case study I: Protein conformation determination

Proteins: large biomolecules comprising multiple amino acid chains.[4]




Generic amino acid



RuBisCO

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[4] RuBisCO (responsible for photosynthesis) has 550 amino acids (smallish).
[5] A coupling which occurs through space, rather than chemical bonds.

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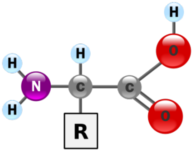


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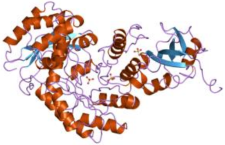

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- Proteins participate in virtually every cellular process !
- Protein structure $\rightarrow$ predicts how functions are performed.
- NMR spectroscopy (Nuclear Overhauser effect[5]) can determine a subset of interatomic distances without damage (under $6$Å ).

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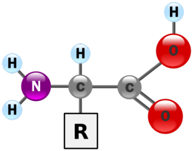


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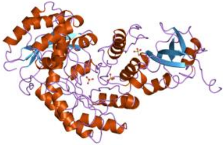

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A *low-rank Euclidean distance matrix completion* problem.

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# Six Proteins

## Numerics if reconstructed using reflection methods

We use only interatomic distances below 6Å typically constituting less than 8% of the total nonzero entries of the distance matrix.

**Table.** **Six Proteins: average (maximum) errors from five replications.**

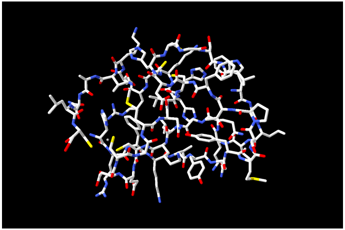
| Protein | # Atoms | Rel. Error (dB) | RMSE | Max Error |
|---|---|---|---|---|
| 1PTQ | 404 | -83.6 (-83.7) | 0.0200 (0.0219) | 0.0802 (0.0923) |
| 1HOE | 581 | -72.7 (-69.3) | 0.191 (0.257) | 2.88 (5.49) |
| 1LFB | 641 | -47.6 (-45.3) | 3.24 (3.53) | 21.7 (24.0) |
| 1PHT | 988 | -60.5 (-58.1) | 1.03 (1.18) | 12.7 (13.8) |
| 1POA | 1067 | -49.3 (-48.1) | 34.1 (34.3) | 81.9 (87.6) |
| 1AX8 | 1074 | -46.7 (-43.5) | 9.69 (10.36) | 58.6 (62.6) |

$$\text{Rel. error}(dB) := 10\log_{10}\left(\frac{\|P_{C_2}P_{C_1}X_N - P_{C_1}X_N\|^2}{\|P_{C_1}X_N\|^2}\right),$$

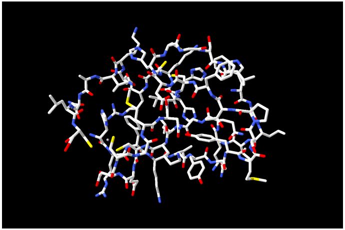
$$\text{RMSE} := \sqrt{\frac{\sum_{i=1}^{m}\|\hat{p}_i - p_i^{true}\|_2^2}{\text{\# of atoms}}}, \qquad \text{Max} := \max_{1\le i\le m}\|\hat{p}_i - p_i^{true}\|_2.$$

- The points $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n$ denote the best fitting of $p_1, p_2, \ldots, p_n$ when rotation, translation and reflection is allowed.

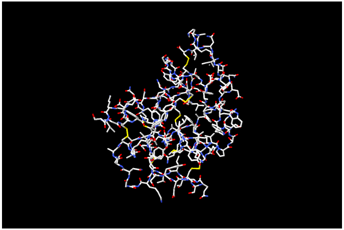
# What do the reconstructions look like?



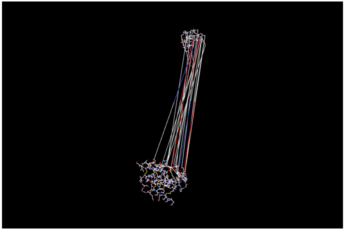
1PTQ (actual)



5,000 steps, -83.6dB (perfect)



1POA (actual)



5,000 steps, -49.3dB (mainly good!)

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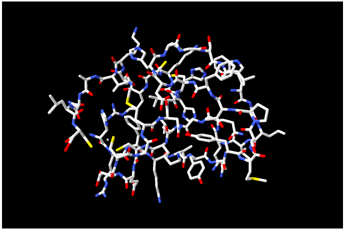
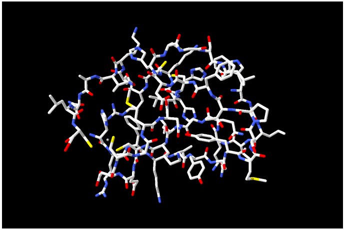


1PTQ (actual)



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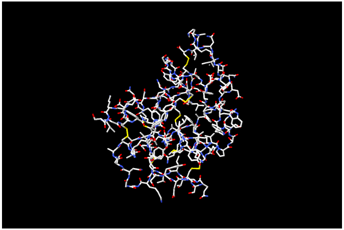
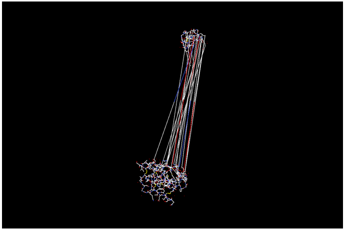


1POA (actual)



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- The picture of 'failure' suggests many strategies

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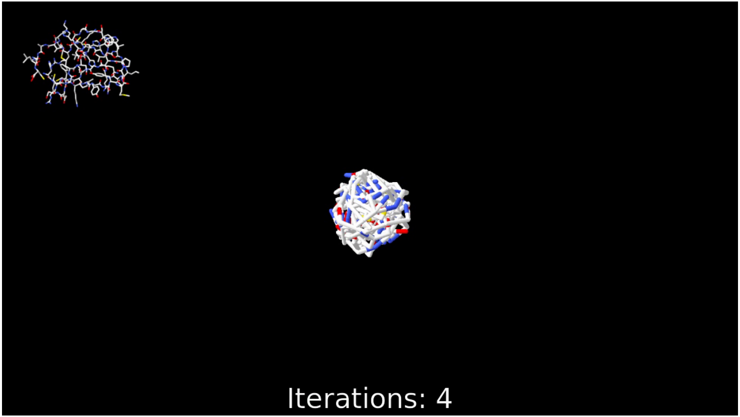


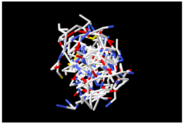

Video: First 3,000 steps of the 1PTQ reconstruction.

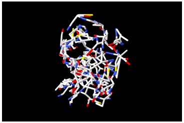
At `http://carma.newcastle.edu.au/DRmethods/1PTQ.html`

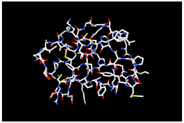
# What do reconstructions look like?

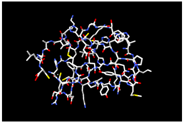
There are many projection methods, so why use Douglas-Rachford?

Douglas–Rachford reflection method reconstruction:

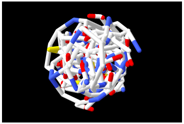

500 steps, -25 dB.

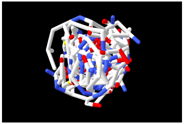

1,000 steps, -30 dB.

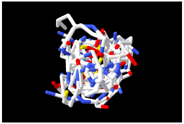

2,000 steps, -51 dB.

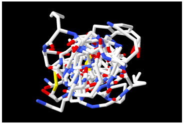

5,000 steps, -84 dB.

Alternating projection method reconstruction:


500 steps, -22 dB.


1,000 steps, -24 dB.

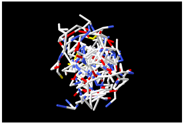

2,000 steps, -25 dB.

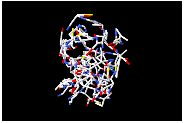

5,000 steps, -28 dB.

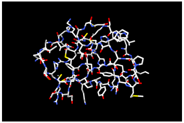
# What do reconstructions look like?

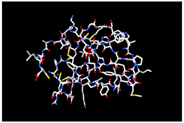
There are many projection methods, so why use Douglas-Rachford?

Douglas–Rachford reflection method reconstruction:

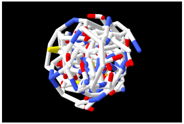

500 steps, -25 dB.

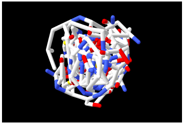

1,000 steps, -30 dB.

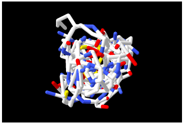

2,000 steps, -51 dB.


5,000 steps, -84 dB.

Alternating projection method reconstruction:


500 steps, -22 dB.


1,000 steps, -24 dB.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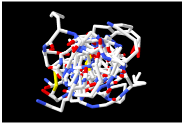
5,000 steps, -28 dB.

- Yet MAP works very well for optical abberation correction (Hubble, amateur telescopes). Why?

# Contents

# How the mathematical software world has changed

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> *"If anyone gets 50 digits in total, I will be impressed."*

- To his surprise, a total of **94** teams, representing 25 different nations, submitted results. Twenty of these teams received a full 100 points (10 correct digits for each problem).
- Bailey, Fee and I quit at 85 digits!

# The hundred digit challenge

▸ SKIP

The problems and solutions are dissected most entertainingly in

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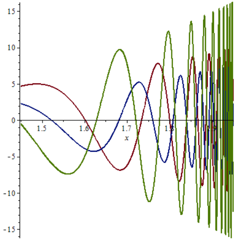
> *Success in solving these problems required a broad knowledge of mathematics and numerical analysis, together with significant computational effort, to obtain solutions and ensure correctness of the results. As described in [1] the strengths and limitations of* Maple, Mathematica, MATLAB *(The 3Ms), and other software tools such as PARI or GAP, were strikingly revealed in these ventures.*
>
> *Almost all of the solvers relied in large part on one or more of these three packages, and while most solvers attempted to confirm their results, there was no explicit requirement for proofs to be provided.*

# Trefethen's problem #9

*The integral*

$$I(\alpha) = \int_0^2 [2+\sin(10\alpha)]x^\alpha \sin\left(\frac{\alpha}{2-x}\right) \mathrm{d}x$$

*depends on the parameter $\alpha$. What is the value $\alpha \in [0,5]$ at which $I(\alpha)$ achieves its maximum?*

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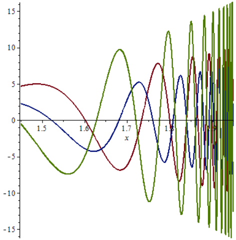



Integrands for some $\alpha$

- $I(\alpha)$ is expressible in terms of a *Meijer-G function* —a special function with a solid history that we use below.

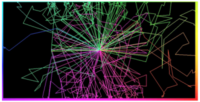
$$I(\alpha) = 4\sqrt{\pi}\,\Gamma(\alpha)\, G^{3,0}_{2,4}\left(\frac{\alpha^2}{16} \,\middle|\, \begin{matrix} \frac{\alpha+2}{2}, \frac{\alpha+3}{2} \\ \frac{1}{2}, \frac{1}{2}, 1, 0 \end{matrix}\right) [\sin(10\,\alpha) + 2].$$

- Unlike most contestants, *Mathematica* and *Maple* will figure this out; help files or a web search then inform the scientist.
- This is another measure of the changing environment. It is usually a good idea—and not at all immoral—to data-mine.

# Trefethen's problem #10 ANIMATION

*A particle at the center of a $10 \times 1$ rectangle undergoes Brownian motion (i.e., 2-D random walk with infinitesimal step lengths) till it hits the boundary. What is the probability that it hits at one of the ends rather than at one of the sides?*

Walking in a $10 \times 5$ box



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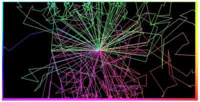
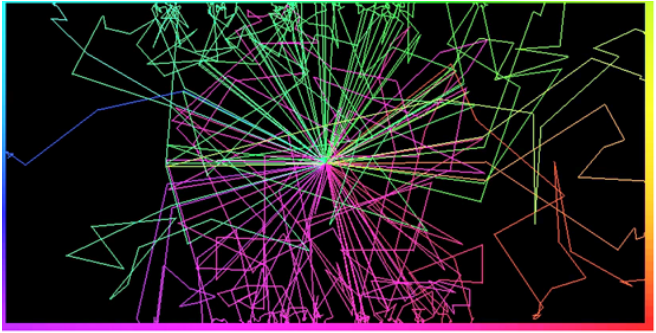


**Hitting the Ends**. Bornemann [1] starts his remarkable solution by exploring *Monte-Carlo methods*, which are shown to be impracticable.

- He reformulates the problem *deterministically* as *the value at the center of a $10 \times 1$ rectangle of an appropriate harmonic measure of the ends*, arising from a 5-point discretization of *Laplace's equation* with Dirichlet boundary conditions.
- This is then solved by a well chosen *sparse Cholesky* solver. A reliable numerical value of $3.837587979 \cdot 10^{-7}$ is obtained and the problem is solved *numerically* to the requisite ten places.
- This is the warm up....

# Walking in a $b \times a$ box ANIMATION



# Trefethen's problem #10

We may proceed to develop two analytic solutions, the *first* using *separation of variables* on the underlying PDE on a general $2a \times 2b$ rectangle. We learn that with $\rho := a/b$

$$p(a,b) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \operatorname{sech}\left(\frac{\pi(2n+1)}{2}\rho\right). \tag{5}$$

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- Three terms yields $50$ correct digits:
  $p(10,1) = \underline{0.000000383758797925}122610340713318620483910079300559407 24\ldots$
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A *second* method using *conformal mappings*, yields

$$\operatorname{arccot} \rho = p(a,b)\,\frac{\pi}{2} + \arg \mathrm{K}\left(e^{ip(a,b)\pi}\right) \tag{6}$$

where $\mathrm{K}$ is the *complete elliptic integral* of the first kind.

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Bornemann et al ultimately show that the answer is

$$p = \frac{2}{\pi} \arcsin\left(k_{100}\right) \tag{7}$$

where

$$k_{100} := \left(\left(3-2\sqrt{2}\right)\left(2+\sqrt{5}\right)\left(-3+\sqrt{10}\right)\left(-\sqrt{2}+\sqrt[4]{5}\right)^2\right)^2,$$

is a *singular value*. [In general $p(a,b) = \frac{2}{\pi}\arcsin\left(k_{(a/b)^2}\right)$.]

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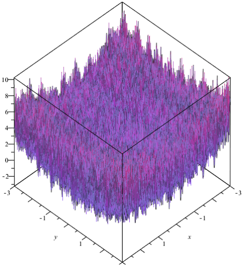
- No one (except harmonic analysts perhaps) anticipated a closed form—let alone one like this.
- Can be done for some other shapes (perhaps, convex with piecewise smooth boundaries, starting at barycentre), and for self-avoiding walks.

# Trefethen's problem #4

... zooming

*What is the global minimum of the function*

$$\exp(\sin(50x)) + \sin(60e^y) + \sin(70\sin x) + \sin(\sin(80y))$$
$$-\sin(10(x+y)) + (x^2+y^2)/4?$$



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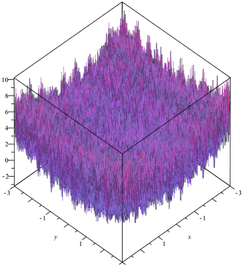
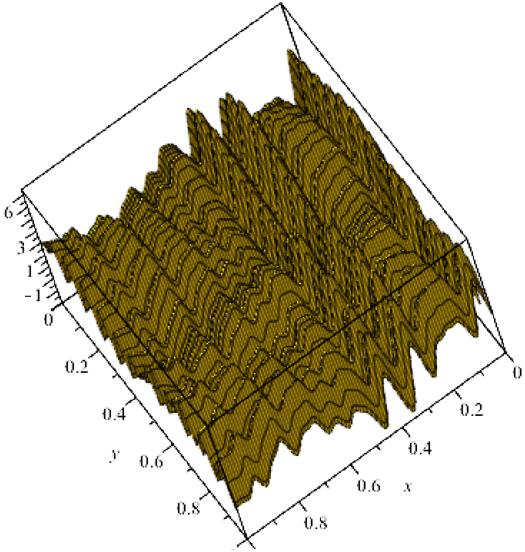


- Can be solved in a global optimization package or by a damped Newton method
- In *Mathematica* by `NMinimize[f[x, y], x, y, Method -> "RandomSearch", "SearchPoints" -> 250, WorkingPrecision -> 20]`
- In *Maple* by `NLPSolve(f(x,y), x = -4 .. 4, y = -4 .. 4, initialpoint = {x = -.4, y = -.1});`
- or by 'zooming' on $[-3,3]\times[-3,3]$.

# Trefethen's problem #4

... zooming on [0,1]



# Contents

# Algorithm performance

a simulated interlude

## Proposition (Polylogarithm computation)

*(a) For $s = n$ a positive integer,*

$$\mathrm{Li}_n(z) = \sum_{m=0}^{\infty}{}' \zeta(n-m)\frac{\log^m z}{m!} + \frac{\log^{n-1} z}{(n-1)!}\left(H_{n-1} - \log(-\log z)\right). \tag{8}$$

*(b) For any complex order $s$ not a positive integer,*

$$\mathrm{Li}_s(z) = \sum_{m\geq 0} \zeta(s-m)\frac{\log^m z}{m!} + \Gamma(1-s)(-\log z)^{s-1}. \tag{9}$$

Here $\zeta(s) := \sum_n n^{-s}$ and continuations, $H_n := 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, and $\sum'$ avoids the singularity at $\zeta(1)$.

In (8), $|\log z| < 2\pi$ precludes use when $|z| < e^{-2\pi} \approx 0.0018674$. For small $|z|$, however, it suffices to use the definition

$$\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}. \tag{10}$$

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- We found (10) faster than (8) whenever $|z| < 1/4$, for precision from $100$ to $4000$ digits. We illustrate for $\mathrm{Li}_2$ in the Figure.

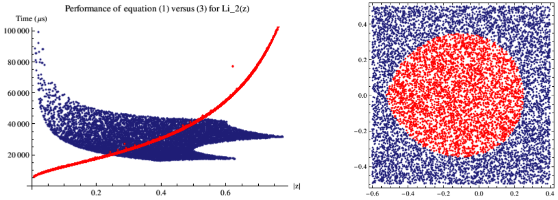


Figure : (L) Timing (8) (blue) and (10) (red).(R) blue region where (8) is faster.

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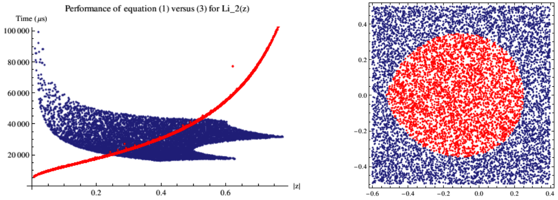
- Timings show microseconds required for $1,000$ digit accuracy as the modulus goes from $0$ to $1$ with blue showing superior performance of (8). The region records $10,000$ trials of random $z$, such that $-0.6 < \Re(z) < 0.4, -0.5 < \Im(z) < 0.5$.

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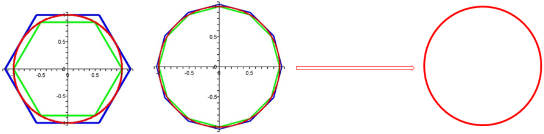

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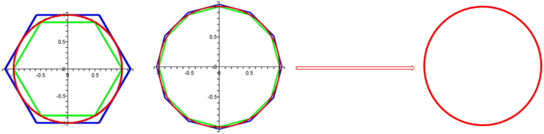

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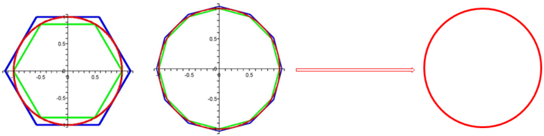
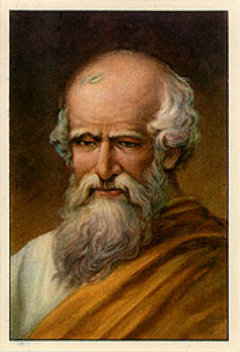


$\mathbf{6} \mapsto \mathbf{12} \mapsto 24 \mapsto 48 \mapsto \mathbf{96}$ to obtain the estimate

$$3\frac{10}{71} < \pi < 3\frac{10}{70}.$$

# Archimedes' "Method of Mechanical Theorems"

Pi movie below



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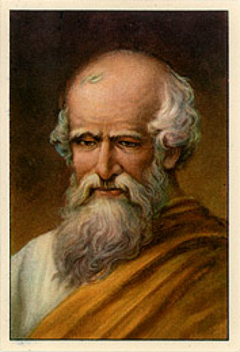

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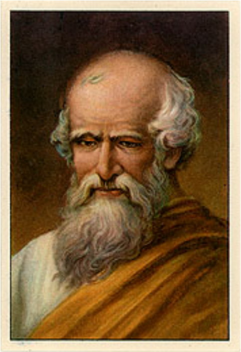

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- Only recently rediscovered and even more recently *reconstructed* ...

# Proving $\pi$ is not $\frac{22}{7}$

Even *Maple* or *Mathematica* ‘knows’ this since

$$0 \quad < \quad \int_0^1 \frac{(1-x)^4 x^4}{1+x^2}\, dx = \frac{22}{7} - \pi, \tag{11}$$

though it would be prudent to ask ‘why’ it can perform the integral and ‘whether’ to trust it?

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- Accidentally, $22/7$ is one of the early continued fraction approximation to $\pi$. These commence:

$$3, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \ldots$$

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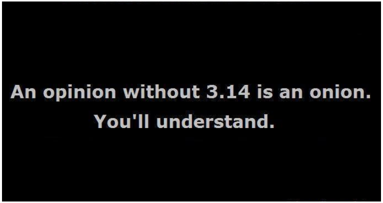
$$\int_0^{t} \frac{x^4(1-x)^4}{1+x^2}\,dx \quad = \quad \frac{1}{7}t^7 - \frac{2}{3}t^6 + t^5 - \frac{4}{3}t^3 + 4t - 4\arctan(t)$$

as differentiation easily confirms, and the fundamental theorem of calculus **proves** (11). **QED**



# Randomness

- The digits expansions of $\pi, e, \sqrt{2}$ appear to be "random":

$$\pi = 3.14159265358979323846264338327950288419716939937 5\ldots$$

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$\boxed{\text{Are they really?}}$

# Randomness

- The digits expansions of $\pi, e, \sqrt{2}$ appear to be "random":

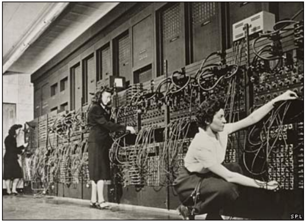
$$\pi = 3.141592653589793238462643383279502884197169399375\ldots$$

$$e = 2.718281828459045235360287471352662497757247093699\ldots$$

$$\sqrt{2} = 1.414213562373095048801688724209698078569671875376\ldots$$

Are they really?

- **1949 ENIAC** (*Electronic Numerical Integrator and Calculator*) computed of $\pi$ to **2,037** decimals (in **70** hours)—proposed by polymath John von Neumann (**1903-1957**) to shed light on distribution of $\pi$ (and of $e$).




# Two continued fractions

Change representations often

**Gauss map.** Remove the integer, invert the fraction and repeat: for 3.1415926 and 2.7182818 to get the fractions below.

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}}}}}$$

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$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{1 + \ldots}}}}}}}}}$$

**Leonhard Euler** (**1707-1783**) named $e$ and $\pi$.

"Lisez Euler, lisez Euler, c'est notre maître à tous." Simon Laplace (**1749-1827**)

# Are the digits of $\pi$ random?

| Digit | Ocurrences |
|---|---|
| 0 | 99,993,942 |
| 1 | 99,997,334 |
| 2 | 100,002,410 |
| 3 | 99,986,911 |
| 4 | **100,011**,958 |
| 5 | **99,998**,885 |
| 6 | 100,010,387 |
| 7 | 99,996,061 |
| 8 | 100,001,839 |
| 9 | 100,000,273 |
| Total | **1,000,000,000** |

Table : Counts of first billion digits of $\pi$. Second half is 'right' for law of large numbers.

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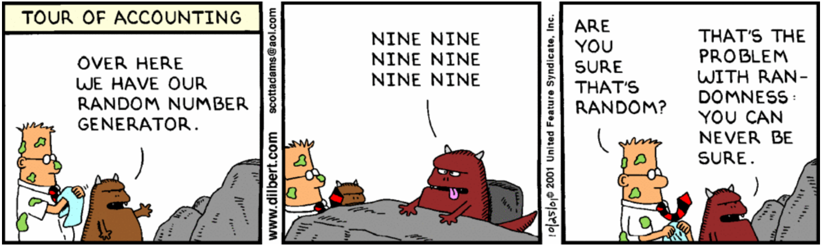
- Or pretty much anything else...

# What is “random”? A hard question



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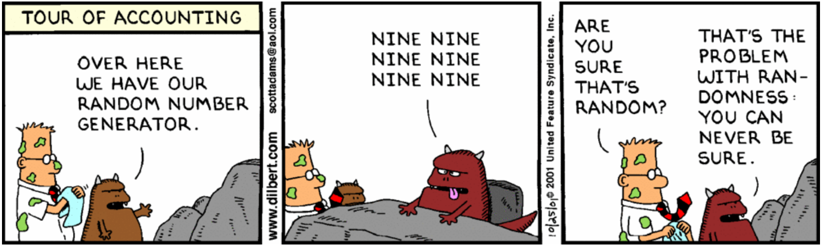


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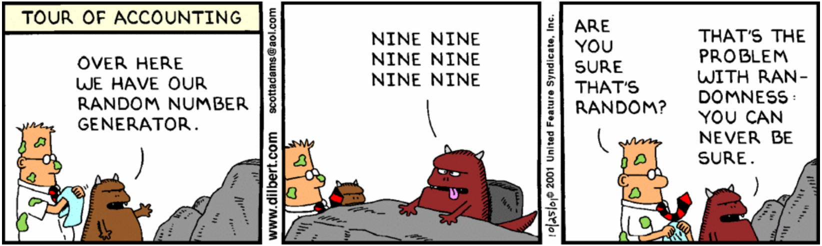


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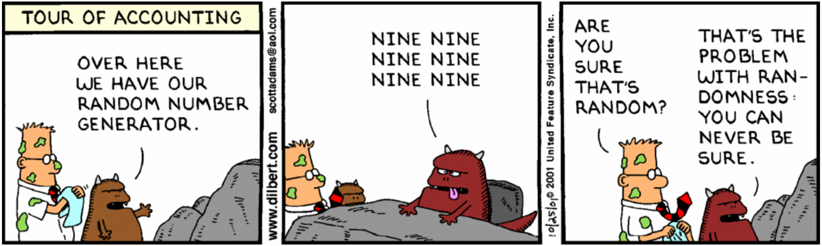


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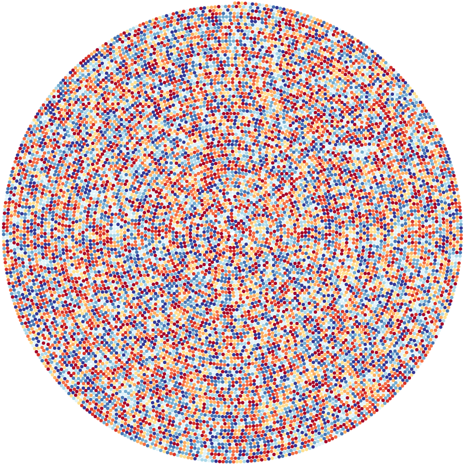
**Best Theorem** [BBCP, 04] (Feeble but hard) Asymptotically all degree $d$ algebraics have at least $n^{1/d}$ ones in binary (should be $n/2$)

# Randomness in Pi?

http://mkweb.bcgsc.ca/pi/art/



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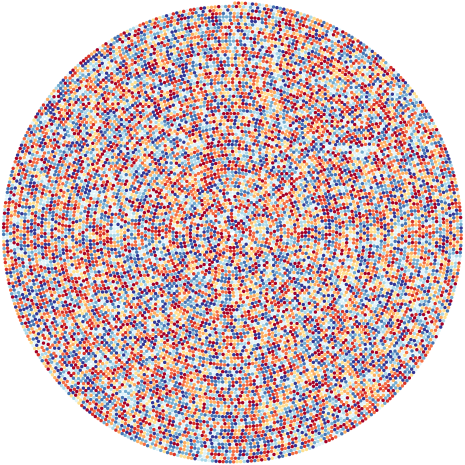

- a better color palette for art if not for science

# Contents

# Normality

A property random numbers must possess

**Definition**

A real constant $\alpha$ is $b$-normal if, given the positive integer $b \geq 2$ (the base), every $m$-long string of base-$b$ digits appears in the base-$b$ expansion of $\alpha$ with precisely the expected limiting frequency $1/b^m$.

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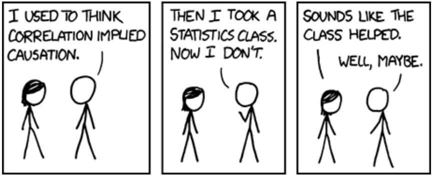

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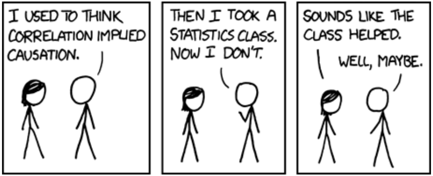
- Indeed, almost all real numbers are $b$-normal simultaneously for all positive integer bases ("absolute normality").



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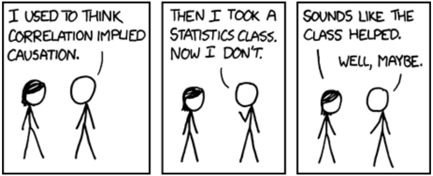
- Indeed, almost all real numbers are $b$-normal simultaneously for all positive integer bases ("absolute normality").
- Unfortunately, it has been very difficult to prove normality for any number in a given base $b$, much less all bases simultaneously.



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$$C_{10} := 0.123456789101112131415161718\ldots$$

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- Normality proofs are not known for $\pi, e, \log 2, \sqrt{2}$ etc.

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**Theorem (Davenport-Erdös (1952))**

Let $p$ be any polynomial positive on the natural numbers. Then the concatenation number

$$0.p(1)p(2)p(3)\ldots p(n)\ldots$$

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- Includes Champernowne's number and $0.1491625\dots$ (Besicovich)
- See H. Davenport and P. Erdös, "Note on normal numbers." *Can. J. Math.*, **4** (1952), 58–63.

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# Is $\pi$ 10-normal?

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| 5 | **99,998,885** | 05 | **10,002,048** | 005 | **1,002,881** |
| 6 | 100,010,387 | 06 | 9,995,451 | 006 | 999,294 |
| 7 | 99,996,061 | 07 | 9,993,703 | 007 | 998,919 |
| 8 | 100,001,839 | 08 | 10,000,565 | 008 | 999,962 |
| 9 | 100,000,273 | 09 | 9,999,276 | 009 | 999,059 |
| | | 10 | 9,997,289 | 010 | 998,884 |
| | | 11 | 9,997,964 | 011 | 1,001,188 |
| | | ⋮ | ⋮ | ⋮ | ⋮ |
| | | 99 | 10,003,709 | 099 | 999,201 |
| | | | | ⋮ | ⋮ |
| | | | | 999 | 1,000,905 |
| **TOTAL** | 1,000,000,000 | **TOTAL** | 1,000,000,000 | **TOTAL** | 1,000,000,000 |

Table : Counts for the first billion digits of $\pi$.

# Is $\pi$ 16-normal

That is, in Hex?

$\hookleftarrow$ Counts of first trillion hex digits

| | |
|---|---|
| 0 | 62499881108 |
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| 2 | 62499924780 |
| 3 | 62500188844 |
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- They are $353CB3F7F0C9ACCFA9AA215F2$

See www.karrels.org/pi/index.html

# Modern $\pi$ Calculation Records: and IBM Blue Gene/L at LBL

| Name | Year | Correct Digits |
|---|---|---|
| Miyoshi and Kanada | 1981 | 2,000,036 |
| Kanada-Yoshino-Tamura | 1982 | 16,777,206 |
| Gosper | 1985 | 17,526,200 |
| Bailey | Jan. 1986 | 29,360,111 |
| Kanada and Tamura | Sep. 1986 | 33,554,414 |
| Kanada and Tamura | Oct. 1986 | 67,108,839 |
| Kanada et. al | Jan. 1987 | 134,217,700 |
| Kanada and Tamura | Jan. 1988 | 201,326,551 |
| Chudnovskys | May 1989 | 480,000,000 |
| Kanada and Tamura | Jul. 1989 | 536,870,898 |
| Kanada and Tamura | Nov. 1989 | 1,073,741,799 |
| Chudnovskys | Aug. 1991 | 2,260,000,000 |
| Chudnovskys | May 1994 | 4,044,000,000 |
| Kanada and Takahashi | Oct. 1995 | 6,442,450,938 |
| Kanada and Takahashi | Jul. 1997 | 51,539,600,000 |
| Kanada and Takahashi | Sep. 1999 | 206,158,430,000 |
| Kanada-Ushiro-Kuroda | Dec. 2002 | 1,241,100,000,000 |
| Takahashi | Jan. 2009 | 1,649,000,000,000 |
| Takahashi | April 2009 | 2,576,980,377,524 |
| Bellard | Dec. 2009 | 2,699,999,990,000 |
| Kondo and Yee | Aug. 2010 | **5,000,000,000,000** |
| Kondo and Yee | Oct. 2011 | **10,000,000,000,000** |
| Kondo and Yee | Dec. 2013 | **12,100,000,000,000** |

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- An algorithm found by computer—now used to check record $\pi$ computations and in some compilers.

# What BBP Is?

## Reverse Engineered Mathematics

This is based on the following then new formula for $\pi$:

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i} \left( \frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6} \right) \tag{12}$$

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- Bailey-Crandall (220) link BBP and normality.

# Edge of Computation Prize Finalist (2005)

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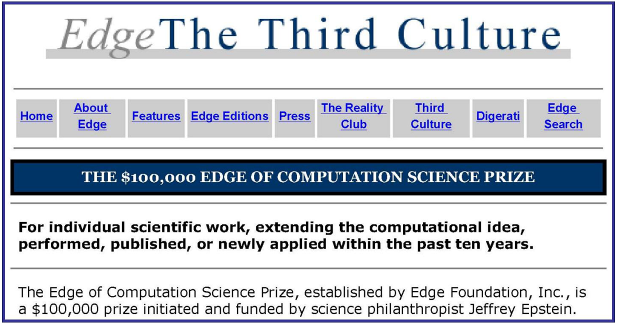
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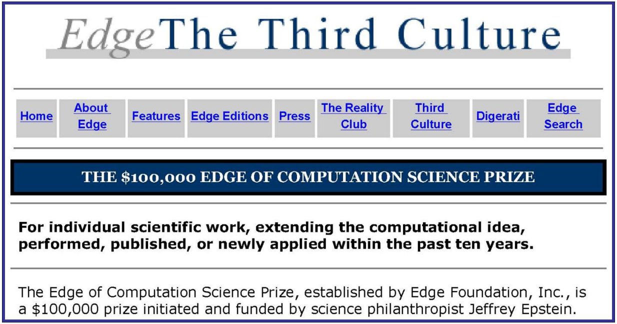


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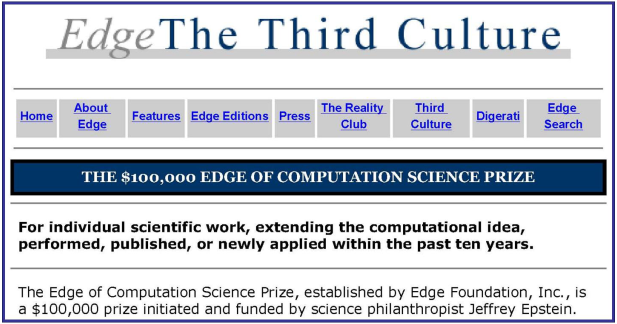


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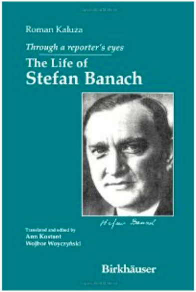
- Won by David Deutsch — discoverer of Quantum Computing.

# Stefan Banach (1892-1945) Another Nazi casuality

*A mathematician is a person who can find analogies between theorems; a better mathematician is one who can see analogies between proofs and the best mathematician can notice analogies between theories.* [6]



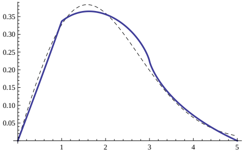


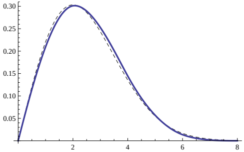
[6] Only the best get stamps. Quoted in `www-history.mcs.st-andrews.ac.uk/Quotations/Banach.html`.

# Contents

# A Little History:

From a vast literature

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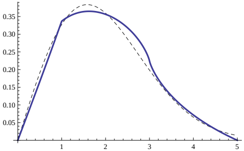


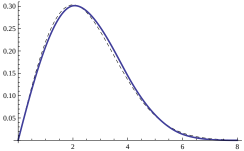


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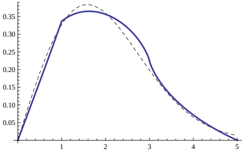


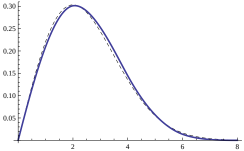


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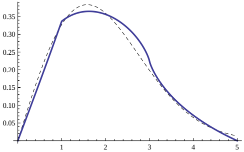


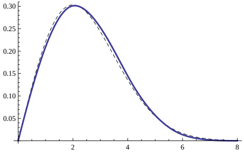


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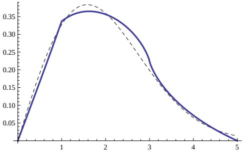


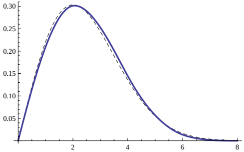


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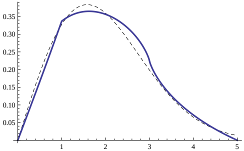


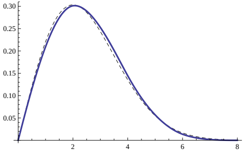


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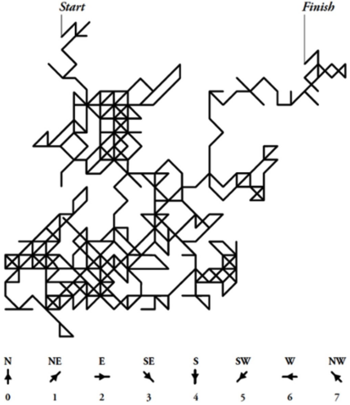
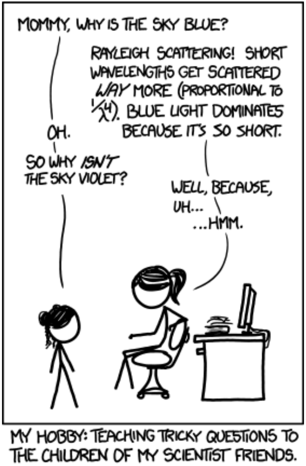
- appear in graph theory, quantum chemistry, in quantum physics as hexagonal and diamond lattice integers, etc ...

# The first walk (Venn)

## Why is the sky blue?

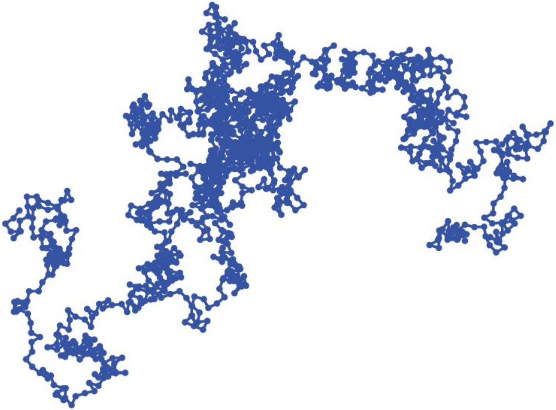


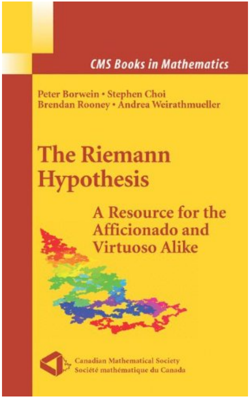

he first person to visualize the random nature of pi's decimal digits was the Victorian nathematician John Venn. In *The Logic of Chance* (1888), he suggested that the digits 0 to 7 epresent eight compass directions, and he followed the path tracked by these digits in pi. He nisses out the initial 3, and starts 14159. Venn's image was the first "random walk", an idea now ised frequently in probability and statistics. (The illustration is taken from my book, Alex's \dventures in Numberland)



# One 1500-step ramble: a familiar picture

Liouville function





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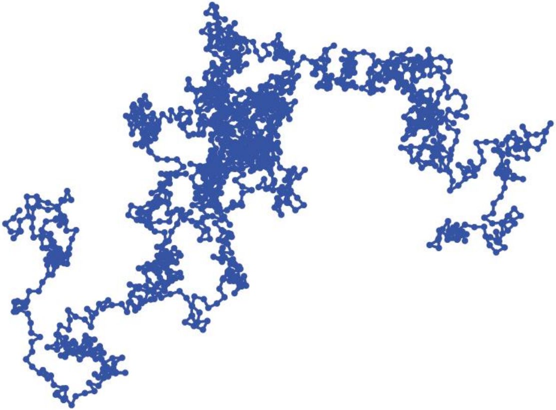

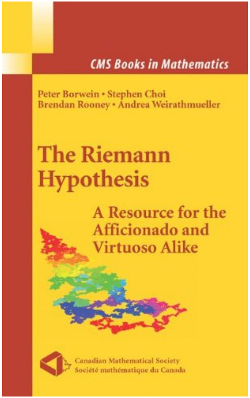


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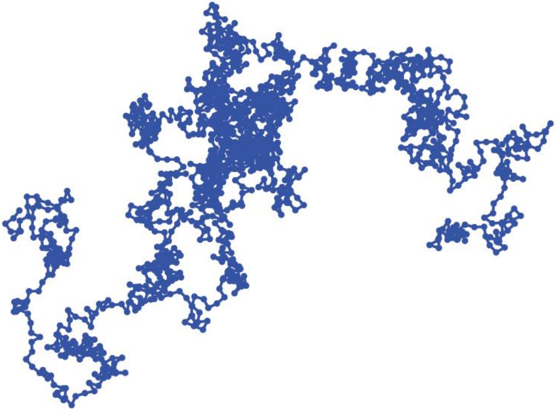
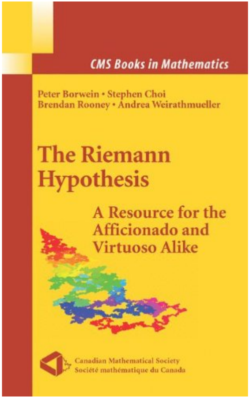


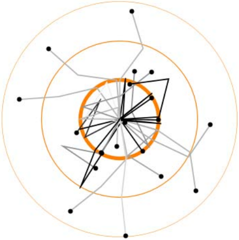



- 1D (and 3D) easy. Expectation of RMS distance is easy ($\sqrt{n}$).
- 1D or 2D *lattice*: probability one of returning to the origin.

# Contents

# Case study II: short rambles

a less familiar picture?



$1000$ three-step uniform planar walks

## The moments of an $n$-step planar walk: $W_n := W_n(1)$

- Second simplest case:

$$W_2 = \int_0^1 \int_0^1 \left| e^{2\pi ix} + e^{2\pi iy} \right| \mathrm{d}x\mathrm{d}y = \ ?$$

[7] Quadrature was our first interest

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$$\begin{aligned} W_n(s) &= \int_{[0,1]^n} \left| \sum_{k=1}^{n} e^{2\pi x_k i} \right|^s \mathrm{d}(x_1, \ldots, x_{n-1}, x_n) \\ &= \int_{[0,1]^{n-1}} \left| 1 + \sum_{k=1}^{n-1} e^{2\pi x_k i} \right|^s \mathrm{d}(x_1, \ldots, x_{n-1}) \end{aligned}$$

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- So $W_2 = 4\int_0^{1/4} \cos(\pi x)\mathrm{d}x = \frac{4}{\pi}$.

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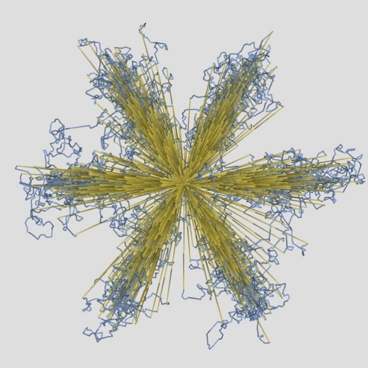
# Art meets science

AAAS & Bridges conference



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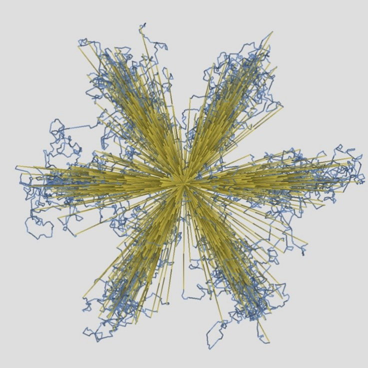

*A visualization of six routes that 1000 ants took after leaving their nest in search of food. The jagged blue lines represent the breaking off of random ants in search of seeds*.

(Nadia Whitehead 2014-03-25 16:15)

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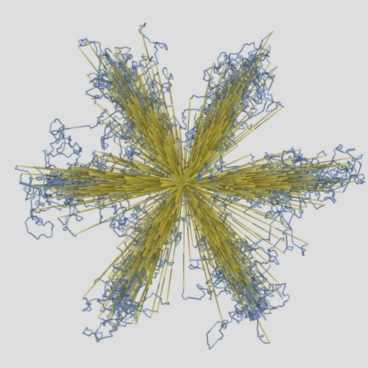

*A visualization of six routes that 1000 ants took after leaving their nest in search of food. The jagged blue lines represent the breaking off of random ants in search of seeds.*

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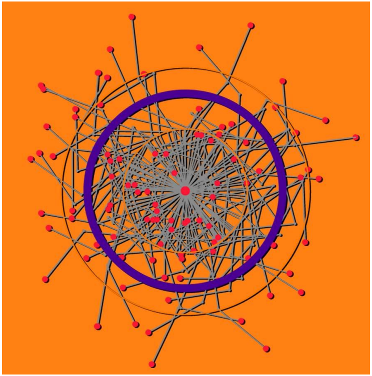
(JonFest 2011 Logo) *Three-step random walks. The (purple) expected distance travelled is **1.57459** ...*

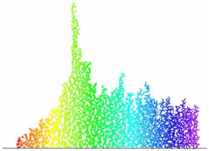
*The closed form $W_3$ is given below.*

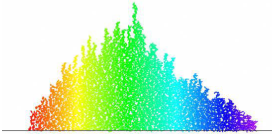


$$\boxed{W_3 = \frac{16\sqrt[3]{4}\,\pi^2}{\Gamma(\frac{1}{3})^6} + \frac{3\Gamma(\frac{1}{3})^6}{8\sqrt[3]{4}\,\pi^4}}$$

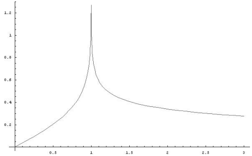
# Simulating the densities for $n = 3, 4$

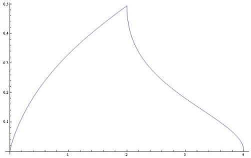
ANIMATION





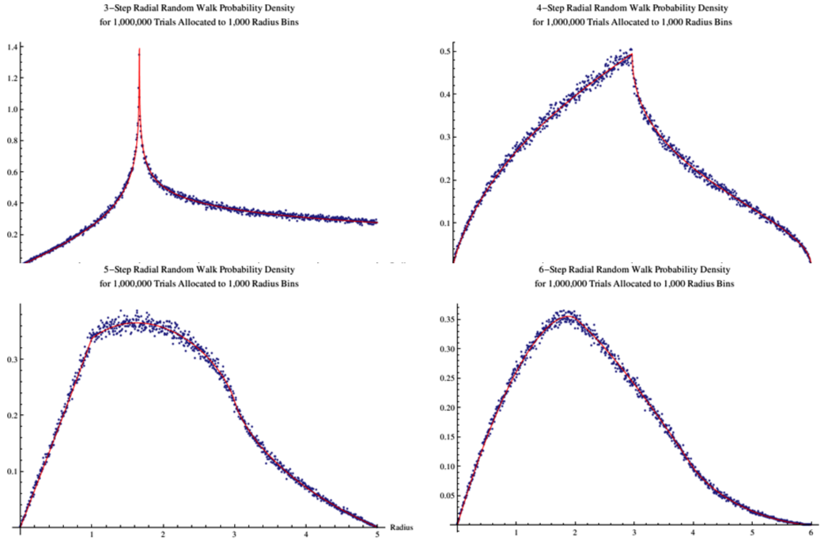
The densities $p_3$ (L) and $p_4$ (R)




Simulation thanks to Cam Rogers

# The radial densities for $3 \leq n \leq 6$ (simulations by A. Mattingly)



# Pearson's original full question

and comment on $p_5$

*A man starts from a point $O$ and walks $l$ yards in a straight line; he then turns through any angle whatever and walks another $l$ yards in a second straight line. He repeats this process $n$ times. I require the probability that after these $n$ stretches he is at a distance between $r$ and $r+\delta r$ from his starting point, $O$.*

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*"the graphical construction, however carefully reinvestigated, did not permit of our considering the curve to be anything but a straight line... Even if it is not absolutely true, it exemplifies the extraordinary power of such integrals of $J$ products to give extremely close approximations to such simple forms as horizontal lines."*

# The radial densities for $n = 3, 4$ are modular functions

Let $\sigma(x) := \frac{3-x}{1+x}$. Then $\sigma$ is an involution on $[0,3]$ sending $[0,1]$ to $[1,3]$:

$$p_3(x) = \frac{4x}{(3-x)(x+1)} p_3(\sigma(x)). \tag{13}$$

So $\frac{3}{4} p_3'(0) = p_3(3) = \frac{\sqrt{3}}{2\pi}$, $p(1) = \infty$.

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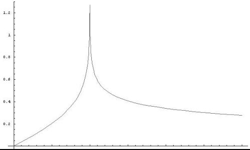
So $\frac{3}{4}p_3'(0) = p_3(3) = \frac{\sqrt{3}}{2\pi}$, $p(1) = \infty$. We found:

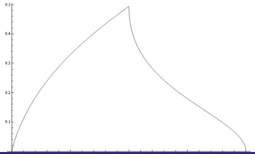
$$p_3(\alpha) = \frac{2\sqrt{3}\alpha}{\pi\,(3+\alpha^2)}\,{}_2F_1\left(\left.{\frac{1}{3}, \frac{2}{3} \atop 1}\right| \frac{\alpha^2\left(9-\alpha^2\right)^2}{\left(3+\alpha^2\right)^3}\right) = \frac{2\sqrt{3}}{\pi}\,\frac{\alpha}{\mathrm{AG}_3\left(3+\alpha^2, 3\left(1-\alpha^2\right)^{2/3}\right)}$$

where $\mathrm{AG}_3$ is the *cubically convergent* mean iteration (1991):

$$\mathrm{AG}_3(a,b) := \frac{a+2b}{3} \bigotimes \left(b \cdot \frac{a^2+ab+b^2}{3}\right)^{1/3}.$$

The densities $p_3$ (L) and $p_4$ (R)





# Formula for the 'shark-fin' $p_4$

SKIP

We ultimately deduce on $2 \leq \alpha \leq 4$ a hyper-closed form:

$$p_4(\alpha) = \frac{2}{\pi^2} \frac{\sqrt{16-\alpha^2}}{\alpha} \, {}_3F_2\left( \left. {\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \atop \frac{5}{6}, \frac{7}{6}} \right| \frac{\left(16-\alpha^2\right)^3}{108\,\alpha^4} \right). \tag{15}$$

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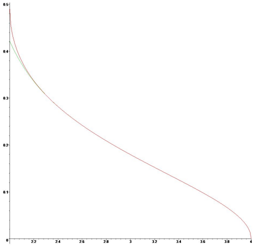



$\leftarrow$ $p_4$ from (15) vs 18-terms of empirical power series

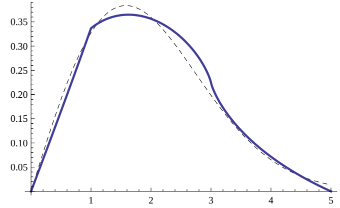
✓ Proves $p_4(2) = \frac{2^{7/3}\pi}{3\sqrt{3}}\,\Gamma\left(\frac23\right)^{-6} = \frac{\sqrt{3}}{\pi}\,W_3(-1) \approx 0.494233 < \frac12$

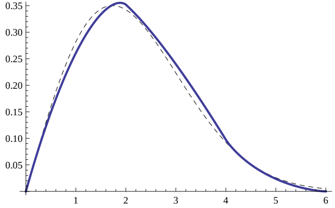
- Empirically, quite marvelously, we found — and proved by a subtle use of distributional Mellin transforms — that on $[0,2]$ as well:

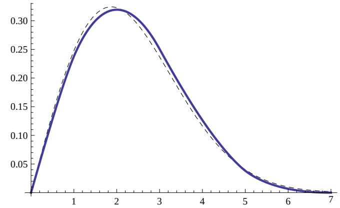
$$p_4(\alpha) \quad \stackrel{?}{=} \quad \frac{2}{\pi^2}\frac{\sqrt{16-\alpha^2}}{\alpha}\,\Re\,{}_3F_2\left(\left.{\frac12,\frac12,\frac12 \atop \frac56,\frac76}\right|\frac{\left(16-\alpha^2\right)^3}{108\,\alpha^4}\right) \tag{16}$$

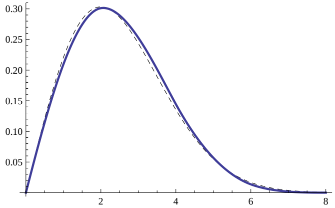
(Discovering this $\Re$ brought us full circle.)

# The radial densities for $5 \leq n \leq 8$ (and large $n$ approximation)



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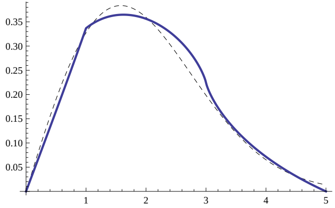
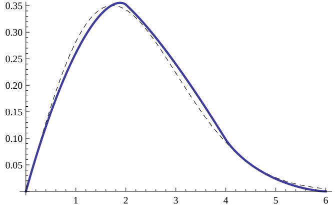
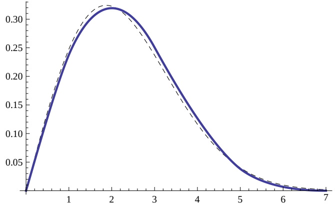
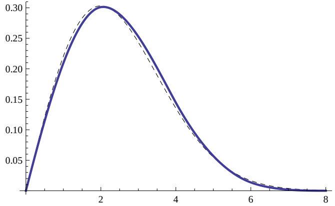



Both $p_{2n+4}, p_{2n+5}$ are $n$-times continuously differentiable for $x > 0$ with $p_n(x) \sim \frac{2x}{n} e^{-x^2/n}$. So "four is small" *but* "eight is large."



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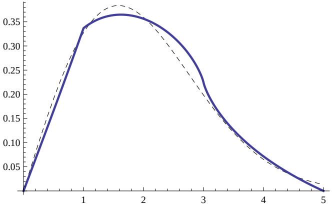


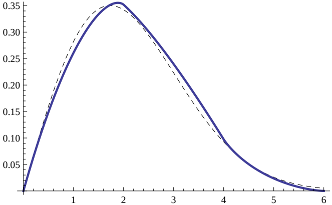


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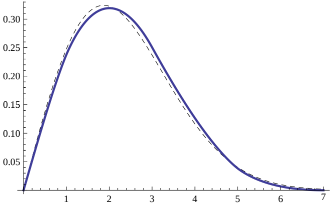
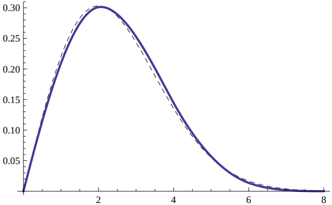


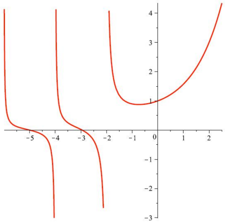

- Pearson wondered if $p_5$ was linear on $[0, 1]$. Only disproven in sixties.

# Meijer-G (1936) form for $W_3$ and graph on real line

**Theorem (Meijer-G form for $W_3$)**

For $s$ not an odd integer

$$W_3(s) = \frac{\Gamma(1+\frac{s}{2})}{\sqrt{\pi}\,\Gamma(-\frac{s}{2})}\, G_{33}^{21}\left(\begin{matrix} 1,1,1 \\ \frac{1}{2},-\frac{s}{2},-\frac{s}{2}\end{matrix} \,\middle|\, \frac{1}{4}\right).$$



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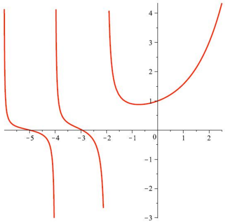

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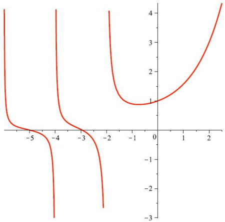
- Proved using residue calculus methods.
- $W_3(s)$ is among the first non-trivial higher order Meijer-G function to be placed in closed form.



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*He [`Gauss (or Mathematica)`] is like the fox, who effaces his tracks in the sand with his tail.*— Niels Abel (1802-1829)

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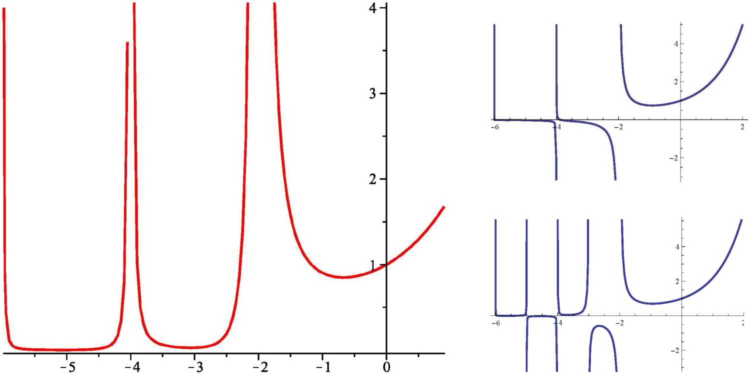
But we really need a formula with $s = 1$, that is an **integer**.

# Visualizing $W_4, W_5$, and $W_6$ on the real line



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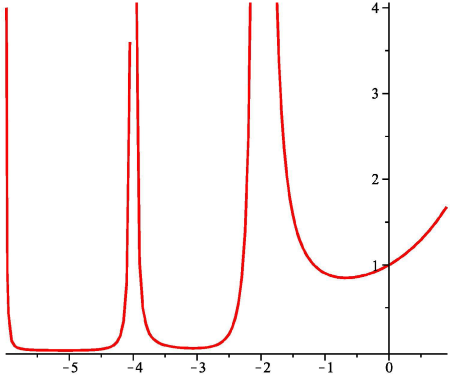


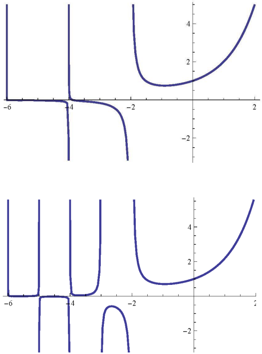


- Use recursion from $s > 1$

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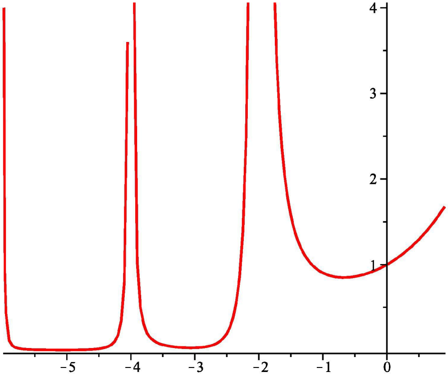
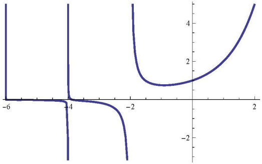
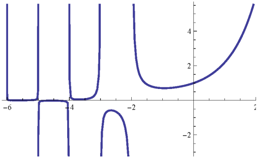
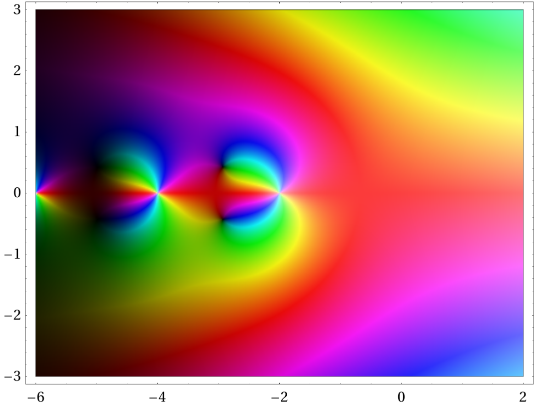



- Use recursion from $s > 1$
- Nonnegativity of $W_4$ was hard to prove (Wan)

# Visualizing $W_4$ in the complex plane



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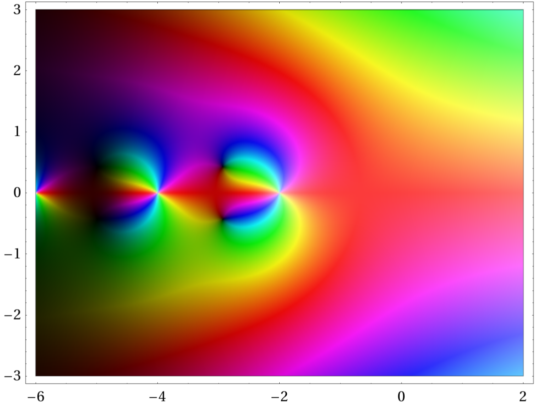


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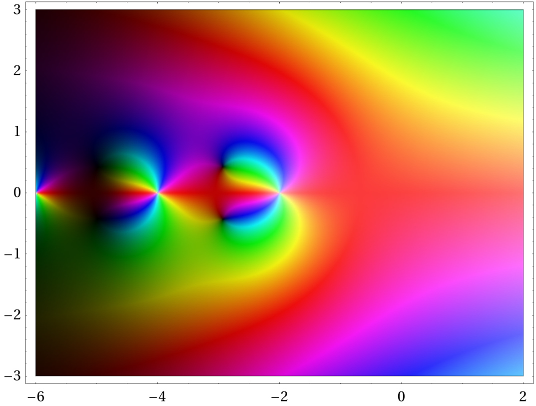


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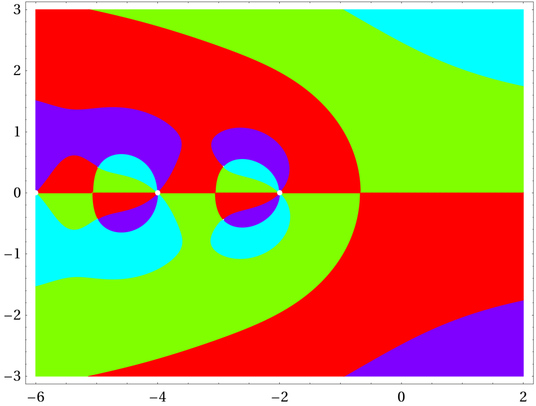
# Visualizing $W_4$ in the complex plane: sometimes less is more




- Easily drawn now in *Mathematica* from the the Meijer-G representation.
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# Simplifying the Meijer integrals for $W_3$ and $W_4$

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**Corollary (Hypergeometric forms for non-integer $s > -2$)**

$$W_3(s) = \frac{\tan\left(\frac{\pi s}{2}\right)}{2^{2s+1}} \binom{s}{\frac{s-1}{2}}^2 {}_3F_2\left( {\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \atop \frac{s+3}{2}, \frac{s+3}{2}} \middle| \frac{1}{4} \right) + \binom{s}{\frac{s}{2}} {}_3F_2\left( {-\frac{s}{2}, -\frac{s}{2}, -\frac{s}{2} \atop 1, -\frac{s-1}{2}} \middle| \frac{1}{4} \right),$$

and

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- We (humans) were able to provably take the limit at $\pm 1$: e.g.,

$$W_4(-1) = \frac{\pi}{4} {}_7F_6\left( \left. \begin{matrix} \frac{5}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{1}{4}, 1, 1, 1, 1, 1 \end{matrix} \right| 1 \right) = \frac{\pi}{4} \sum_{n=0}^{\infty} \frac{(4n+1)\binom{2n}{n}^6}{4^{6n}}$$

$$= \frac{\pi}{4} {}_6F_5\left( \left. \begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1, 1, 1 \end{matrix} \right| 1 \right) + \frac{\pi}{64} {}_6F_5\left( \left. \begin{matrix} \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \\ 2, 2, 2, 2, 2 \end{matrix} \right| 1 \right).$$

# Hypergeometric values of $W_3$: from Meijer-G values.

With much work involving moments of elliptic integrals we obtain:

**Theorem (Tractable hypergeometric form for $W_3$)**

(a) For $s \neq -3, -5, -7, \dots$ , we have

$$W_3(s) = \frac{3^{s+3/2}}{2\pi} \beta\left(s+\frac{1}{2}, s+\frac{1}{2}\right) {}_3F_2\left( {\frac{s+2}{2}, \frac{s+2}{2}, \frac{s+2}{2} \atop 1, \frac{s+3}{2}} \middle| \frac{1}{4} \right). \tag{18}$$

(b) For every natural number $k = 1, 2, \dots$,

$$W_3(-2k-1) = \frac{\sqrt{3}\binom{2k}{k}^2}{2^{4k+1} 3^{2k}} {}_3F_2\left( {\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \atop k+1, k+1} \middle| \frac{1}{4} \right).$$

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**Theorem (Tractable hypergeometric form for $W_3$)**

(a) For $s \neq -3, -5, -7, \dots$ , we have

$$W_3(s) = \frac{3^{s+3/2}}{2\pi} \beta\left(s+\frac{1}{2}, s+\frac{1}{2}\right) {}_3F_2\left(\left.{\frac{s+2}{2}, \frac{s+2}{2}, \frac{s+2}{2} \atop 1, \frac{s+3}{2}}\right| \frac{1}{4}\right). \qquad (18)$$

(b) For every natural number $k = 1, 2, \dots$,

$$W_3(-2k-1) = \frac{\sqrt{3}\binom{2k}{k}^2}{2^{4k+1}3^{2k}} {}_3F_2\left(\left.{\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \atop k+1, k+1}\right| \frac{1}{4}\right).$$

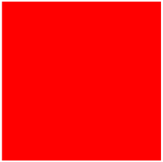
- The following formula hints at the role played by Bessel functions (Kluywer 1906):

$$W_n = n\int_0^\infty J_1(x)J_0(x)^{n-1}\frac{\mathrm{d}x}{x} \approx \frac{\sqrt{n\pi}}{2}.$$

# Contents

# What is a (base four) random walk ?

Pick a random number in $\{0,1,2,3\}$ and move according to $0 = \rightarrow$, $1 = \uparrow$, $2 = \leftarrow$, $3 = \downarrow$



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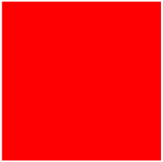

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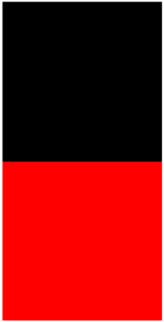

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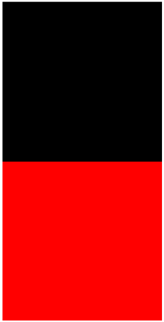

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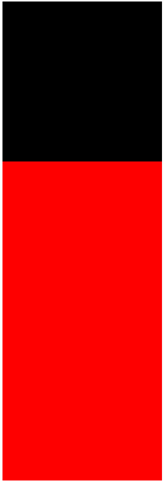

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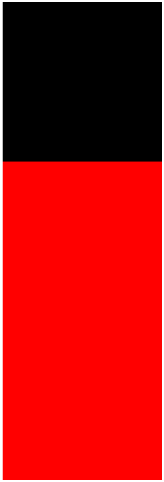

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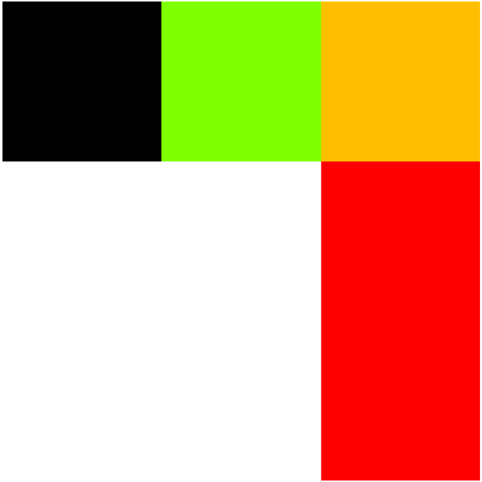

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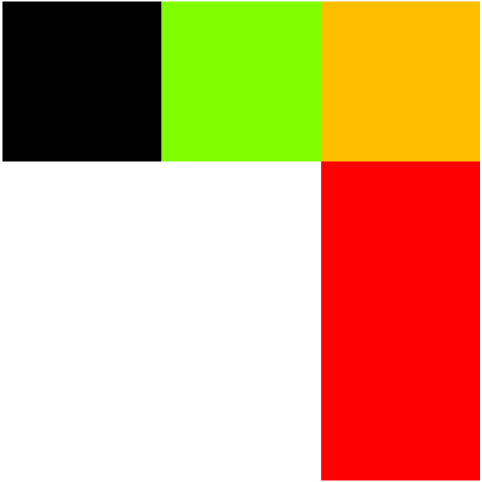

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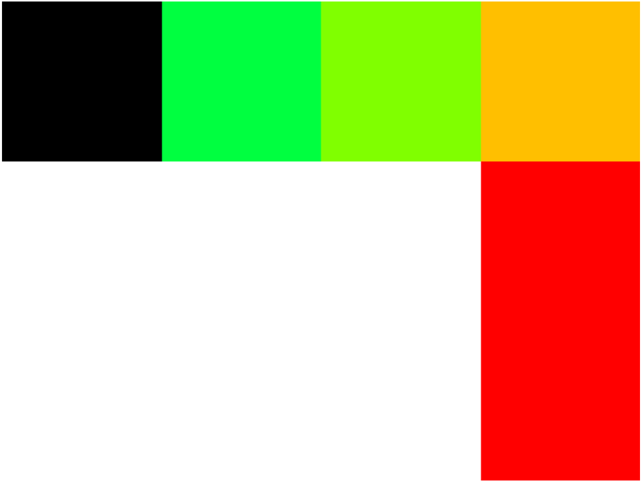

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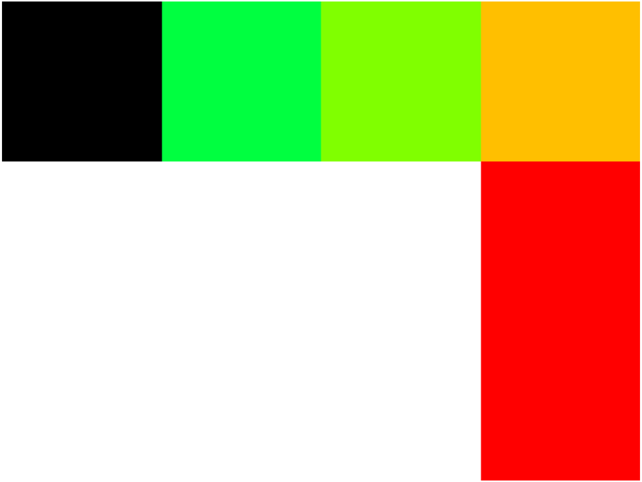

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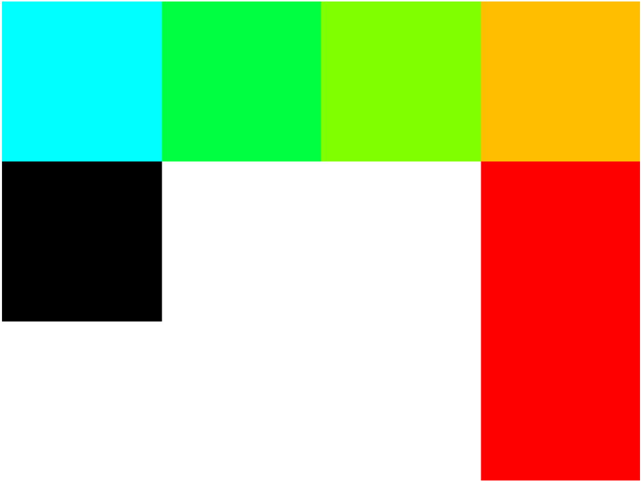

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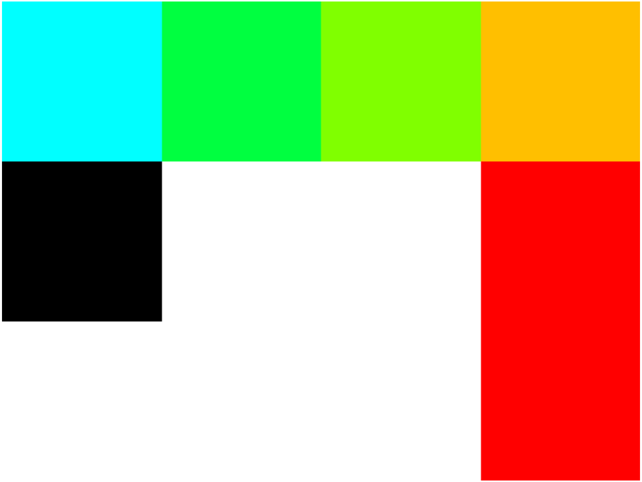

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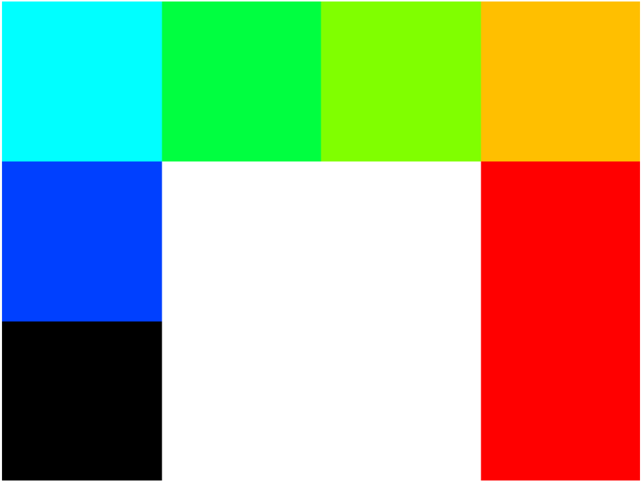

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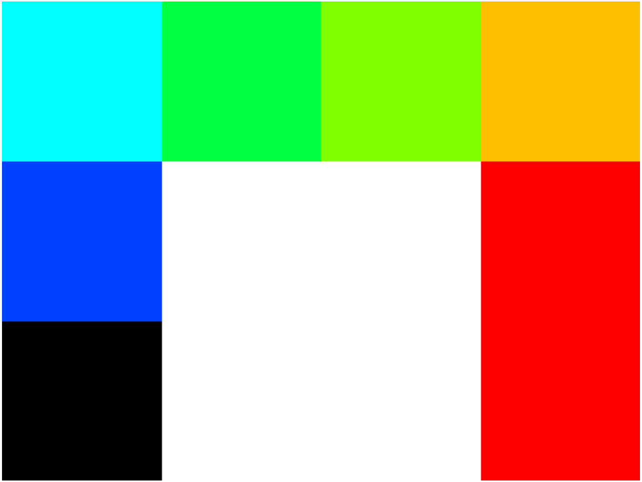

$$0 = \rightarrow$$

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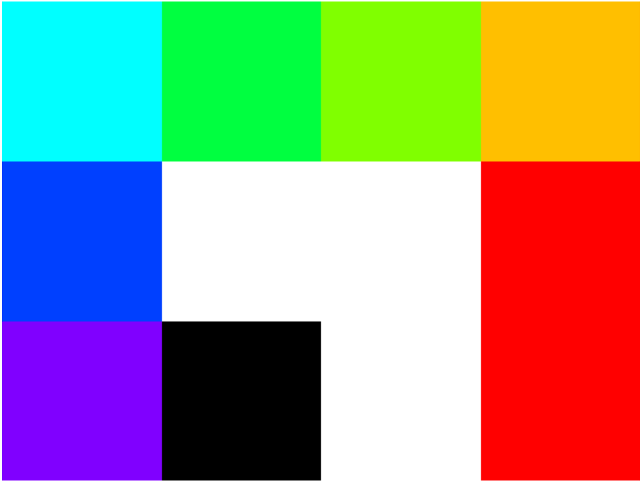

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# What is a random walk (base 4)?

Pick a random number in $\{0,1,2,3\}$ and move $0 = \rightarrow,\ 1 = \uparrow,\ 2 = \leftarrow,\ 3 = \downarrow$ ANIMATION

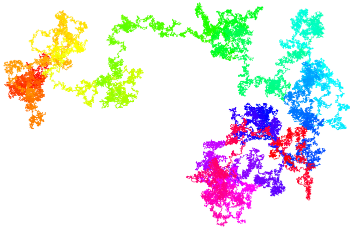

Figure : A million step base-4 pseudorandom walk. We use the spectrum to show when we visited each point (ROYGBIV and R).

# Random walks look similarish

Chaos theory (order in disorder)

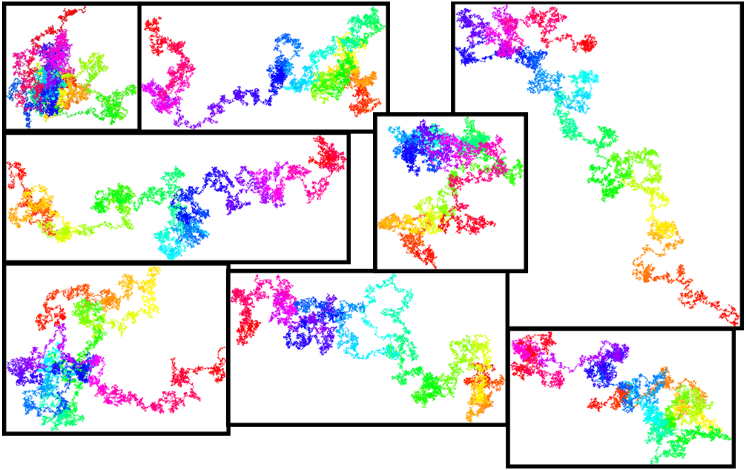

Figure : Eight different base-4 (pseudo)random[8] walks of one million steps.

[8] Python uses the *Mersenne Twister* as the core generator. It has a period of $2^{19937} - 1 \approx 10^{6002}$.

# Base-$b$ random walks: Our direction choice

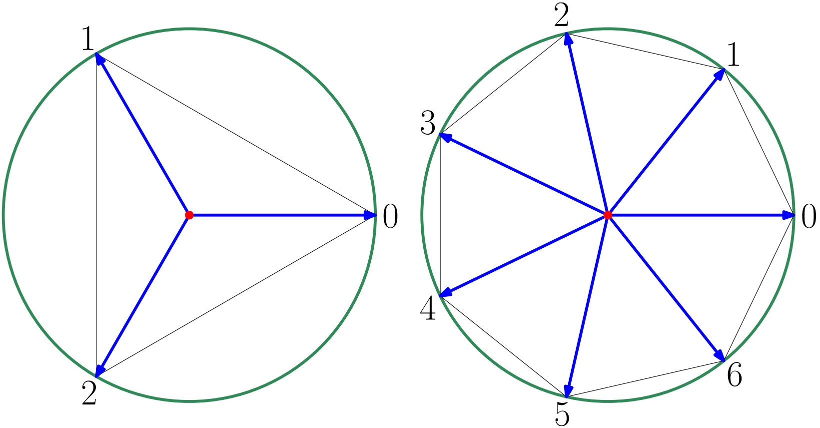


Figure : Directions for base-3 and base-7 random walks.

# Contents

# III: Two rational numbers

ANIMATION

The base-4 digit expansion of $Q1$ and $Q2$:

```
Q1=
0.2212210122321212001221012231210012221000112321231210001222100012221
0001222100012221000012221000122201103010122010012010311033333333333
3333333333333333011111111111111111111111111111001000000003003003200032
0032003022300032220300032223000302222030003222300032223000322230000032
2232000023222300032223003222133002332123302321323211211212122232323233
3330300000100032300323003220303203011033301110330110310111101133233 3
32323223212212112111211223222222122...
```

```
Q2=
0.2212210122321212001221012231210012221000112321231210001222100012221
0001222100012221000012221000122201103010122010012010311033333333333
3333333333333333011111111111111111111111111111001000000003003003200032
0032003022300032220300032223000302222030003222300032223000322230000032
2232000023222300032223003222133002332123302321323211211212122232323233
333030000010003230032300322030320301103330111033011031011110110000000
000000...
```

# III: Two rational numbers ANIMATION

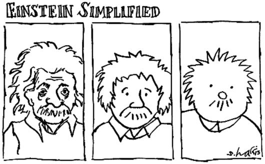


Figure : Self-referent walks on the rational numbers $Q1$ (top) and $Q2$ (bottom).

# Two more rationals

Hard to tell from their decimal expansions

The following relatively small rational numbers [G. Marsaglia, 2010]

$$Q3 = \frac{3624360069}{7000000001} \quad \text{and} \quad Q4 = \frac{123456789012}{1000000000061},$$

have base-10 periods with huge length of **1,750,000,000** digits and **1,000,000,000,060** digits, respectively.

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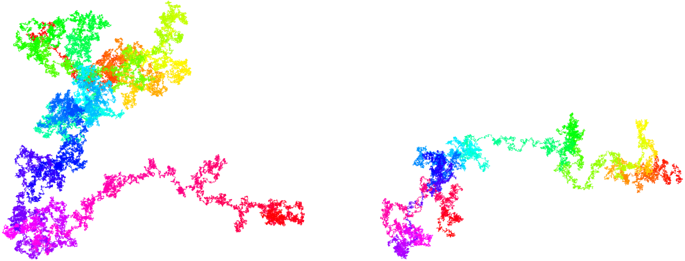

(a) $Q3$ (b) $Q4$

Figure : Walks on the first million base-10 digits of the rationals $Q3$ and $Q4$.

# Walks on the digits of numbers

ANIMATION

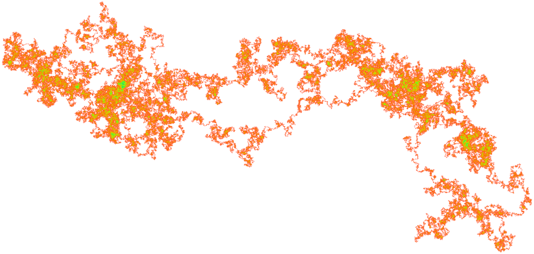

Figure : A walk on the first 10 million base-4 digits of $\pi$.

# Walks on the digits of numbers

Coloured by hits (more pink is more hits)

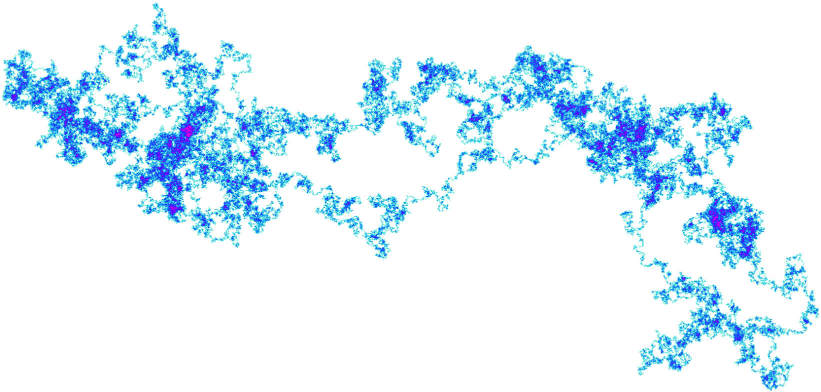

Figure : 100 million base-4 digits of $\pi$ coloured by number of returns to points.

# Contents

# The Stoneham numbers $\alpha_{b,c} = \sum_{n=1}^{\infty} \frac{1}{c^n b^{c^n}}$

**1973** Richard Stoneham proved some of the following (nearly 'natural') constants are $b$-normal for relatively prime integers $b, c$:

$$\alpha_{b,c} := \frac{1}{cb^c} + \frac{1}{c^2 b^{c^2}} + \frac{1}{c^3 b^{c^3}} + \ldots.$$

Such super-geometric sums are Stoneham constants. To 10 places

$$\alpha_{2,3} = \frac{1}{24} + \frac{1}{3608} + \frac{1}{3623878656} + \ldots$$

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- Since $3 < 2^{3-1} = 4$, $\alpha_{2,3}$ is 2-normal and 6-nonnormal !

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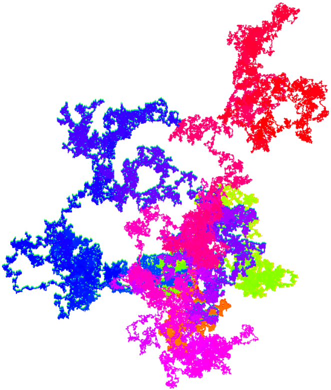

Figure : $\alpha_{2,3}$ is 2-normal (top) but 6-nonnormal (bottom). Is seeing believing?

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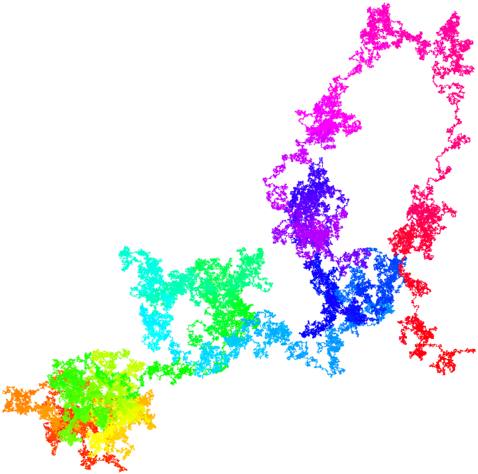

Figure : Is $\alpha_{2,3}$ 3-normal or not?

# Contents

# The expected distance to the origin $\frac{\sqrt{\pi N}}{2\,d_N} \to 1$

**Theorem**

The expected distance $d_N$ to the origin of a base-$b$ *random walk* of $N$ steps behaves like to $\sqrt{\pi N}/2$.

# The expected distance to the origin

$\frac{\sqrt{\pi N}}{2\,d_N} \to 1$

**Theorem**

The expected distance $d_N$ to the origin of a base-$b$ *random walk* of $N$ steps behaves like to $\sqrt{\pi N}/2$.

| Number | Base | Steps | Average normalized dist. to the origin: $\frac{1}{\text{Steps}}\sum_{N=2}^{\text{Steps}}\frac{\text{dist}_N}{\frac{\sqrt{\pi N}}{2}}$ | Normal |
|---|---|---|---|---|
| Mean of 10,000 random walks | 4 | 1,000,000 | **1.00315** | Yes |
| Mean of 10,000 walks on the digits of $\pi$ | 4 | 1,000,000 | **1.00083** | ? |
| $\alpha_{2,3}$ | 3 | 1,000,000 | 0.89275 | ? |
| $\alpha_{2,3}$ | 4 | 1,000,000 | 0.25901 | Yes |
| $\alpha_{2,3}$ | 6 | 1,000,000 | 108.02218 | No |
| $\pi$ | 4 | 1,000,000 | 0.84366 | ? |
| $\pi$ | 6 | 1,000,000 | 0.96458 | ? |
| $\pi$ | 10 | 1,000,000 | 0.82167 | ? |
| $\pi$ | 10 | 1,000,000,000 | 0.59824 | ? |
| $\sqrt{2}$ | 4 | 1,000,000 | 0.72260 | ? |
| Champernowne $C_{10}$ | 10 | 1,000,000 | 59.91143 | Yes |

# Contents

# Number of points visited

For a 2D lattice

- The expected number of distinct points visited by an $N$-step random walk on a two-dimensional lattice behaves for large $N$ like $\pi N/\log(N)$ (Dvoretzky–Erdős, **1951**).

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$$\left(\frac{\pi(N+0.84)}{1.16\pi-1-\log 2+\log(N+2)}, \frac{\pi(N+1)}{1.066\pi-1-\log 2+\log(N+1)}\right).$$

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- For example, for $N=10^6$ these bounds are $(199256.1, 203059.5)$, while $\pi N/\log(N)=227396$, which overestimates the expectation.

# Catalan's constant

$G = 1 + 1/4 + 1/9 + 1/16 + \cdots$

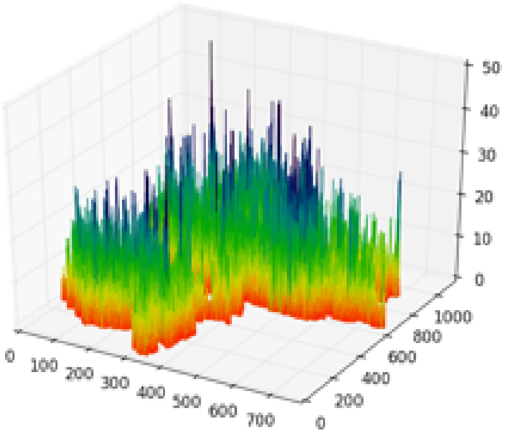
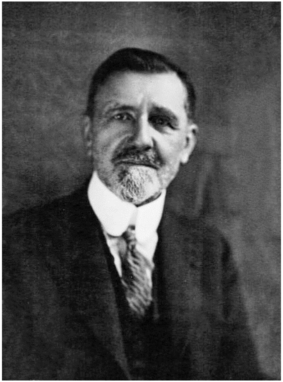



Figure : A walk on one million quad-bits of $G$ with height showing frequency

# Paul Erdős (1913-1996) "My brain is open"

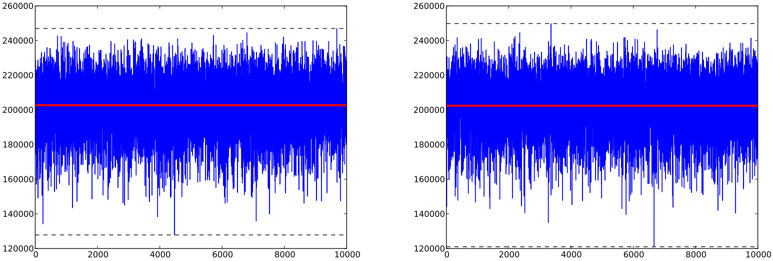
(a) Paul Erdős (Banff 1981. I was there)



(b) Émile Borel (1871–1956)

Figure : Two of my favourites. Consult MacTutor.

# Number of points visited: Again $\pi$ looks random




(a) (Pseudo)random walks.

(b) Walks built by chopping up 10 billion digits of $\pi$.

Figure : Number of points visited by $10,000$ million-steps base-4 walks.

# Points visited by various base-4 walks

| Number | Steps | Sites visited | Bounds on the expectation of sites visited by a random walk | |
|---|---|---|---|---|
| | | | Lower bound | Upper bound |
| Mean of 10,000 random walks | 1,000,000 | **202,684** | 199,256 | 203,060 |
| Mean of 10,000 walks on the digits of $\pi$ | 1,000,000 | **202,385** | 199,256 | 203,060 |
| $\alpha_{2,3}$ | 1,000,000 | 95,817 | 199,256 | 203,060 |
| $\alpha_{3,2}$ | 1,000,000 | 195,585 | 199,256 | 203,060 |
| $\pi$ | 1,000,000 | **204,148** | 199,256 | 203,060 |
| $\pi$ | 10,000,000 | 1,933,903 | 1,738,645 | 1,767,533 |
| $\pi$ | 100,000,000 | 16,109,429 | 15,421,296 | 15,648,132 |
| $\pi$ | 1,000,000,000 | 138,107,050 | 138,552,612 | 140,380,926 |
| $e$ | 1,000,000 | 176,350 | 199,256 | 203,060 |
| $\sqrt{2}$ | 1,000,000 | **200,733** | 199,256 | 203,060 |
| $\log 2$ | 1,000,000 | **214,508** | 199,256 | 203,060 |
| Champernowne $C_4$ | 1,000,000 | 548,746 | 199,256 | 203,060 |
| Rational number $Q_1$ | 1,000,000 | 378 | 199,256 | 203,060 |
| Rational number $Q_2$ | 1,000,000 | 939,322 | 199,256 | 203,060 |

# Normal numbers need not be so "random" ...

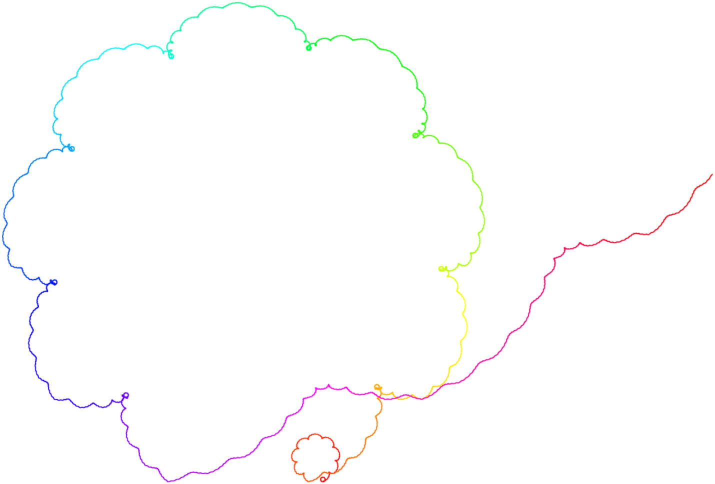


Figure : Champernowne $C_{10} = 0.123456789101112\ldots$ (normal).
Normalized distance to the origin: 15.9 (50,000 steps).

# Normal numbers need not be so "random" ...

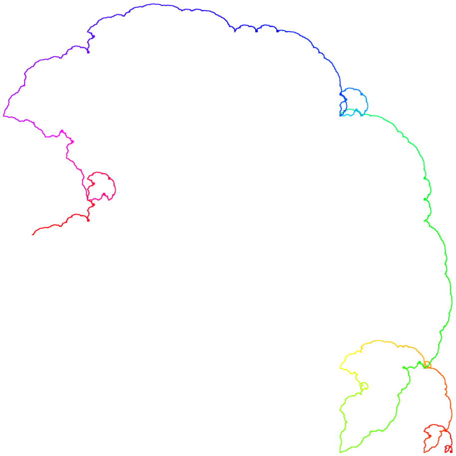


Figure : Champernowne $C_4 = 0.123101112132021\ldots$ (normal).
Normalized distance to the origin: 18.1 (100,000 steps).
Points visited: 52760. Expectation: (23333, 23857).

# Normal numbers need not be so "random" ...

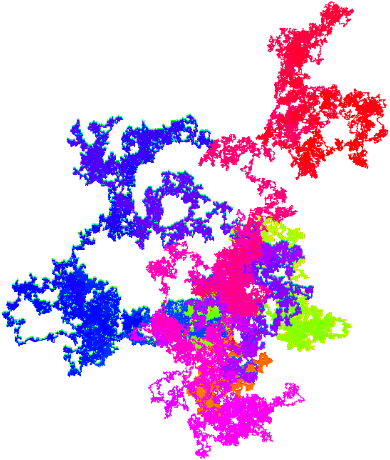


Figure : Stoneham $\alpha_{2,3} = 0.0022232032\ldots_4$ (normal base 4).
Normalized distance to the origin: 0.26 (1,000,000 steps).
Points visited: 95817. Expectation: (199256, 203060).

# Normal numbers need not be so "random" ...

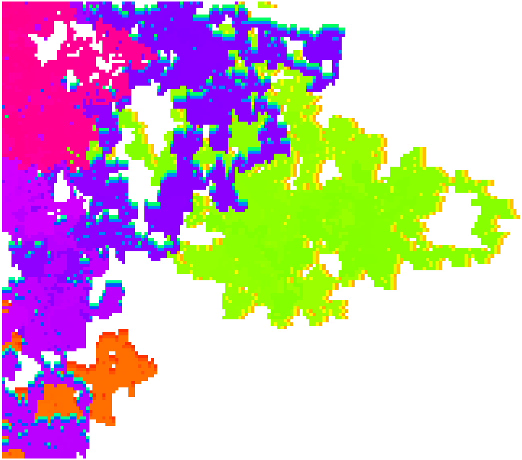
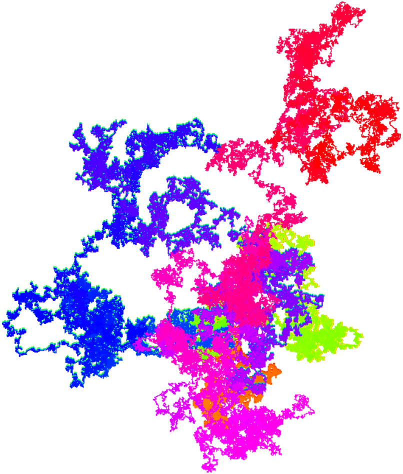


Figure : Stoneham $\alpha_{2,3} = 0.0022232032\ldots_4$ (normal base 4).
Normalized distance to the origin: 0.26 (1,000,000 steps).
Points visited: 95817. Expectation: (199256, 203060).

# $\alpha_{2,3}$ is 4-normal but not so "random" ANIMATION



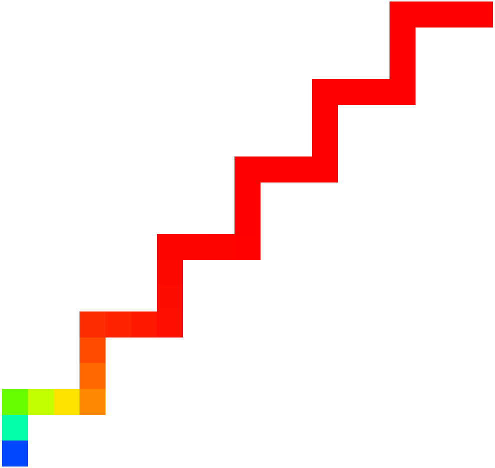


Figure : A pattern in the digits of $\alpha_{2,3}$ base 4. We show only positions of the walk after $\frac{3}{2}(3^n+1), \frac{3}{2}(3^n+1)+3^n$ and $\frac{3}{2}(3^n+1)+2\cdot 3^n$ steps, $n=0,1,\ldots,11$.

# *Experimental* conjecture

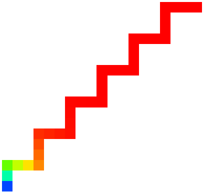
Proven 12-12-12 by Coons

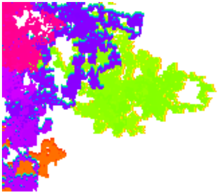
**Theorem (Base-4 structure of Stoneham $\alpha_{2,3}$)**

Denote by $a_k$ the $k^{\text{th}}$ digit of $\alpha_{2,3}$ in its base 4 expansion: $\alpha_{2,3} = \sum_{k=1}^{\infty} a_k/4^k$, with $a_k \in \{0,1,2,3\}$ for all $k$. Then, for all $n = 0,1,2,\ldots$ one has:

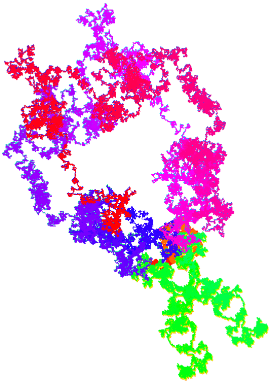
(i) $$\sum_{k=\frac{3}{2}(3^n+1)}^{\frac{3}{2}(3^n+1)+3^n} e^{a_k \pi i/2} = \begin{cases} -i, & \text{n odd} \\ -1, & \text{n even} \end{cases};$$

(ii) $a_k = a_{k+3^n} = a_{k+2\cdot 3^n}$ if $k = \dfrac{3(3^n+1)}{2}, \dfrac{3(3^n+1)}{2}+1, \ldots, \dfrac{3(3^n+1)}{2}+3^n-1$.





# Likewise, $\alpha_{3,5}$ is 3-normal ... but not very "random" ANIMATION



# Contents

# Box-dimension: Tends to ‘2’ for a planar random walk

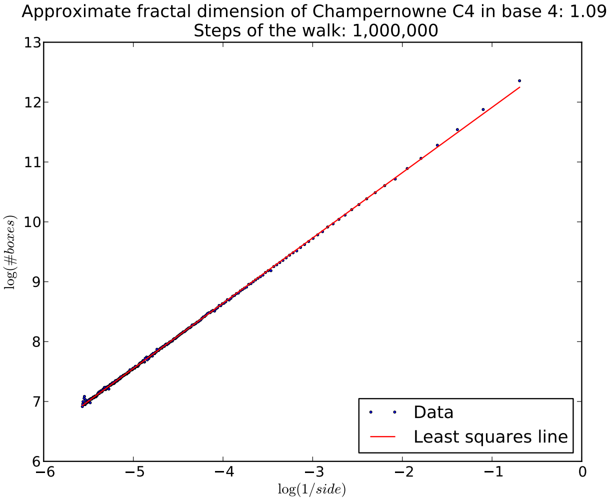
▸ SKIP


$$\text{Box-dimension} = \lim_{\text{side}\to 0} \frac{\log(\#\ \text{boxes})}{\log(1/\text{side})}$$

Norway is “frillier” — *Hitchhiker’s Guide to the Galaxy*

# Box-dimension:

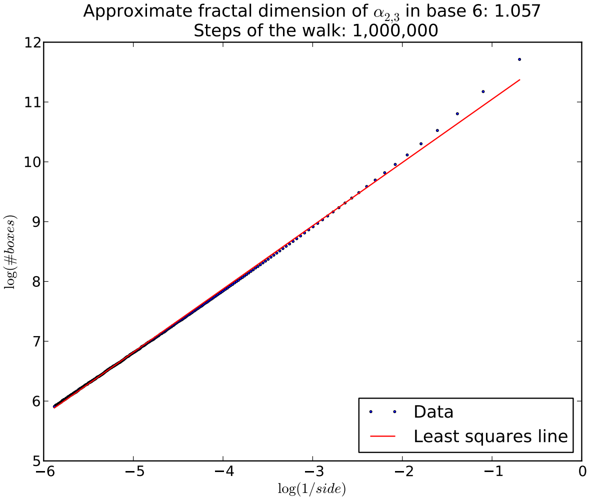
Tends to ‘2’ for a planar random walk ▸ SKIP




Fractals: self-similar (zoom invariant) partly space-filling shapes (clouds & ferns not buildings & cars). Curves have dimension 1, squares dimension 2

# Box-dimension:

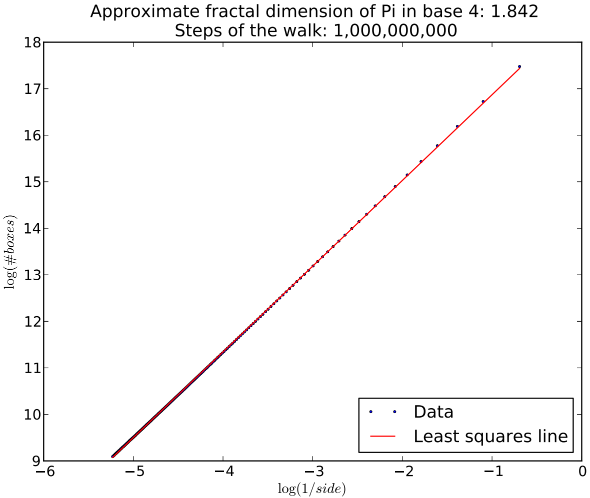
Tends to '2' for a planar random walk ▸ SKIP




Fractals: self-similar (zoom invariant) partly space-filling shapes (clouds & ferns not buildings & cars). Curves have dimension 1, squares dimension 2

# Box-dimension:

Tends to '2' for a planar random walk ‣ SKIP

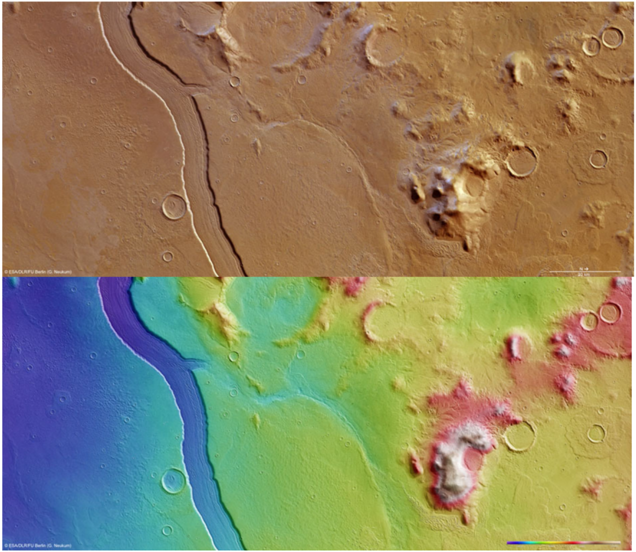



Fractals: self-similar (zoom invariant) partly space-filling shapes (clouds & ferns not buildings & cars). Curves have dimension 1, squares dimension 2

# Contents

# Fractals everywhere — From Mars

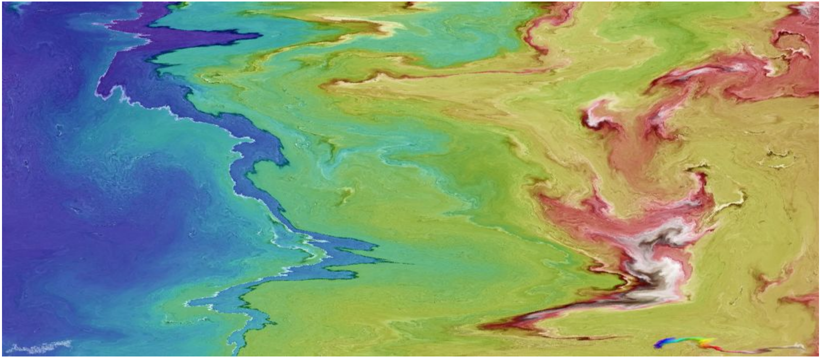
SKIP



# Fractals everywhere

## From Mars

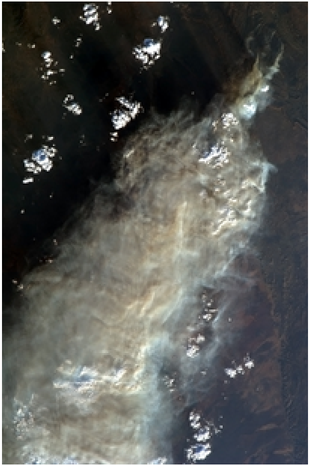
▸ SKIP

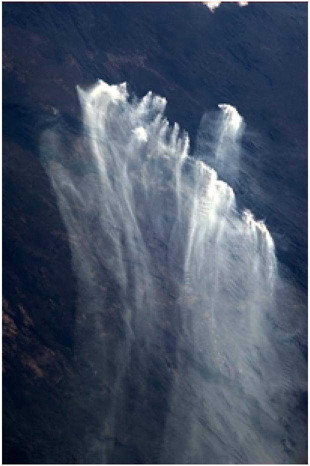


The picture fractalized by the Barnsley's
`http://frangostudio.com/frangocamera.html`

# Fractals everywhere

## From Space





# Fractals everywhere

$1 \mapsto 3$ or $1 \mapsto 8$ or ...

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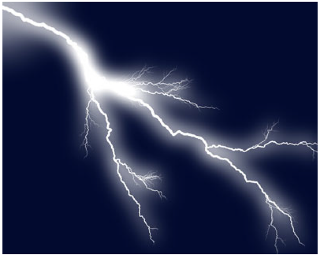
$1 \mapsto 3$ or $1 \mapsto 8$ or ...



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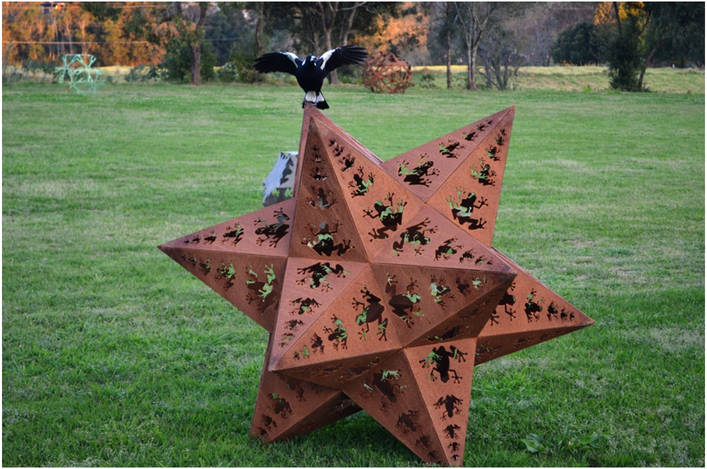


# Fractals everywhere

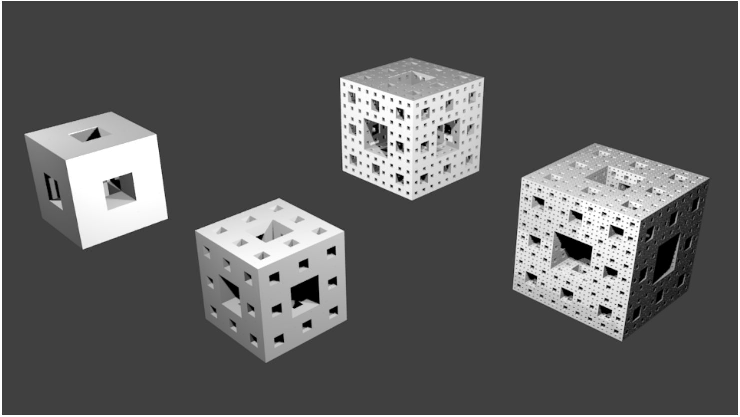
$1 \mapsto 3$ or $1 \mapsto 8$ or ...

Pascal triangle modulo two

[1] [1,1] [1,2,1] [1,3,3,1,] [1,4,6,4,1] [1,510,10,5,1] ...

# Fractals everywhere

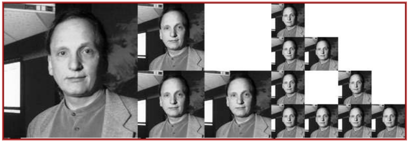
$1 \mapsto 3$ or $1 \mapsto 8$ or ...



Steps to construction of a Sierpinski cube

# Fractals everywhere The Sierpinski Triangle

$1 \mapsto 3 \mapsto 9$



# Fractals everywhere — The Sierpinski Triangle

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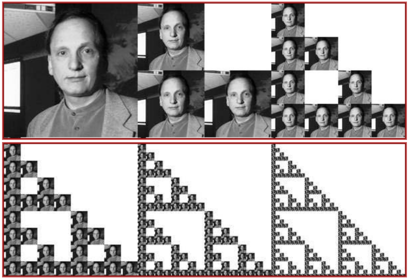
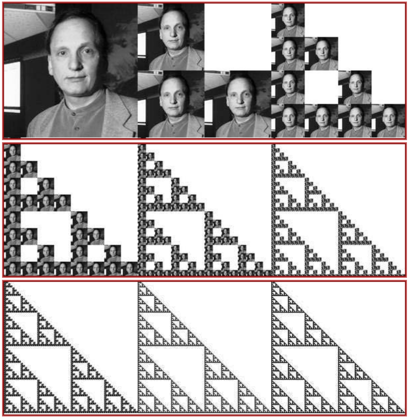


# Fractals everywhere

## The Sierpinski Triangle

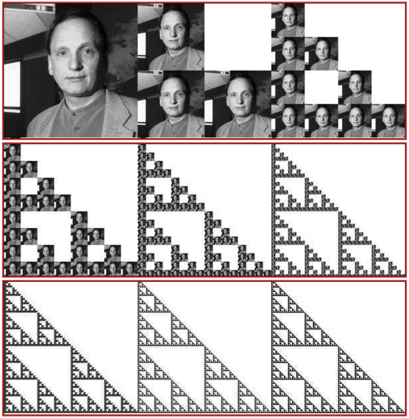
$$1 \mapsto 3 \mapsto 9$$

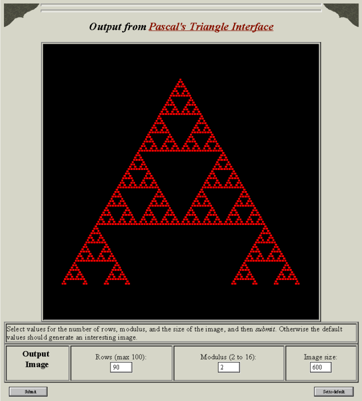


# Fractals everywhere

## The Sierpinski Triangle

$1 \mapsto 3 \mapsto 9$






http://oldweb.cecm.sfu.ca/cgi-bin/organics/pascalform

# Contents

# Three dimensional walks: Using base six — soon on 3D screen

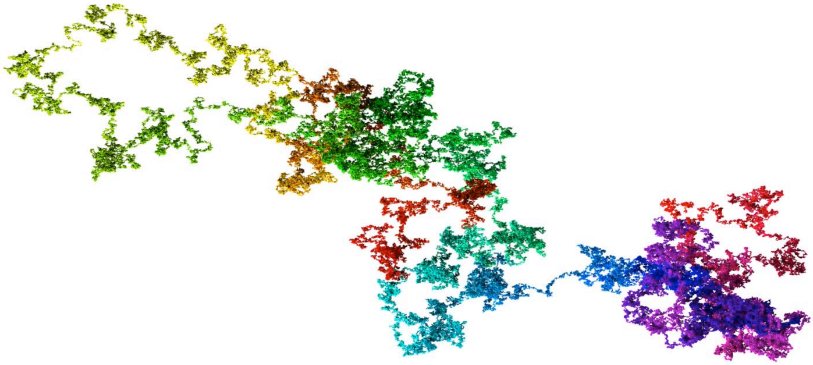


Figure : Matt Skerritt's 3D walk on $\pi$ (base 6), showing one million steps. But 3D random walks are not recurrent.

# Three dimensional walks: Using base six — soon on 3D screen

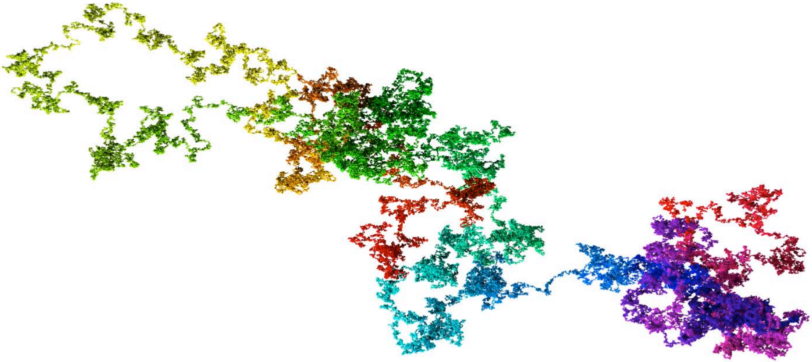
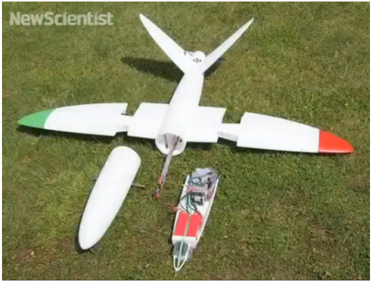


Figure : Matt Skerritt's 3D walk on $\pi$ (base 6), showing one million steps. But 3D random walks are not recurrent.

"A drunken man will find his way home, a drunken bird will get lost forever." (Kakutani)

# Three dimensional printing:

3D everywhere



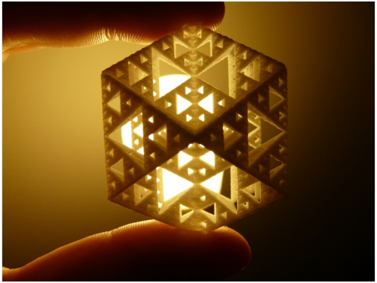



Figure : The future is here ...

www.digitaltrends.com/cool-tech/the-worlds-first-plane-created-entirely-by-3d-printing-takes-flight/

www.shapeways.com/shops/3Dfractals

# Contents

# Chaos games:

Move half-way to a (random) corner

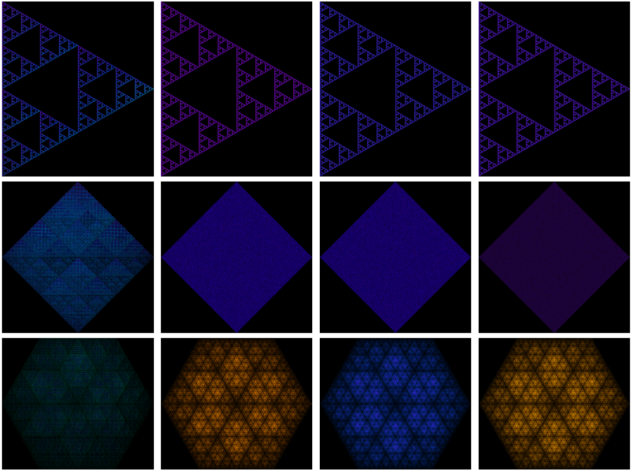


Figure : Coloured by frequency — leads to random fractals.
Row 1: Champernowne $C_3$, $\alpha_{3,5}$, random, $\alpha_{2,3}$. Row 2: Champernowne $C_4$, $\pi$, random, $\alpha_{2,3}$. Row 3: Champernowne $C_6$, $\alpha_{3,2}$, random, $\alpha_{2,3}$.

# Contents

# Automatic numbers: Thue–Morse and Paper-folding

Automatic numbers are never normal. They are given by simple but fascinating rules...giving structured/boring walks:

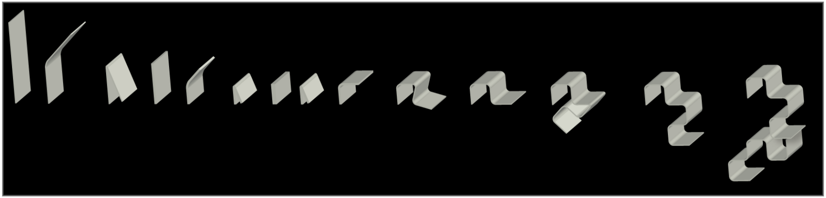


Figure : **Paper folding**. The sequence of left and right folds along a strip of paper that is folded repeatedly in half in the same direction. Unfold and read 'right' as '1' and 'left' as '0': 1 0 1 1 0 0 1 1 1 0 0 1 0 0

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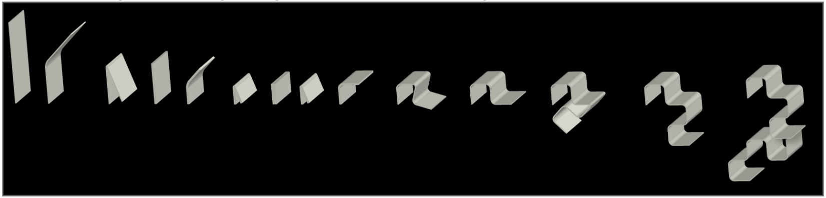

Figure : **Paper folding**. The sequence of left and right folds along a strip of paper that is folded repeatedly in half in the same direction. Unfold and read 'right' as '1' and 'left' as '0': 1 0 1 1 0 0 1 1 1 0 0 1 0 0

**Thue–Morse constant** (transcendental; 2-automatic, hence nonnormal):

$$TM_2 = \sum_{n=1}^{\infty} \frac{1}{2^{t(n)}} \text{ where } t(0) = 0, \text{ while } t(2n) = t(n) \text{ and } t(2n+1) = 1 - t(n)$$

$$0.01101001100101101001011001101001\ldots$$

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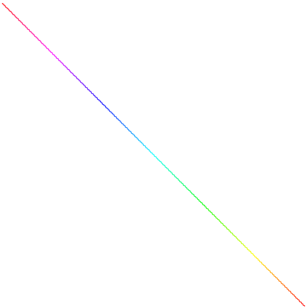
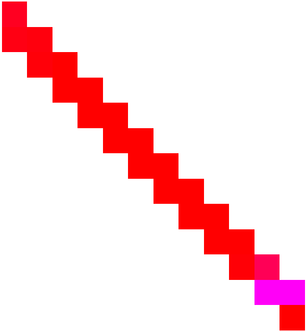


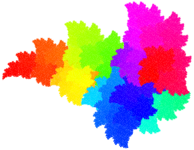
(a) 1,000 bits of Thue–Morse sequence.



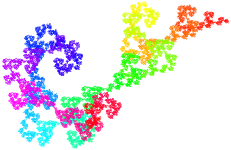
(b) 10 million bits of paper-folding sequence.

Figure : Walks on two automatic and so nonnormal numbers.

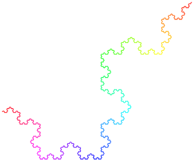
# Automatic numbers: Turtle plots look great!



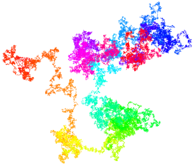
(a) Ten million digits of the paper-folding sequence, rotating $60^\circ$.



(b) One million digits of the paper-folding sequence, rotating $120^\circ$ (a dragon curve).



(c) 100,000 digits of the Thue–Morse sequence, rotating $60^\circ$ (a Koch snowflake).



(d) One million digits of $\pi$, rotating $60^\circ$.

Figure : Turtle plots on various constants with different rotating angles in base 2—where '**0**' yields forward motion and '**1**' rotation by a fixed angle.

# Contents

# Genomes as walks:

We are all base 4 numbers (ACGT/U)

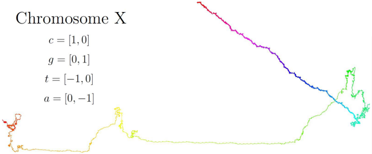
Chromosome X

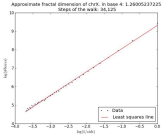
$c = [1, 0]$

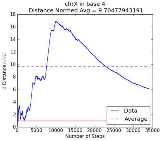
$g = [0, 1]$

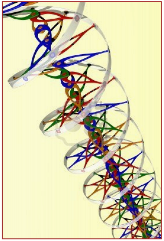
$t = [-1, 0]$

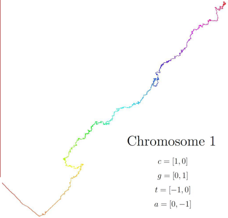
$a = [0, -1]$







Chromosome 1

$c = [1, 0]$

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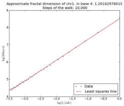
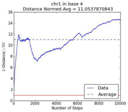
$a = [0, -1]$



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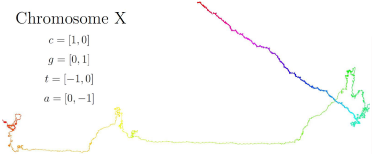

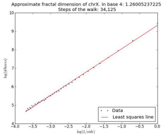


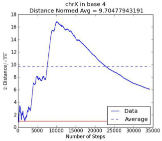


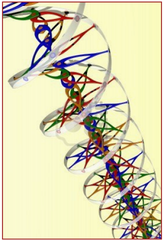

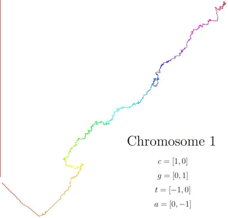

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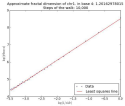
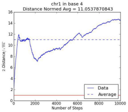
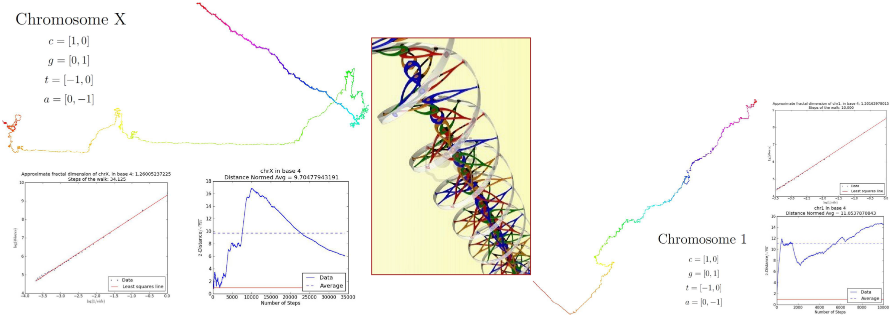



The X Chromosome (34K) and Chromosome One (10K).

# Genomes as walks:

## We are all base 4 numbers (ACGT/U)




The X Chromosome (34K) and Chromosome One (10K).

® Chromosomes look less like $\pi$ and more like concatenation numbers?

# DNA for Storage:

## We are all base 4 numbers (ACGT/U)

News › Science › Biochemistry and molecular biology

### Shakespeare and Martin Luther King demonstrate potential of DNA storage

All 154 Shakespeare sonnets have been spelled out in DNA to demonstrate the vast potential of genetic data storage

**Ian Sample**, science correspondent
The Guardian, Thursday 24 January 2013
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When written in DNA, one of Shakespeare's sonnets weighs 0.3 millionths of a millionth of a gram. Photograph: Oli Scarff/Getty

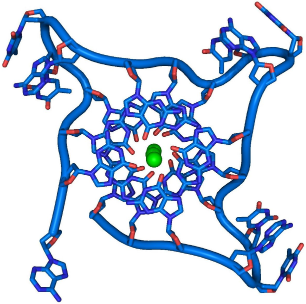

Figure : The potential for DNA storage (L) and the quadruple helix (R)

## The end with some fractal dessert

# The end with some fractal dessert

Thank you

# Contents

# Other References

http://carma.newcastle.edu.au/walks/


- M. Barnsley: *Fractals Everywhere*, Academic Press, Inc., Boston, MA, 1988.
- **F.J. Aragón Artacho, D.H. Bailey, J.M. Borwein, P.B. Borwein: Walking on real numbers, The Mathematical Intelligencer 35 (2013), no. 1, 42–60.**
- D.H. Bailey and J.M. Borwein: Normal numbers and pseudorandom generators, Proceedings of the Workshop on Computational and Analytical Mathematics in Honour of JMB's 60th Birthday. Springer Proceedings in Mathematics and Statistics **50**, pp. 1–18.
- D.H. Bailey and R.E. Crandall: Random generators and normal numbers, Experimental Mathematics 11 (2002), no. 4, 527–546.
- D.G. Champernowne: The construction of decimals normal in the scale of ten, Journal of the London Mathematical Society 8 (1933), 254–260.
- A.H. Copeland and P. Erdős: Note on normal numbers, Bulletin of the American Mathematical Society 52 (1946), 857–860.
- D.Y. Downham and S.B. Fotopoulos: The transient behaviour of the simple random walk in the plane, J. Appl. Probab. 25 (1988), no. 1, 58–69.
- A. Dvoretzky and P. Erdős: Some problems on random walk in space, Proceedings of the 2nd Berkeley Symposium on Mathematical Statistics and Probability (1951), 353–367.
- G. Marsaglia: On the randomness of pi and other decimal expansions, preprint (2010).
- R. Stoneham: On absolute $(j,\varepsilon)$-normality in the rational fractions with applications to normal numbers, Acta Arithmetica 22 (1973), 277–286.